PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

ECHAHID CHEIKH LARBI TEBESSI UNIVERSITY

FACULTY OF SCIENCES AND TECHNOLOGY

FIELD OF TRAINING: SCIENCES AND TECHNOLOGY (ST)

Course

Mathematics 4

Matter : Mathematics 4
Speciality : Common trunk science and technology
Level : Second year S4

Realized by : Dr. Benseghir Roumaissa

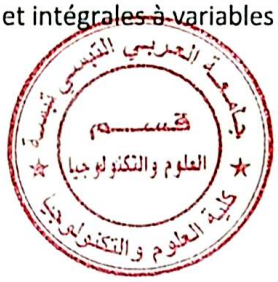
College year: 2025/2026

Contents

Subject	Page
Course Syllabus	
Introduction.	1
Chapter 1: Complex Numbers.	3
1.1 Fundamental Concepts and Graphical Representation.	4
1.2 Polar and Exponential Forms of Complex Numbers.	10
1.3 Euler's Formula and De-Moivre's Theorem.	11
Chapter 2: Holomorphic Functions and Cauchy Riemann Equations.	18
2.1 Complex-valued Functions	19
2.2 Holomorphic Functions.	21
2.3 Cauchy Riemann Equations.	23
2.4 Harmonic Functions.	27
Chapter 3: Power Series.	32
3.1 Sequences	33
3.2 Power Series.	35
3.3 Power Series Expansions: Taylor Series and Laurent Series.	37
Chapter 4: Cauchy Theory.	45
3.1 Cauchy's Integral Theorem and Cauchy's Integral Formula.	46
3.1 Singularities.	52
3.1 Complex Integral.	54
Chapter 5: Applications.	58
5.1 Equivalence between Holomorphy and Analycity	59
5.2 Some Fundamental Theorems.	60
5.3 Cauchy's Residue Theorems.	61
Chapter 6: Special Functions.	67
6.1 Gamma Function.	68
6.2 Beta Function	69
6.2 Application to integral	70
Solved Exercises.	73
Bibliographical References	84

Semestre: 4
Unité d'enseignement: UEF2.2.1
Matière 1: Mathématique 4
VHS: 45h00 (Cours: 1h30,TD : 1h30**)**
Crédits: 4
Coefficient:2

Objectifs de l'enseignement :
Ce cours porte sur le calcul différentiel et intégral des fonctions complexes d'une variable complexe. L'étudiant doit maîtriser les différentes techniques de résolution des fonctions et intégrales à variables complexes et spéciales.



Connaissances préalables recommandées :
Mathématiques 1, Mathématiques 2 et Mathématiques 3.

Contenu de la matière :
Fonctions à variables complexes et Fonctions Spéciales

Chapitre 1 : Fonctions holomorphes. Conditions de Cauchy Riemann 3 semaines

Chapitre 2 : Séries entières 3 semaines
Rayon de convergence. Domaine de convergence. Développement en séries entières. Fonctions Analytiques. Séries de Laurent et développement en séries de Laurent

Chapitre 3 : Théorie de Cauchy 3 semaines
Théorème de Cauchy ; Formules de Cauchy. Point singulier de fonctions, méthode générale de calcul des intégrales complexes

Chapitre 4 : Applications 4 semaines
Equivalence entre holomorphie et Analyticité. Théorème du Maximum. Théorème de Liouville. Théorème de Rouché. Théorème des Résidus. Calcul d'intégrales par la méthode des Résidus.

Chapitre 5 : Fonctions Spéciales 2 semaines
Fonctions spéciales d'Euler : fonctions Gamma, Béta, applications aux calculs d'intégrales

Mode d'évaluation :
Contrôle continu : 40%; Examen: 60%.

Références bibliographiques:
1- Henri Catan, Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes. Editeur Hermann, Paris 1985.
2- Jean Kuntzmann, Variable complexe. Hermann, Paris, 1967.Manuel de premier cycle.
3- Herbert Robbins Richard Courant. What is Mathematics ?, Oxford University Press, Toronto,1978. Ouvrage classique de vulgarisation.
4- Walter Rudin, Analyse réelle et complexe. Masson, Paris, 1975. Manuel de deuxième cycle.

Introduction:

Complex Analysis is an important area of mathematics that focuses on functions of complex variables and their unique properties. At first, it may seem like just an extension of real analysis, but it quickly reveals a much richer structure and more powerful results. What makes this field special is not only its theoretical depth, but also its wide range of applications in science and engineering.

We usually start with complex numbers, learning how to represent them and perform basic operations. Moving to polar and exponential forms helps us understand them in a more intuitive way. Concepts like Euler's formula and De-Moivre's theorem are especially useful, as they connect different areas of mathematics and make many calculations simpler.

As we go further, we study holomorphic functions and the Cauchy–Riemann equations, which are key to understanding differentiability in the complex plane. These functions behave in a very "well-organized" way, which allows us to obtain strong and elegant results. Closely related to this are harmonic functions, which appear naturally in many physical problems such as heat transfer and fluid flow.

Another important part of the subject is power series, including Taylor and Laurent series. These tools allow us to express complex functions as infinite sums, which makes it easier to study their behaviour, especially near certain points. They are also very useful in approximation methods used in engineering.

Cauchy's theory is one of the most powerful parts of Complex Analysis. It provides fundamental results that help us evaluate complex integrals and understand the behaviour of functions in a deeper way. Building on this, the residue theorem gives us a very efficient method for solving integrals that would otherwise be very difficult using real-variable techniques.

We also encounter special functions like the Gamma and Beta functions, which extend familiar concepts and appear in many applications, especially in probability and advanced mathematics.

In practice, Complex Analysis is not just a theoretical subject, but a very useful tool in both mechanical and civil engineering. Engineers often use it to better understand real-world problems such as fluid flow, stress distribution, and structural behaviour. For example, it can help simplify complicated geometries, making calculations easier and more efficient. Whether it is analysing airflow around an object, studying how forces are distributed in a structure, or modelling water flow through soil, complex analysis provides clear and effective solutions. This is why it remains an essential part of the mathematical background for many engineering fields.

Chapter 1

Complex Numbers

The main goal of this chapter is to introduce complex numbers, their modulus ,conjugates and operations such as addition, subtraction, multiplication, and division, as well as their polar and exponential forms. Closely related to this is Euler's formula which establishes a deep connection between exponential and trigonometric functions.

Finally, we conclude this chapter by presenting De Moivre's theorem which constitutes a fundamental principle in the study of complex numbers. These fundamental concepts play a big role to understand the upcoming chapters.

1.1 Fundamental Concepts and Graphical Representation.

1.1.1 Fundamental concepts:

Definition 1: A complex number is defined by: $z = x + iy$, where x and y are real numbers, and i is an imaginary number such that $i^2 = -1$.

- x is called the real part of z, and it is denoted by $Re(z)$;
- y is called the imaginary part of z and it is denoted by $Im(z)$;

Notes:

- If z is purely real, then $Im(z) = 0$;
- If z is purely imaginary, then $Re(z) = 0$.

Example:

Determine the real part and the imaginary part of the following complex numbers:

1. $z = 2 + 8i$.

2. $z = 1 - 5i$.

3. $z = 4i$.

4. $z = 7$.

5. $z = -10 + i$.

Solution:

Statement	Complex number	Real Part	Imaginary Part
1	$z = 2 + 8i$	$Re(z) = 2$	$Im(z) = 8$
2	$z = 1 - 5i$	$Re(z) = 1$	$Im(z) = -5$
3	$z = 4i$	$Re(z) = 0$	$Im(z) = 4$
4	$z = 7$	$Re(z) = 7$	$Im(z) = 0$
5	$z = -10 + i$	$Re(z) = 7$	$Im(z) = 1$
6	$z = -3 - 6i$	$Re(z) = -3$	$Im(z) = -6$

Definition 2: The complex conjugate of a complex number is obtained by changing the imaginary part, for example if $z = x + iy$, then its complex conjugate, $\bar{z}$, is defined by $\bar{z} = x - iy$.

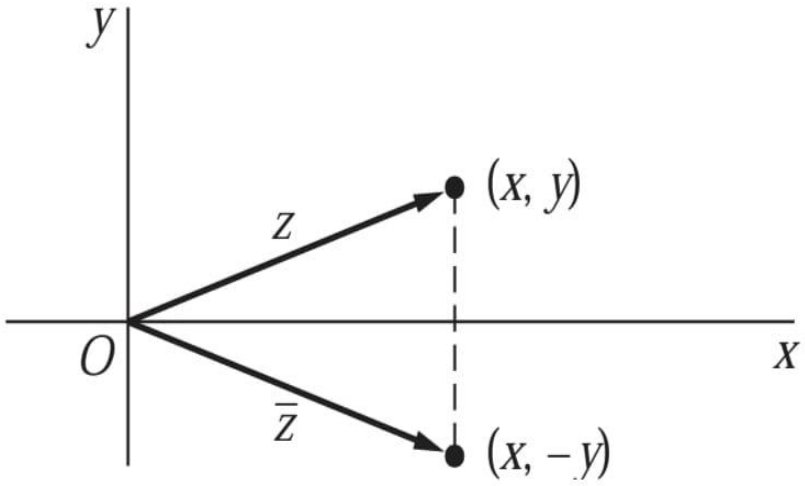


Figure 1: Graphical representation of a complex number and its conjugate

Properties: Let z, w be two complex numbers, then:

1. $\bar{\bar{z}} = z$

2. $\overline{z + w} = \bar{z} + \bar{w}$

3. $\overline{z - w} = \bar{z} - \overline{w}$

4. $\overline{z.w} = \bar{z}.\overline{w}$

5. $\overline{z^n} = \bar{z}^n$

6. $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

7. $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$

8. $z + \bar{z} = 2\,Re(z) \Rightarrow Re(z) = \frac{z+\bar{z}}{2}$

9. $z - \bar{z} = 2i\,Im(z) \Rightarrow Im(z) = \frac{z-\bar{z}}{2i}$

10. $z = \bar{z} \Leftrightarrow z \in \mathbb{R}$ (z is purely real).

11. $z = -\bar{z} \Leftrightarrow z \in i\mathbb{R}$ (z is purely imaginary).

Definition 3: The modulus of a complex number $z = x + iy$ is given by: $|z| = \sqrt{x^2 + y^2}$.

It represents the distance of z from the origin in the complex plane.

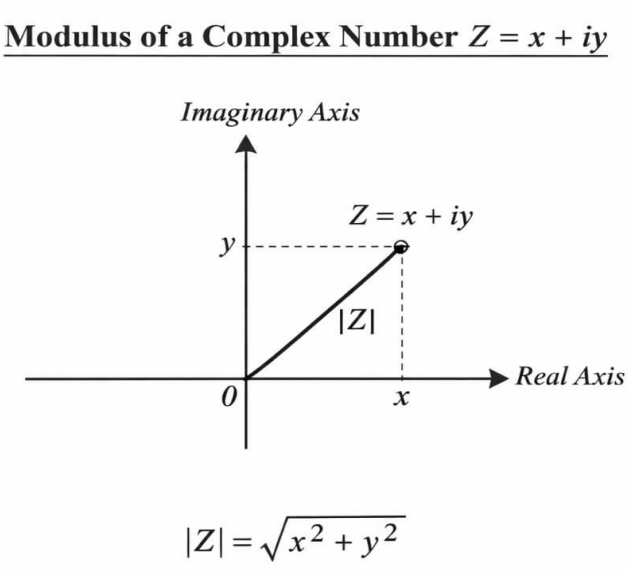


Figure 2: Graphical representation of modulus of a complex number

Properties:

1. $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$

2. $z.\bar{z} = \bar{z}.z = |z|^2 = x^2 + y^2$

3. $|z| = |\bar{z}| = |-z|$

4. $|z.w| = |z|.|w|$

5. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$ if $w \neq 0$.

6. $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ if $z \neq 0$.

7. $|z^n| = |z|^n, n \in \mathbb{Z}$.

8. $|z + w| \leq |z| + |w|$ (Triangle inequality).

9. $|z| - |w| \leq |z \pm w| \leq |z| + |w|$.

Remark: The set of complex numbers is defined as:

$$\mathbb{C} = \{(x, y) \in \mathbb{R}^2 : z = x + iy\}.$$

Since is isomorphic to $\mathbb{R}^2$, every complex number can be uniquely represented as a point in the plane $\mathbb{R}^2$.

1.1.2 Graphical representation of complex numbers:

Every complex number has a location on the complex plane, which is a two-dimensional plane obtained by setting the real numbers on the horizontal axis, and the purely imaginary numbers on the vertical axis.

A complex number $z = x + iy$ is represented by the point (x, y) in this plane, where x is the real part and y is the imaginary part.

This representation allows complex numbers to be visualized geometrically, making it easier to interpret their properties. In particular, the position of a complex number can also be viewed as a vector from the origin to the point (x, y).

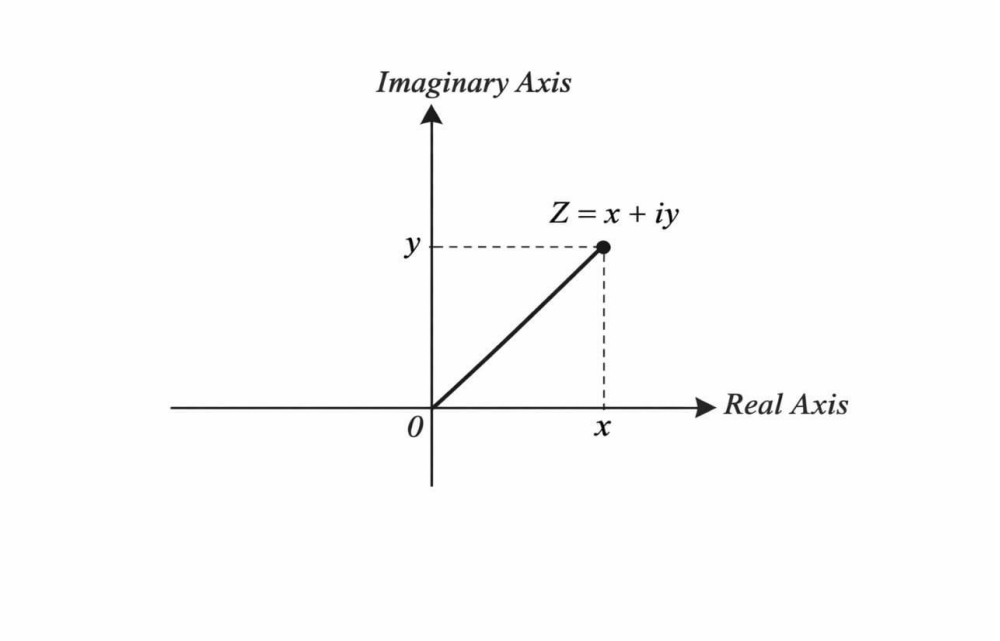


Figure 3: Graphical representation of a complex number $Z = x + iy$

Example:

Represent the following complex numbers in the complex plane:

1. $z = -5 + 3i$

2. $z = -1 + 4i$

3. $z = 1 + i.$

Solution:

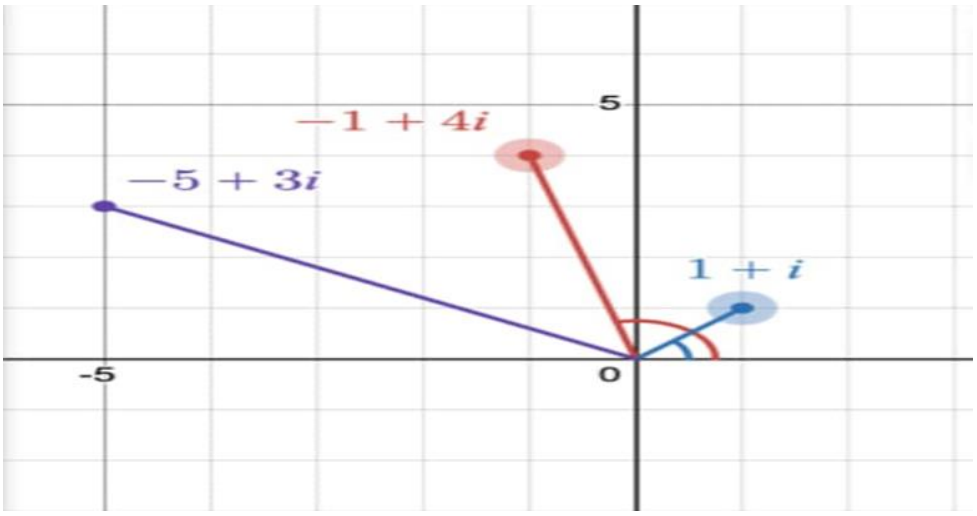


Figure 4: Graphical representation of given complex numbers in example

1.1.3 Basic operations of complex numbers:

Let $z = x_1 + iy_1$ and $w = x_2 + iy_2$ be two complex numbers, then we can have the following operations:

- **Equality:**

  $z = w$ if and only if $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

- **Addition:**

$$z + w = (x_1 + iy_1) + (x_2 + iy_2) = x_1 + x_2 + i(y_1 + y_2) = x_2 + x_1 + i(y_2 + y_1)$$
$$= (x_2 + iy_2) + (x_1 + iy_1) = w + z.$$

Note: If $w = 0$, then $z + 0 = 0 + z = z$.

- **Subtraction:**

$$z - w = (x_1 + iy_1) - (x_2 + iy_2) = x_1 + iy_1 - x_2 - iy_2 = x_1 - x_2 + i(y_1 - y_2).$$

- **Multiplication:**

$$z \times w = (x_1 + iy_1) \times (x_2 + iy_2) \quad = x_1x_2 + iy_2x_1 + iy_1x_2 + iy_1iy_2$$
$$= x_1x_2 - y_1y_2 + i(y_2x_1 + y_1x_2).$$

Note: $z \times w = w \times z$.

- **Division:**

  If $w \neq 0$ ($x_2 \neq 0$ and $y_2 \neq 0$).

$$\frac{z}{w} = \frac{z \times \overline{w}}{w \times \overline{w}} = \frac{(x_1 + iy_1) \times (x_2 - iy_2)}{(x_2 + iy_2) \times (x_2 - iy_2)} = \frac{x_1x_2 - ix_1y_2 + ix_2y_1 + y_1y_2}{x_2^2 + y_2^2}$$

$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2}\, i.$$

1.2 Polar and Exponential Forms of Complex Numbers.

1.2.1 Polar form of a complex number

Definition: The polar form of a complex number is $z = r(\cos(\theta) + i\sin(\theta))$,

Where $r = |z|$ and θ called the argument of $z =$ within $\cos(\theta) = \frac{x}{r}$ and $\sin(\theta) = \frac{y}{r}$.

Notes:

1. $-\pi < principle\ Arg(z) \leq \pi$

2. $\theta = principle\ Arg(z) \pm 2k\pi, k = 0,1,2, \ldots$

1.2.2 Exponential form of a complex number

Definition: The exponential form of a complex number is $z = re^{i\theta}$.

Note:

$\boldsymbol{\theta = Arg\ (z)}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
	0°	30°	45°	60°	90°	120°	135°	150°	180°
$\mathbf{Sin(\theta)}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\mathbf{Cos(\theta)}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1

Table 1: Trigonometric values ($\mathrm{Sin}(\theta)$ *and* $\mathrm{Cos}(\theta)$) for selected arguments $\boldsymbol{\theta} \in [\mathbf{0}, \boldsymbol{\pi}]$

Properties:

1. $Arg(z^n) = n\ Arg(z), n \in \mathbb{Z}$;

2. $Arg(z \times w) = Arg(z) + Arg(w) \pm 2k\pi;\ \ k = 0,1,2, \ldots$

3. If $w \neq 0$, then $Arg\left(\frac{z}{w}\right) = Arg(z) - Arg(w) \pm 2k\pi;\ \ k = 0,1,2, \ldots$

Example:

Find polar and exponential form of the following complex numbers:

1. $z = 4i$;

2. $z = 1 + i$.

Solution:

1. We have

$$\theta = \frac{\pi}{2} \pm 2k\pi, k = 0,1,2, \ldots \quad \text{and} \quad r = |z| = 4$$

$$\Rightarrow \ z = 4\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right) \quad \text{(Polar form)}$$

$$\Rightarrow \ z = 4e^{i\frac{\pi}{2}} \quad \text{(Exponential form)}$$

2. We have

$$\theta = \frac{\pi}{4} \pm 2k\pi, k = 0,1,2, \ldots \quad \text{and} \quad r = |z| = \sqrt{2}$$

$$\Rightarrow \ z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \quad \text{Polar form}$$

$$\Rightarrow \ z = \sqrt{2}e^{i\frac{\pi}{4}} \quad \text{Exponential form}$$

1.3 Euler's Formula and De-Moivre's Theorem.

1.3.1 Euler's formula:

Euler's Formula is a fundamental mathematical equation connecting complex exponential functions to trigonometry functions. It represents a point on the unit circle in the complex plane, where e is the natural log base, i is the imaginary unit, and is the angle measured in radians.

Euler's formula, states that for any real number x:

$$e^{ix} = \cos(x) + i\sin(x),$$

where i is the imaginary unit.

The Euler's formula can be proved using a series expansion:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n (x)^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x).$$

Note: $e^{-ix} = \cos(x) - i\sin(x)$.

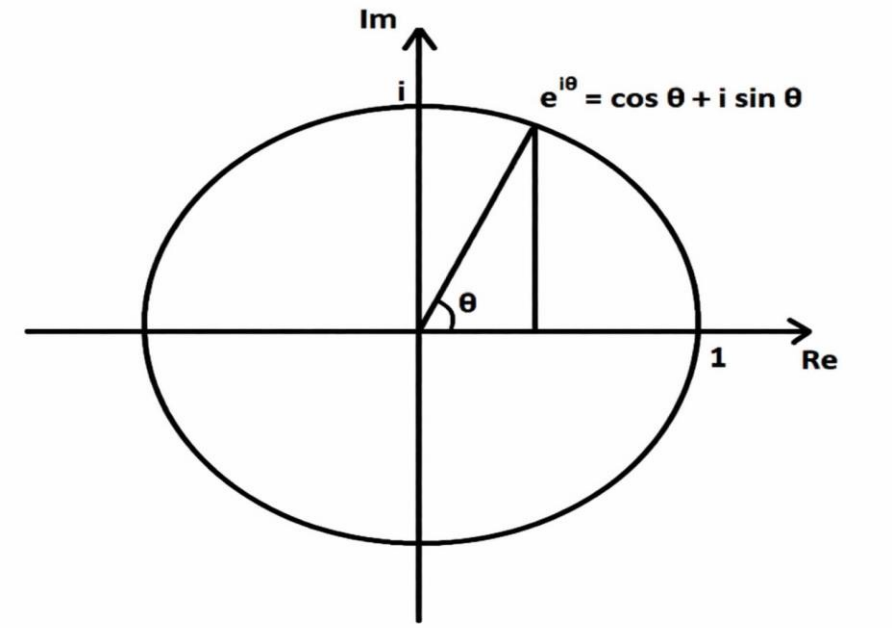


Figure 5: Geometric interpretation of Euler's formula on the unit circle in the complex plane

Properties: For any real number x and for any integer n:

1. $\cos(x) = \dfrac{e^{ix} + e^{-ix}}{2}$;

2. $\sin(x) = \dfrac{e^{ix} - e^{-ix}}{2i}$;

3. $\cos(nx) = \dfrac{e^{inx} + e^{-inx}}{2}$;

4. $\sin(nx) = \dfrac{e^{inx} - e^{-inx}}{2i}$.

Example 01:

Evaluate the following:

1. $e^{i\pi}$.

2. $e^{i2\pi}$.

Solution 01:

By Euler's formula, we get

1. $e^{i\pi} = \cos(\pi) + i\sin(\pi) = -1 + i(0) = -1.$

2. $e^{i2\pi} = \cos(2\pi) + i\sin(2\pi) = 1 + i(0) = 1.$

Example 02:

Using Euler's formula to prove that:

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

Solution 02:

We have:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \dots \dots (1)$$

$$e^{-i\theta} = \cos(\theta) - i\sin(\theta) \dots \dots (2)$$

By addition, we get:

$$\cos(\theta) + i\sin(\theta) + \cos(\theta) - i\sin(\theta) = e^{i\theta} + e^{-i\theta}$$

$$\Rightarrow 2\cos(\theta) = e^{i\theta} + e^{-i\theta}$$

$$\Rightarrow \cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

1.3.2 De-Moivre's theorem:

Theorem: Let $z = r(\cos(\theta) + i\sin(\theta))$, and let n be a positive integer. Then

$$z^n = \left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos(n\theta) + i\sin(n\theta)), n = 1, 2, 3, \dots$$

Proof: If $z = r(\cos(\theta) + i\sin(\theta))$, then

$$z^n = \left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos(\theta) + i\sin(\theta))^n = r^n\left(e^{i\theta}\right)^n = r^n e^{in\theta}$$

$$= r^n(\cos(n\theta) + i\sin(n\theta)), n = 1, 2, 3, \dots$$

Since, $\cos(\theta) + i\sin(\theta) = e^{i\theta}$ (Euler's formula).

Example:

Express $(1-i)^6$ in the form $x+iy$.

Solution:

Step 1: Change to polar form:

$$1-i=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i.\sin\left(-\frac{\pi}{4}\right)\right)=\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)-i.\sin\left(\frac{\pi}{4}\right)\right)$$

$$\Rightarrow\ (1-i)^6=\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)-i.\sin\left(\frac{\pi}{4}\right)\right)\right)^6;$$

Step 2: Using De-Moivre's theorem:

$$(1-i)^6=\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)-i.\sin\left(\frac{\pi}{4}\right)\right)\right)^6=8\left(\cos\left(\frac{6\pi}{4}\right)-i.\sin\left(\frac{6\pi}{4}\right)\right)$$

$$=8\left(\cos\left(\frac{3\pi}{2}\right)-i.\sin\left(\frac{3\pi}{2}\right)\right)=8\big(0-i(-1)\big)=8i.$$

Note: $\cos(-\theta)=\cos(\theta)$ and $\sin(\theta)=-\sin(\theta)$.

Exercises:

Exercise 01: Compute the following:

1. i^3;

2. i^4;

3. i^5;

4. i^6;

5. i^{98};

6. i^{99};

7. i^{598};

8. i^{999}.

Exercise 02: Simplify the following complex numbers:

1. $\dfrac{i}{\sqrt{2}+i}$;

2. $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$;

3. $\dfrac{4+3i}{2-i}$.

Exercise 03: Solve the following equations with complex numbers:

1. $8+15i \;=\; 2x+3yi$;

2. $12+24i \;=\; 6x+6yi$.

Exercise 04: Solve the following equations:

1. $z^2-10z-29=0$;

2. $z^2+1=0$;

3. $z^2-(3+2i)z+5+i=0$.

Exercise 05: Using Euler's formula to prove that:

$$\cos(x) = \frac{e^{x} + e^{-x}}{2}$$

Exercise 06: Let $z = 2 - i$ and $w = 1 + 3i$

1. Write the following complex numbers in the form $x + iy$,

a) $\bar{z}$ and $\bar{w}$;

b) $z - w, z + w$ and $w - z$;

c) $z.w$;

d) $\frac{z}{w}$;

e) $\frac{z.w}{z + w}$;

f) $\frac{wz}{w - z}$.

2. Calculate:

a) $|z|$ and $|w|$;

b) $|z + w|, |w - z|$ and $|z - w|$.

3. Conclude:

a) $|z.w|$;

b) $\left|\frac{z}{w}\right|$;

c) $|z^2|, |z^8|\ and\ |z^{-1}|$;

d) $|w^2|, |w^3|\ and\ |w^{-1}|$.

Exercise 07: Sketch the following region in the complex plane:

1. $\{z: -1 \leq Re(z) \leq 1\}$;

2. $\{z: 0 \leq Im(z) \leq 2\}$;

3. $\{z: |z - i - 1| = 3\}$;

4. $\{z: 0 \leq Re(z) \leq 1 \text{ and } -1 \leq Im(z) \leq 2\}$

Exercise 08: Calculate

1. $|cos(\theta) - i\cos(\theta)|$;

2. $|2\ cos(\theta) + 2i\cos(\theta)|$.

Exercise 09: Rewrite the following complex numbers in polar and exponential forms:

1. $z = -2i$;

2. $z = 4 + 2i$.

Exercise 10: Using De-Moivre's theorem (De-Moivre's formula), prove that:

$$\begin{cases} \cos(3\theta) = \cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \\ \sin(3\theta) = 3\cos^2(\theta)\sin(\theta) - \sin^3(\theta) \end{cases}$$

Exercise 11: Use De-Moivre's formula to deduce formulas for $\cos(2\theta)$ and $\sin(2\theta)$.

Chapter 2

Holomorphic Functions and Cauchy-Riemann Equations

The main goal of this chapter is to introduce holomorphic (complex differentiable) functions which play a central role in complex analysis. To better understand holomorphicity, we present the Cauchy-Riemann equations (Cauchy-Riemann conditions) which provide necessary and sufficient conditions for a complex-valued function to be holomorphic.

In addition, the chapter introduces harmonic functions, which are closely related to holomorphic functions, and they play an important role in physical phenomena such as heat transfer, fluid flow, and electrostatics.

2.1 Complex-Valued Functions.

The study of complex-valued functions is essential for understanding key concepts such as continuity, differentiability, and analyticity in the complex sense. These ideas lead to powerful results and applications, including contour integration, and residue theory.

2.1.1 Complex-valued functions

Definition: A complex-valued function, $f(z): D \subset \mathbb{C} \to \mathbb{C}$ maps a complex number $z = x + iy$. to another complex number $w = u + iv$.

For example, $f(z) = z: \mathbb{C} \to \mathbb{C}$

2.1.2 Domain of a complex-valued function:

Definition: A domain D of a complex function is the set of all complex numbers for which the function is defined.

i.e $D = \{z \in \mathbb{C}: f(z) \text{ is defined}\}$.

Example 01:

Find the domain of the following complex-valued functions:

1. $f(z) = z^2$;

2. $f(z) = Im(z)$;

3. $f(z) = \dfrac{z+1}{z^2+1}$;

4. $f(z) = \dfrac{1}{z-2i}$.

Solution 01:

Function	Domain
$f(z) = z^2$	$D_f = \mathbb{C}$
$f(z) = Im(z)$	$D_f = \mathbb{C}$
$f(z) = \dfrac{z+1}{z^2+1}$	$D_f = \mathbb{C} \setminus \{-i, i\}$
$f(z) = \dfrac{1}{z-2i}$	$D_f = \mathbb{C} \setminus \{2i\}$

Example 02: Consider the function

$$f(z) = z^3 - 1.$$

1. Find the domain of definition of the function f.

2. Calculate $f(2), f(2i)$.

Solution 02:

1. The domain of definition of the function f is $D_f = \mathbb{C}$.

2.

- $f(2) = 2^3 - 1 = 8 - 1 = 7$.
- $f(2i) = (2i)^3 - 1 = -8i - 1 = -1 - 8i$.

2.1.3 Zeros of a complex-valued function:

Definition: The zeros of a function $f(z)$ are values of the complex variable z such that the values satisfy the equation $f(z) = 0$.

Note: The zeros of a function are also called the roots of a function.

Example 01: Find the zeros of the complex valued-function: $f(z) = z^2 + 1$.

Solution 01: $f(z) = 0 \Rightarrow z^2 + 1 = 0 \Rightarrow z^2 = -1 \Rightarrow z = \pm i$.

Example 02: Find the zeros of the complex valued-function: $f(z) = e^z - 1$.

Solution 02: $f(z) = 0 \Rightarrow e^z - 1 = 0 \Rightarrow e^z = 1 \Rightarrow z = 2k\pi i, k = 0,1,2,\ldots$

because $e^{2k\pi i} = \cos(2k\pi) + i\sin(2k\pi) = 1 + i(0) = 1$.

2.2 Holomorphic Functions.

Definition 1: Let D be an open set in $\mathbb{C}$ and f a complex-valued function on D. The function $f: D \to \mathbb{C}$ is holomorphic at the point $z_0 \in U$ if the quotient

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

converges to a limit when $\Delta z \to 0$. Here $\in \mathbb{C}$ and $\Delta z \neq 0$ with $z_0 + \Delta z \in \mathbb{C}$, so that the quotient is well defined. The limit of the quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}.$$

Definition 2: If a function $f(z)$ has a derivative at z_0 and at every point in some neighborhood of z_0, then $f(z)$ is said to be holomorphic at z_0. A function $f(z)$ is said to holomorphic in a domain D, if it is holomorphic at every point of D.

Notes:

1. The basic differentiation formulas given below can be derived from the previous definition by essentially the same steps as the ones used in calculus. In these formulas, the derivative of a function f at a point z is denoted by either: $\frac{d}{dz}$ or $f'(z)$.

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

2. The function f is said to be holomorphic on $\mathbb{C}$ if it is holomorphic at every point at every point $z \in \mathbb{C}$.

Example 01: Consider the function $f(z) = z^2$, Show that f is holomorphic on $\mathbb{C}$ and find $f'(z)$ (using definition).

Solution 01:

We have: $f(z) = z^2$. Then

$$\begin{aligned}\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z+\Delta z)^2 - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z^2 + \Delta z^2 + 2z.\Delta z - z^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z^2 + 2z.\Delta z}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z(\Delta z + 2z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} (\Delta z + 2z) = 2z, \forall z \in \mathbb{C}\end{aligned}$$

Thus,

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = 2z;$$

So, f is holomorphic on $\mathbb{C}$ and $f'(z) = 2z$.

Example 02: Consider the function $f(z) = \bar{z}$, Is it holomorphic (complex differentiable) on $\mathbb{C}$.

Solution 02:

We have: $z = x + iy \Rightarrow \Delta z = \Delta x + i\Delta y$

$$\begin{aligned}f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}\end{aligned}$$

If $\Delta y \to 0 \Rightarrow f'(z) = 1$

If $\Delta x \to 0 \Rightarrow f'(z) = -1$

So, f is not holomorphic on $\mathbb{C}$.

Properties: Suppose that f and g are holomorphic in U. Let $\mathcal{C}$ be a constant, then

1. $\frac{d}{dz}(\mathcal{C}) = 0$.

2. $f + g$ is holomorphic in U and $(f + g)' = f' + g'$.

3. fg is holomorphic in U and $(fg)' = f'g + fg'$.

4. $(\mathcal{C}.f)' = \mathcal{C}.f'$

5. If $g(z) \neq 0$, then f/g is holomorphic at z and $(f/g)' = \frac{f'g - fg'}{g^2}$

6. If $f: D_1 \to D_2$ is a holomorphic function and $g: D_2 \to \mathbb{C}$, then $\Big(f\big(g(z)\big)\Big)' = f'\big(g(z)\big)g'(z)$.

2.3 Cauchy Riemann Equations (Cauchy Riemann Conditions).

The Cauchy Riemann equations are necessary conditions for a function $f(z)$ to be holomorphic in a domain D.

2.3.1 Cauchy Riemann equations in cartesian form:

The Cauchy Riemann equations for a function $f(z) = u(x, y) + iv(x, y)$ in cartesian form are:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \Leftrightarrow \begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

Where $f(z) = f(x + iy) = u(x, y) + iv(x, y)$.

Moreover,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

2.3.2 Cauchy-Riemann equations in polar form:

The Cauchy-Riemann equations for a function $f(z) = u(r, \theta) + iv(r, \theta)$ in polar form are:

$$\begin{cases} \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}; \\ \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \end{cases}$$

Where $f(z) = f\left(re^{i\theta}\right) = u(r,\theta) + iv(r,\theta)$.

Moreover,

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right).$$

Theorem: A complex-valued function $f(z)$ is said to be holomorphic in a domain D if and if the function has continuous partial derivatives that satisfies Cauchy-Riemann equations for all points in D.

Example 01:

Show that $f(z) = -4z$ is holomorphic on $\mathbb{C}$.

Solution 01:

Cauchy-Riemann equations in cartesian form:

Step 1: Rewrite f in Cartesian form $i.e\ f = u(x,y) + iv(x,y)$

Let $z = x + iy$, then $f(z) = f(x+iy) = -4(x+iy) = -4x - 4iy = -4x + i(-4y)$.

Thus $u(x,y) = -4x$ and $v(x,y) = -4y$

Step 2: Calculate partial derivatives

$$\frac{\partial u}{\partial x} = -4,$$

$$\frac{\partial u}{\partial y} = 0,$$

$$\frac{\partial v}{\partial x} = 0,$$

$$\frac{\partial v}{\partial y} = -4.$$

Step 3: Verify Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = -4; \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} = 0. \end{cases}$$

So**,** Cauchy-Riemann equations are satisfied everywhere and derivatives are continuous.

Therefore, $f(z) = -4z$ is holomorphic on $\mathbb{C}$.

Moreover,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -4 + i(0) = -4$$

Cauchy-Riemann equations in polar form:

Step 1: Rewrite f in polar form $i.e\ f = u(r, y) + iv(r, y)$

Let $z = re^{i\theta}$ then $f(z) = f\left(re^{i\theta}\right) = -4re^{i\theta} = -4r(\cos(\theta) + i\sin(\theta)) = -4r\cos(\theta) + i(-4r\sin(\theta))$ Thus $u(r, \theta) = -4r\cos(\theta)$ and $v(r, \theta) = -4r\sin(\theta)$

Step 2: Calculate partial derivatives

$$\frac{\partial u}{\partial r} = -4\cos(\theta),$$

$$\frac{\partial u}{\partial \theta} = -4r(-\sin(\theta)) = 4r\sin(\theta),$$

$$\frac{\partial v}{\partial r} = -4\sin(\theta),$$

$$\frac{\partial v}{\partial \theta} = -4r\cos(\theta).$$

Step 3: Verify Cauchy-Riemann equations

$$\begin{cases} \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} = -4\cos(\theta); \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} = 4r\sin(\theta). \end{cases}$$

So**,** Cauchy-Riemann equations are satisfied everywhere and derivatives are continuous.

Therefore, $f(z) = -4z$ is holomorphic on $\mathbb{C}$.

Moreover,

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}\big(-4\cos(\theta) + i(-4\sin(\theta))\big)$$

$$= -4e^{-i\theta}(\cos(\theta) + i\sin(\theta)) = -4e^{-i\theta}e^{+i\theta} = -4.$$

Example 02: Show that the function $f(z) = z^2$ is holomorphic on $\mathbb{C}$.

Solution 02: We have

$$f(z) = z^2 \Rightarrow f(x+iy) = (x+iy)^2 = x^2 + (iy)^2 + i2xy$$

$$\Rightarrow f = x^2 - y^2 + i2xy$$

$$\Rightarrow \begin{cases} u(x,y) = x^2 - y^2, \\ v(x,y) = 2xy. \end{cases}$$

Then

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2)$$

$$\Rightarrow \quad u_x = 2x;$$

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(x^2 - y^2)$$

$$\Rightarrow \quad u_y = -2y;$$

$$v_x = \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(2xy)$$

$$\Rightarrow \quad v_x = 2y;$$

$$\text{and} \quad v_y = \frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(2xy)$$

$$\Rightarrow \quad v_y = 2x.$$

u_x, u_y, v_x and v_y are continuous on $\mathbb{C}$.

Now, check the C-R equations: $u_x = 2x = v_y$ and $u_y = -2y = -v_x$

Thus, Cauchy Riemann equations are satisfied for all $(x, y) \in \mathbb{R}^2$.

Hence the given function is holomorphic at every point $z \in \mathbb{C}$.

Moreover,

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = 2x + i(2y) = 2(x + iy) = 2z.$$

2.4 Harmonic Functions.

Definition: A real-valued function $u(x, y)$ is said to be harmonic in a domain D if it obeys Laplace's equation, which means:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Remark: The harmonic conjugate of u will satisfy the Cauchy-Riemann equations and have continuous partials of all orders.

Theorem: A complex-valued function $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is holomorphic in a domain D if and only if v is a harmonic conjugate of u.

Example 01: Check that the function $u(x, y) = x^3 - 3xy^2 - 5y$ is harmonic, then find its conjugate harmonic function $v(x, y)$.

Solution 01:

We have $u(x, y) = x^3 - 3xy^2 - 5y$ then

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3y^2) = 6x$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-6\text{xy} - 5) = -6x$$

Clearly,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0.$$

Therefore, $u(x,y) = x^3 - 3xy^2 - 5y$ is harmonic.

Since $v(x,y)$ is a conjugate harmonic of the given harmonic function $u(x,y)$, then

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -6\text{xy} - 5 \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \\ \frac{\partial v}{\partial x} = -(-6\text{xy} - 5) \end{cases} \Rightarrow \begin{cases} \frac{\partial v}{\partial y} = 3x^2 - 3y^2 \ldots\ldots (1) \\ \frac{\partial v}{\partial x} = 6\text{xy} + 5 \quad \ldots \ldots \quad (2) \end{cases}$$

From equation (1) we have $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$ so by integration we get:

$$v(x,y) = \int (3x^2 - 3y^2)\, dy = 3x^2y - y^3 + K(x)$$

From equation (2), we have $\frac{\partial v}{\partial x} = 6\text{xy} + 5 = \frac{\partial}{\partial x}\big(3x^2y - y^3 + K(x)\big) = 6xy + K'(x) \Rightarrow$

$$K'(x) = 5$$

Thus $K(x) = \int 5\, dx = 5x + C,\ C \in \mathbb{R}$.

Therefore, $v(x,y) = 3x^2y - y^3 + 5x + C,\ C \in \mathbb{R}$.

Example 02: Check that the function $u(x,y) = e^{2x}(x\cos(2y) - y\sin(2y))$ is harmonic.

Solution 02:

We have $u(x,y) = e^{2x}(x\cos(2y) - y\sin(2y))$ then

$$\frac{\partial u}{\partial x} = 2e^{2x}(x\cos(2y) - y\sin(2y)) + \cos(2y)\, e^{2x} = e^{2x}\big((2x+1)\cos(2y) - 2y\sin(2y)\big)$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\Big(e^{2x}\big((2x+1)\cos(2y) - 2y\sin(2y)\big)\Big)$$

$$= e^{2x}\big((4x+4)\cos(2y) - 4y\sin(2y)\big).$$

$$\frac{\partial u}{\partial y} = e^{2x}(-2x\sin(2y) - \sin(2y) - 2y\cos(2y))$$

$$\Rightarrow \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(-2x\sin(2y)) = e^{2x}\big((-4x-4)\cos(2y) + 4y\sin(2y)\big)$$

Thus,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^{2x}\big((-4x-4)\cos(2y)+4y\sin(2y)+(4x+4)\cos(2y)-4y\sin(2y)\big)=0.$$

Therefore, $u(x,y)=e^{2x}(x\cos(2y)-y\sin(2y))$ is harmonic.

Exercises:

Exercise 01:

1. State the Cauchy-Riemann equations in algebraic form?
2. State the Cauchy-Riemann equations in polar form?

Exercise 02:

Using the definition, prove that $f(z) = z^3$ is holomorphic and deduce $f'(z)$.

Exercise 03:

Consider the complex-valued function

$$f(z) = e^z$$

1. Rewrite f as the form $f(z) = u(x,y) + iv(x,y)$;
2. Show that f is holomorphic on $\mathbb{C}$.

Exercise 04:

1. Rewrite the function $f(z) = z^3$ as the form $f(z) = u(x,y) + iv(x,y)$
2. Show that f is holomorphic on $\mathbb{C}$.

Exercise 05:

1. Rewrite the function $f(z) = z^3$ as the form $f(z) = u(r,\theta) + iv(r,\theta)$
2. Show that f is holomorphic on $\mathbb{C}$.

Exercise 06:

1. Rewrite the function $f(z) = z^m, m \in \mathbb{N}$ as the form $f(z) = u(r,\theta) + iv(r,\theta)$
2. Show that f is holomorphic on $\mathbb{C}$.

Exercise 07: Consider the following functions:

1. $u(x,y) = e^x cosy$;

2. $u(x,y) = x^2 - y^2 + 2x$;

3. $u(x,y) = \mathrm{x(y-1)} + y$.

In each case, verify that u is harmonic function.

Exercise 08:

Consider the following functions:

1. $u(x,y) = 3x^2\mathrm{y} + 2x^2 - y^3 - 2y^2$;

2. $u(x,y) = y^3 + 3x^2y$.

In each case, verify that u is harmonic, then find a conjugate harmonic function v (i.e. v such that $u + iv$ is holomorphic).

Exercise 09:

Check that the function $u(x,y) = e^{2x}(x\cos(2y) - y\sin(2y))$ is harmonic, then find its conjugate harmonic function $v(x,y)$.

Exercise 10:

Find the values of α, β and γ for which the function $u = \alpha x^2 + \beta xy + \gamma y^2$ is harmonic

Exercise 11:

Consider the function

$$u(x,y) = x^3 - 3xy^2,$$

Show that u is a harmonic function.

Exercise 12: Consider the complex-valued function $f(z) = z^2 - 2Im(z)$.

Write f in algebraic form ($f = u + iv$).

Chapter 3

Sequences and Power Series

In this chapter, we introduce sequences which help us to analyze infinite processes, define series and develop more advanced topics like power series, then we study power series in details, focusing on their convergence and their role in representing analytic functions.

In addition, the chapter explores power series expansion, which make it possible to represent functions in an infinite series representation about a given point.

3.1 Sequences.

Definition 1 (Infinite sequence): An infinite sequence $\{z_n\}_{n\in\mathbb{N}}$ is a function $f:\mathbb{N}\to\mathbb{C}$, whose domain is the set of positive integers.

$$\{z_n\}_{n\in\mathbb{N}} = \{z_1, z_2, z_3, \dots \dots\ \}.$$

$z_1, z_2, z_3, \dots \dots$ are the terms of the sequence.

Example:

Consider the sequence $z_n = \frac{i^n}{n}, n = 1, 2, 3, \dots$

Find z_1, z_2, z_3, z_4 and z_5.

Solution:

$$z_1 = \frac{i^1}{1} = i;$$

$$z_2 = \frac{i^2}{2} = \frac{-1}{2};$$

$$z_3 = \frac{i^3}{3} = \frac{-i}{3};$$

$$z_4 = \frac{i^4}{4} = \frac{1}{4};$$

$$z_5 = \frac{i^5}{5} = \frac{i}{5}.$$

Notes:

1. A sequence can be finite or infinite.

2. A sequence can be defined by recurrence relation where z_{n+1} is given as a function of lower (earlier) terms.

Definition 2 (Convergent sequence): A sequence z_n is said to be convergent if its limit given exists and is finite, which means:

$$\lim_{n\to\infty} z_n = L < \infty$$

Notes:

1 If a sequence $\{z_n\}_{n\in\mathbb{N}}$ has a limit, the limit is unique.

2 The sequence $\{z_n\}_{n\in\mathbb{N}}$ is divergent if it is not convergent.

Theorem: A sequence $\{z_n\}_{n\in\mathbb{N}}$ converges to a complex number *L* if and only if $Re(z_n)$ converges to $Re(z)$

$Im(z_n)$ converges to $Im(z)$.

Example: Consider the sequence $z_n = \frac{3+ni}{n+2ni}$

1. Calculate z_1, z_2 and z_5

2. Calculate $\lim_{n\to+\infty} Re(z_n)$ and $\lim_{n\to+\infty} Im(z_n)$.

3. Conclude $\lim_{n\to+\infty} z_n$.

Solution:

$$z_n = \frac{3+ni}{n+2ni} \times \frac{n-2ni}{n-2ni} = \frac{2n^2+3n+(n^2-6n)i}{n^2+(2n)^2} = \frac{2n^2+3n}{5n^2} + \frac{n^2-6n}{5n^2}i$$

$$z_1 = \frac{5-5i}{5} = 1-i$$

$$z_2 = \frac{14-8i}{20} = \frac{7}{10} - \frac{4}{10}i$$

$$z_5 = \frac{65 - 5i}{125} = \frac{13}{25} - \frac{1}{25}i$$

Now,

$$Re(z_n) = \frac{2n^2 + 3n}{5n^2} \xrightarrow[n\to+\infty]{} \frac{2}{5}$$

$$Im(z_n) = \frac{n^2 - 6n}{5n^2} \xrightarrow[n\to+\infty]{} \frac{1}{5}$$

Then by the previous theorem, we conclude that $\{z_n\}_{n\in\mathbb{N}}$ converges to $\frac{2}{5} + \frac{1}{5}i$, and we can write:

$$z_n = Re(z_n) + Im(z_n)i \xrightarrow[n\to+\infty]{} \frac{2}{5} + \frac{1}{5}i.$$

3.2 Power Series

Definition (power series): A power series is an infinite series having the form:

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n = a_0 + a_1(z - z_0)^1 + a_2(z - z_0)^2 + \cdots$$

Where,

- The complex constants a_n are the coefficients of the power series and they depend on n,
- z is the variable of the power series,
- The complex point $z_0 \in \mathbb{C}$ is referred to the center of the given power series.

Example:

The following infinite series is a power series:

$$\sum_{n=0}^{\infty} \frac{(z - 0)^n}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Where,

- The complex constants $\frac{1}{n!}$ are the coefficients of the given power series

- z is the variable of the given power series,
- The complex point $z_0 = 0 \in \mathbb{C}$ is referred to the center of the given power series.

3.2.1 Convergence

Definition: The power series $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ is said to be converge if the sequence of partial sums $S_N(z) = \sum_{n=0}^{N} a_n(z - z_0)^n$ converges to a finite limit as $N \to \infty$. That limit is the sum of the series at z.

Some Tests for Convergence:

Ratio Test:
Let $L = \lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert$, then • If $L < 1$ then the series is absolutely convergent, and therefore convergent. and so, the radius of convergence is $R = \frac{1}{L}$ in this case. • If $L < 1$ then the series is divergent. • If $L < 1$ then the test fails, the series may converge or diverge (Inconclusive).

Root Test:
Let $L = \lim_{n\to\infty} \sqrt[n]{\lvert a_n \rvert}$, then • If $L < 1$ then the series is absolutely convergent, and therefore convergent. and so, the radius of convergence is $R = \frac{1}{L}$ in this case. • If $L < 1$ then the series is divergent. • If $L < 1$ then the test fails, the series may converge or diverge (Use another test).

Notes:

1. The Ratio test is good test to use when the series contains sequences such as: $n!, c^n$
2. The root test is convenient to apply when n^{th} powers occur.

1. Radius of Convergence:

Definition: A complex power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is said to have $0 \leq R \leq \infty$ as its radius of convergence, such that the given series converges absolutely if $|z-z_0| < R$, and diverges if $|z-z_0| > R$.

Notes: Consider a power series, with radius R. Then:

1 $\frac{d}{dz}(\sum_{n=0}^{\infty} a_n(z-z_0)^n) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}((z-z_0)^{n-1})$.

2 $\int \sum_{n=0}^{\infty} a_n(z-z_0)^n \, dz = \sum_{n=0}^{\infty} \int a_n(z-z_0)^n \, dz$.

3 The power series and its derivative have the same radius of convergence R.

4 The power series and its integral have the same radius of convergence R.

2. Domain of Convergence:

Definition: We call domain of convergence of a complex power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$, the set of all complex numbers where the power series is convergent.

We define his domain of convergence by:

$$D_R(z_0) = \{z \in \mathbb{C}: |z-z_0| < R\}.$$

1. If $R = 0$, then $D_{R=0}(z_0) = \{z_0\}$ (in which case the power series converges only at the center point $z = z_0$).
2. If R is a finite positive number, then $D_R(z_0) = \{z \in \mathbb{C}: |z-z_0| < R\}$ (in which case the power series converges at all interior points of the circle $|z-z_0| = R$).
3. If $R = \infty$, then $D_R(z_0) = \{\mathbb{C}\}$ (in which case the power series converges for all $z \in \mathbb{C}$).

3.3 Power Series Expansions: Taylor Series and Laurent Series.

Definition: Let f be a function defined in a neighborhood of $z_0 \in \mathbb{C}$. We say that f admits a power series expansion at the point z_0 if there exists a power series.

3.3.1 Taylor Series:

If f is analytic (holomorphic) at z_0, then the series:

$$f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \frac{f'''(z_0)}{3!}(z - z_0)^3 + \frac{f^{(4)}(z_0)}{4!}(z - z_0)^4 + \frac{f^{(5)}(z_0)}{5!}(z - z_0)^5 + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, n = 0, 1, 2, \ldots$$

is called the Taylor series for f at z_0.

In other words, the Taylor series expansion of f at z_0 is given by:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n, \quad n = 0, 1, 2, \ldots$$

Notes:

1. The Taylor series are used to approximate functions by polynomials near a point z_0;
2. The Maclaurin series is a special case of the Taylor series expansion of a function at $z_0 = 0$.

Properties: Let f and g be analytic functions with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ and $g(z) = \sum_{n=0}^{\infty} c_n(z - z_0)^n$ at a point z_0. Then:

1. The Taylor series for $\alpha f(z) = \alpha \sum_{n=0}^{\infty} a_n(z - z_0)^n = \sum_{n=0}^{\infty} \alpha a_n(z - z_0)^n$;
2. The Taylor series for $f(z) \pm g(z) = \sum_{n=0}^{\infty}(a_n \pm c_n)(z - z_0)^n$.

Example: Find the Taylor series expansion of $f(z) = e^z$ at $z_0 = 0$.

Solution: We have $f(z) = e^z$,then:

If $n = 0$:

$$f^{(0)}(z) = f(z) = e^z \Rightarrow f^{(0)}(z_0 = 0) = f(0) = e^0 = 1;$$

If $n = 1$:

$$f^{(1)}(z) = f'(z) = e^z \Rightarrow f^{(1)}(z_0 = 0) = f'(0) = e^0 = 1;$$

If $n = 2$:

$$f^{(2)}(z) = f''(z) = e^z \;\Rightarrow\; f^{(2)}(z_0 = 0) = f''(0) = e^0 = 1;$$

If $n = 3$:

$$f^{(3)}(z) = f'''(z) = e^z \;\Rightarrow\; f^{(3)}(z_0 = 0) = f'''(0) = e^0 = 1;$$

$$\vdots$$

$$\vdots$$

Then, the Taylor series expansion of $f(z)$ at $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z-0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n, \;\; n = 0, 1, 2, \ldots$$

$$\Rightarrow \; f(z) = f(0) + f'(0)z + \frac{f''(0)}{2}z^2 + \frac{f'''(z_0)}{3}z^3 + \cdots$$

$$= 1 + 1.z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$$

Therefore:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

Some Special Series:

The following list shows some special series with their regions of convergence:

1. $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$ $\qquad |z| < \infty$

2. $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots$ $\qquad |z| < \infty$

3. $\cos(z) = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots$ $\qquad |z| < \infty$

4. $\ln(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots$ $\qquad |z| < 1$

5. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots$ $\qquad |z| < 1$

3.2.2 Laurent Series:

Suppose that $f(z)$ is analytic on the annular $r < |z - z_0| < R$ then $f(z)$ can be expressed as a series:

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

This is the Laurent series of a function $f(z)$,

where the coefficients a_n and b_n are defined by contour integrals:

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz,\ n \geq 0.$$

and

$$b_n = \frac{1}{2\pi i} \oint_C f(z)(z - z_0)^{n-1} dz, n \geq 1.$$

Furthermore,

- The series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges to an analytic function for $|z - z_0| < R$.
- The series $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ converges to an analytic function for $|z - z_0| > r$.
- Together, the series both converges both converges on the annular $r < |z - z_0| < R$ where f is analytic.

Notes:

1. C is positively oriented closed contour around z_0.
2. The series is valid in an annulus (ring domain), not necessarily a full disk.
3. The part $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ is called the analytic part.
4. The part $\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$ is called the principal part.

Example: Expand the function $f(z) = \frac{1}{(z-2)(z-1)}$ in Laurent series valid for the annular region given by:

a. $|z| < 1$.

b. $1 < |z| < 2$.

c. $|z| > 2$.

Solution: Given:

$$f(z) = \frac{1}{(z-2)(z-1)} = \frac{1}{z-2} - \frac{1}{z-1}$$

a. $|z| < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

1. $\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{1}{2}.\frac{1}{\frac{z}{2}-1} = -\frac{1}{2}.\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$;

2. $-\frac{1}{z-1} = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$;

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty}(1 - 2^{-n-1})z^n.$$

b. $1 < |z| < 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

1. $\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{1}{2}.\frac{1}{\frac{z}{2}-1} = -\frac{1}{2}.\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$;

2. $-\frac{1}{z-1} = \frac{-1}{z\left(1-\frac{1}{z}\right)} = \frac{-1}{z}.\frac{1}{1-\frac{1}{z}} = \frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = -\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$;

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$$

c. $|z| > 2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

1. $\frac{1}{z-2} = \frac{1}{z\left(1-\frac{2}{z}\right)} = \frac{1}{z}.\frac{1}{1-\frac{2}{z}} == \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$;

2. $-\frac{1}{z-1}=\frac{-1}{z\left(1-\frac{1}{z}\right)}=\frac{-1}{z}.\frac{1}{1-\frac{1}{z}}=\frac{-1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n}=-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}$;

$\Rightarrow f(z)=\sum_{n=0}^{\infty}\frac{2^{n}}{z^{n+1}}-\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=\sum_{n=0}^{\infty}\frac{2^{n}-1}{z^{n+1}}$.

Exercises:

Exercise 01:

What is the power series of $cos(z)$ about $z = 0$? What is its radius of convergence?

Exercise 02:

What is the power series of $\sinh(z)$ about $z = 0$? What is its radius of convergence?

Exercise 03:

Find the power series of $cosh(z)$ about $z = 0$? What is its radius of convergence?

Exercise 04:

Find power series expansion of the following functions:

1. $f(z) = \frac{1}{1-z^2}$

2. $f(z) = \frac{1}{1+z}$

3. $\frac{1}{1+z^2}$

Exercise 05:

Find power series expansion of the following functions

1. $f(z) = \arctan(z)$

2. $f(z) = \frac{1}{(1-z)^2}$

Exercise 06:

Determine whether the following power series converges, then find the radius and domain of convergence.

1. $\sum_{n=0}^{\infty} \frac{(z-2)^n}{3}$

2. $\sum_{n=0}^{\infty} \frac{z^n}{n!}$

3. $\sum_{n=0}^{\infty} \frac{z^n}{2^n}$

4. $\sum_{n=0}^{\infty} \frac{n!}{(2n)!} z^n$

5. $\sum_{n=0}^{\infty} \left(\frac{6n+1}{2n+5}\right)^n z^n$

Chapter 4

Cauchy Theory

Cauchy theory is a fundamental part of complex analysis that studies functions and their behavior in the complex plane.

It introduces the concept of complex integrals, extending integration to complex-valued functions.

The Cauchy Integral Theorem shows that the integral of a function over a closed contour is zero under certain conditions.

The Cauchy Integral Formula allows us to compute function values using integrals along a simple closed curve.

The chapter also examines singularities, where functions are not well-defined.

4.1 Cauchy's Integral Theorem and Cauchy's Integral Formula.

4.1.1 Cauchy's Integral Theorem:

Let $f(z)$ be analytic function inside and on a simple closed contour C (with no exception, not even a single point), then

$$\oint_C f(z)dz = 0.$$

Example 01: Evaluate the following integral

$$\oint_C \frac{e^z}{z-4}dz,$$

Where $C = \{z: |z-0| = 3\}$

Solution 01:

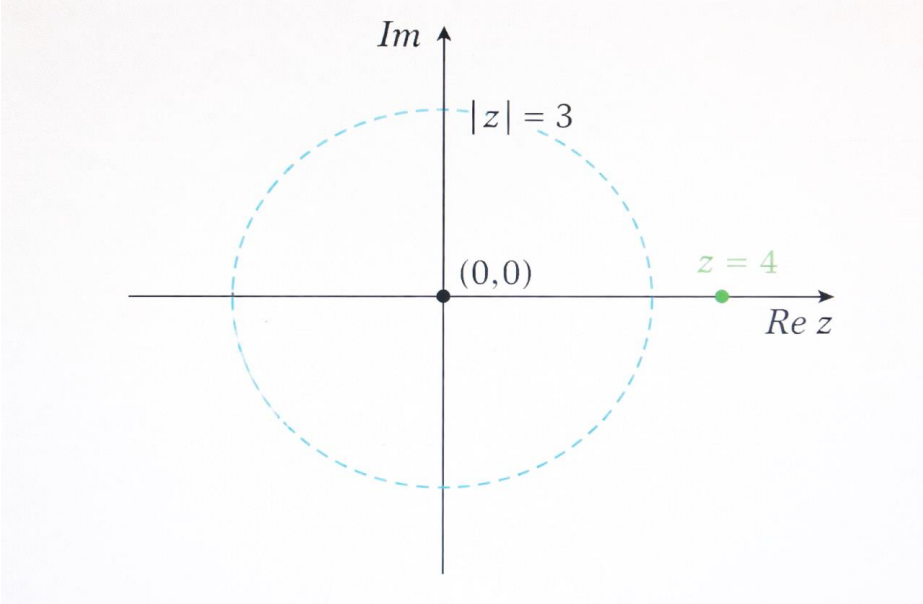


Figure 6: Circle of radius 3 centered at point (0,0) in the complex plane

$f(z) = \frac{e^z}{z-4}$ is analytic inside and on a simple closed contour C, then by Cauchy's integral theorem

$$\oint_C \frac{e^z}{z-4} dz = 0.$$

Example 02: Evaluate the following integral

$$\oint_C \frac{1}{z-2} dz,$$

where C is a circle with center $(0,0)$ and radius 1.

Solution 02:

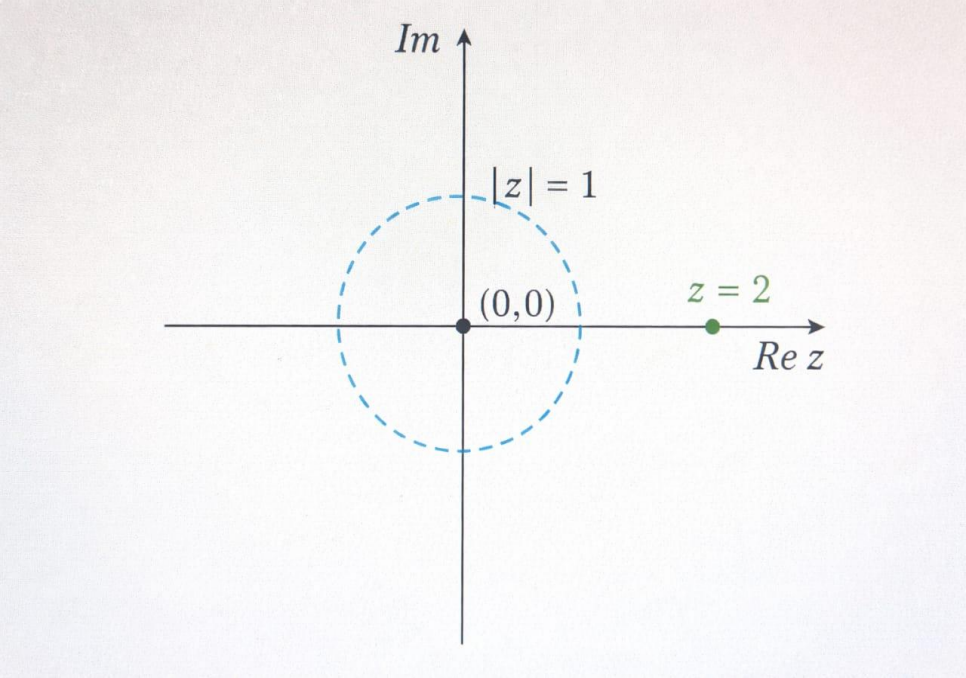


Figure 7: The unit circle in the complex plane

$f(z) = \frac{1}{z-2}$ is analytic inside and on a simple closed contour C, then by Cauchy's integral theorem

$$\oint_C \frac{1}{z-2} dz = 0.$$

4.1.1 Cauchy's integral formula:

Let $f(z)$ be analytic function inside and on simple closed curve (contour) C. Let z_0 is any point inside C, then

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Note: If $n = 0$, $f(z)$ be analytic inside and on simple closed curve (contour) C, and z_0 is any point inside C, then

$$\oint_C \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0).$$

Example 01: Evaluate the following integral:

$$\oint_C \frac{e^{2z}}{z^4} dz,$$

where $C = \{z: |z| = 1\}$.

Solution 01:

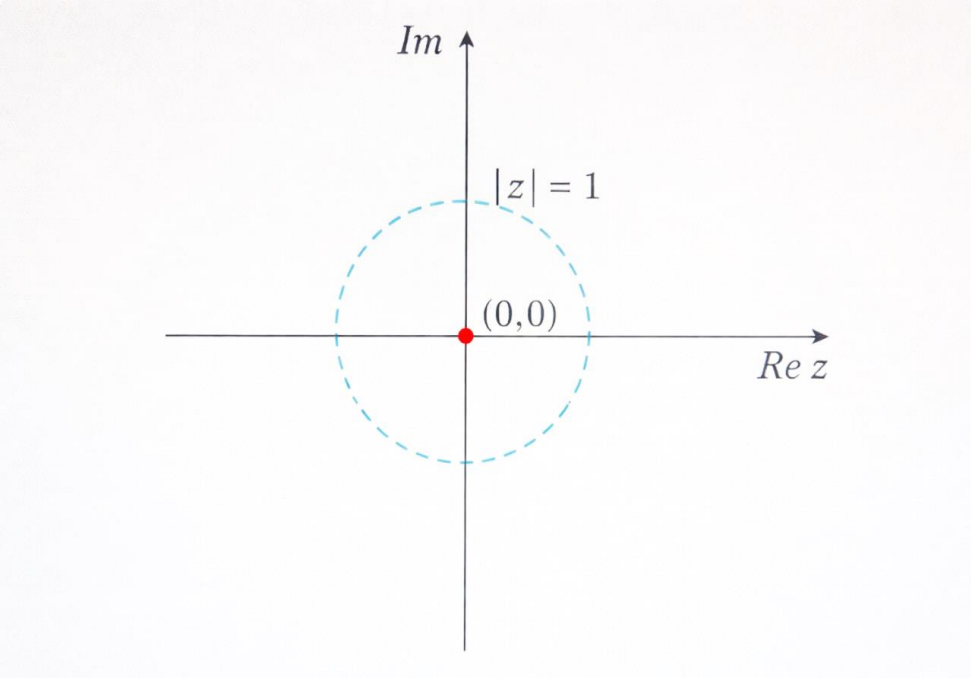


Figure 8: The unit circle in the complex plane

The function $\frac{e^{2z}}{z^4}$ is analytic in C except at 0, since $z^4 = 0 \Rightarrow z = 0 \in C$.

The function $f(z) = e^{2z}$ is analytic inside and on C. Then, by Cauchy's integral formula,

$$\oint_C \frac{e^{2z}}{z^4} dz = \frac{2\pi i}{3!} f^{(3)}(0) = \frac{2\pi i}{6}(8e^0) = \frac{16}{6}\pi i$$

$f(z) = e^{2z} \Rightarrow f^{(1)}(z) = 2e^{2z} \Rightarrow f^{(2)}(z) = 4e^{2z} \Rightarrow f^{(3)}(z) = 8e^{2z}$.

Example 02: Evaluate the following integral:

$$\oint_C \frac{e^z}{z} dz,$$

where $C = \{z: |z - 1| = 2\}$

Solution 02: The function $\frac{e^z}{z^3}$ is analytic in C except at 0, since $z^3 = 0 \Rightarrow z = 0 \in C$

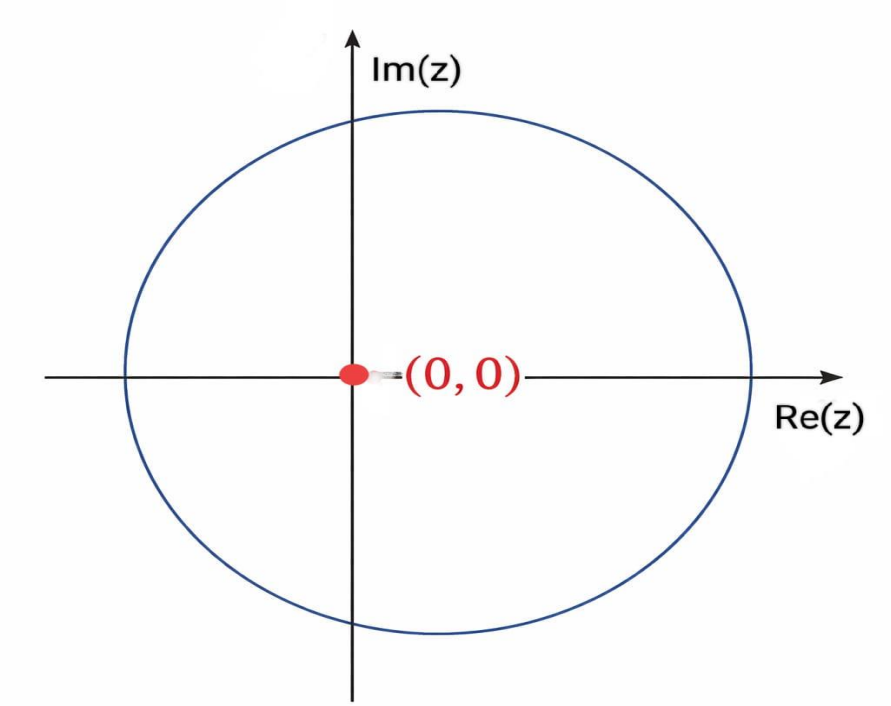


Figure 9: Circle of radius 2 centered at point (1,0) in the complex plane

Let $f(z) = e^z$, clearly f is analytic inside and on C. Then, by Cauchy's integral formula

$$\oint_C \frac{e^z}{z} dz = 2\pi i e^0 = 2\pi i.$$

Example 03: Evaluate the following integral:

$$\oint_C \frac{1}{z(z+2)} dz,$$

where $C = \{z: |z| = 3\}$

Solution 03:

The function $\frac{1}{z(z+2)}$ is analytic in C except at 0 and -2, since $z(z+2)=0 \Rightarrow \begin{cases} z=0 \in C \\ \text{or} \\ z=-2 \in C \end{cases}$

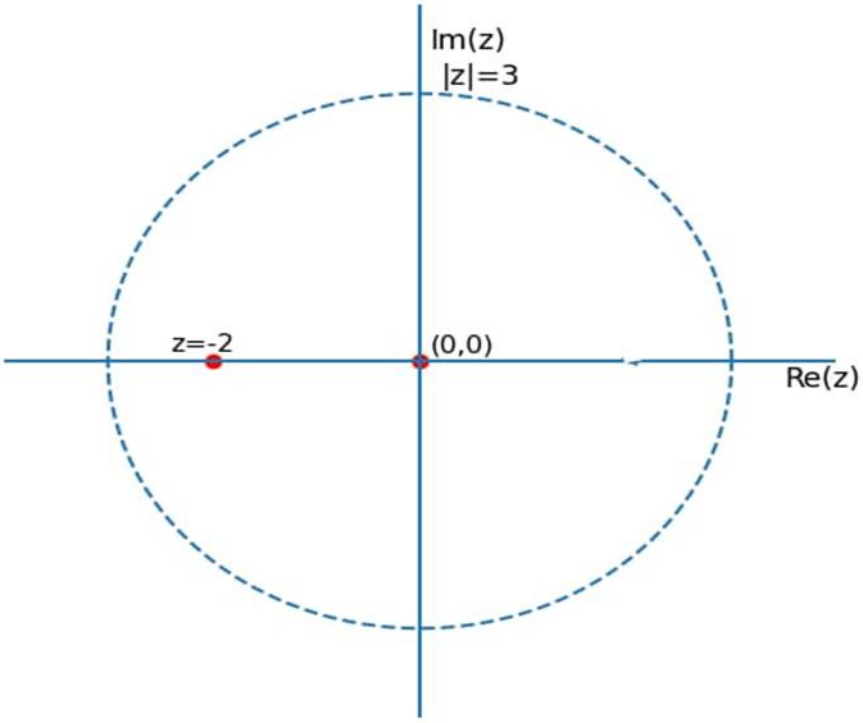


Figure 10: Circle of radius 3 centered at point (0,0) in the complex plane

Now, we need to perform partial fraction decomposition before integrating the function.

$$\frac{1}{z(z+2)}=\frac{A}{z}+\frac{B}{z+2}=\frac{A(z+2)}{z(z+2)}+\frac{Bz}{z(z+2)}=\frac{Az+2A+Bz}{z(z+2)}=\frac{(A+B)z+2A}{z(z+2)}$$

So

$$\begin{cases} A+B=0 \\ 2A=1 \end{cases} \Rightarrow \begin{cases} B=-A=-\frac{1}{2} \\ A=\frac{1}{2} \end{cases}$$

Therefore,

$$\frac{1}{z(z+2)}=\frac{1}{2}\frac{1}{z}-\frac{1}{2}\frac{1}{z+2};$$

Then, by Cauchy's integral formula

$$\oint_C \frac{1}{z(z+2)}dz = \frac{1}{2}\oint_C \frac{1}{z}dz - \frac{1}{2}\oint_C \frac{1}{z+2}dz = \frac{2\pi i}{2}(1) - \frac{2\pi i}{2}(1) = 0.$$

4.2 Singularities

4.2.1 Singularity:

Definition: A point z_0 is said to be a singular point of a complex-valued function $f(z)$ is a value of z at which the function f fails to be analytic.

In other words, f is analytic except at $z = z_0$.

Examples:

1. $f(z) = \frac{1}{z-5}$, $z = 5$ is a singular point of $f(z)$
2. $f(z) = \frac{z}{z^2+4}, z = -2i$ and $z = 2i$ are singular points of $f(z)$
3. $f(z) = \frac{z-3}{z(z+2)(z-1)}, z = 0, z = -2$ and $z = 1$ are singular points of $f(z)$
4. $f(z) = e^{\frac{1}{z-2}}, z = 2$ is a singular point of $f(z)$.

4.2.2 Types of Singularity:

1. **Isolated singularity:** A singular point z_0 is said to be isolated singular point of a function $f(z)$ if there exist some neighborhood of z_0 which does not contain any singular point of $f(z)$.

   For example, $f(z) = \frac{1}{z-5}, z = 5$ is an isolated singular point of f.

2. **Non-isolated singularity:** A singular point z_0 is said to be non-isolated singular point (limit point) of a function $f(z)$ if every neighborhood of z_0 contains at least one singular point of $f(z)$ other than z_0.

For example, the singular point $z = 0$ is a non-isolated singular point of the function

$f(z) = \ln(z)$

since every neighborhood of $z = 0$ contains points on the negative real axis.

4.2.3 Types of Isolated Singularity:

Let $f(z)$ be a complex-valued function and let z_0 be an isolated singular point

1. **Removable:** A point $z = z_0$ is called removable isolated singular point of a complex-valued function $f(z)$, if principal part of $f(z)$ does not exist.

   i.e $b_1 = b_2 = \cdots = 0$

   In other words, All power of z are non-negative in Laurent expansion of $f(z)$.

Note: In this case, $\lim_{z \to z_0} f(z)$ exist and finite, and we can remove this singularity by redefining the function.

Example: Consider the function $f(z) = \frac{\sin(z)}{z}$, $f(z)$ is analytic except at $z = 0$, so 0 is isolated singular point.

Now,

$$f(z) = \frac{\sin(z)}{z} = \frac{\sum_{n=0}^{+\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}}{z} = \sum_{n=0}^{+\infty}(-1)^n \frac{z^{2n}}{(2n+1)!} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

Then, $z = 0$ is removable isolated singular point of $f(z) = \frac{\sin(z)}{z}$.

2. **Pole:**

A point $z = z_0$ is called pole of order m, if principal part in Laurent series of $f(z)$ about z_0 has finite number m of terms.

i.e $f(z) = \sum_{n=0}^{+\infty} a_n (z - z_0)^n + b_1(z - z_0)^{-1} + b_2(z - z_0)^{-2} + \cdots + b_m(z - z_0)^{-m}$.

Notes:

1. $\lim_{z \to z_0} f(z)$ exist and infinite (i.e $\lim_{z \to z_0} f(z) = \infty$).

2. If $m = 1$, then z_0 is called simple pole.

Example:

Consider the function $f(z) = \frac{1}{z^2}$, $f(z)$ is not analytic if $z^2 = 0 \Rightarrow z = 0$ is isolated singular point.

Now,

$$f(z) = \frac{1}{z^2} = z^{-2} = (z-0)^{-2}$$

Therefore, $z = 0$ is pole of order 2.

3. Essential:

A point $z = z_0$ is called essential isolated singular point of a complex-valued function $f(z)$, if principal part in Laurent series of $f(z)$ about z_0 has infinite number of terms.

Note: In this case, $\lim_{z \to z_0} f(z)$ does not exist.

Example:

Consider the function $f(z) = e^{\frac{1}{z}}$, $f(z)$ is not analytic if $z = 0 \Rightarrow z = 0$ is isolated singular point.

Now,

$$f(z) = e^{\frac{1}{z}} = \sum_{n=0}^{+\infty} \frac{\left(\frac{1}{z}\right)^n}{n!} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!\,z^3} + \cdots = 1 + z^{-1} + \frac{1}{2!}z^{-2} + \frac{1}{3!}z^{-3} + \cdots$$

Then, there are infinite number of terms in principal part.

Therefore, $z = 0$ is essential isolated singular point.

4.3 Complex Integral

Definition: A contour integral is a way of evaluating complex integral along a contour (or path) C in complex plane from a point $z = z_1$ to a point $z = z_2$.

Properties of complex integral:

1. $\int_C \big(af(z)+bg(z)\big)\,dz = a\int_C f(z)dz + b\int_C g(z)dz\,, a, b$ are constant.

2. $\int_{z=a}^{z=b} f(z)\,dz = F(b)-F(a).$

3. $\int_{-C} f(z)\,dz = -\int_C f(z)dz.$

4. $\int_C f(z)\,dz = \int_{C_1\cup C_2} f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$

Theorem: If f is continuous on a smooth contour $C: z(t) = x(t) + iy(t)$, parametrize as $z = z(t), a \le t \le b$. Then

Example: Calculate the following integral:

$$\int_0^2 (t^2+it)\,dt$$

Solution:

$$\int_0^2 (t^2+it)\,dt = \int_0^2 t^2\,dt + i\int_0^2 t\,dt = \frac{8}{3}+2i$$

4.3.1 Contour integrals along a circle

In this case, we use parametrization: $z = z_0 + re^{it},\ a \le t \le b, a, b\ (in\ radian), z_0$ is the center of the given circle C and r is its radius.

Example 01: Calculate the following integral:

$$\int_C z\,dz\,, C = \{z: |z-0| = 1\}$$

Solution 01: C is the unit circle, it is centered by $z_0 = 0$ within radius

$z(t) = z_0 + re^{it} = 0 + 1e^{it}, 0 \le t \le 2\pi \ \Rightarrow\ z'(t) = ie^{it}dt$

Then

$$\int_C z\,dz = \int_0^{2\pi} e^{it} ie^{it} dt = \int_0^{2\pi} e^{it} ie^{it} dt = i\int_0^{2\pi} e^{2it} dt = \frac{1}{2}\left(e^{2i\pi} - e^{0}\right) = 0$$

Example 02: Evaluate the following integral:

$$\int_C \bar{z}\,dz,$$

where C is the right-hand half of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$ (Figure 11)

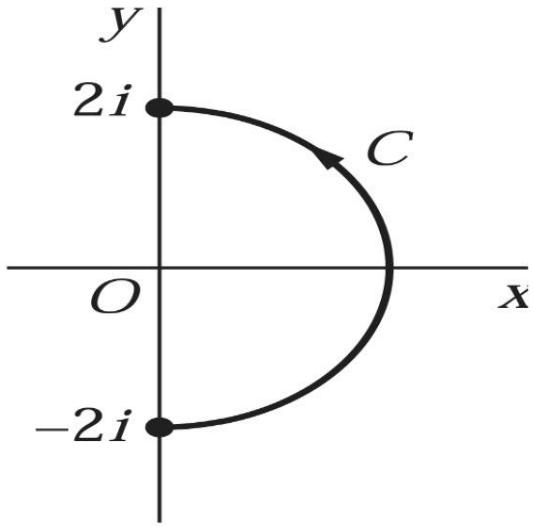


Figure 11: the right-hand half of the circle $|z| = 2$ from $z = -2i$ to $z = 2i$

Solution 02:

Let: $z(\theta) = z_0 + re^{i\theta} = 0 + 2e^{i\theta} = 2e^{i\theta} \;\Rightarrow z'(\theta) = 2ie^{i\theta} d\theta, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

So, $\overline{z(\theta)} = \overline{2e^{\iota\theta}} = 2e^{-i\theta}$.

Then

$$\int_C \bar{z}\,dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2e^{-i\theta} 2ie^{i\theta} d\theta = 4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-i\theta} e^{i\theta} = 4i\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\,d\theta = 4i\,\theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 4i\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$= 4i\pi.$$

Exercises

Exercise 01: Calculate the following integral:

$$\oint_C \frac{1}{z-5} dz$$

where $C = \{z: |z| = 1\}$

Exercise 02: Calculate the following integral:

$$\oint_C \frac{1}{z-5} dz$$

where $C = \{z: |z| = 5\}$

Exercise 03: Calculate the following integral:

$$\oint_C z^2 + z \, dz$$

where $C = \{z: z(t) = 2e^{it},\ 0 \leq t \leq 2\pi\}$

Exercise 04: Calculate the following integral:

$$\oint_C \frac{\cos(z)}{(z-\pi)^2} \, dz$$

where $C = \{z: z(t) = \pi + re^{it},\ 0 \leq t \leq 2\pi,\ r > 0\}$

Chapter 5

Applications

This chapter presents fundamental results in complex analysis, focusing on the equivalence between holomorphy and analyticity. It also introduces key theorems such as the Maximum Modulus Theorem, Liouville's Theorem, Rouche's Theorem, and Cauchy's Residue Theorem, which describe the behavior of holomorphic functions and provide powerful tools for analyzing complex integrals and zeros of functions.

5.1 Equivalence between Holomorphy and Analyticity.

Definition: A function f is analytic in a neighborhood of z_0 if it is holomorphic everywhere in U.

In other words, A function f is analytic, if for every point $z_0 \in U$, it admits a convergent power series expansion of the form:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Theorem: A function f is holomorphic on domain D if and only if it is analytic on D.

Notes:

1. Holomorphy and analicity are equivalent concepts. Many mathematicians use them as synonyms.
2. It is clear that on a domain D, if a function f is analytic, then f is holomorphic.
3. The region of analyticity contains only interior points; therefore, analytic functions are defined only on domains (open and connected sets).
4. Since the derivative of a polynomial exists for all $z \in \mathbb{C}$, every polynomial is an entire function.

5.2 Some Fundamental Theorems.

5.2.1 Maximum modulus:

Maximum modulus theorem: If $f(z)$ is analytic and non-constant in a domain D, then $|f(z)|$ does not attain its maximum value inside D, it occurs only on the boundary.

5.2.2 Liouville's theorem:

Definition (Bounded function): An analytic function $f(z)$ in a domain D is said to be bounded, if there exists a positive number $M > 0$ such that

$$|f(z)| \leq M \, , \forall \, z \in \mathbb{C}.$$

Liouville's theorem: If a complex-valued function $f(z): \mathbb{C} \to \mathbb{C}$ is entire and bounded in the complex plane, then $f(z)$ is constant throughout the complex plane.

In other words, every bounded entire function is constant.

5.2.3 Rouche's Theorem:

Rouche's Theorem: Let $f(z)$ and $g(z)$ be analytic inside and on a simple closed contour C and let $|g(z)| < |f(z)|$ on C, then $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Example: Determine the number of roots of the following equation:

$$z^7 - 4z^3 + z - 1 = 0,$$

inside the unit circle $|z| = 1$.

Solution:

Let $f(z) = -4z^3$ be the biggest term on the unit circle, and let $g(z) = z^7 + z - 1$.

On the unit circle, then

$$\begin{aligned} |g(z)| &= |z^7 + z - 1| \\ &\leq |z^7| + |z| + |1| \\ &\leq |z|^7 + |z| + 1 \\ &\leq 1 + 1 + 1 \leq 3 \end{aligned}$$

So,

$$|g(z)| = 3 \leq |f(z)| = |-4z^3| = 4|z^3| \leq 4 \times 1 = 4;$$

By Rouche's theorem, $f(z)$ and $f(z) + g(z) = z^7 - 4z^3 + z - 1$ has the same number of zeros on the unit circle.

As $f(z)$ has three zeros, so $z^7 - 4z^3 + z - 1$ has three zeros which means

$z^7 - 4z^3 + z - 1 = 0$ has three roots on the unit circle.

5.3 Cauchy's Residue Theorem:

Before we embark on the study of the Cauchy's Residue Theorems, we recall the residue of a complex-valued function $f(z)$.

5.3.1 Cauchy's Residue Theorem:

Residue at an isolated singular point:

Theorem: If $f(z)$ has an isolated singularity at z_0, then

$$Res(f, z_0) = \operatorname*{Res}_{z=z_0} f(z) = b_1$$

Where

$$f(z) = \sum_{n=1}^{+\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{+\infty} a_n(z-z_0)^n = \frac{b_1}{(z-z_0)^1} + \sum_{n=2}^{+\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{+\infty} a_n(z-z_0)^n$$

The Laurent series of $f(z)$ about z_0 on the region $0 < |z - z_0| < r$.

Example: Find the residue of $f(z) = e^{\frac{1}{2z}}$ at $z = 0$.

Solution:

$f(z) = e^{\frac{1}{2z}}$ has an isolated singularity at 0. From the Laurent series we obtain that $Res(f, 0) = \frac{1}{2}$

$$f(z) = e^{\frac{1}{2z}} = 1 + \frac{1}{2z} + \frac{1}{2(2z)^2} + \cdots = 1 + \frac{1}{2(z-0)} + \frac{1}{4(z-0)^2} + \cdots$$

Residue at finite poles:

Theorem: If $f(z)$ has a pole of order n at z_0, then

$$\operatorname*{Res}_{z=z_0} f(z) = \frac{1}{(n-1)!}\lim_{z=z_0}\frac{d^{n-1}}{dz^{n-1}}(z-z_0)^n f(z).$$

Example 01: Find the residue of the following function at $z = i$.

$$f(z) = \frac{1}{(z^2+1)^3}$$

Solution 01:

Step 1: Given

$$f(z) = \frac{1}{(z^2+1)^3} = \frac{1}{\big((z+i)(z-i)\big)^3} = \frac{1}{(z+i)^3(z-i)^3} = \frac{\frac{1}{(z+i)^3}}{(z-i)^3} = \frac{g(z)}{(z-i)^3}$$

Where $g(z) = \frac{1}{(z+i)^3}$;

Step 2: The residue of $f(z)$ at $z = i$ which is a pole of order $n = 3$ is:

$$\operatorname*{Res}_{z=i} f(z) = \frac{1}{(3-1)!}\lim_{z\to i}\frac{d^{3-1}}{dz^{3-1}}[(z-i)^3 f(z)]$$

$$= \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\left[(z-i)^3\frac{1}{(z+i)^3(z-i)^3}\right]$$

$$= \frac{1}{2!}\lim_{z\to i}\frac{d^2}{dz^2}\frac{1}{(z+i)^3}$$

$$= \frac{1}{2}\lim_{z\to i}\frac{12}{(z+i)^5} = \frac{1}{2}\cdot\frac{12}{(2i)^5} = \frac{6}{32i} = \frac{3}{16i}$$

$$= -\frac{3i}{16}$$

Example 02: Find the residues of the following function at poles.

$$f(z) = \frac{1}{(z^2+1)^2}$$

Solution 02:

We have

$$f(z)=\frac{1}{(z^2+1)^2}=\frac{1}{(z^2-(-1))^2}=\frac{1}{\left(z^2-(i^2)\right)^2}=\frac{1}{\left((z+i)(z-i)\right)^2}=\frac{1}{(z+i)^2(z-i)^2}$$

$\Rightarrow$ $z=i$ (pole of order 2) and $z=-i$ (pole of order 2)

➢ Residue of f at $z=i$ $(n=2)$

$$\operatorname*{Res}_{z=-i} f(z)=\frac{1}{(2-1)!}\lim_{z\to -i}\frac{d^{2-1}}{dz^{2-1}}[(z-i)^2 f(z)]$$

$$=\frac{1}{1!}\lim_{z\to -i}\frac{d}{dz}\left((z-i)^2\frac{1}{(z+i)^2(z-i)^2}\right).$$

$$=\lim_{z\to -i}\frac{d}{dz}\left(\frac{1}{(z+i)^2}\right)=\lim_{z\to -i}\frac{-2}{(z+i)^3}$$

$$=\frac{-2}{(2i)^3}=\frac{2}{8i}=\frac{-i}{4}.$$

➢ Residue of f at $z=-i$ $(n=2)$

$$\operatorname*{Res}_{z=-i} f(z)=\frac{1}{(2-1)!}\lim_{z\to -i}\frac{d^{2-1}}{dz^{2-1}}\left((z+i)^2 f(z)\right)$$

$$=\frac{1}{1!}\lim_{z\to -i}\frac{d}{dz}\left((z+i)^2\frac{1}{(z+i)^2(z-i)^2}\right).$$

$$=\lim_{z\to -i}\frac{d}{dz}\left(\frac{1}{(z-i)^2}\right)=\lim_{z\to -i}\frac{-2}{(z-i)^3}$$

$$=\frac{-2}{(-2i)^3}=\frac{-2}{8i}=\frac{i}{4}.$$

5.3.2 Cauchy's Residue Theorem:

Theorem: If $f(z)$ is analytic in and on a closed contour C except at a finite number of points $z_1,\ z_2,\ldots,\ z_n$within C, then

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z) \text{ inside } C$$

Example: Use Cauchy's Residue theorem to evaluate:

$$\oint_C \frac{1}{z(z-1)(z-2)^2}\,dz$$

where $C = \{z: |z-0| = 3\}$

Solution:

$$z(z-1)(z-2)^2 = 0 \Rightarrow \begin{cases} z = 0 \\ z-1 = 0 \\ (z-2)^2 = 0 \end{cases} \Rightarrow \begin{cases} z = 0 \ (simple\ root) \\ z = 1 \ (simple\ root) \\ z = 2 \ (double\ root) \end{cases}$$

So, $z = 0, z = 1$ are simple poles inside C, and $z = 2$ is a pole of order $n = 2$ inside C.

Residue of f at $z = 2$ $(n = 2)$.

$$\operatorname*{Res}_{z=2} f(z) = \frac{1}{(2-1)!}\lim_{z\to 2}\frac{d^{2-1}}{dz^{2-1}}\left((z-2)^2 f(z)\right)$$

$$= \frac{1}{1!}\lim_{z\to 2}\frac{d}{dz}\left((z-2)^2\frac{1}{z(z-1)(z-2)^2}\right).$$

$$= \lim_{z\to 2}\frac{d}{dz}\left(\frac{1}{z(z-1)}\right) = \lim_{z\to -2}\frac{2z-1}{\left(z(z-1)\right)^2} = \frac{-3}{4};$$

Residue of f at $z = 1$.

$$\operatorname*{Res}_{z=1} f(z) = \frac{1}{(1-1)!}\lim_{z\to 1}\frac{d^{1-1}}{dz^{1-1}}\left((z-1) f(z)\right)$$

$$= \frac{1}{0!}\lim_{z\to 1}\left(\frac{1}{z(z-2)^2}\right).$$

$$= \lim_{z\to 1}\left(\frac{1}{z(z-2)^2}\right) = 1;$$

Residue of f at $z = 0$.

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{(1-1)!}\lim_{z\to 2}\frac{d^{1-1}}{dz^{1-1}}\big(zf(z)\big)$$

$$= \frac{1}{0!}\lim_{z\to 0}\left(z\frac{1}{z(z-1)(z-2)^2}\right).$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{1}{(z-1)(z-2)^2}\right) = \frac{-1}{4}.$$

Therefore,

$$\oint_C \frac{1}{z(z-1)(z-2)^2}\,dz = 2\pi i\sum_{k=1}^{n}\operatorname*{Res}_{z=z_k} f(z) \text{ inside } C = 2\pi i\left(1-\frac{1}{4}-\frac{3}{4}\right) = 0.$$

Exercises:

Exercise 01:

Show that

$$z^7 - 5z^2 + 12 = 0$$

has no zeros inside $C = \{z: |z - 0| = 1\}$.

Exercise 02:

Let

$$f(z) = \frac{2 + z + z^2}{(z-2)(z-3)(z-4)(z-5)}$$

Show that all poles are simple and compute their residues.

Exercise 03:

Consider the following function:

$$f(z) = \frac{1}{z(z^2+1)},$$

1. Find all isolated points, then compute their residues.
2. Calculate the following integral

$$\oint_C \frac{1}{z(z^2+1)} dz,$$

a. $C = \{z: |z - 0| = 2\}$;

b. $C = \{z: |z - 3| = 1\}$.

Chapter 6

Special Functions

6.1 Euler's Gamma Function.

Definition: Let $n > 0$ then the Euler gamma function is defined by:

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx$$

Properties:

There are several properties of Euler gamma function such as

1. $\Gamma(n) = (n-1)!, \forall n \in \mathbb{N}$.
2. $\Gamma(n+1) = n! = n\,\Gamma(n)$.
3. $\Gamma(n)\Gamma(1-n) = \frac{\pi}{sin(n\pi)}, n \neq 0$.

Examples:

Note that from the above Theorem, we get:

1. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
2. $\frac{\Gamma(6)}{2\Gamma(3)} = \frac{5!}{2.2!} = \frac{120}{4} = 30$.
3. $\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$.
4. $\int_0^{\infty} x^4 e^{-x} dx = \Gamma(5) = 4! = 24$.

6.1 Euler's Beta Function.

Definition: The beta function, denoted as $B(m,n)$ is a special function defined as the Euler integral of the first kind:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \;\; m > 0 \text{ and } n > 0$$

It is closely related to the gamma function (Γ).

Properties:

1. $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

2. $B(m,n) = B(n,m)$.

3. $B(m,n) = 2\int_0^{\frac{\pi}{2}} \text{Sin}^{2m-1}(\theta).\cos^{2n-1}(\theta)d\theta$.

Examples:

1. $B(1,2) = \dfrac{\Gamma(1)\,\Gamma(2)}{\Gamma(1+2)} = \dfrac{\Gamma(1)\,\Gamma(2)}{\Gamma(3)} = \dfrac{1.1}{2} = \dfrac{1}{2}$.

2. $B(2,3) = \dfrac{\Gamma(2)\,\Gamma(3)}{\Gamma(2+3)} = \dfrac{\Gamma(2)\,\Gamma(3)}{\Gamma(5)} = \dfrac{1.2}{24} = \dfrac{1}{12}$.

Since $\Gamma(n) = (n-1)!$

6.3 Application to Evaluate Integrals

6.3.1 Using gamma function to evaluate integrals:

Example 01: Evaluate the following integral:

$$I = \int_0^{\infty} x^6 e^{-2x} dx.$$

Solution 01: Let $y = 2x \Rightarrow \frac{y}{2} = x \Rightarrow \frac{1}{2}dy = dx$, and $\frac{y^6}{64} = x^6$

Then

$$I = \frac{1}{2}.\frac{1}{64}\int_0^{\infty} y^6 e^{-y} dy = \frac{1}{128}\Gamma(7) = \frac{6!}{128} = \frac{720}{128} = \frac{45}{8}.$$

Example 02: Evaluate the following integral:

$$I = \int_0^{\infty} \sqrt{X} e^{-x^3} dx.$$

Solution 02: Let $y = x^3 \Rightarrow y^{\frac{1}{3}} = x \Rightarrow dx = \frac{1}{3}y^{-\frac{2}{3}}dy$

$$I = \int_0^{\infty} \left(y^{\frac{1}{3}}\right)^{\frac{1}{2}} e^{-y} \frac{1}{3} y^{-\frac{2}{3}} dy = \frac{1}{3}\int_0^{\infty} y^{\frac{1}{3}\cdot\frac{1}{2}} y^{-\frac{2}{3}} e^{-y} dy = \frac{1}{3}\int_0^{\infty} y^{-\frac{1}{2}} e^{-y} dy = \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{3}.$$

6.3.2 Using beta function to evaluate integrals:

Example 01: Evaluate the following integral:

$$I = \int_0^2 \frac{x^2}{\sqrt{(2-x)}} dx$$

Solution 01: Let $x = 2y \Rightarrow dx = 2dy$, and $x^2 = 4y^2$

Then

$$I = \int_0^2 \frac{4y^2}{\sqrt{(2-2y)}} 2dy = 8\int_0^2 \frac{y^2}{\sqrt{2(1-y)}} dy = \frac{8}{\sqrt{2}}\int_0^2 y^2(1-y)^{-\frac{1}{2}} dy = \frac{8}{\sqrt{2}} B\left(3, \frac{1}{2}\right) = \frac{64\sqrt{2}}{15}.$$

Because

$$\begin{cases} m-1=2 \\ n-1=-\dfrac{1}{2} \end{cases} \Rightarrow \begin{cases} m=3 \\ n=\dfrac{1}{2} \end{cases}$$

Example 02: Evaluate the following integral:

$$I=\int_0^2 x\sqrt{(8-x^3)}dx$$

Solution 02: Let $x^3=8y \Rightarrow x=8^{\frac{1}{3}}y^{\frac{1}{3}}=2y^{\frac{1}{3}} \Rightarrow dx=\frac{2}{3}y^{-\frac{2}{3}}dy$.

Then

$$I=\int_0^2 2y^{\frac{1}{3}}(8-8y)^{\frac{1}{2}}\frac{2}{3}y^{-\frac{2}{3}}dy=\frac{4}{3}.8^{\frac{1}{2}}\int_0^2 y^{-\frac{1}{3}}(1-y)^{\frac{1}{2}}dy=\frac{8}{3}\int_0^2 y^{-\frac{1}{3}}(1-y)^{\frac{1}{2}}dy=\frac{8}{3}B\left(\frac{2}{3},\frac{3}{2}\right).$$

Because

$$\begin{cases} m-1=-\dfrac{1}{3} \\ n-1=\dfrac{1}{2} \end{cases} \Rightarrow \begin{cases} m=\dfrac{2}{3} \\ n=\dfrac{3}{2} \end{cases}$$

Exercises:

Exercise 01:

Prove that:

$$B(m,n) = B(n,m).$$

Exercise 02:

Evaluate the following integral

$$\int_0^2 \sqrt{x(2-x)}$$

Exercise 03:

Evaluate the following integral using beta function:

$$\int_0^2 x^2(1-x)^3 dx$$

Exercise 04:

Evaluate the following integral

$$\int_0^1 \frac{1}{\sqrt{x(1-x)}} dx$$

Solved Exercises

Solved Exercises

Exercise 01: Simplify the following complex numbers:

1. $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5i}$.

2. $\dfrac{5i}{(1-i)(2-i)(3-i)}$.

3. $(1-i)^4$.

4. $(3+i)(4-i)$

Solution 01:

1. $\dfrac{1+2i}{3-4i}+\dfrac{2-i}{5}=\dfrac{(1+2i)(3+4i)}{(3-4i)(3+4i)}+\dfrac{(2-i)(-5i)}{(5i)(-5i)}=\dfrac{-5+10i}{25}+\dfrac{-5-10i}{25}=-\dfrac{2}{5}$.

2. $\dfrac{5i}{(1-i)(2-i)(3-i)}=\dfrac{5i}{(1-3i)(3-i)}=\dfrac{5i}{-10i}=-\dfrac{1}{2}$.

3. $(1-i)^4=[(1-i)^2]^2=\big((1-i)(1-i)\big)^2=(-2i)^2=-4$.

4. $(3+i)(4-i)=12-3i+4i-i^2=12+i-(-1)=13+i$.

Exercise 02: Solve the following equations in $\mathbb{C}$.

1. $z^2+9=0$.

2. $4x+15yi=8+15i$.

Solution 02:

1. $z^2+9=0 \Rightarrow z^2=-9=(\pm 3i)^2 \Rightarrow z=\pm 3i$.

2. $4x+15yi=8+15i \Rightarrow \begin{cases} 4x=8 \\ 15y=15 \end{cases} \Rightarrow \begin{cases} x=\dfrac{8}{4}=2 \\ y=\dfrac{15}{15}=1 \end{cases}$

Exercise 03:

1. Express $z = 9i$ in the form $z = re^{i\theta}$ (exponential form) where $r > 0$ and $-\pi < \theta \leq \pi$

2. Express $z = 9i$ in polar form.

Solution 03:

1. We have: $z = 9i \Rightarrow r = |z| = |9i| = \sqrt{0^2 + 9^2} = 9 > 0$.

$-\pi < \theta = \dfrac{\pi}{2} \leq \pi$

Then: $z = 9e^{i\frac{\pi}{2}}$

2. By Euler's formula:

$$z = 9e^{i\frac{\pi}{2}} = 9\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right).$$

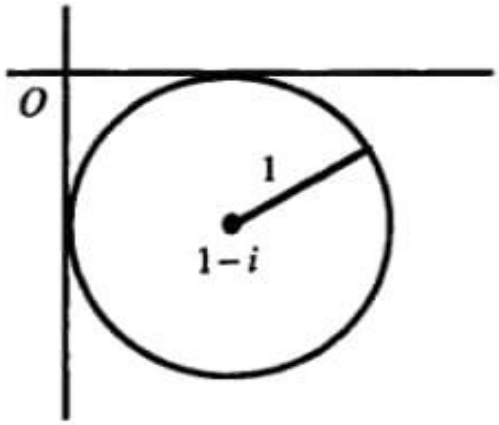
Exercise 04:

Describe and sketch the set of complex numbers satisfying $|z - 1 + i| = 1$.

Solution 04:

Rewrite $|z - 1 + i| = 1$ as $|z - (1 - i)| = 1$. This is the circle centered at $1 - i$ with radius 1.

It is shown below.



Exercise 05:

Determine whether the function $f(z) = 2xy - y + i(y^2 - x^2 + x)$ is holomorphic on $\mathbb{C}$.

Solution 05:

We have: $f(z) = 2xy - y + i(y^2 - x^2 + x) = u(x,y) + iv(x,y)$ where $u(x,y) = 2xy - y$ and

$v(x,y) = y^2 - x^2 + x.$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2xy - y) = 2y + 0 = 2y.$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2xy - y) = 2x - 1.$

$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(y^2 - x^2 + x) = 0 - 2x + 1 = -2x + 1.$

$\frac{\partial v}{\partial y} = \frac{\partial}{\partial y}(y^2 - x^2 + x) = 2y + 0 + 0 = 2y.$

So,

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2y \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 2x - 1 \end{cases}$$

Partial derivatives are continuous in $\mathbb{C}$.

Then, the Cauchy-Riemann equations are satisfied. Therefore, the function is holomorphic on $\mathbb{C}$.

Exercise 06:

Using Euler's formula to prove that:

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

Solution 06:

We have:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta) \dots \dots (1)$$

$$-e^{-i\theta} = -\cos(\theta) + i\sin(\theta) \dots \dots (2)$$

By addition, we obtain:

$$\Rightarrow \cos(\theta) + i\sin(\theta) - \cos(\theta) - i\sin(\theta) = e^{i\theta} - e^{-i\theta}$$

$$\Rightarrow 2i\sin(\theta) = e^{i\theta} + e^{-i\theta}$$

$$\Rightarrow \sin(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Exercise 07: Calculate:

1. $|1 + i|$

2. $|4 - 3i|$

Solution 07:

If $z = x + iy$ then $|z| = |x + iy| = \sqrt{x^2 + y^2}$

1. $|1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$.

2. $|4 - 3i| = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$.

Exercise 08:

Find isolated singular points for the following complex valued functions:

1. $f(z) = \frac{1}{1 - z}$

2. $f(z) = \frac{1}{z^2 + 1}$

Solution 08:

1. $f(z) = \frac{1}{1 - z}$,

Assume that $1 - z = 0 \Rightarrow z = 1$ is isolated singular point of $f(z) = \frac{1}{1-z}$.

2. $f(z) = \dfrac{1}{z^2 + 1}$,

Assume that $z^2 + 1 = 0 \Rightarrow z^2 = -1 = (\pm i)^2 \Rightarrow z = \pm i$ are isolated singular point of $f(z) = \frac{1}{z^2+1}$.

Exercise 09: Consider the following function:

$$f(z) = z^2 - z,$$

Find the Taylor series expansion of $f(z)$ at $z_0 = 3$.

Solution 09:

$$f(z) = z^2 - z = (z - 3 + 3)^2 - (z - 3 + 3)$$

$$= (z-3)^2 - 2 \times 3(z-3) + 3^2 - (z-3) - 3$$

$$= (z-3)^2 - 6(z-3) + 9 - (z-3) - 3 = (z-3)^2 - 5(z-3) + 6.$$

Exercise 10:

Consider the following function:

$$f(z) = \frac{1}{1-z}$$

Find the Taylor series expansion of $f(z)$ at $z_0 = 2$.

Solution 10:

$$f(z) = \frac{1}{1-z} = \frac{1}{1-z-2+2} = \frac{1}{1-2-(z-2)} = \frac{1}{-1-(z-2)} \times \frac{-1}{-1} = \frac{-1}{1+(z-2)}$$

$$= -\left(\frac{1}{1+(z-2)}\right) = -\sum_{n=0}^{\infty}(-1)^n(z-2)^n = \sum_{n=0}^{\infty}(-1)^{n+1}(z-2)^n.$$

Exercise 11:

Find the Taylor series expansion for the following:

1. $f(z) = \frac{1}{1-z^2}$

2. $f(z) = \dfrac{1}{(1-z)^2}$

3. $f(z) = \arctan(z)$

Solution 11:

1. Let $u = z^2$, then

$$f(z) = \frac{1}{1-z^2} = \frac{1}{1-u} = \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} (z^2)^n = \sum_{n=0}^{\infty} z^{2n}.$$

2. We have

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = f(z) \Rightarrow \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n.\right) = \frac{1}{(1-z)^2} = f(z)$$

Since $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$.

So,

$$f(z) = \frac{d}{dz}\left(\sum_{n=0}^{\infty} z^n.\right) = \sum_{n=0}^{\infty} \frac{d}{dz}(z^n) = \sum_{n=0}^{\infty} nz^{n-1}\,.$$

3. We have

$$\frac{d}{dz}\arctan(z) = \frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n}$$

By integration, we get:

$$\arctan(z) = \int \sum_{n=0}^{\infty} (-1)^n z^{2n}\, dz = \sum_{n=0}^{\infty} \int (-1)^n z^{2n}\, dz = \sum_{n=0}^{\infty} (-1)^n \int z^{2n}\, dz$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}.$$

Exercise 12:

Determine the poles and the poles order for each of the following functions:

1. $f(z) = \frac{1}{z^2(z-1)}$

2. $f(z)=\frac{1}{z(z-2)}$

Solution 12:

1. $f(z)=\frac{1}{z^2(z-1)}$, $z^2(z-1)=0 \Rightarrow \begin{cases} z^2=0 \\ (z-1)=0 \end{cases} \Rightarrow \begin{cases} z=0 \\ z=1 \end{cases}$ are isolated singular points

- Laurent series about $z=1$:

$$f(z)=\frac{1}{z^2(z-1)}=\frac{A}{(z-1)}+\frac{B}{z}+\frac{C}{z^2}=\frac{Az^2}{z^2(z-1)}+\frac{Bz(z-1)}{z^2(z-1)}+\frac{C(z-1)}{z^2(z-1)}$$

$$\Rightarrow \begin{cases} A=1 \\ B=-1 \\ C=1 \end{cases}$$

So

$$f(z)=\frac{1}{z^2(z-1)}=\frac{1}{(z-1)}+\frac{-1}{z}+\frac{1}{z^2}=(z-1)^{-1}+\frac{-1}{z}+\frac{1}{z^2}.$$

- $\frac{-1}{z}$,

$$\frac{-1}{z}=\frac{-1}{z-1+1}=\frac{-1}{1+(z-1)}=-\left(\frac{1}{1+(z-1)}\right)=-\sum_{n=0}^{+\infty}(-1)^n(z-1)^n$$

$$=\sum_{n=0}^{+\infty}(-1)^{n+1}(z-1)^n$$

since $\frac{1}{1+w}=\sum_{n=0}^{+\infty}(-1)^n w^n$.

- $\frac{1}{z^2}$,

$$\frac{1}{z^2}=\frac{1}{(z-1+1)^2}=\frac{1}{\left(1+(z-1)\right)^2}=-\left(\frac{-1}{\left(1+(z-1)\right)^2}\right)=-\sum_{n=0}^{+\infty}(-1)^n n(z-1)^{n-1}$$

$$=\sum_{n=0}^{+\infty}(-1)^{n+1}n(z-1)^{n-1}$$

Therefore

$f(z) = (z-1)^{-1} + \frac{-1}{z} + \frac{1}{z^2} = (z-1)^{-1} + \sum_{n=0}^{+\infty}(-1)^{n+1}(z-1)^n + \sum_{n=0}^{+\infty}(-1)^{n+1}n(z-1)^{n-1}$

$= (z-1)^{-1} + \sum_{n=0}^{+\infty}(-1)^{n+1}(z-1)^n + 0 + (z-1)^0 - 2(z-1) + 3(z-1)^2 + \cdots$

$z = 1$ is simple pole (pole of order 1)

- Laurent series about $z = 0$:

$$f(z) = \frac{1}{z^2(z-1)} = \frac{1}{(z-1)} - \frac{1}{z} + \frac{1}{z^2} = \frac{1}{(z-1)} - z^{-1} + z^{-2}$$

- $\frac{1}{(z-1)} = \frac{1}{-1+z} \times \frac{-1}{-1} = \frac{-1}{1-z} = -\left(\frac{1}{1-z}\right) = -\sum_{n=0}^{+\infty} z^n = -1 - z - z^2 - z^3 - \cdots$

$\Rightarrow f(z) = z^{-2} - z^{-1} - 1 - z - z^2 - z^3 - \cdots \Rightarrow z = 0$ is pole of order 2.

2. $f(z) = \frac{1}{z(z-2)}$, f is not analytic if $z(z-2) = 0 \Rightarrow \begin{cases} z = 0 \\ z - 2 = 0 \end{cases} \Rightarrow \begin{cases} z = 0 \\ z = 2 \end{cases}$ are isolated singular points

$$f(z) = \frac{1}{z(z-2)} = \frac{A}{z} + \frac{B}{z-2} = \frac{A(z-2)}{z(z-2)} + \frac{Bz}{z(z-2)} = \frac{(A+B)z - 2A}{z(z-2)}$$

$$\Rightarrow \begin{cases} A = -\frac{1}{2} \\ B = \frac{1}{2} \end{cases}$$

$$\Rightarrow f(z) = -\frac{1}{2}\left(\frac{1}{z}\right) + \left(\frac{1}{2}\right)\frac{1}{z-2}.$$

- Laurent series about $z = 0$:

$$f(z) = -\frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z-2}\right) = -\frac{1}{2}z^{-1} + \frac{1}{2}\left(\frac{1}{-2+z}\right) = -\frac{1}{2}z^{-1} - \frac{1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n$$

Since

$$\frac{1}{-2+z} = \frac{1}{-2+z} \times \frac{-1}{-1} = \frac{-1}{2-z} = \frac{-1}{2\left(1-\frac{z}{2}\right)} = \frac{-1}{2} \times \frac{1}{1-\frac{z}{2}} = \frac{-1}{2}\sum_{n=0}^{+\infty}\left(\frac{z}{2}\right)^n = -\sum_{n=0}^{+\infty}\frac{-1}{2^{n+1}}z^n.$$

Therefore, $z = 0$ is simple pole.

- Laurent series about $z = 2$:

$$f(z) = -\frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z-2}\right) = +\frac{1}{2}\left(\frac{1}{z-2}\right) = +\frac{1}{2}(z-2)^{-1}$$

Since

$$\frac{1}{z} = \frac{1}{z-2+2} = \frac{1}{2+(z-2)} = \frac{1}{2\left(1+\left(\frac{z-2}{2}\right)\right)} = \frac{1}{2} \times \frac{1}{1+\left(\frac{z-2}{2}\right)} = \frac{1}{2}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{z-2}{2}\right)^n = \sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{n+1}}(z-2)^n.$$

Therefore, $z = 2$ is simple pole.

Exercise 13:

Determine the number of zeros, counting multiplicities, of the following polynomials:

1. $z^6 - 5z^4 + z^3 - 2z$.

2. $2z^4 - 2z^3 + 2z^2 - 2z + 9$.

Inside the circle $|z - 0| = 1$.

Solution 13:

1. Let $f(z) = -5z^4$ be the biggest term on the unit circle, and let $g(z) = z^6 + z^3 - 2z$.

On the unit circle, then

$$\begin{aligned} |g(z)| &= |z^6 + z^3 - 2z| \\ &\leq |z^6| + |z^3| + |2z| \\ &= |z|^6 + |z|^3 + 2|z| \\ &= 1 + 1 + 2 \\ &= 4 \end{aligned}$$

So,

$$|g(z)| = 4 \leq |f(z)| = |-5z^4| = 5|z^4| = 5|z|^4 \leq 5 \times 1 = 5;$$

By Rouche's theorem, $f(z)$ and $f(z) + g(z) = z^6 - 5z^4 + z^3 - 2z$ has the same number of zeros on the unit circle.

As $f(z)$ has 4 zeros, so $z^6 - 5z^4 + z^3 - 2z$ has 4 zeros on the unit circle.

2. Let $f(z) = 9 = 9z^0$ be the biggest term on the unit circle, and let $g(z) = 2z^4 - 2z^3 + 2z^2 - 2z$.

On the unit circle, then

$$\begin{aligned}|g(z)| &= |2z^4 - 2z^3 + 2z^2 - 2z| \\ &\leq |2z^4| + |2z^3| + |2z^2| + |2z| \\ &= 2|z|^6 + 2|z|^3 + 2|z^2| + 2|z| \\ &= 2 + 2 + 2 + 2 \\ &= 8\end{aligned}$$

So,

$$|g(z)| = 8 \leq |f(z)| = |9| = 9;$$

By Rouche's theorem, $f(z)$ and $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ has the same number of zeros on the unit circle.

As $f(z)$ has 0 zeros, so $f(z) + g(z) = 2z^4 - 2z^3 + 2z^2 - 2z + 9$ has 0 zeros on the unit circle.

Exercise 14:

Determine the number of zeros, counting multiplicities, of the polynomial $z^4 + 3z^3 + 6$ inside the circle $|z| = 2$.

Solution 14:

3. Let $f(z) = 3z^3$ be the biggest term on the circle $|z| = 2$, and let $g(z) = z^4 + 6$.

On the circle $|z| = 2$

$$\begin{aligned}|g(z)| &= |z^4 + 6| \\ &\leq |z^4| + 6 \\ &= |z|^4 + 6 \\ &= 2^4 + 6 \\ &= 22.\end{aligned}$$

So,

$$|g(z)| = 22 \leq |f(z)| = |3z^3| = 3|z|^3 = 3 \times 2^3 = 24;$$

By Rouche's theorem, $f(z)$ and $f(z) + g(z) = z^4 + 3z^3 + 6$ has the same number of zeros on the circle $|z| = 2$.

As $f(z)$ has 3 zeros, so $f(z) + g(z) = z^4 + 3z^3 + 6$ has 3 zeros on the circle $|z| = 2$.

Bibliographical References

Bibliographical References

[1] Zill, Dennis, and Patrick Shanahan. A first course in complex analysis with applications. Jones & Bartlett Learning, 2009.

[2] Brown, James Ward, and Ruel V. Churchill. Complex variables and applications ‘’7th ed’’. McGraw-Hill,, 2009.

[3] Ahlfors, Lars V. Complex analysis: an introduction to the theory of analytic functions of one complex variable"3rd ed". 1997.

[4] Henri Catan, Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes. Editeur Hermann, Paris 1985.

[5] Jean Kuntzmann, Variable complexe. Hermann, Paris, 1967.Manuel de premier cycle.

[6] Herbert Robbins Richard Courant. What is Mathematics?, Oxford University Press, Toronto, 1978. Ouvrage classique de vulgarisation.

[7] Walter Rudin, Analyse réelle et complexe. Masson, Paris, 1975. Manuel de deuxième cycle.

[8] Saff, Edward B., and Arthur David Snider. Fundamentals of Complex Analysis with Applications to Engineering and Science. 3rd ed. Upper Saddle River, NJ: Prentice Hall, 2003.

[9] Juan Carlos Ponce Campuzano. Complex Analysis: Problems with Solutions. August 2016. Self-published.

[10] Howie, John Mackintosh. Complex Analysis. London: Springer, 2003