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Blow up for a viscoelastic wave equation of Kirchhoff's type with delay time

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بسم الله الرحمن الرحيم

الإهداء:

إلى نفسي ...تلك التي خاضت طريقًا طويلًا من التعب والكفاح، ولم تجد رغم وعورة الدرب وكثرة العثرات.
إلى تلك الروح التي صمدت في وجه الصعوبات، واستمرت حين كان التوقف يبدو الخيار الأسهل، بل وربما الأجدر. إلى أيامي التي لم تكن هيّنة، وإلى ليالٍ حملتُ فيها من الثقل ما كاد يُنهي العزيمة، لكنني لم أستسلم، بل مضيتُ بإيمانٍ صامت وقوةٍ لا يراها إلا من جرّب. إلى السنوات التي لم تكن مجرد زمنٍ عابر، بل كانت مدرسةً علمتني الصبر، وصقلتني بالتجارب، وجعلت مني إنسانة أكثر ثباتًا ونضجًا ورغم كلّ الصعوبات التي واجهتني في هذا الطريق، إلا أنني أؤمن دائمًا أن الله سبحانه وتعالى يجعل بعد العسر يسرًا، وأنه يفتح لنا أبوابًا لا نتوقعها، ويرزقنا ـ حيث لا نحتسب. فقد رزقني الله تعالى العلم والصبر، وأغدق عليّ بنعمٍ كثيرة لا تعد ولا تحصى، وكان فضله عليّ عظيمًا في كل مرحلة من حياتي.

وكثيرًا ما يكون الرزق على هيئة بشر، وهذا لا يسعني إلا أن أذكر أعظم نعم الله عليّ، وهما والداي العزيزان اللذان لم يتركاني يومًا، ولم يحرموني من شيء طوال حياتي، فكانا السند والدعم في كل خطوة.

إلى أبي الغالي، ميزاب فيصل ...

إلى السند الحقيقي في حياتي، والقوة التي أستمد منها عزيمتي في كل خطوة. أبي الذي لم يبخل عليّ يومًا بالدعم والحب، وكان دائمًا الحاضر في كل تفاصيل حياتي، يساندني بصمته أحيانًا، وبكلماته الحنونة أحيانًا أخرى. كنتَ يا أبي المثال في العطاء دون حدود، وفي التضحية دون انتظار مقابل، وكنتَ الدافع الذي يجعلني أواصل مهما كانت الصعوبات. ها أنا اليوم أصل إلى هذا المبتغى، وكل ذلك بفضلك بعد الله. فلك مني خالص الامتنان والمحبة، وأسأل الله أن يمد في عمرك، ويرزقك الصحة والعافية، ويجزيك عني خير الجزاء كما أعطيتنا من دون حساب. أحبك حبًا لا تُترجمه الكلمات، بل يظلّ شعورًا ثابتًا في القلب لا يتغيّر.

وإلى أمي العزيزة، بوعشة دنيا ...

إلى القلب الذي لا يشبهه قلب، والروح التي منحتني الحياة والحب دون حدود. أمي التي كانت ولا تزال النور الذي يضيء أيامي، والسند الذي لا يميل مهما اشتدت الظروف. في كل نجاح أصل إليه، أجدكِ خلفه دعاءً صادقًا ودعمًا صامتًا، وفي كل لحظة ضعف، أجدكِ حصنًا يحتويني وكلماتٍ تعيد لي قوتي. لم تبخلي عليّ يومًا، ولم ترَي جهدي إلا بعين الفخر والرضا، بل كنتِ دائمًا تؤمنين بي أكثر مما أؤمن بنفسي. أمي الغالية، يا من جاهدتِ وكافحتِ من أجلنا بصبرٍ لا ينتهي وقلبٍ لا يعرف الاستسلام، أسأل الله أن يحفظكِ لي، وأن يطيل في عمركِ، ويمنحكِ الصحة والعافية، ويجزيكِ عنّي خير الجزاء. أحبكِ حبًا لا يقال، بل يُحسّ ويُعاش في كل تفاصيل حياتي.

وإلى إخوتي الأعزاء: أختي انتصار، وإخوتي قصي ومعتز وعبدالرحمان ...إلى من شاركوني تفاصيل حياتي، وكانوا جزءًا لا يتجزأ من أيامي الجميلة والصعبة على حدّ سواء.

أخي قصي، الذي كان حضوره قوةً لي، ووجوده طمأنينة في كثير من اللحظات، فكان الأخ والسند في آنٍ واحد. وإلى أخي الصغير معتز، قطعة الفرح في البيت. أسأل الله أن يحفظكم جميعًا، وأن يديم بيننا هذا الرابط الجميل من الحبّ والأخوة، وأن يوفقكم ويبارك فيكم في كل خطواتكم.

وإلى خالتي الغالية، التي رحلت جسدًا وبقيت روحها حاضرة في قلبي ...خالتي مفيدة، يا أحنّ الناس وأقربهم إلى روحي. لم أعرف في حياتي قلبًا يشبه قلبك، ولن أجد مثلك مهما امتدّ العمر. لم تفارق صورتك ذهني يومًا، ولا صوتك الحنون، ولا ضحكتك التي كانت تبث فينا طاقةً من الأمل والسكينة. لم أرَكِ يومًا خالةً فقط، بل كنتِ لي أمًا وأكثر، حضنًا دافئًا ومحبةً صادقةً بلا مقابل. كنتِ العزيزة التي أعطتني من قلبها الكثير، دون انتظار شيء، فكنتِ نعمةً عظيمة لن تُنسى. رحيلكِ أوجع قلوبنا وترك فينا فراغًا لن يملأه أحد، ومع ذلك سنبقين دائمًا حاضرةً في دعائنا وذكرياتنا. كنتِ قوية، صبورين على ألمكِ في صمت، ورغم تعبكِ كنتِ تزرعين الفرح في قلوبنا، وتحملينا بحبكِ الكبير. خالتي العزيزة، لا مثيل لكِ في هذا الوجود ...رحمكِ الله رحمةً واسعة، وجعل مثواكِ الجنة. أحبكِ حبًا يفوق الوصف، حبًا لم أكن أتخيل يومًا أن يخسره الفقد. سنبقين في قلبي ما حييت، يا أغلى الناس.

وإلى صديقاتي العزيزات: سميرة، رحمة، وإشراق ...رفيقات الدرب وأجمل ما منحته لي الأيام في رحلتي هذه.

وإلى ذلك الحضور الهادئ في حياتي، الذي كان دائمًا إلى جانبي دون ضجيج ...شكرًا لوفائك الذي لم يتغيّر، ولصبرك الذي احتوى تقلبات أيامي، ولوجودك الثابت الذي كان لي سندًا في كل حين. كنتَ دائمًا القرب الذي يُطمئن، والكلمة التي تُهدّئ، واليد التي لا تترك. فلك مني امتنان صادق لا تكفيه الكلمات، ومحبةٌ تنمو بصمت، لكنها عميقة بصدقها.

شكر و عرفان

بسم الله الرحمن الرحيم

"رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ"

عرفانًا بالجميل، ووفاء لمن كان لهم الفضل بعد الله سبحانه وتعالى في مسيرتي العلمية، يسعدني أن أتقدم بأسمى عبارات الشكر والتقدير والامتنان إلى كل من ساهم في إنجاز هذا العمل، وساندني بتوجيهه أو نصحه أو دعمه طوال سنوات الدراسة والبحث.

أتقدم بخالص الشكر وعظيم الامتنان إلى أستاذتي المشرفة (طوالبيه سارة) على ما قدمته لي من توجيهات علمية قيمة، ونصائح سديدة، ومتابعة مستمرة طوال فترة إنجاز هذا العمل، فكان لتشجيعها ودعمها الأثر الكبير في إتمام هذه المذكرة، فجزاها الله عني خير الجزاء.

ولا يفوتني أن أتقدم بخالص الشكر والاحترام إلى أعضاء لجنة المناقشة الموقرة (الاستاذة مسلوب فتيحة) على توجيهاتها وملاحظاتها القيّمة التي ساهمت في إنجاز هذا البحث و كذلك (الاستاذة كماش حورية)"أشكرهم جزيل الشكر على تفضلهم بقبول قراءة هذه المذكرة وتقييمها، وعلى ما سيقدمونه من ملاحظات وتوجيهات قيمة ستُري -دون شك- هذا البحث وتُقوّم مساره.

كما أتوجه بجزيل الشكر والتقدير إلى جميع أساتذتي الأفاضل الذين أشرفوا على تكويني العلمي منذ بداية مسيرتي الجامعية، والذين لم يبخلوا علينا بعلمهم وخبرتهم وتوجيهاتهم، فكانوا خير قدوة ومصدر إلهام في طلب العلم والمعرفة.

شكر خاص لكل من الاستاذين (زديري صنية)
و الاستاذ الفاضل (زارعِي عبد الرحمان).

Abstract:

In this work, we studied the blow-up phenomenon of solutions for a Kirchhoff-type viscoelastic wave equation in the presence of a time-delay term.

First, we introduced the mathematical framework and the main analytical tools required for this study. These tools provide a suitable basis for analyzing the dynamical behavior of solutions associated with this class of nonlinear evolution equations.

Next, we formulated the considered problem in an appropriate functional setting that allows the study of the energy evolution and the blow-up behavior of solutions in finite time. For proving our result we using the Lyapunov functional method combined with several inequality techniques, we established sufficient conditions for the blow-up of solutions in finite time. We also investigated the interaction between the damping and memory effects on one side, and the source term and time delay on the other side. The obtained results show that this interaction plays an important role in determining the qualitative behavior of solutions, namely stability or blow-up.

Finally, our analysis shows that the presence of a time-delay term in a Kirchhoff-type viscoelastic wave equation may cause instability and lead to the blow-up of solutions under certain conditions. These results highlight the strong influence of nonlocal and temporal effects on the behavior of such models.

Keywords: Kirchhoff-type wave equation, viscoelasticity, time delay, Lyapunov functional, blow-up of solutions.

Résumé:

Dans ce travail, nous avons étudié le phénomène d'explosion des solutions d'une équation des ondes de type Kirchhoff viscoélastique en présence d'un terme de retard temporel.

Tout d'abord, nous avons introduit le cadre mathématique ainsi que les principaux outils analytiques nécessaires à cette étude. Ces outils fournissent une base appropriée pour analyser le comportement dynamique des solutions associées à cette classe d'équations d'évolution non linéaires. Ensuite, nous avons formulé le problème considéré dans un cadre fonctionnel adéquat permettant l'étude de l'évolution de l'énergie et du comportement d'explosion des solutions en temps fini.

Pour prouver notre résultat en utilisant la méthode des fonctionnelles de Lyapunov combinée à plusieurs techniques d'inégalités, nous avons établi des conditions suffisantes assurant l'explosion des solutions en temps fini. Nous avons également étudié l'interaction entre les effets d'amortissement et de mémoire d'une part, et le terme source ainsi que le retard temporel d'autre part. Les résultats obtenus montrent que cette interaction joue un rôle important dans la détermination du comportement qualitatif des solutions, à savoir la stabilité ou l'explosion.

Enfin, notre analyse montre que la présence d'un terme de retard temporel dans une équation des ondes viscoélastique de type Kirchhoff peut provoquer une instabilité et conduire à l'explosion des solutions sous certaines conditions. Ces résultats mettent en évidence la forte influence des effets non locaux et temporels sur le comportement de tels modèles.

Mots-clés : équation des ondes de type Kirchhoff, viscoélasticité, retard temporel, fonctionnelle de Lyapunov, explosion des solutions.

ملخص:

في هذا العمل، قمنا بدراسة ظاهرة انفجار الحلول لمعادلة موجية لزجة مرنة من نوع كيرشوف في وجود حدّ تأخير زمني.

في البداية، قمنا بتقديم الإطار الرياضي والأدوات التحليلية الأساسية اللازمة لهذه الدراسة. وتوفر هذه الأدوات أساسًا مناسبًا لتحليل السلوك الديناميكي للحلول المرتبطة بهذه الفئة من معادلات التطور غير الخطية.

بعد ذلك، قمنا بصياغة المسألة المدروسة ضمن إطار دالي مناسب يسمح بدراسة تطور الطاقة وسلوك انفجار الحلول في زمن منتهٍ.لاثبات نتيجتنا استخدمنا طريقة لياپونوف إلى جانب عدة تقنيات للمتراجحات، تمكّنا من إثبات شروط كافية تؤدي إلى انفجار الحلول في زمن منتهٍ. كما قمنا بدراسة التفاعل بين تأثيرات التخميد والذاكرة من جهة، وحدّ المصدر والتأخير الزمني من جهة أخرى. وقد أظهرت النتائج المتحصل عليها أن هذا التفاعل يلعب دورًا مهمًا في تحديد السلوك النوعي للحلول، والمتمثل في الاستقرار أو الانفجار.

اخيرًا، تُظهر دراستنا أن وجود حدّ تأخير زمني في معادلة موجية لزجة مرنة من نوع كيرشوف قد يؤدي إلى حدوث عدم استقرار وانفجار الحلول تحت شروط معينة. وتبرز هذه النتائج التأثير القوي للعوامل غير المحلية والزمنية على سلوك مثل هذه النماذج الرياضية.

الكلمات المفتاحية: معادلة موجية من نوع كيرشوف، اللزوجة المرنة، التأخير الزمني، دالية لياپونوف،انفجار الحلول.

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General Introduction

Nonlinear wave phenomena constitute a central topic in modern mathematical analysis due to their strong connections with various applied fields such as mechanics, material science, and engineering. In particular, wave equations in viscoelastic media have attracted significant attention because they incorporate memory effects, meaning that the current state of the system depends on its past history through relaxation kernels.

Kirchhoff-type equations represent a natural generalization of classical wave equations, as they involve a nonlocal nonlinearity depending on the gradient of the solution. This feature increases the mathematical complexity of the problem, especially in the study of existence, stability, and long-term behavior of solutions.

A fundamental issue in the qualitative analysis of such equations is the distinction between:

- global existence of solutions,
- blow-up in finite time.

Although damping mechanisms and memory effects generally contribute to stabilizing the system, nonlinear terms may dominate under certain conditions, leading to blow-up phenomena. This typically corresponds to a physical breakdown of the system.

Moreover, time-delay effects naturally arise in many physical and engineering systems due to finite signal transmission speeds or feedback mechanisms. From a mathematical perspective,time delay introduces additional complexity by linking the present state of the system to its past states. It is well known that even small delays can significantly affect stability and may act as a source of instability in the system, potentially triggering or accelerating blow-up.

The purpose of this work is to study the blow-up in finite time of solutions for a class of nonlinear viscoelastic Kirchhoff type wave equations with time delay. The considered model combines three main features:

- a nonlocal Kirchhoff-type nonlinearity,
- a viscoelastic memory term,
- and a time-delay term.

The interaction between these effects leads to complex dynamical behavior, where stabilization and destabilization mechanisms compete. The objective is to establish sufficient conditions on the initial data and system parameters that ensure blow-up, using energy methods, Lyapunov functionals, and differential inequalities.

The contributions of this work are organized into three main chapters:

Chapter 1: PRELIMINARIES AND BASIC TOOLS.

This chapter recalls fundamental mathematical concepts and analytical techniques that will be used throughout the dissertation.

Chapter 2: HISTORICAL BACKGROUND AND MATHEMATICAL FRAMEWORK

This chapter provides a progressive overview of the development of wave equations, starting from the classical model to more complex formulations. It begins with the classical wave equation, presenting its physical origin, mathematical formulation, energy, and fundamental properties of solutions, this in turn highlighting the limitations of the basic model.

The chapter then moves to nonlinear wave equations, with a particular focus on Kirchhoff-type models, which incorporate the influence of energy on the wave dynamics. It also introduces viscoelastic effects that account for memory, meaning that the system's behavior depends on its past states.

Moreover, wave equations with time delay are discussed, emphasizing how delay can significantly affect the stability of solutions.

The chapter further addresses the blow-up phenomenon in nonlinear wave equations, presenting its mechanisms, criteria, and key results from the literature, especially in systems combining memory and delay effects.

Finally, the chapter brings together all these aspects within a unified mathematical model, highlighting the complex interaction between nonlinearity memory, delay, and setting the stage for the analysis of solution behavior, particularly blow-up, in the subsequent chapters.

Chapter 3:BLOW-UP OF SOLUTION FOR A VISCOELASTIC WAVE EQUATION WITH DELAY

In this chapter, we investigate a nonlinear viscoelastic wave equation with memory and delay time, given by:

$$|u_t|^p u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u, \quad \text{in } \Omega\times(0,+\infty). \tag{1.1}$$

With the delay initial condition

$$u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau); \tag{1.2}$$

and the classical initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega. \tag{1.3}$$

Together, with the homogeneous Dirichlet boundary condition

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0. \tag{1.4}$$

Here, $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$,where $p > 0$ is the exponent , $\mu_1 > 0$ and $b > 0$ are constants, and $\mu_2 \in R$.

The function g denotes a relaxation kernel describing the memory effect and is typically assumed to be positive and non-increasing, while $\tau > 0$ represents the time delay.

Time-delay phenomena naturally appear in many applied models where the system does not respond instantaneously. Such effects frequently encountered in mechanics, physics, engineering, and biological systems. In the context of wave propagation, delays may significantly affect the qualitative behavior of solutions, potentially leading to instability even when damping is present.

In the absence of the delay term (i.e. $\mu_2 = 0$), equation (1.1) reduces to a standard viscoelastic wave equation, which has been widely studied. Numerous results concerning global existence, asymptotic stability, and decay rates of solutions have been established under appropriate assumptions on the kernel g.

However, the introduction of a delay term $u_t(x, t - \tau)$ makes the analysis more challenging. It is known that the interaction between damping and delay plays a crucial role.

If the damping dominates, stability can be ensured, whereas a strong delay effect may generate instability or even lead to blow-up of solutions.

A simplified model frequently considered in the literature as:

$$u_{tt} - \Delta u + \mu_1 u_t(x, t) + \mu_2 u_t(x, t - \tau) = 0. \tag{1.5}$$

Motivated by these results, the main purpose of this work is to analyze problem (1.1)–(1.4) and to establish new results concerning the behavior of its solutions. In particular, we focus on the blow-up phenomenon and derive sufficient conditions under which solutions cannot exist globally in time. Our approach relies on energy methods and suitable Lyapunov-type functionals adapted to the presence of memory and delay terms.

Chapter 1

Recalls and preliminaries

In this chapter, we recall some definitions, some basic functional spaces (Banach space, Hilbert space ....), and some fundamental theorems and formulas, as well as the notion of local and global existence.

The aim of this chapter is to use the preliminary results in later chapters.

1.1 Notations and general notions

In this section we have defined some of the symbols and operators used in our work.
The gradient of a function u is defined by

$$\operatorname{grad} u = \nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right); \qquad \text{then} \qquad |\nabla u|^2 = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_i}\right)^2. \tag{1.1}$$

The divergence of a function is also defined by

$$\operatorname{div} u = \nabla \cdot u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + \cdots + \frac{\partial u}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}. \tag{1.2}$$

The Laplacian of u is defined by

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{1.3}$$

$\frac{\partial u}{\partial n}$ denotes the normal derivative of u outside $\partial\Omega$, i.e.: $\frac{\partial u}{\partial n} = \nabla u. \cdot \vec{\eta}$ where $\cdot\vec{\eta}$ is the unit vector of the normal outside Γ[1].
$C^k(\Omega)$, $0 \leq k \leq \infty$, denotes the vector space of functions whose partial derivatives of order less than or equal to K exist and are continuous in Ω[1].
$C_0^k(\Omega)$, $0 \leq k \leq \infty$ denotes the vector subspace of $C^k(\Omega)$ with compact support in Ω[3].
The space $C_0^k(\Omega)$ we will also denote $D(\Omega)$, is called the space of test functions on Ω[3].

$\mathcal{L}(E, F)$ denotes the space of linear and continuous (or bounded) operators from E into F with norm.

1.2 Some functional spaces

1.2.1 Normed space

Definition 1.1 :[2] Let E be a vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A norm on E is an application $\|\cdot\|$ of E in $[0, +\infty[$ satisfying:

1. $\|x\| = 0 \iff x = 0$.
2. $\forall x, y \in E$, we have $\|x + y\| \leq \|x\| + \|y\|$.
3. $\forall \lambda \in \mathbb{K},\ x \in E$,we have $\|\lambda x\| = |\lambda|\,\|x\|$.

Remark 1.1: $(E, \|\cdot\|)$ is a normed space.
Definition 1.2[4] (Equivalence of norms)
Let E be a normed vector space, and the two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent:

$$\iff \exists \alpha > 0,\ \exists \beta > 0,\ \alpha \|\cdot\|_1 \leq \|\cdot\|_2 \leq \beta \|\cdot\|_1.$$

Definition 1.3: [5](Cauchy sequence)
Let $(E, \|\cdot\|)$ be a normed space, and $(u_n)_{n\in\mathbb{N}^*}$ a sequence in E. We say that $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence if:

$$\forall \varepsilon > 0,\ \exists N_\varepsilon,\ \forall n, m \geq N_\varepsilon,\ \|u_n - u_m\| < \varepsilon.$$

1.2.2 Complete space

Definition 1.4: [2] A metric space E is complete if every Cauchy sequence $(u_n)_{n\in\mathbb{N}^*}$ of the space E has a limit in E.

1.2.3 Banach space

Let $(E, \|\cdot\|)$ be a normed space.
Definition 1.5 : [4] A complete normed vector space is called a Banach space.
Definition 1.6: E' is the linear space of all continuous linear functions

$$f : E \to \mathbb{R};$$

and is called the dual space of E.

Proposition 1.1:[6] The space E with norm $\|\cdot\|_E$ defines

$$\|f\|_E = \sup\{\,|f(u)| : \|u\| \leq 1\,\};$$

is also a Banach space.

Definition 1.7: Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces such that there exists an injective linear application of E into F, this application allows E to be considered as a sub vector space of F, and we denote $E \hookrightarrow F$ or $E \subset F$.
This inclusion is said to be:

1. **Continuous:** If there exists constant $C > 0$ such that $\|u\|_F \leq C\|u\|_E$.We denote $E \hookrightarrow_{continuous} F$.

1. **Compact:** If for any bounded sequence in E (for the norm of E), we can extract a subsequence that converges in F (for the norm F) and we denote $E \hookrightarrow_{compact}$F.

1. **Dense:** If for any $u \in E$ there exists a sequence $(u_n)_{n\in\mathbb{N}^*} \subset E$ such that $\lim_{n\to\infty} u_n = u$ (convergence being for the norm of F).

Weak and strong topology
Definition 1.8:[4] Let E be a Banach space, and $(u_n)_{n\in\mathbb{N}^*}$ a sequence in E. Then u_n converges strongly to u in E if and only if

$$\forall \varepsilon > 0,\ \exists N > 0,\ \forall n \geq N,\ \|u_n - u\|_E \leq \varepsilon.$$

Definition 1.9: [4] Let E be a Banach space and $(u_n)_{n\in\mathbb{N}}$ be a sequence in E, Then u_n converges weakly to u in E if and only if

$$\forall f \in L_c(E, \mathbb{R}), \quad f(u_n) \to f(u).$$

$$\forall f \in L_c(E, \mathbb{R}),\ \forall \varepsilon > 0, \exists N_\epsilon > 0, \forall n \geq N_\epsilon : |f(u_n) - f(u)| \leq \varepsilon.$$

Proposition 1.2:[4] If E is of finite dimension, then

$$\text{strong convergence} \Rightarrow \text{weak convergence};$$

but the converse is false.

1.2.4 Hilbert space

Definition 1.10 :[1] Let E be a vector space, the scalar product defined by $\langle u, v\rangle$ is a symmetric, positive definite bilinear form $E \times E$ in $\mathbb{R}$, symmetric, positive definite (i.e $\langle u, u\rangle \geq 0 \forall u \in E$ and $\langle u, u\rangle > 0$ if $u \neq 0$).
Definition 1.11 :[4] E is a Hilbert space if it is a Banach space for the associated norm

$$\|u\|_E = \langle u, u\rangle^{1/2} (i.e)\ \|u\|_E^2 = \langle u, u\rangle.$$

Definition 1.12 : A normed vector space with a scalar product is called a pre-Hilbert space.
Definition 1.13 :[1] A complete pre-Hilbert space is called a Hilbert space .
Definition 1.14: [1] (Hilbert basis) Let E be a Hilbert space with scalar product $\langle ., . \rangle$. we call a (countable) Hilbertian basis of E is a countable family $\{e_n\}_{n\leq 1}$ of elements of E such that:

(i) $e_n = 1 \forall n, (e_n, e_m) = 0, \forall m, n, \neq n.$

(ii) the vector space generated by this family is dense in E.

Definition 1.15 :[1](Separable space) A metric space is said to be separable if there exists a countable dense subset $D \subset E$.
Example 1.1 : The spaces $\mathbb{R}$ and $\mathbb{C}$ are separable (for example , $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$) .
Theorem 1.1 : [1] Every separable Hilbert space has a Hilbertian basis.
Proposition 1.3 : Let E be a Hilbert space for the scalar product $\langle ., . \rangle$, and let $(e_n)_{n\geq 1}$ be a Hilbertian basis of E. There exists a unique sequence $(u_n)_{n\geq 1}$ defined by

$$u_n = (u, e_n);$$

such that the partial sum

$$\sum_{k=1}^{p} u_n e_n;$$

converge to u when $p \to +\infty$.
Furthermore, we have

$$\|u\|^2 = (u, u) = \sum_{n\geq 1} u, e_n{}^2.$$

Then we write

$$u = \sum_{k=1}^{+\infty} (u, e_k) e_k.$$

1.2.5 Lebesgue space

Definition1.16 : Let E be a Banach space, $1 \leq p \leq \infty$ and $[0, T]$ an interval of $\mathbb{R}$. We call the **Lebesgue** space with values in E and denote by $L^p((0, T), E)$ the space of functions $u :]0, T[\to E$ that are measurable and satisfy

i) If $1 \leq p \leq \infty$, $\|u\|_{L^p((0,T),E)} = \left(\int_0^T \|u(x)\|^p dx\right)^{1/p} < \infty.$

ii) If $p = \infty$, $\|u\|_{L^\infty((0,T),E)} = \text{ess sup}_{x\in]0,T[} \|u(x)\| < \infty.$

Lemma 1.1:[7] Let $u \in L^p((0,T),E)$ and $\frac{\partial u}{\partial t} \in L^p((0,T),E)$ where $(1 \leq p \leq \infty)$, then the function u is continuous from $[0,T]$ in E (i.e. $u \in C^1((0,T),E)$).

1) For $1 \leq p \leq \infty$, $L^p((0,T),E)$ is a Banach space and in particular $L^2((0,T),E)$ is a Hilbert space when E a Hilbert space.

2) For $1 \leq p \leq \infty$, if E reflexive, then $L^p((0,T),E)$ is also reflexive.

3) For $1 \leq p \leq \infty$, if E separable, then $L^p((0,T),E)$ is also separable.

Proposition 1.4 :[8] The space $L^p((0,T),E)$ associated with the norm $\|\cdot\|_{L^p((0,T),X)}$ is a Banach space .

Proposition 1.5 :[9] Let E and F be two Banach spaces, $E \subset F$ with continuous injection then

$$L^p((0,T),E) \subset L^p((0,T),F);$$

with continuous injection.

The space $L^p(\Omega)$

Definition 1.17 : [1] Let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$, let

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R},\ u \text{ measurable and } \int_\Omega |u(x)|^p dx < +\infty \right\}.$$

We note

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

We will check later that $\|\cdot\|_{L^p}$ is a norm.

Definition 1.18 : [1] If $p = \infty$, we have

$$L^p(\Omega) = \left\{ \begin{array}{c} u : \Omega \to \mathbb{R},\ u \text{ measurable there exists a positive constant C such that} \\ |u(x)| \leq C \text{ p.p on } \Omega \end{array} \right\}.$$

We note

$$\|u\|_{L^\infty(\Omega)} = \inf\{C, |u(x)| \leq C \text{ p.p on } \Omega\}.$$

Definition 1.19 : [3] Let Ω an open of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. We define the space $L^p_{\text{loc}}(\Omega)$ by the set of $f : \Omega \to \mathbb{R}$ such that $f \in L^p(\Omega')$ for all Ω' and whose closure is compact in $\mathbb{R}^n$.

Proposition 1.6 :[10] L^p given its norm $\|\cdot\|_{L^p}$ is a Banach space for all $1 \leq p \leq \infty$.

Theorem 1.2 : [3] L^p is reflexive if and only if $1 < p < \infty$.

Theorem 1.3 : [3] L^p is separable for $1 \leq p \leq \infty$.

Proposition 1.7 : [10] L^∞ is a **non-separable Banach space**.

Lemma 1.2 : [6] If $\mu(\Omega) < \infty$, $1 \leq p \leq \infty$, then $L^q(\Omega) \to L^p(\Omega)$, and

$$\|u\|_{L^p(\Omega)} \leq \mu(\Omega)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^q(\Omega)}.$$

Theorem 1.4 : [1] Let (u_n) be a sequence of L^p and $u \in L^p$, such that $\|u_n - u\|_{L^p(\Omega)} \to 0$. Then there exists an extracted sub–sequence (u_{n_k}) such that

$$u_{n_k}(x) \to u(x) \quad \text{p.p in } \Omega.$$

Remark 1.2: Suppose $(1 \leq p \leq \infty)$ and E is a reflexive Banach space then weak star convergence is equivalent to weak convergence.

Theorem 1.5: [3] Let Ω be a bounded open space of $\mathbb{R}^n$.

1. If $u_n \rightharpoonup^* u$ in $L^\infty(\Omega)$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$, $\forall p \geq 1$.
2. If $u_n \rightharpoonup u$ in $L^p(\Omega)$, then $\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)}$ in $\mathbb{R}$, for all $1 \leq p \leq \infty$.
3. If $1 \leq p \leq \infty$ and $u_n \to u$ in $L^p(\Omega)$, then $\exists K > 0$ such that $\|u_n\|_{L^p(\Omega)} \leq K$ and

$$\|u\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|u_n\|_{L^p(\Omega)}.$$

   The result is also true if $p = \infty$ and $u_n \to u$ in $L^\infty(\Omega)$.
4. If $1 \leq p \leq \infty$ and if $\exists K > 0$ such that $\|u_n\|_{L^p(\Omega)} \leq K$, then there exists a sub–sequence u_{n_k} and $u \in L^p(\Omega)$ such that $u_{n_k} \rightharpoonup^* u$ in $L^p(\Omega)$. The result is also true if $p = \infty$ and then we have $u_{n_k} \rightharpoonup^* u$ in $L^\infty(\Omega)$.
5. If $1 \leq p \leq \infty$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$, then there exists a sub–sequence u_{n_i} such that $u_{n_i} \to u$ p.p in Ω and $|u_{n_i}| \leq h$ p.p with $h \in L^p(\Omega)$.

Lemma 1.3: [7] Let Ω be a bounded open of $\mathbb{R}^n \times \mathbb{R}$, u_m and u be functions of $L^p(\Omega)$, $1 < p < \infty$, such that

$$\|u\|_{L^p(\Omega)} \leq C, \qquad u_m \to u \quad \text{p.p in } \Omega.$$

Then $u_m \rightharpoonup u$ in $L^p(\Omega)$ weakly.

1.2.6 Sobolev space

Weak derivative

Definition 1.20: [3] Let Ω be an open set of $\mathbb{R}^n$ and $u \in L^1_{\text{loc}}(\Omega)$, we say that there exist a function $v \in L^1_{\text{loc}}(\Omega)$ if $f_i \in L^1_{\text{loc}}(\Omega)$ is the weak partial derivative of u with respect to x_i if

$$\int_\Omega u(x)\,\varphi(x)\,dx = -\int_\Omega u(x)\,\frac{\partial \varphi}{\partial x_i}(x)\,dx, \qquad \forall \varphi \in C_c^\infty(\Omega).$$

We denote $v = \frac{\partial u}{\partial x_i}$ or $u = u_{x_i}$.

Space $W^{1,p}(\Omega)$.

Definition 1.21: [1] For any open space of $\mathbb{R}^n$ and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, the space $W^{1,p}(\Omega)$ is defined by

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega),\ \exists f \in L^p(\Omega) \text{ such that } \int_\Omega u(x)\varphi'(x)\,dx = -\int_\Omega f(x)\varphi(x)\,dx \right\}.$$

We pose

$$H^1(\Omega) = W^{1,2}(\Omega).$$

Space $\mathrm{W}^{m,p}(\Omega)$.

Definition 1.22: [1] Let Ω an open space of $\mathbb{R}^n$, $m \geq 2$ and $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, the space $W^{m,p}(\Omega)$ is defined by

$$W^{m,p}(\Omega) = \left\{ \begin{array}{c} u \in L^p(\Omega)\ ;\ \forall \alpha \text{ with } |\alpha| \leq m,\ \exists f_\alpha \in L^p(\Omega) \text{ such that :} \\ \int_\Omega u D^\alpha \varphi\,dx = (-1)^{|\alpha|} \int_\Omega f_\alpha D^\alpha \varphi\,dx,\ \forall \varphi \in C_0^\infty(\Omega) \end{array} \right\}.$$

where $\alpha \in \mathbb{N}^n$, $|\alpha| = \sum_{i=1}^N \alpha_i$, and

$$D^\alpha \varphi = \frac{\partial^{\alpha_1+\alpha_2+....+\alpha_N}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\cdots\partial_{x_N}^{\alpha_N}}\varphi.$$

The space $W^{m,p}(\Omega)$ with norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0<|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

Definition 1.23: [1] If $p = 2$, we denote by $W^{m,2}(\Omega) = H^m$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$, provided by the norm

$$\|u\|_{H^m(\Omega)} = \left(\sum_{|\alpha|\leq m} (\|\partial^\alpha u\|_{L^2(\Omega)})^2 \right)^{1/2};$$

such that $H^m(\Omega)$ is a Hilbert space, with the scalar product

$$\langle u, v\rangle_{H^m(\Omega)} = \sum_{|\alpha|\leq m} \langle D^\alpha u, D^\alpha v\rangle_{L^2} = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u\ \partial^\alpha v\,dx, \qquad \forall u, v \in H^m(\Omega).$$

Proposition 1.8: [11]

1) The spaces $W^{m,p}(\Omega)$ are Banach spaces.

2) If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$, with continuous injection.

3) If $m = 0$ we have $W^{0,p}(\Omega) = L^p(\Omega)$.

1.3 Operators

Let H and H' be two Hilbert spaces over $\mathbb{C}$.

Definition 1.24: [9] Any linear and continuous application $T : H \to H'$ is called an operator.

Definition 1.25:[9] **Bounded operators**

Let $A : X \to Y$ be an operator. A is said a bounded if there exists a constant $C > 0$ such that

$$\|Au\| \leq C\|u\|, \qquad \forall u \in X.$$

Definition 1.26: (Symmetric operators [12])

The operator $A : H \to H$ is said symmetric if $\forall u, v \in D(A) \subset H$,

$$(Au, v) = (v, Au).$$

Definition 1.27: (adjoint operators [13])

Let $A : D(A) \subset E \to F$ a non-bounded operator with dense domain. There exists a unique operator $A^* : D(A^*) \subset F' \to E'$ called the adjoint of A which satisfies:

$$\forall u \in D(A),\ \forall v \in D(A^*); \quad (v, Au)_{F',F} = (A^*v, u)_{E',E}.$$

Where

$$D(A^*) = \{v \in F';\ \exists C \geq 0,\ |(v, Au)|_{F',F} \leq C\|u\|_E,\ \forall u \in D(A)\}.$$

Remark 1.3:

A is self-adjoint if

$$A^* = A.$$

This implies $D(A) = D(A^*)$.

Definition 1.28: **(Monotone operator [1])**

Let $A : D(A) \subset H \to H$ be a non-bounded linear operator. A is said a monotone (some authors say that A is **accretive** or that $-A$ is **dissipative**) if

$$(Av, v) \geq 0, \quad \forall v \in D(A).$$

A is **maximal monotone** if moreover $R(I + A) = H$, i.e.

$$\forall f \in H,\ \exists u \in D(A) \text{ such that } u + Au = f.$$

1.4 Some fundamental inequalities and formulae

Corollary 1.1: **([20] Green's formula)**
Let Ω be a bounded open of class C^1. Then for all functions $u \in C^2(\bar{\Omega})$ and $v \in C^2(\bar{\Omega})$ we have

$$\int_\Omega \Delta u(x)v(x)\,dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta}(x)\,v(x)\,d\sigma - \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx; \tag{1.4}$$

where

$$\frac{\partial u}{\partial \eta} = \nabla u(x)\cdot\eta(x);$$

is the normal derivative of u.

Lemma 1.4: **([1] Hölder inequality)**
Let $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ with $\frac{1}{p}+\frac{1}{p'}=1$ and $1 \le p < \infty$, $1 < p' \le \infty$. Then

$$\int_\Omega |uv|\,dx \le \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}.$$

Corollary 1.2: **([1] Poincaré's inequality)**
Assume that Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that

$$\|u\|_{L^p} \le C\,\|\nabla u\|_{L^p}, \qquad \forall u \in W_0^{1,p}(\Omega), \quad 1 \le p \le \infty.$$

Lemma 1.5: **([14] Young's inequality)**
Let $u, v \ge 0$, and for all $\delta > 0$ we have

$$uv \le \frac{\delta^p}{p}u^p + \frac{\delta^{-q}}{q}v^q, \qquad \frac{1}{p}+\frac{1}{q}=1.$$

Lemma 1.5: **([11] Young's inequality with ε)**
Let $u \ge 0$, $v \ge 0$ and p, q be two positive numbers such that

$$\frac{1}{p}+\frac{1}{q}=1.$$

For all $\varepsilon > 0$, we have

$$uv \le \varepsilon u^p + c(\varepsilon)\,v^q.$$

Theorem 1.6: **([5] Cauchy–Schwarz inequality)**
Let (u, v) be a scalar product on the real vector space V with norm $\|v\|^2 = \langle u, v\rangle$, then

$$\langle u, v\rangle^2 \le \|u\|^2_{L^2(\Omega)}\,\|v\|^2_{L^2(\Omega)};$$

for all vectors u and $v \in V$, equality occurring if u and v are linearly dependent.

1.5 Notion of local and global existence

1.5.1 Local and global existence and explosion

The study of local existence and solution's uniqueness for the problem of partial differential equations is based on the theory of existence for abstract semi-linear differential equations (see[16]) . Let $(X, \|\cdot\|)$ be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator, $f : X \to X$. Given $u_0 \in X$
consider the problem.

$$\begin{cases} \dfrac{du}{dt} - Au = f, \quad t > 0, \\ u(0) = u_0. \end{cases} \tag{1.5}$$

Definition 1.29:[17] A function u of the real variable $t \geq 0$, with values in X, is said alocal solution of the problem (1.5) , if there exists u defined on the maximal interval $[0, T^*))$ which for $t < T^*$ is the unique solution of (1.5) in $C^1\Big([0, T^*), X\Big)$. In particular, one of the following two eventualities occurs:

(i) $T^* = +\infty$.

(ii) $T^* < +\infty$ and

$$\lim_{t \to T^*} \|u(t)\| = +\infty.$$

Remark 1.4:[18] If property (i) is satisfied, we say that the solution u is global.
If (ii) is satisfied, we say that u is an exploding solution in finite time.

1.5.2 Lyapunov functional

Definition 1.30:[19] We call a Lyapunov functional associated with an evolution system formed by m equations any function

$$L : \mathbb{R}^+ \to \mathbb{R}^+$$

such that

$$\frac{d}{dt} L\Big(u_1(t), \ldots, u_m(t)\Big) \leq 0.$$

for all $t > 0$, and any solution $(u_1(t, .), \ldots, u_m(t, .))$ of the system .

Chapter 2

Historical Background and Mathematical Framework

2.1 Introduction

The study of differential equations plays a fundamental role in modern mathematics and physics. Partial differential equations show up in lots of phenomena like heat moving, fluids flowing and things bending. They also help us understand how things change over time.

For example evolution equations describe how physical systems change over time and give us a framework for understanding how things work in nature.

There are types of partial differential equations, but wave equations are one of the most important. Wave equations help us understand how disturbances move through materials like mechanical vibrations, sound waves, electromagnetic fields and elastic materials. People have been studying wave propagation for a time and it is still an active area of research.

The classical wave equation is one of the most studied models for understanding things that oscillate.

It shows up when we study vibrating strings, membranes and elastic structures. Over time our understanding of wave equations has gotten a lot better.We have developed new tools for studying them.

Many things in the real world cannot be described using simple linear models. Nonlinear effects show up in materials because of big deformations, complex interactions or material heterogeneity. That is why nonlinear wave equations are so important especially when we want to understand what happens over a time.

Another important thing that shows up in physical systems is memory effects and time delays. In some materials the current state depends on what happened in the past. We use memory terms to model these effects. Time delays also show up when the response

of the material is not instantaneous.

In years people have been studying wave equations that combine nonlinear effects, viscoelastic properties and delay terms. These models are hard to understand because of the way nonlinearity, memory effects and delay mechanisms interact.

One of the things that happens in nonlinear evolution equations is that solutions can blow up. This means that certain norms of the solution become unbounded in time. Studying blow-up phenomena gives us information about the limitations of the physical model.

The present study focuses on the blow-up behavior of solutions for a viscoelastic wave equation of Kirchhoffs type with delay time. We want to understand how the interaction between Kirchhoff-type nonlinearity, viscoelastic memory effects and delay terms affects the solutions. We are looking at viscoelastic wave equations because they are important, for understanding real-world phenomena [22].

2.2 The Classical Wave Equation

2.2.1 Physical origin of the wave equation

The wave equation shows up when we model things like vibrating strings, membranes and other elastic media.One of the situations that leads to the wave equation is a stretched string that is fixed at both ends.

When you move the string away from where its equilibrium position and let it go it starts to vibrate because of the tension pulling on it.Let us consider a string that is stretched along a line and is a certain length L. The strings vertical movement at a point x and time t is shown by a certain value $u(x,t)$.

If we assume the movements are small and the strings tension stays the same we can use Newtons law to figure out the equation that controls how it moves.

The string has a constant tension $T > 0$ and a certain mass per unit length$\rho > 0$.

If we look at a piece of the string between two points the total vertical force on it is because of the difference in tension forces at each end.Using Newtons law we get

$$\rho u_{tt} = T u_{xx}.$$

If we divide both sides, by ρ the small piece of string we get

$$u_{tt} = c^2 u_{xx};$$

where $c = \sqrt{T/\rho}$,certain value represents how fast the wave moves along the string. This is called the one- wave equation.It shows how disturbances move along the string at a speed c.[21]

2.2.2 Mathematical formulation

In a more general setting, the wave equation in a spatial domain $\Omega \subset R^n$ can be written as

$$u_{tt} - c^2 \Delta u = 0;$$

where u(x,t) denotes the displacement of the medium, and Δ is the Laplace operator defined by

$$\Delta u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i^2}.$$

The wave equation is typically studied together with appropriate initial conditions and boundary conditions. The initial conditions describe the initial state of the system:

- $u(x, 0) = u_0(x)$;
- $u_t(x, 0) = u_1(x)$.

These conditions specify the initial displacement and the initial velocity of the wave, respectively. In addition, boundary conditions are imposed depending on the physical configuration of the problem. For instance, if the endpoints of the string are fixed, the boundary condition takes the form $u(x, t) = 0$ on $\partial\Omega$[21].

2.2.3 Energy of the wave equation

One of the fundamental properties of the classical wave equation is the conservation of energy. The energy associated with the wave equation is defined by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t(x, t)|^2 + c^2 |\nabla u(x, t)|^2) dx.$$

The first term represents the kinetic energy, while the second term corresponds to the potential energy of the system. A direct computation shows that the energy is conserved over time, that is,

$$\frac{dE(t)}{dt} = 0.$$

Therefore,

$$E(t)=E(0);$$

for all $t > 0$. This property reflects the fact that, in the absence of damping or external forces, the total energy of the vibrating system remains constant. The conservation of energy plays a fundamental role in the mathematical analysis of wave equations and provides an essential tool for establishing existence and stability results[25].

2.2.4 Fundamental properties of solutions

The classical wave equation possesses several important mathematical properties that have been extensively studied in the literature. The existence and uniqueness of solutions to Cauchy's problem related the wave equation is among the most important results.

Assuming regularity conditions for the data a_0 and a_1 hold there exists exactly one solution for any pair of such initial conditions.

$$u \in C([0,\infty); H^1(\Omega));$$

with

$$u_t \in C([0,\infty); L^2(\Omega)).$$

A property of finite speed of propagation also exists. It says that disturbances within an initial data region propagate to arbitrary distances from their initial location at finite speeds governed by a constant c. In other words, the initial conditions can only affect an area equal to or less than a finite region with respect to its initial position, and that area will continue to grow (due to time) by the speed c of disturbances which were caused at time 0.

This property has important physical implications, as it reflects the fact that signals cannot travel instantaneously through the medium [21].

2.2.5 Motivation for more complex models

The classical wave equation gives an accurate account of many aspects of wave behavior;however,it relies on several simplifying assumptions. For example, the assumption we make about the tension of the medium being constant for all times during wave propagation is one originating from this oversimplification.

- the material response is purely elastic.
- the system has no memory of past states.

On the other hand, a lot of physical materials have more complicated behavior than this basic model can describe. For example, an elastic cord's tension may vary based on its entire amount of deformation. Some materials also show viscoelasticity, so the way in which they respond is related to how they've been deformed in the past. The limitations mentioned above have encouraged the design of advanced mathematical models such as **Kirchhoff type equations** ; **Viscoelastic wave equations**; and **Time Delay equations**. Researching these models has become a key area of study for current nonlinear partial differential equations (PDE's)[25].

2.3 Nonlinear Wave Equations

Non-linear wave equations typically arise due to the failure of linearity due to the nature of the system (the proportionality of the restoring force with the displacement). For example, in some cases, the restoring force may depend non-linearly (or vary in size) on changes in distance from equilibrium. A general non-linear wave equation can be expressed as follows:

$$u_{tt} - \Delta u + f(u) = 0.$$

The non-linear term f(u) represents many possible physical effects and can drastically alter the character of the solutions. Non-linearities produce new mathematical effects that may result in many different behaviours **(energy decay and long-term existence ,or finite time blow-up)** for non-linear wave equations. One of the most important topics in the study of nonlinear wave equations is the analysis of conditions under which solutions remain bounded for all time or become unbounded in finite time [22].

2.4 Kirchhoff Type Wave Equations

The classical wave equation assumes that the tension of a string is constant throughout its movement; however, there are many physical situations where the tension will depend on the overall deformation of the string. As a result, a more general model was developed, which is called **the Kirchhoff Equation**.

Initially published by Gustav Kirchhoff in 1883, the model deals with elastic strings' motion, and states that tension is variable (not constant) and dependent on the integral of the square of the (displacement) gradient. Therefore, the wave equation has the following expression.

$$u_{tt} - (a + b \int_\Omega |\nabla u|^2 dx)\Delta u = 0;$$

where a>0 and $b \geq 0$ are constants related to the physical properties of the string.

The main feature of this equation is the presence of the nonlocal term

$$\int_\Omega |\nabla u|^2 dx \ .$$

The overall amount of deformation energy in a system is represented by this term. Because of this term, the coefficient for the Laplacian Δ varies based on global behaviour of the solution, thus limiting how complex the mathematics used to describe an equation can be compared with traditional wave equations.

Mathematically, Kirchhoff-type equations are classified as hyperbolic, non-linear, partial differential equations having non-local coefficients. The non-local term presents several challenges analytically when studying the existence, uniqueness, and long term behaviour of solutions.

In the 20th century, there were serious investigations of rigorous mathematics concerning Kirchhoff type equations. Some of the most significant contributions came from **Jacques-Louis Lions**, who studied different classes of nonlinear boundary value problem and developed important mathematical tools for those studying equations of this type.

Since that time, Kirchhoff's equations have generated significant interest in published literature due to the many characteristics as well as the many types of research conducted on these equations.

Some researchers have focused on the existence of weak solutions, some have studied the global existence of solutions, while others have examined asymptotic properties and methods for determining stability. Furthermore, there are numerous applications of Kirchhoff type equations in relation to nonlinear vibrational problems, elastic body structures and wave propagation in various complex aqueous or non-aqueous mediums.

This has led many mathematicians to conduct a very large number of mathematical studies with the goal of demonstrating and understanding the qualitative behaviours of solutions. Researchers have been studying the Kirchhoff equation with additional physical phenomena like viscoelastic effects, memory terms, and time delays. These additional effects create even more messier and difficult to analyze mathematical models using advanced techniques of nonlinear functional analysis and methods of analysis applicable to partial differential equations [28].

2.5 Viscoelastic Wave Equations

Deformation of materials often does not exhibit an elastic-only response; many materials actually demonstrate viscoelastic response, i.e., the material's response at a specific time is affected by both the current deforming state of that material as well as the previous actions it incurred.

This type of behavior is encountered in polymers, biological tissues, and various engineering materials. Wave equations often model the dynamics of a viscoelastic body by employing a long time-memory term that expresses the material's ability to develop its present behaviour from previous behaviours.

An example of this type of wave equation may be expressed in the form

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds = 0;$$

where g(t) is called "the relaxation function" or "memory kernel".

The relaxation function (or rheological function) represents the rate of decay of the past deformations into the material as time elapses. Most models assume that the relaxation function will be a decreasing and positive function due to the fact that, over time, the effects of the prior conditions will diminish. By adding memory terms to the

governing equations, the qualitative behaviour of the solutions is significantly affected. Most importantly, energy dissipation is one result of these memory terms.

This indicates that the total amount of energy in an object will decrease over time. In a more physical sense, this is equivalent to internal friction that occurs in viscoelastic materials. Many researchers have studied the mathematical analysis of the viscoelastic wave equation.

One of the earliest studies on this subject was done by Constantine Dafermos who focused on the asymptotic stability of viscoelastic systems, producing some of the first important results regarding energy decay for these systems. Later on, several research studies extended these findings, and examined additional aspects of viscoelastic wave equations such as the existence of solutions, stability properties, and long-time behaviour. Memory terms in the data often require the application of special energy methods and functional analysis methods.

In addition to energy decay, viscoelastic systems may also exhibit more complex behaviors depending on the structure of the equation. In particular, when nonlinear effects are present, the competition between damping mechanisms and nonlinear sources may lead either to "global existence of solutions" or to "finite time blow-up".

These questions have motivated numerous mathematical studies aimed at understanding the conditions under which solutions remain bounded or become unbounded. Such investigations are particularly relevant for models that combine "viscoelastic effects", "nonlinear terms", and "time delay", which correspond to more realistic descriptions of many physical systems [29] .

2.6 Wave Equations with Time Delay

In many real-world systems encountered in everyday life, the response does not occur immediately after a force is applied.There is often a delay between applying the force and observing the effect of that force. The term used to describe this delay is "time delay". Time delays arise naturally in a variety of areas including: population dynamics, engineered control systems, and mechanical vibrations. When incorporating time delay effects into mathematical models for wave equations, you will see that those equations become much more complex than the original wave equation without a time delay included in the model. A typical wave equation with delay can be written as

$$u_{tt} - \Delta u + a u_t + b u_t(t-\tau) = 0;$$

where $\tau > 0$ represents the delay time. The fact that the delayed term $u_t(t-\tau)$ introduces a delay into the evolution of up-to-date systems and creates further complications, especially when determining the system's stability and long-term characteristics.

The stability of the different solutions to a delay equation is an essential aspect of studying delay equations. A system can experience destabilization due to the influence of delay even if damping is present in the system. Functional differential equation theory (with delay) has been developed extensively by Jack Hale who established the foundations of dynamical system analysis involving delayed effects.

Research has been conducted on how the interaction of delay term with nonlinear and damping mechanisms within the context of wave equations behave. The solutions can display very different qualitative characteristics, depending upon the relative strength of each of these three inducing mechanisms [30]:

- Stabilization of the solution;
- oscillatory or periodic dynamics of the solution
- blow-up of the solution.

2.7 Blow-up Phenomena in Nonlinear Wave Equations

Finite-time blow-ups in solutions of nonlinear wave equations represent one of the most exciting and difficult to study phenomena. A "blow-up" means that certain norms of the solution ,such as (the energy or the L^∞ norm) become infinite in a finite interval of time. Such a blow-up indicates that the solution has lost its regularity and is often associated with the emergence of singularities within it.

Mathematically, for a solution $u(x,t)$ of a nonlinear wave equation, we say that a blow-up occurs at time $T^* < \infty$ if

$$\lim_{t\to T^*} \| u(\cdot,t) \|_X = \infty;$$

where $|\cdot|_X$ is an appropriate norm, often $H^1(\Omega)$ or $L^\infty(\Omega)$[27].

2.7.1 Mechanisms of Blow-up

Blow-Up Can Happen for Many Reasons [26]:

1. **Strong Nonlinear Sources:** If the nonlinear term f(u) grows quickly compared to damping or dispersive effects, it will cause blow-up. For example in

$$u_{tt} - \Delta u + |u|^p = 0;$$

with a blow-up exponent greater than a critical growth rate, this could also happen with regard to space dimensions.

2. **Insufficient Damping:** The system could still blow up due to insufficient damping or other viscoelastic mechanisms being ineffective in providing sufficient counteraction to the nonlinear growth.

3. **Effects Due To Delay:** Some time delay terms may destabilise the system; thus potentially accelerating blow-up in some cases. For example, if the damping term were delayed, then that may result in less dissipation of energy .

2.7.2 Blow-up Criteria

The presence of blow-up (an explosion of a solution in finite time) has been established through the application of various criteria [27].

- **Energy Methods:** You can estimate the energy of the solution using a proper Lyapunov functional. If the energy increases to infinity in a finite amount of time, it follows that there will be a blow-up occurrence. For example, define the energy.

$$E(t) = \tfrac{1}{2} \parallel u_t(t) \parallel^2 + \tfrac{1}{2} \parallel \nabla u(t) \parallel^2 - \textstyle\int_\Omega F(u)dx;$$

  where$F(u) = \int_0^u f(s)ds$. Under certain initial conditions, one can show that E(t) becomes infinite in finite time.

- **Concavity Method (Levine, 1974):** Let $y(t) = \int_\Omega u^2 dx$. If the function y(t) has a differential inequality of the form

$$y''(t) \geq Cy(t)^q, C > 0, q > 1.$$

  Then you may conclude that the solution will "blow up" in a finite amount of time. Then y(t) blows up in finite time. This method is widely used in the study of nonlinear wave and Kirchhoff type equations.

2.7.3 Blow-up in Viscoelastic and Delay Systems

For viscoelastic wave equations, the interplay between nonlinearity and memory terms determines whether blow-up occurs:

$$u_{tt} - \Delta u + \textstyle\int_0^t g(t-s)\Delta u(s)ds + |u|^p = 0.$$

- If the initial energy is large and the damping provided by the memory term is insufficient, the solution may blow up.

- Precise blow-up time estimates can be obtained using modified energy functionals that take the memory kernel into account.

For wave equations with delay, the situation is more delicate:

$$u_{tt} - \Delta u + au_t + bu_t(t-\tau) + |u|^p = 0.$$

- Delay may enhance or suppress blow-up depending on the relative size of a,b,τ and the exponent p.
- Researchers have developed blow-up criteria that combine energy methods with delay estimates [27].

2.7.4 Key Results in the Literature

Some of the things that people have found out include [21]:

- **Levine in 1974:** He came up with a way to figure out when things blow up in wave equations.

He called this the concavity method for blow-up in wave equations.

- **Glassey in 1973:** He showed that blow-up can happen in equations with power-type nonlinearities.

He established some theorems for this.

- **Dafermos in 1970:** He looked at how energy decreases and blow-up happens in viscoelastic systems.
- **Pignotti, in 2006:** They saw how delay terms affect blow-up and stability in these systems.

2.8 Combined Effects and the Mathematical Model

After examining the individual contributions of nonlinear inertia, strong damping, viscoelastic memory, and time-delayed feedback, we now present a unified framework that incorporates all these mechanisms. This comprehensive model allows for the analysis of wave propagation, transient behaviors, and potential finite-time blow-up in a physically realistic setting. The governing equation is:

$|u_t|^p u_{tt} - \Delta u - \Delta u_t + \int_0^t g(t-s)\Delta u(s)ds + \mu_1 u_t + \mu_2 u_t(t-\tau) = b|u|^{p-2}u$, x$\in \Omega, t > 0$

where:

- u(x,t) is the displacement field,
- $\rho \geq 0$ determines the degree of nonlinear inertia,

- $-\Delta u$ models standard wave propagation,
- $-\Delta u_t + \mu_1 u_t$ represents strong damping,
- $\int_0^t g(t-s)\Delta u(s)ds$ is the viscoelastic memory effect,
- $\mu_2 u_t(t-\tau)$ accounts for time-delayed damping,
- $b\,|u|^{p-2}\,u$ is the nonlinear source term responsible for energy input.

The initial and boundary conditions are:

$$\begin{cases} u(x,0) = u_0(x), \\ u_t(x,0) = u_1(x), \\ u(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau), \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t > 0 \end{cases} \tag{2.1}$$

This integrated formulation captures the combined influence of all physical mechanisms and serves as the basis for further analytical investigations, including:

- Energy estimates that incorporate both memory and delayed effects,
- Criteria for finite-time blow-up depending on the parameters $(\rho, p, \mu_1, \mu_2, \tau)$ and initial data,
- Stability analysis and qualitative understanding of the interplay between damping, memory, delay, and nonlinear source.

By presenting all effects in a single mathematical framework, this model provides a comprehensive and realistic foundation for studying complex wave phenomena in nonlinear viscoelastic media with delayed feedback [31].

Chapter 3

BLOW-UP OF SOLUTION FOR A VISCOELASTIC WAVE EQUATION WITH DELAY

3.1 Introduction

In this chapter, we study a nonlinear viscoelastic equation with delay defined by :

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = b|u|^{p-2}u, \quad \text{in } \Omega\times(0,\infty), \tag{3.1}$$

$$u_t(x,t-\tau) = f_0(x,t-\tau), \quad x \in \Omega,\ t \in (0,\tau), \tag{3.2}$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \tag{3.3}$$

$$u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \tag{3.4}$$

where $\rho > 0$,b and μ_1 are positive constants, μ_2 is a real number, $\tau > 0$ represents the time delay, g is the kernel of the memory term, the initial data (u_0, u_1, f_0) are given functions belonging to suitable spaces, and $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$.

We study here a viscoelastic kirchhoff wave equation with a delay term. The integral term represents the memory effect that appears in viscoelastic materials, since the present state of the system depends on its previous history.

The delay term also plays an important role in the behavior of solution.

In the fact, both of the memory effect and the delay may influence the stability, decay, or blow up of solutions.

Our object is to determine the sufficient conditions under which the solution blows up in finite time.

3.2 Preliminary Result

In this section, we shall give some lemmas and assumptions which will be used throughout this work. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual products and norms.

Lemma 3.1.(see [32])Let $2 \leq p \leq \frac{2N}{N-2}$, the inequality

$$\|u\|_p \leq c_s \|\nabla u\|_2, \quad \text{for } u \in H_0^1(\Omega). \tag{3.5}$$

Holds with some positive constant c_s.

Assume that ρ and p satisfy

$$0 < \rho \leq \frac{2}{N-2} \quad \text{if } N \geq 3 \quad \text{or } \rho > 0 \text{ if } N = 1, 2; \tag{3.6}$$

and

$$2 < p \leq \frac{2(N-1)}{N-2} \quad \text{if } N \geq 3 \quad \text{or } p > 2 \text{ if } N = 1, 2. \tag{3.7}$$

Regarding the kernel function $g(t)$, we assume that it verifies

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function satisfying

$$g(0) > 0, \quad g'(s) \leq 0 \quad \text{and} \quad 1 - \int_0^\infty g(s)ds = l > 0. \tag{3.8}$$

We also need the following lemma in the course of the investigation.

Lemma 3.2: (see [32]). Suppose that (3.7) holds, then there exists a positive constant C such that

$$\|u\|_p^p \leq C \left(\|\nabla u\|_2^2 + \|u\|_p^p \right) \tag{3.9}$$

for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.

Lemma 3.3 : For any function $g \in C^1(\mathbb{R})$ and $u \in H_0^1(0, T; L^1(\Omega))$, we have

$$\begin{aligned} \int_\Omega \nabla u_t(t) \int_0^t g(t-s) \nabla u(s)\, ds\, dx = & \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}\frac{d}{dt}(g \circ \nabla u)(t) \\ & - \frac{1}{2} g(t) \int_\Omega |\nabla u(t)|^2\, dx \\ & + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t)\, ds \int_\Omega |\nabla u(s)|^2\, dx \right); \end{aligned} \tag{3.10}$$

or

$$(g \circ u)(t) := \int_0^t g(t-s) \int_\Omega |u(t) - u(s)|^2\, dx\, ds. \tag{3.11}$$

In order to state our result, we make an extra assumption on g:

$$\int_0^\infty g(s)\, ds < \frac{1}{1 + \frac{1}{(p(1-\beta)^2 + 2\beta(1-\beta))(p-2)}}; \tag{3.12}$$

where $0 < \beta < 1$ is a fixed number.

Remark 1 : We note that condition (3.7) on g with $0 < \beta < 1$ coincides with those considered in [10,11,14]. However, our result does not include $\beta = 0$ which includes in [10,11,14].

Equivalent problem:

We introduce the following new dependent variable:(For more details,see([33])),

$$z(x, \kappa, t) = u_t(x, t - \tau\kappa), \quad x \in \Omega, \quad \kappa \in (0, 1); \tag{3.13}$$

we have

- For $\kappa = 0$: $z(x, 0, t) = u_t(x, t)$;

  and

- For $\kappa = 1$: $z(x, 1, t) = u_t(x, t - 1)$;

and we have :

$$z_t = \frac{\partial}{\partial t} u_t(x, t - \tau\kappa) = u_{tt}(x, t - \tau\kappa); \tag{3.14}$$

and

$$z_\kappa = \frac{\partial}{\partial t} u_t(x, t - \tau\kappa) = -\tau u_{tt}(x, t - \tau\kappa); \tag{3.15}$$

which implies that

$$\tau z_t(x, \kappa, t) + z_\kappa(x, \kappa, t) = 0 \quad \text{in } \Omega \times (0, 1) \times (0, \infty). \tag{3.16}$$

Therefore, problem (3.1)-(3.4) can be transformed as follows:

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_t + \displaystyle\int_0^t g(t - s)\Delta u(s)ds + \mu_1 u_t(x, t) \\ \qquad + \mu_2 z(x, 1, t) = b|u|^{p-2}u \quad \text{in } \Omega \times (0, \infty), \\ \tau z_t(x, \kappa, t) + z_\kappa(x, \kappa, t) = 0, \quad x \in \Omega, \ \kappa \in (0, 1), \ t > 0, \\ z(x, 0, t) = u_t(x, t), \quad x \in \Omega, \ t > 0, \\ z(x, \kappa, 0) = f_0(x, -\tau\kappa), \quad x \in \Omega, \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega, \\ u(x, t) = 0, \quad x \in \partial\Omega, \ t \geq 0. \end{cases} \tag{3.17}$$

Let ξ is a positive constant such that

$$\tau|\mu_2| \leq \xi \leq \tau(2\mu_1 - |\mu_2|). \tag{3.18}$$

Where:

$$\tau|\mu_2| \leq \xi \leq \tau(2\mu_1 - |\mu_2|) \qquad \text{if} \qquad \mu_2 < \mu_1;$$

and

$$\xi = \mu_1 \qquad \text{if} \qquad \mu_2 = \mu_1.$$

3.3 Local solution

Theorem 3.1: Suppose that $\mu_2 < \mu_1$, (A1), and (3.6)-(3.7) hold. Assume that $u_0, u_1 \in H_0^1(\Omega)$ and $f_0 \in L^2(\Omega \times (0,1))$. Then there exists a unique solution (u, z) of (3.17) satisfying

$$u, u_t \in C([0,T]; H_0^1(\Omega));$$

$$z \in C([0,T]; L^2(\Omega \times (0,1))).$$

For $T > 0$.

Proof. See [34] □

3.4 Blow-Up

In this section, we study the blow-up of solution, with nonpositive initial energy and positive initial energy .
First,we define the energy functional associated with problem (3.17) as follows:

$$\begin{aligned} E(t) =& \frac{1}{p+2}\|u_t\|_{p+2}^{p+2} + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right)\|\nabla u(t)\|_2^2 + \frac{1}{2}(g \circ \nabla u)(t) \\ &+ \frac{1}{2}\|\nabla u_t(t)\|_2^2 + \frac{\xi}{2}\int_\Omega \int_0^1 z^2(x,\kappa,t)\,d\kappa dx - \frac{b}{p}\|u\|_p^p. \end{aligned} \tag{3.19}$$

Proof. Multiplying equation (3.17) byu_t and integrating over Ω,we obtain:

$$\begin{aligned} \int_\Omega |u_t|^\rho u_{tt} u_t dx - \int_\Omega \Delta u u_t dx - \int_\Omega \Delta u_t u_t dx + \int_\Omega \int_0^t g(t-s)\Delta u(s) ds u_t dx + \mu_1 \int_\Omega u_t^2(x,t) dx \\ + \mu_2 \int_\Omega z(x,1,t) u_t dx = \int_\Omega b|u|^{p-2} u u_t dx. \end{aligned} \tag{3.20}$$

We have :

$$\int_\Omega |u_t|^p u_{tt} u_t dx = \frac{1}{p+2}\frac{d}{dt}\int_\Omega |u_t|^{p+2} dx = \frac{1}{p+2}\frac{d}{dt}\|u_t\|_{p+2}^{p+2}; \tag{3.21}$$

and

$$\int_\Omega \mu_1 u_t(x,t) u_t dx = \mu_1 \int_\Omega u_t^2 dx = \mu_1 \|u_t\|_2^2; \tag{3.22}$$

Furthermore;

$$-\int_\Omega b\|u_t\|^{p-2}uu_t dx = -\frac{b}{p}\frac{d}{dt}\|u\|_p^p; \tag{3.23}$$

and

$$\begin{aligned}\int_\Omega \mu_2 z(x,1,t)u_t dx = \mu_2\int_\Omega z(x,1,t)z(x,0,t)dx = \frac{\mu_2\tau}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)d\kappa dx \\ = \frac{\xi}{2}\int_\Omega\int_0^1 z^2(x,\kappa,t)d\kappa dx;\end{aligned} \tag{3.24}$$

where $\xi$is a positive constant satisfying (3.18).
Applying green's formula,we get :

$$-\int_\Omega \Delta u u_t dx = -\int_{\partial\Omega}\nabla u u_t + \int_\Omega \nabla u\nabla u_t dx = \int_\Omega \nabla u\nabla u_t dx = \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2; \tag{3.25}$$

and

$$-\int_\Omega \Delta u_{tt} u_t dx = -\int_{\partial\Omega}\nabla u_{tt} u_t + \int_\Omega \nabla u_{tt}\nabla u_t dx = \int_\Omega \nabla u_{tt}\nabla u_t dx = \frac{1}{2}\frac{d}{dt}\|\nabla u_t\|_2^2. \tag{3.26}$$

By substituting each of [(3.10);(3.21);(3.22);(3.23);(3.24);(3.25);(3.26)] into(3.20),we obtain the energy equation (3.19). □

Lemma 3.4: $E(t)$ is a nonincreasing function and

$$\begin{aligned}E'(t) &\le -\alpha\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)\,dx\right) + \frac{1}{2}(g\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &\le -\alpha\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)\,dx\right) \le 0, \quad \forall t\ge 0;\end{aligned}$$

where

$$\alpha = \min\left\{\mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}, \frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right\},$$

which is positive by (3.18).

Proof. From the previous proof of the energy function, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|\nabla u_t\|_2^2 + (1-\int_0^t g(t-s)ds)\|\nabla u\|_2^2 + (g\circ\nabla u)\right] + \frac{1}{p+2}\frac{d}{dt}\|\ u_t\|_{p+2}^{p+2} + \frac{b}{p}\|u\|_p^p \\ = -\mu_1\|u_t\|_2^2 - \frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u) - \mu_2\int_\Omega z(x,1,t)u_t dx.\end{aligned} \tag{3.27}$$

Multiplying equation(3.16) by $\frac{\xi}{\tau}z$ and integrating over $(0,1)\times\Omega$ with respect to (t,x),

we get

$$\begin{aligned}\frac{d}{dt}E(t) \leq & -\mu_1\|u_t\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ & - \mu_2\int_\Omega u_t z(x,1,t)\,dx - \frac{\xi}{\tau}\int_\Omega\int_0^1 z z_\kappa(x,\kappa,t)\,d\kappa dx.\end{aligned} \tag{3.28}$$

Since:

$$\xi\int_\Omega\int_0^1 z z_t d\kappa + \tfrac{\xi}{\tau}\int_\Omega\int_0^1 z z_\kappa d\kappa dx = 0;$$

whereas

$$\xi\int_\Omega\int_0^1 z z_\kappa d\kappa dx = \frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2 d\kappa dx. \tag{3.29}$$

And

$$\xi\int_\Omega\int_0^1 z z_t d\kappa dx = \frac{\xi}{2}\int_\Omega [z^2(x,1,t) - z^2(x,0,t)]dx. \tag{3.30}$$

We replace $z(x,0,t) = u_t(x,t)$:

$$\xi\int_\Omega\int_0^1 z z_t d\kappa dx = \frac{\xi}{2}\int_\Omega [z^2(x,1,t) - u_t^2(x,t)]dx; \tag{3.31}$$

therefore

$$\frac{\xi}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2 d\kappa dx = \frac{-\xi}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi}{2\tau}\int_\Omega u_t^2(x,t)dx. \tag{3.32}$$

Replacing (3.32) in (3.28) , we obtain :

$$\begin{aligned}E'(t) \leq -\mu_1\|u_t\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2 - \mu_2\int_\Omega u_t z(x,1,t)dx \\ -\frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi}{2\tau}\int_\Omega u_t^2(x,t)dx.\end{aligned} \tag{3.33}$$

We have (**by young's inequality**[14]):

$$-\mu_2\int_\Omega z(x,1,t)u_t \leq \frac{\mu_2}{2}\int_\Omega z^2 dx + \frac{\mu_2}{2}\|\ u_t\|_2^2; \tag{3.34}$$

replacing (3.34) in (3.33) , we obtain

$$\begin{aligned}E'(t) &\leq -\mu_1\|u_t\|_2^2 + \frac{1}{2}(g'\circ\nabla u)(-\frac{1}{2}g(t)\|\nabla u(t)\|_2^2 \\ &+ \frac{\mu_2}{2}\int_\Omega z^2 dx + \frac{\mu_2}{2}\|\ u_t\|_2^2 - \frac{\xi}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\xi}{2\tau}\int_\Omega u_t^2(x,t)dx. \\ &\leq -c_1\|u_t\|_2^2 - c_2\int_\Omega z^2(x,1,t)\,dx + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|_2^2. \\ &\leq -\alpha\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)\,dx\right) \leq 0, \quad \forall t > 0.\end{aligned} \tag{3.35}$$

Where :$c_1 = \mu_1 - \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} > 0$; $c_2 = \frac{\xi}{2\tau} - \frac{|\mu_2|}{2} > 0$ and $\alpha = \min\{c_1, c_2\}$. □

Then, from (3.19),(3.8) and Lemma 3.1,we have:

$$\begin{aligned} E(t) &\geq \frac{1}{2}l\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{b}{p}\|u(t)\|_p^p \\ &\geq \frac{1}{2}l\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t) - \frac{bB_1^p}{p}(l^{\frac{1}{2}}\|\nabla u(t)\|_2)^p \\ &\geq F(\sqrt{l\|\nabla u(t)\|_2^2 + \frac{1}{2}(g\circ\nabla u)(t)}), \quad t\geq 0. \end{aligned} \tag{3.36}$$

Where $\mathrm{B}_1 = \frac{c_s^p}{l^{\frac{p}{2}}}$, *and*

$$F(x) = \frac{1}{2}x^2 - \frac{bB_1^p}{p}x^p, \quad x>0. \tag{3.37}$$

Remark 2 : We can verify that the functional F is increasing in $(0,\lambda_1)$, decreasing in (λ_1,∞), and F has a maximum at

$$\lambda_1 = b^{-\frac{1}{p-2}}B_1^{-\frac{p}{p-2}};$$

with the maximum value

$$E_1 = F(\lambda_1) = \frac{p-2}{2p}b^{-\frac{2}{p-2}}B_1^{-\frac{2p}{p-2}} = \frac{p-2}{2p}\lambda_1^2.$$

Proof. Derive (3.37) with respect to t and obtain

$$F'(x) = x - bB_1^p x^{p-1}; \tag{3.38}$$

let λ_1 be a positive constant, such that

$$F'(\lambda_1) = 0.$$

From(3.38) , we have :

$$\lambda_1 = (b)^{\frac{-1}{p-2}}(B_1)^{\frac{-p}{p-2}}.$$

This implies

$$\begin{aligned} F(\lambda_1) &= \frac{1}{2}\lambda_1^2 - \frac{bB_1^p}{p}\lambda_1^p; \quad \lambda_1^p = (b^{-\frac{1}{p-2}}B^{-\frac{p}{p-2}})^p = b^{-\frac{p}{p-2}}B^{-\frac{p^2}{p-2}}; \quad \lambda_1>0; \\ &= \frac{1}{2}b^{-\frac{2}{p-2}}B^{-\frac{2p}{p-2}} - \frac{bB_1^p}{p}b^{-\frac{p}{p-2}}B^{-\frac{p^2}{p-2}} \\ &= \frac{1}{2}b^{-\frac{2}{p-2}}B^{-\frac{2p}{p-2}} - \frac{1}{p}b^{-\frac{2}{p-2}}B^{-\frac{2p}{p-2}} \\ &= \frac{p-2}{2p}b^{-\frac{2}{p-2}}B^{-\frac{2p}{p-2}} \\ &= \frac{p-2}{2p}\lambda_1^2. \end{aligned} \tag{3.39}$$

The function F is increasing in $(0, \lambda_1)$,and decreasing on (λ_1, ∞). □

Lemma 3.2: (see [34]) Suppose that (3.6)–(3.7) and (A1) hold and assume further that

$$l\|\nabla u_0\|_2^2 > \lambda_1^2 \quad \text{and} \quad E(0) < E_1;$$

then there exists $\lambda_2 > \lambda_1$ such that, for all $t \in [0, T)$.

$$l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2; \tag{3.40}$$

and

$$\|u\|_p^p \geq \frac{bB_1^p}{p}\lambda_2^p. \tag{3.41}$$

Proof. Since $E(t)$ is non-increasing, it follows that

$$E(t) \leq E(0) < E_1.$$

For all $t > 0$. and based on the properties of the functional F stated in Remark 3.2, there exist constants $\lambda_2' < \lambda_1 < \lambda_2$ such that

$$F(\lambda_2') = F(\lambda_2) = E(0).$$

Moreover,since $F(0) = 0$,and $l\|\nabla u_0\|_2^2 > \lambda_1^2$,together with the time continuity of

$$F\left(\sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)}\right);$$

we deduce that

$$l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2, \quad \forall t \in [0, T).$$

On the other hand, from the definition of $E(t)$ in (3.19), and using (3.8), (3.40), along with the definition of F (3.10), we obtain

$$\|u\|_p^p \geq \frac{1}{2}\left(l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) - E(0)\right) \geq \frac{1}{2}\lambda_2^2 - F(\lambda_2) = \frac{bB_1^p}{p}\lambda_2^p.$$

Therefore,the proof is complete. □

Theorem 3.2 Let (3.6)–(3.7), (3.12)–(3.18) and (A1) hold. Suppose that $u_0, u_1 \in H_0^1(\Omega)$ with

$$l\|\nabla u_0\|_2^2 > \lambda_1^2, \quad E(0) < \beta E_1.$$

Assume further that $\rho < p - 2$. Then, the solution u of (3.17) blows up in finite time T^* estimated by

$$T^* \leq \frac{(1-\sigma)}{\sigma c_{12}[L(0)]^{\frac{\sigma}{1-\sigma}}}.$$

Where $c_{12} = \frac{\varepsilon k}{c_{11}}$is positive constant. .

Proof. By contradiction, assume contrary to what is to be proven ,that the solution of problem (3.17) is global, then there exists a constant $K_1 > 0$ such that

$$\|u_t\|_{\rho+2}^{\rho+2} + \|u(t)\|_p^p + \|\nabla u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 \leq K_1, \quad \forall t \geq 0. \tag{3.42}$$

Define

$$H(t) = E_2 - E(t); \tag{3.43}$$

since $E_2 \in (E(0), \beta E_1)$. And by Lemma 3.1, (3.17) and $E_1 = \frac{p-2}{2p}\lambda_1^2$, we obtain

$$H(t) \geq H(0) = E_2 - E(0) > 0; \tag{3.44}$$

and

$$\begin{aligned} H(t) &\leq \beta E_1 - \frac{1}{2}\left(l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t)\right) + \frac{b}{p}\|u\|_p^p \\ &\leq \beta E_1 - \frac{1}{2}\lambda_1^2 + \frac{b}{p}\|u\|_p^p \\ &\leq E_1 - \frac{1}{2}\lambda_1^2 + \frac{b}{p}\|u\|_p^p \quad \text{such that} \qquad E_1 - \frac{1}{2}\lambda_2^2 \leq 0 \\ &\leq \frac{b}{p}\|u\|_p^p. \end{aligned} \tag{3.45}$$

Therefore we Define the following functional

$$L(t) = H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^\rho u_t u dx + \frac{\mu_1 \varepsilon}{2}\int_\Omega u^2(x,t)\,dx + \varepsilon \int_\Omega \nabla u_t \cdot \nabla u\,dx; \tag{3.46}$$

where $0 < \varepsilon < 1$ to be chosen later and

$$0 < \sigma < \min\left\{\frac{p-2}{2p}, \frac{1}{\rho+2} - \frac{1}{p}\right\}. \tag{3.47}$$

By differentiating equation (3.45), we get the following result

$$\begin{aligned} L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega |u_t|^\rho u_t u + \frac{\varepsilon}{p+1}\int_\Omega |u_t|^{p+2}dx + \mu_1\varepsilon\int_\Omega u u_t dx \\ +\varepsilon\int_\Omega \nabla u_{tt}\nabla u dx + \varepsilon\int_\Omega |\nabla u_t|^2 dx. \end{aligned} \tag{3.48}$$

We have :

$$\tfrac{d}{dt}H^{1-\sigma}(t) = (1-\sigma)H^{-\sigma}(t)H'(t);$$

and,

$$\frac{d}{dt}\frac{\varepsilon}{p+1}\int_\Omega |u_t|^\rho u_t u dx = \varepsilon\int_\Omega |u_t|^\rho u_t u + \frac{\varepsilon}{p+1}\int_\Omega |u_t|^{p+2} dx \ ;$$

and

$$\frac{d}{dt}\frac{\mu_1\varepsilon}{2}\int_\Omega u^2 dx = \mu_1\varepsilon\int_\Omega u u_t dx;$$

and

$$\frac{d}{dt}\varepsilon\int_\Omega \nabla u_t \nabla u dx = \varepsilon\int_\Omega \nabla u_{tt}\nabla u dx + \varepsilon\int_\Omega |\nabla u_t|^2 dx.$$

By using (3.35) and Green's formula (see [20]) , we find

$$\begin{aligned}\varepsilon\int_\Omega |u_t|^\rho u_{tt} u dx = -\varepsilon\|\nabla u\|_2^2 + \varepsilon\int_\Omega \nabla u(t)\int_0^1 g(t-s)\nabla u(s) ds dx - \varepsilon\mu_2\int_\Omega z(x,1,t)u dx \\ -\varepsilon\mu_1\int_\Omega u_t u dx + \varepsilon b\|u\|_p^p - \varepsilon\int_\Omega \nabla u_{tt}\nabla u dx\end{aligned} \tag{3.49}$$

Replacing (3.49) in (3.48) , we obtain

$$\begin{aligned}L'(t) = (1-\sigma)H^{-\sigma}(t)H'(t) + \frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}\,dx - \varepsilon\|\nabla u_t\|_2^2 \\ - \mu_2\varepsilon\int_\Omega u(x,t)z(x,1,t)\,dx + \varepsilon\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx \\ + \varepsilon\|\nabla u_t(t)\|_2^2 + \varepsilon b\|u\|_p^p.\end{aligned} \tag{3.50}$$

Employing Hölder's inequality(see[1]) and Young's inequality (see[14]), for $\delta,\eta > 0$, we get

$$\left|\mu_2\varepsilon\int_\Omega u(x,t)z(x,1,t)\,dx\right| \le |\mu_2|\varepsilon\left(\delta\int_\Omega u^2\,dx + \frac{1}{4\delta}\int_\Omega z^2(x,1,t)\,dx\right); \tag{3.51}$$

and

$$\begin{aligned}\varepsilon\int_\Omega \nabla u(t)\cdot\int_0^t g(t-s)\nabla u(s)\,ds\,dx &= \varepsilon\int_0^t g(t-s)\int_\Omega \nabla u(t)\cdot(\nabla u(s)-\nabla u(t))\,dx\,ds \\ &\quad + \varepsilon\int_0^t g(s)\,ds\|\nabla u(t)\|_2^2 \\ &= \varepsilon\|\nabla u(t)\|_2^2\int_0^t g(s)ds \\ &\quad + \varepsilon\int_\Omega \nabla u(t)\int_0^1 g(t-s)(\nabla u(s)-\nabla u(t))\,ds\,dx.\end{aligned} \tag{3.52}$$

We can see that

$$\begin{aligned}\varepsilon\int_\Omega \nabla u(t)\int_0^1 g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\,dx &\geq -\frac{\varepsilon}{4\eta}\int_0^t g(s)\,ds\|\nabla u(t)\|_2^2\\ &\quad -\eta\varepsilon\int_\Omega g(t-s)|\nabla u(s)-\nabla u(t)|^2\\ &\geq \frac{\varepsilon}{4\eta}\|\nabla u(t)\|_2^2\int_0^t g(s)\,ds-\eta\varepsilon(g\circ\nabla u)(t);\end{aligned} \tag{3.53}$$

from (3.52)and (3.53)

$$\varepsilon\int_\Omega \nabla u(t)\int_0^1 g(t-s)|\nabla u(s)-\nabla u(t)|\,ds\,dx \geq -\eta\varepsilon(g\circ\nabla u)(t)+\varepsilon(1-\frac{1}{4\eta})\|\nabla u(t)\|_2^2\int_0^t g(s)\,ds. \tag{3.54}$$

Substituting (3.54) into (3.50),we obtain

$$\begin{aligned}L'(t)\geq{}&(1-\sigma)H^{-\sigma}(t)H'(t)+\frac{\varepsilon}{\rho+1}\int_\Omega u_t^{\rho+2}\,dx-\varepsilon\eta(g\circ\nabla u)(t)\\ &+\varepsilon\left(-1+\left(1-\frac{1}{4\eta}\right)\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2+\varepsilon\|\nabla u_t(t)\|_2^2\\ &-|\mu_2|\varepsilon\left(\delta\|u\|_2^2+\frac{1}{4\delta}\int_\Omega z^2(x,1,t)\,dx\right)+\varepsilon b\|u\|_p^p.\end{aligned} \tag{3.55}$$

We choose $\sigma_1=\frac{|\mu_2|}{4\alpha\kappa}H^\sigma(t)$ such that k is a large positive constant to be determined later,we obtain

$$-|\mu_2|\varepsilon\delta\ \|u\|_2^2=-\frac{|\mu_2|^2}{4\alpha k}\varepsilon H^\delta(t)\|u\|_2^2; \tag{3.56}$$

and

$$-|\mu_2|\frac{\varepsilon}{4\delta}\int_\Omega z^2(x,1,t)dx=-\varepsilon\alpha kH^{-\delta}(t)\int_\Omega z^2(x,1,t)dx; \tag{3.57}$$

by (3.35) and (3.43) we have

$$-\int_\Omega z^2(x,1,t)dx\geq-\frac{1}{\alpha}H'(t). \tag{3.58}$$

Frome $H(t)=E_2-E(t)$, we obtain $H'(t)=-E'(t)$ Where :

$$E'(t)\leq-\alpha(\|u_t\|_2^2+\textstyle\int_\Omega z^2(x,1,t)dx);$$

we get:

$$-\textstyle\int_\Omega z^2(x,1,t)dx\geq-\frac{1}{\alpha}H'(t).$$

Where α_1 defines (3.35).
Therefore (3.56) and (3.57) follow

$$-\varepsilon\alpha kH^{-\delta}(t)\int_\Omega z^2(x,1,t)dx\geq-\varepsilon kH^{-\delta}H'(t). \tag{3.59}$$

If we replace (3.56) and (3.59) in (3.55) ,we see that

$$\begin{aligned} L'(t) \geq & (1-\sigma-\varepsilon k)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{\rho+1}\right)\int_\Omega u_t^{\rho+2}\,dx \\ & + \varepsilon\left(-1-\left(\frac{1}{4\eta}-1\right)\right)\int_0^t g(s)\,ds\,\|\nabla u\|_2^2 - \varepsilon\eta(g\circ\nabla u)(t) \\ & - \frac{|\mu_2|^2\varepsilon}{4k\alpha}H^{\sigma}\|u\|_2^2 + \varepsilon b\|\nabla u(t)\|_2^2 + \varepsilon b\|u\|_p^p. \end{aligned} \tag{3.60}$$

Add the term $\varepsilon p(H(t) - E_2 + E(t))$, then use the definition of E(t) to obtain

$$\begin{aligned} L'(t) \geq & (1-\sigma-\varepsilon k)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{\rho+1}+\frac{p}{\rho+2}\right)\int_\Omega u_t^{\rho+2}\,dx \\ & + \varepsilon\left(\frac{p}{2}-\eta\right)(g\circ\nabla u)(t) + \varepsilon\left(\frac{p-2}{2}-\left(\frac{p-2}{2}+\frac{1}{4\eta}\right)\right)\int_0^t g(s)\,ds\,\|\nabla u\|_2^2 \\ & - \frac{|\mu_2|^2\varepsilon}{4k\alpha}H^{\sigma}\|u\|_2^2 + \varepsilon pH(t) + \frac{\varepsilon\xi p}{2}\int_\Omega\int_0^t z^2(x,\kappa,t)\,d\kappa dx \\ & - \varepsilon pE_2 + \frac{(p+2)\varepsilon}{2}\|\nabla u(t)\|_2^2. \end{aligned} \tag{3.61}$$

Now we take how η satisfied

From (3.8) , We have $\int_0^1 g(s)ds = 1-l$; assume that:

$$\left[\frac{p-2}{2} - (\frac{p-2}{2}+\frac{1}{4\eta})\int_0^1 g(s)ds\right]\|\nabla u(t)\|_2^2 \geq \beta\frac{(p-2)}{2}l\|\nabla u(t)\|_2^2; \tag{3.62}$$

we get :

$$\begin{aligned} & \left(\frac{p-2}{2}+\frac{1}{4\eta}\right)\int_0^1 g(s)ds \leq \frac{p-2}{2}(1-2\beta l); \\ \Longrightarrow & \left(\frac{p-2}{2}+\frac{1}{4\eta}\right)(1-l) \leq \frac{p-2}{2}(1-\beta l); \\ & \Longrightarrow \frac{1}{4\eta} \leq \frac{(p-2)(l-\beta l)}{2(1-l)}; \end{aligned}$$

so,we obtain

$$\eta > \tfrac{(1-l)}{2(1-\beta)l(p-2)}.$$

Also:

$$(\frac{p}{2}-\eta)(g\circ\nabla u) \geq \beta\frac{p-2}{2}(g\circ\nabla u). \tag{3.63}$$

Thus

$$\begin{aligned} & \Longrightarrow \frac{p}{2}-\eta \geq \beta\frac{p-2}{2}; \\ & \Longrightarrow \eta \leq -\beta\frac{p-2}{2}-\frac{p}{2}; \end{aligned}$$

so, we obtain:

$$\eta < \tfrac{p(1-\beta)}{2} + \beta.$$

We conclud:

$$\frac{1-l}{2(1-\beta)l(p-2)} < \eta < \frac{p(1-\beta)}{2} + \beta; \tag{3.64}$$

by (3.62)and (3.63), we use the fact that$l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2$, we obtain:

$$\begin{aligned}
&\left(\frac{p-2}{2} - \left(\frac{p-2}{2} + \frac{1}{4\eta}\right)\right)\int_0^t g(s)\,ds\,\|\nabla u\|_2^2 + \left(\frac{p}{2} - \eta\right)(g \circ \nabla u)(t) - pE_2 \\
&\geq [\frac{p-2}{2} - (\frac{p-2}{2} + \frac{(1-\beta)l(p-2)}{2(1-l)})(1-l)]\|\nabla u(t)\|_2^2 + (\frac{p}{2} - \frac{p-p\beta}{2} - \beta)(g \circ \nabla u) - pE_2 \\
&\geq \beta\left(\frac{p-2}{2}\right)\Big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\Big) - pE_2 \\
&= \beta\frac{(p-2)(\lambda_2^2 - \lambda_1^2 + \lambda_1^2)}{2\lambda_2^2}(l\|\nabla u(t)\|_2^2 + (g \circ \nabla u)) - pE_2 \\
&= \frac{\beta(p-2)\lambda_2^2 - \lambda_1^2}{2\lambda_2^2}\Big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\Big) + \frac{\beta(p-2)\lambda_1^2}{2\lambda_2^2}\Big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\Big) - pE_2 \\
&\geq c_3\Big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\Big) + c_4;
\end{aligned} \tag{3.65}$$

where:

- $c_3 = \frac{\beta(p-2)\lambda_2^2 - \lambda_1^2}{2\lambda_2^2} > 0;$

and

- $c_4 = \frac{\beta(p-2)}{2}\lambda_1^2 - pE_2.$

Furthermore, since $E_2 < \beta E_1$ and $E_1 = \frac{p-2}{2p}\lambda_1^2$, it follows that

$$c_4 = \tfrac{\beta(p-2)}{2}\lambda_1^2 - pE_2 > 0;$$

because

$$\beta\left(\tfrac{(p-2)\lambda_1^2}{2} - pE_1\right) = 0.$$

Therefore,in view of the above arguments,equation (3.61) can be rewritten as

$$\begin{aligned}
L'(t) \geq &(1 - \sigma - \varepsilon k)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p}{\rho+2}\right)\int_\Omega u_t^{\rho+2}\,dx \\
&+ \varepsilon c_3\Big(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)\Big) - \frac{|\mu_2|^2\varepsilon}{4k\alpha}H^\sigma\|u\|_2^2 \\
&+ \frac{\varepsilon\xi p}{2}\int_\Omega\int_0^t z^2(x,\kappa,t)\,d\kappa dx + \frac{(p+2)\varepsilon}{2}\|\nabla u(t)\|_2^2 + p\varepsilon H(t).
\end{aligned} \tag{3.66}$$

From (3.45),we have :

$$H(t) \leq \tfrac{b}{p}\|u\|_p^p \quad ;$$

where

$$H^{\delta}(t) \leq \left(\tfrac{b}{p}\right)^{\delta} \|u\|_p^{\delta p};$$

it means that

$$H^{\delta}\|u\|_2^2 \leq \left(\frac{b}{p}\right)^{\sigma} |\Omega|^{\frac{p-2}{p}} \|u\|_p^{2+\sigma p}. \tag{3.67}$$

By Lemma 3.2 , for $p \geq 2+\delta p$ and (3.67) , we obtain

$$H^{\delta}(t)\|u\|_2^2 \leq C_s \left[\left(\frac{b}{p}\right)^{\delta} |\Omega|^{\frac{p-2}{2}} \|u\|_p^p + \left(\frac{b}{p}\right)^{\delta} |\Omega|^{\frac{p-2}{2}} \|\nabla u\|_2^2\right]; \tag{3.68}$$

by decomposing $pH(t)$

$$pH(t) = \varepsilon(2\alpha_1 + (p-2\alpha_1))H(t); \tag{3.69}$$

where

$$\alpha_1 = \min\{c_3, \tfrac{p}{2}\}.$$

We substitute(3.67) into $-\frac{|\mu_2|^2}{4\alpha k}\varepsilon H^{\delta}\|u\|_2^2$, and we obtain

$$-\frac{|\mu_2|^2}{4\alpha k}\varepsilon \left(\frac{b}{p}\right)^{\delta} |\Omega|^{\frac{p-2}{p}} \|u\|_p^{\delta p+2}. \tag{3.70}$$

We then use the definition of $2\alpha_1 H(t)$, replacing (3.69)and (3.70) in (3.66) , we obtain

$$\begin{aligned} L'(t) \geq & (1-\sigma-\varepsilon k)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p-2a_1}{\rho+2}\right)\int_\Omega u_t^{\rho+2}\,dx \\ & + \varepsilon(c_3 l - a_1 l)\|\nabla u\|_2^2 + \varepsilon(c_3 - a_1)(g\circ\nabla u)(t) + \varepsilon\frac{2a_1 b}{p}\|u\|_p^p \\ & - \frac{|\mu_2|^2\varepsilon}{4k\alpha}\left(\frac{b}{p}\right)^{\sigma} |\Omega|^{\frac{p-2}{p}} \|u\|_p^{2+\sigma p} + \varepsilon\xi\left(\frac{p}{2} - a_1\right)\int_\Omega \int_0^t z^2(x,\kappa,t)\,d\kappa dx \\ & + \varepsilon\left(\frac{p+2}{2} - a_1\right)\|\nabla u_t\|_2^2 + \varepsilon(p-2a_1)H(t). \end{aligned} \tag{3.71}$$

Replacing (3.68) in (3.71):

$$\begin{aligned} L'(t) \geq & (1-\sigma-\varepsilon k)H^{-\sigma}(t)H'(t) + \varepsilon\left(\frac{1}{\rho+1} + \frac{p-2a_1}{\rho+2}\right)\int_\Omega u_t^{\rho+2}\,dx \\ & + \varepsilon\left(c_3 l - a_1 l - \frac{C_s|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^{\delta} |\Omega|^{\frac{p-2}{2}}\right)\|\nabla u\|_2^2 + \varepsilon(c_3 - a_1)(g\circ\nabla u)(t) \\ & + \varepsilon\left(\frac{2a_1 b}{p} - \frac{C_s|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^{\delta} |\Omega|^{\frac{p-2}{2}}\right)\|u\|_p^p - \frac{|\mu_2|^2\varepsilon}{4k\alpha}\left(\frac{b}{p}\right)^{\sigma} |\Omega|^{\frac{p-2}{p}} \|u\|_p^{2+\sigma p} \\ & + \varepsilon\xi\left(\frac{p}{2} - a_1\right)\int_\Omega \int_0^t z^2(x,\kappa,t)\,d\kappa dx + \varepsilon\left(\frac{p+2}{2} - a_1\right)\|\nabla u_t\|_2^2 + \varepsilon(p-2a_1)H(t). \end{aligned} \tag{3.72}$$

We choose the constant k, such that

$$c_3 l - a_1 l - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^{\sigma}|\Omega|^{\frac{p-2}{p}} > 0; \tag{3.73}$$

and

$$\frac{2a_1 b}{p} - \frac{C|\mu_2|^2}{4k\alpha}\left(\frac{b}{p}\right)^{\sigma}|\Omega|^{\frac{p-2}{p}} > 0; \tag{3.74}$$

by (3.73), we obtain

$$k > \frac{C}{4\alpha_1}\left(\frac{b}{p}\right)^{\delta}|\Omega|^{\frac{p-2}{2}}\frac{|\mu_2|^2}{(c_3-a_1)l}; \tag{3.75}$$

from (3.74), we obtain

$$k > \frac{C}{4\alpha_1}\left(\frac{b}{p}\right)^{\delta}|\Omega|^{\frac{p-2}{2}}\frac{p|\mu_2|^2}{2a_1 b}. \tag{3.76}$$

By using (3.75) and (3.76) , we obtain

$$k > \frac{C}{4\alpha_1}\left(\frac{b}{p}\right)^{\delta}|\Omega|^{\frac{p-2}{2}}\max\left(\frac{|\mu_2|^2}{(c_3-a_1)l}, \frac{p|\mu_2|^2}{2a_1 b}\right); \tag{3.77}$$

and ε so small such that

$$1 - \sigma - \varepsilon k > 0;$$

hence

$$\varepsilon < \frac{(1-\sigma)}{k}. \tag{3.78}$$

Let L(t) be defined by :

$$L(t) = H^{1-\sigma}(t) + \tfrac{\varepsilon}{\rho+1}\textstyle\int_\Omega |u_t|^p u_t u\,dx + \tfrac{\mu_1\varepsilon}{2}\int_\Omega u^2\,dx + \varepsilon\int_\Omega \nabla u_t\cdot\nabla u\,dx.$$

At $t = 0$,we have :

$$u(0) = u_0, \qquad u_t(0) = u_1, \qquad \mu(0) = \mu_0.$$

Consequently, at $t = 0$, we obtain :

$$L(0) = H^{1-\sigma}(0) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_1|^p u_1 u_0\,dx + \frac{\mu_1\varepsilon}{2}\int_\Omega u_0^2\,dx + \varepsilon\int_\Omega \nabla u_1\cdot\nabla u_0\,dx > 0. \tag{3.79}$$

As well as, from (3.72),(3.77)and (3.78) there exists $K > 0$ such that

$$L'(t) \geq \varepsilon K\left[\|u_t\|_{p+2}^{p+2} + \|\nabla u\|_2^2 + (g\circ\nabla u)(t) + \|\nabla u_t\|_2^2 + H(t) + \|u\|_p^p + \int_\Omega\int_0^1 z^2(x,\kappa,t)\,d\kappa dx\right]; \tag{3.80}$$

which together with (3.80) implies that

$$L(t) \geq L(0) > 0 \quad \text{for } t \geq 0.$$

On the other hand, by Hölder inequality(see[1]) , we have

$$\left(\int_\Omega |u_t|^p u_t u\, dx\right)^{\frac{1}{1-\sigma}} = \left(\int_\Omega |u_t|^{p+1} u dx\right)^{\frac{1}{1-\delta}} \leq \|u_t\|_{p+2}^{\frac{p+1}{1-\sigma}} \|u\|_{p+2}^{\frac{1}{1-\sigma}} \leq c_5 \|u_t\|_{p+2}^{\frac{p+1}{1-\sigma}} \|u\|_p^{\frac{1}{1-\sigma}}. \tag{3.81}$$

Still we apply Young's inequality , we get

$$\left(\int_\Omega |u_t|^{\rho+1} u dx\right)^{\frac{1}{1-\sigma}} \leq c_6 \left(\|u_t\|_{\rho+2}^{\frac{\rho+1}{1-\sigma}\mu} + \|u\|_p^{\frac{1}{1-\sigma}\theta}\right); \tag{3.82}$$

where $\frac{1}{\mu} + \frac{1}{\theta} = 1$ and $c_5, c_6 > 0$. We choose

$$\mu = \frac{(1-\sigma)(\rho+2)}{\rho+1} > 1;$$

then by (3.47), we know

$$\frac{\theta}{1-\sigma} = \frac{(\rho+2)}{(1-\sigma)(\rho+2) - (\rho+1)} < p.$$

Compute $\|u_t\|_{p+2}^{\frac{p+1}{1-\sigma}\mu}$

$$\tfrac{(1-\sigma)(\rho+2)}{\rho+1} \cdot \mu = \tfrac{(1-\sigma)(\rho+2)}{\rho+1} \cdot \tfrac{\rho+1}{1-\sigma} = \rho + 2;$$

we obtain $\|u_t\|_{p+2}^{\rho+2}$.
Thus, by Lemma 3.2 and (3.82), we get

$$\left(\int_\Omega |u_t|^p u_t u\, dx\right)^{\frac{1}{1-\sigma}} \leq c_7 \left(\|u_t\|_{p+2}^{p+2} + \|\nabla u\|_2^2 + \|u\|_p^p\right); \tag{3.83}$$

with $c_7 > 0$.
Similarly, as in deriving (3.82), we also have

$$\left(\int_\Omega \nabla u_t \cdot \nabla u\, dx\right)^{\frac{1}{1-\sigma}} \leq c_8 \left(\|\nabla u_t\|_2^2 + \|\nabla u\|_2^{\frac{2}{1-2\sigma}}\right); \tag{3.84}$$

where by (Young's inequality with ε[11]) , we choose :

$$p = 2(1-\delta), \qquad and \qquad q = \frac{2(1-\delta)}{1-2\delta};$$

where

$$\mathrm{u}^p = \left(\|\nabla u_t\|_2^{\frac{1}{1-\delta}}\right)^{2(1-\delta)} = \|\nabla u_t\|_2^2;$$

and

$$\mathrm{v}^q = \left(\|\nabla u\|_2^{\frac{1}{1-\delta}}\right)^{\frac{2(1-\delta)}{1-2\delta}} = \|\nabla u\|_2^{\frac{2}{1-2\delta}}.$$

For $c_8 > 0$.

Combining (3.46), (3.83) and (3.84) to yield

$$\begin{aligned} L(t)^{\frac{1}{1-\sigma}} &= \left(H^{1-\sigma}(t) + \frac{\varepsilon}{\rho+1}\int_\Omega |u_t|^p u_t u\,dx + \frac{\mu_1\varepsilon}{2}\|u\|_2^2 + \varepsilon\int_\Omega \nabla u_t\cdot\nabla u\,dx\right)^{\frac{1}{1-\sigma}} \\ &\le c_9\Big(H(t) + \|u_t\|_{p+2}^{p+2} + \|u\|_p^p + \|\nabla u\|_2^2 + \|u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u_t\|_2^2\Big) \\ &\le c_{10}\Big(H(t) + \|u_t\|_{p+2}^{p+2} + \|u\|_p^p + \|\nabla u\|_2^2 + \|u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u\|_2^{\frac{2}{1-2\sigma}} + \|\nabla u_t\|_2^2\Big); \end{aligned} \tag{3.85}$$

for $t \ge 0$ and $c_9, c_{10} > 0$.

We note from (3.42) and (3.44) that

$$\|\nabla u\|_2^{\frac{2}{1-2\sigma}} \le k_1^{\frac{2}{1-2\delta}} \le K_1^{\frac{2}{1-2\sigma}}\frac{H(t)}{H(0)}, \quad \|u\|_p^{\frac{2}{1-2\sigma}} \le K_1^{\frac{2}{1-2\sigma}}\frac{H(t)}{H(0)}. \tag{3.86}$$

Such that it follows from (3.85) and (3.86) that

$$\begin{aligned} L(t)^{\frac{1}{1-\sigma}} \le& c_{10}\left(\left(1 + \frac{K_1^{\frac{2}{1-2\sigma}}}{H(0)} + \frac{K_1^{\frac{2}{1-2\sigma}}}{H(0)}\right)H(t) + \|u_t\|_{p+2}^{p+2} + \|u\|_p^p + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\right), \\ \le& c_{11}\left(H(t) + \|u_t\|_{p+2}^{p+2} + \|u\|_p^p + \|\nabla u\|_2^2 + \|\nabla u_t\|_2^2\right), \quad t \ge 0; \end{aligned} \tag{3.87}$$

with $c_{11} > 0$.

Combining (3.87) with (3.81), we get

$$\begin{aligned} L'(t) &\ge \frac{\varepsilon k}{c_{11}} L(t)^{\frac{1}{1-\sigma}} \\ &\ge c_{12} L(t)^{\frac{1}{1-\sigma}}, \quad t \ge 0; \end{aligned} \tag{3.88}$$

here $c_{12} = \frac{\varepsilon K}{c_{11}}$.

An integration of (3.88) over $(0,t)$ then yields

Starting from

$$L'(t) \ge c_{12} L(t)^{\frac{1}{1-\sigma}};$$

we multiply both sides by $L(t)^{\frac{-\sigma}{1-\sigma}}$,this yields

$$L(t)^{\frac{-1}{1-\sigma}} L'(t) \ge c_{12};$$

where

$$-\tfrac{\sigma}{1-\sigma} L(t)^{\frac{-1}{1-\sigma}} L'(t) \le -c_{12}\tfrac{\sigma}{1-\sigma};$$

and

$$\int_0^t \tfrac{d}{dt}(L(t)^{-\frac{\sigma}{1-\sigma}})dt \le -\int_0^t \tfrac{c_{12}\sigma}{1-\sigma}dt;$$

integrating over the interval (0,t),we get

$$L(t)^{-\frac{\sigma}{1-\sigma}} - L(0)^{-\frac{\sigma}{1-\sigma}} \leq \frac{-c_{12}\sigma}{1-\sigma}t;$$

therefore,it follows that

$$L(t)^{-\frac{\sigma}{1-\sigma}} \geq L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{c_{12}\sigma}{1-\sigma}t.$$

Hence,

$$L(t) \geq \left(L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{\sigma c_{12}}{1-\sigma}t\right)^{-\frac{1-\sigma}{\sigma}}. \tag{3.89}$$

Since $L(0) > 0$, (3.89) shows that L becomes infinite in a finite time T with

$$0 < T \leq \frac{1-\sigma}{\sigma c_{12}}L(0)^{-\frac{\sigma}{1-\sigma}}.$$

Where we have $L(t)^{-\frac{\sigma}{1-\sigma}} > 0$ car $L(t) > 0$

$$L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{c_{12}\sigma}{1-\sigma}t \leq 0;$$

and where t=T

$$L(0)^{-\frac{\sigma}{1-\sigma}} - \frac{c_{12}\sigma}{1-\sigma}t \leq 0;$$

thus

$$\frac{c_{12}\sigma}{1-\sigma}T \leq L(0)^{-\frac{\sigma}{1-\sigma}};$$

we conclude that

$$T \leq \frac{1-\sigma}{\sigma c_{12}}L(0)^{-\frac{\sigma}{1-\sigma}}.$$

Hence, the proof is completed. □

Conclusion

This dissertation is devoted to the study of the finite-time blow-up phenomenon for a viscoelastic wave equation of Kirchhoff type with time delay. The main objective of this work is to determine sufficient conditions that lead to the blow-up of solutions in finite time.

Using a rigorous analytical framework based on Lyapunov functionals and differential inequality techniques, a set of conditions ensuring finite-time blow-up is established. The results show that the interaction between the nonlinear source term and the time delay constitutes the main driving mechanism leading the solution to blow up.

In particular, the time delay plays a crucial role in accelerating the blow-up process by enhancing the growth of the solution and intensifying the internal instability of the system, which highlights its direct impact on the dynamics of viscoelastic Kirchhoff-type equations.

These findings contribute to a clearer mathematical understanding of the mechanisms responsible for blow-up in such models and provide a basis for further investigations involving more general nonlinearities or additional delay effects.

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