People's Democratic Republic of Algeria
Ministry Of Higher Education and Scientific Research
Echahid Cheikh Larbi Tebessi University - Tebessa
Faculty of Science and Technology
Electrical Engineering Department

Multivariable Linear Systems

Master 1 Automatation and systemes

Master 1 Automation and Industrial Computing

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Senior Lecturer Class A

2025/2026

Preface

This course handout is an educational resource designed for first-year Master's students specializing in Automation and Systems or Automation and Industrial Computing, in accordance with the official academic curriculum. This document concerns the Multivariable Linear Systems module and may also serve students enrolled in related fields of study.

This course is structured around the fundamental concepts of multivariable linear systems, building upon students' prior knowledge of single-variable linear systems.

This course material is divided into two parts. The first part includes the complete course chapters. The second is a supplementary section containing a variety of exercises corresponding to each chapter.

This manuscript is divided into five chapters.

The first chapter is devoted to general concepts of linear automation and single-variable linear systems. In addition, at the end of the chapter, we provide a review of matrix calculus. The objective of the second chapter is the different types of representations of linear systems and the solution of state equations. The third chapter presents the principles of controllability and observability of multivariable linear systems and the various criteria used to verify these characteristics. The fourth chapter addresses the problem of transforming a transfer matrix representation of multivariable systems into a state-space representation using Gilbert's method, the invariant method (Smith-McMillan form), and the realization reduction method. The fifth chapter is dedicated to state-feedback control of multivariable linear systems.

This document contains several examples with solutions in each chapter to facilitate understanding of the theoretical concepts introduced.

Dr. Yousfi Laatra

Semestre: 1
Unité d'enseignement: UEF 1.1.1
Matière: Systèmes linéaires multivariables
VHS: 67h30 (Cours: 3h00, TD: 1h30)
Crédits: 6
Coefficient: 3

Objectifs de l'enseignement:

L'objectif du cours est de donner une méthodologie pour la conception des différentes lois de commande pour les systèmes linéaires invariants multivariables, dans le contexte de l'approche d'état.

Connaissances préalables recommandées:

L'étudiant devra posséder les connaissances suivantes :

- Systèmes asservis linéaires
- Systèmes échantillonnés ;

Contenu de la matière:

Chapitre 1. Introduction **(2 Semaines)**
Objectifs de ce cours, Rappel sur le calcul matriciel, Rappel des notions de l'approche d'état, Différence entre SISO et MIMO.

Chapitre 2. Représentation d'état des systèmes multivariables (SM). **(2 Semaines)**
Définitions, Différentes représentations des systèmes, Résolution de l'équation d'état, Exemples d'applications

Chapitre 3. Commandabilité et Observabilité. **(2 Semaines)**
Introduction, Critère de commandabilité de Kalman, Commandabilité de la sortie, Critère d'observabilité, Dualité entre la commandabilité et l'observabilité, Etude de quelques formes canoniques.

Chapitre 4. Représentation des SM par matrice de transfert. **(3 Semaines)**
Introduction, Passage d'une représentation d'état à la représentation par matrice de transfert, Méthode de Gilbert, Méthode des invariants : forme de Smith-McMillan, Méthode par réduction d'une réalisation

Chapitre 5. Commande par retour d'état des SM. **(4 Semaines)**
Formulation du problème de placement de pôles par retour d'état, Méthodes de calculs pour les systèmes multivariables, Observateur d'état et commande par retour de sortie (i.e. avec observateur d'état) des SM. Commande non interactives des SM , Implémentation.

Mode d'évaluation:

Contrôle continu: 40% ; Examen: 60%.

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GENERAL NTRODUCTION

This course provides fundamental principles on multivariable linear systems and the techniques used for their analysis and control. It moves from theoretical foundations to practical numerical applications, using multiple worked examples to reinforce your understanding.

The knowledge required to read these notes is primarily linear control theory and basic concepts of state representation, stability, controllability, observability, and state feedback control of single-variable linear systems.

Multivariable linear systems (MIMO) have multiple inputs and outputs. They are essential in engineering for modeling complex processes where different variables interact simultaneously, thus ensuring the stability, accuracy, and robustness of industrial control that single-input systems cannot handle..

These systems are crucial in advanced fields that demand precise, concurrent control of multiple physical variables:

• **Aeronautics and Space:** Piloting aircraft (control surfaces and tand thrust work together to determine attitude, altitude, and trajectory) or satellite orientation.

• **Process Industry (Advanced Automation):** Control of distillation columns or chemical reactors where pressure, flow rate, and temperature are regulated simultaneously**.**

• **Robotics and Mechatronics:** Drone stabilization and control of articulated robotic arms.

• **Telecommunications:** Use of MIMO technology in cellular and wireless networks to maximize throughput and range.

This manuscript is organized into several chapters. The first chapter reviews general concepts of linear control and single-variable linear systems, and concludes with a review of matrix calculus. The second chapter deals with the different types of linear systems representations and the solution of state equations. In the third chapter, we extend the concepts of controllability and observability from single-variable to multivariable linear systems, as well as the various criteria for verifying these characteristics. The fourth chapter focuses on converting matrix transfer representations of multivariable systems into state-space form, using Gilbert's method, the invariant method (Smith‑McMillan form), and realization reduction. In the final chapter, we introduce the principle of state feedback control and output feedback control.

CHAPER I : INTRODUCTION

(Review: Single-Variable Linear Systems and State Representation)

1 INTRODUCTION:

A system operates in an open loop when the control signal is generated without knowledge of the output values: there is no feedback. In contrast, the system is considered closed loop. The control signal then depends on the set point and the actual output. In this case, sensors are used to observe the output values, and the information they provide is used to adjust and generate the control signal.

Therefore, an automatic control system may function either as an open-loop (O.L.) or a closed-loop (C.L.) system.

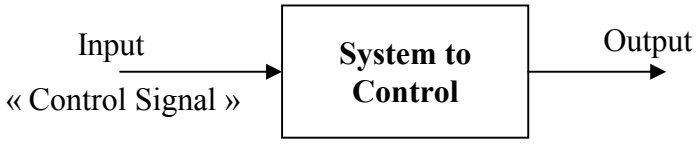


Figure 1.1: Open-loop system diagram

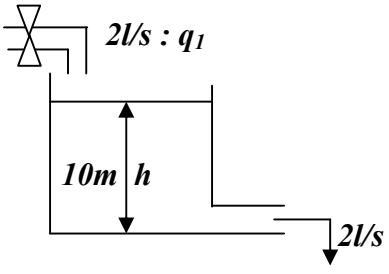
Major drawbacks of open-loop systems:

- Sensitivity to disturbances and uncertainties in internal parameters;
- Inability to modify dynamic performance;
- Inability to stabilize a naturally unstable system;
- Need to control the output and adapt the control.

  Process automation requires the use of a closed loop, which is necessary to:

  - Stabilize an unstable open-loop system,
  - Compensate for external disturbances,
  - Compensate for uncertainties related to the system itself (aging, model inaccuracies, etc.).

Example: A reservoir (level control)



Grandeur de commande : ***flow rate « q_1 »***

Grandeur à commander : ***level « h »***

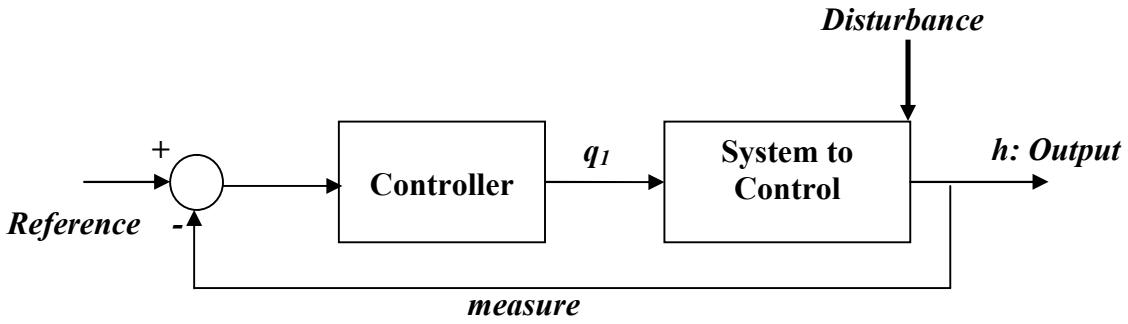


Figure 1.2: Closed-loop system diagram

Aim of control: A control system can perform two distinct functions:

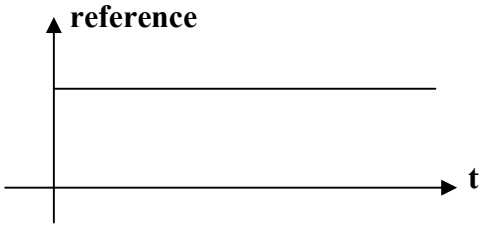
- Regulation, i.e., compensating for the effect of variable disturbances on the output (the setpoint remains constant).
- Control, or tracking a time-varying setpoint at the output.

Therefore, we can say that:

Regulation: In this case, the quantity to be controlled must be kept constant.

Reference = Constant

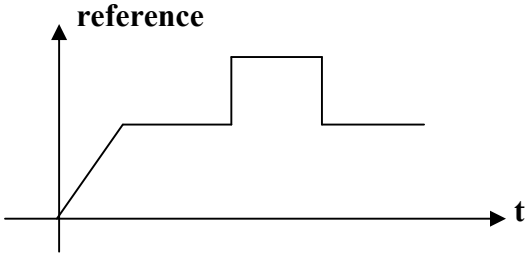
Example: Ovens (temperature regulation)



Tracking: In this case, the quantity to be controlled must follow a given trajectory.

Reference = Variable

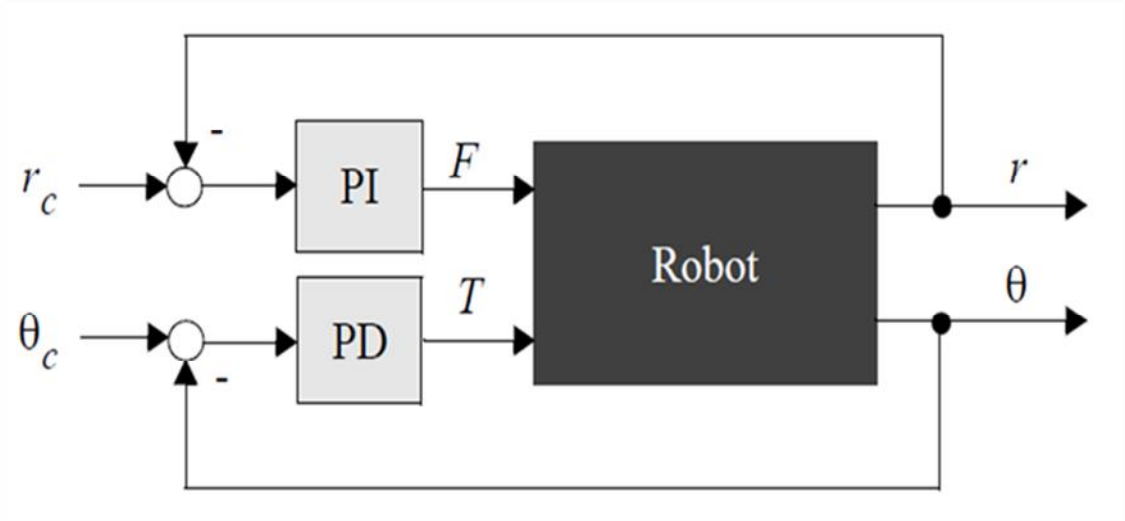
Example: Robots



For linear systems, we can distinguish the following four types::

- System **SISO** : The system has an input and an output. (Single Input Single Output).
  (single-variable system).
- System **MIMO** : The system has multiple inputs and multiple outputs (Multiple Input Multiple Output) (multivariable system).
- System **SIMO** : The system has single input and multiple outputs (Single Input, Multiple Output).
- System **MISO** : The system has multiple inputs and single output (Multiple Output, Single Input)

Example of MIMO systems: Two-degree-of-freedom robot



Inputs : Force : ***F*** and Torque : ***T***

Outputs : Déplacement : ***r*** et Position angulaire : ***θ***

Consignes : Displacement : r_c and Angular position

Figure 1.3: Example of a MIMO system

2 METHODS FOR ANALYZING CONTINUOUS SYSTEMS:

Classical analysis methods of continuous systems are based on the transfer function (TF).

2.1 *Modelisation:*

For the design of a control system, the first step is to establish its mathematical model, that's to say; the equations that give the relationships between the input and the output.

Based on the mathematical model $\xrightarrow{L.T}$ The transfer function (**T.F**)
(Differential Equations)

2.2 *Stability and Accuracy analysis:*

Dynamic characteristic: **Stability**

If the system is described by the transfer function ***G(s)***:

$$G(s) = \frac{N(s)}{D(s)}$$

The poles are the roots of the characteristic equation: ***D(s) = 0***.

A system is stable if all poles of the system lie strictly in the left half of the complex (s-plane) (i.e., they have negative real parts).

Stable → poles with negative real parts ***ℜe(s) < 0***.

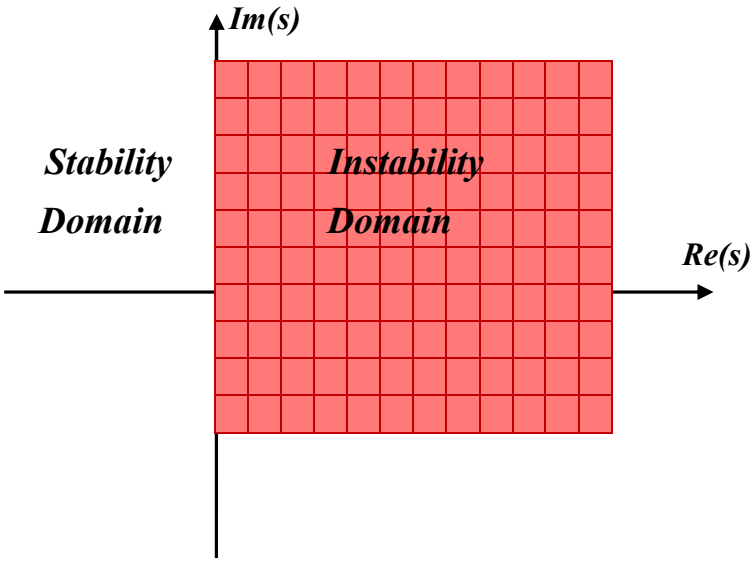


Figure 1.4: Stability domain (in frequentiel domain)

Static characteristic: **Accuracy**

Accuracy defines how well a system tracks a reference input (e.g., step, ramp, or parabolic signals) without steady-state error.

Accuracy is the error in steady state: $\lim_{t\to\infty} e(t) = 0$

(Is the error zero? that is; what is the number of integrators in the **T.F** of the system: the number of (***1/s***)).

The accuracy of a closed-loop system in tracking these reference signals is directly determined by its *system type* (the number of pure integrators, or poles at the origin).

Example:

$$G(s) = \frac{K}{s(s+1)} \rightarrow a\ single\ integrator \rightarrow zero\ positional\ error$$

$$G(s) = \frac{K}{s^2(s+1)} \rightarrow two\ integrators \rightarrow zero\ speed\ error$$

3 CONTROLLER SYNTHESIS:

After analyzing the system to be controlled (in **O.L**) and defining the performance requirements that must be achieved in **C.L** with a regulator.

In control engineering, the methodology generally follows several key steps:

1. Specification (Requirement Analysis): The control engineer must fully understand the problem and its requirements. This means clearly defining the system, identifying its inputs and outputs, and setting performance objectives (e.g., stability, response speed and accuracy).

2. Modeling: The next step is to describe the system's open-loop behavior using the relevant physical laws. This results in algebraic and differential equations, then convert them into a standard control representation such as a transfer function or a state-space model.

3. Analysis: Evaluate the modeled system with control tools to assess stability, settling time, oscillatory behavior, and accuracy. This step helps determine whether the system meets the desired specifications.

4. Synthesis (Controller Design): The final step is to design a control law—an intelligent feedback mechanism that ensures the closed-loop system achieves the desired performance, if possible.

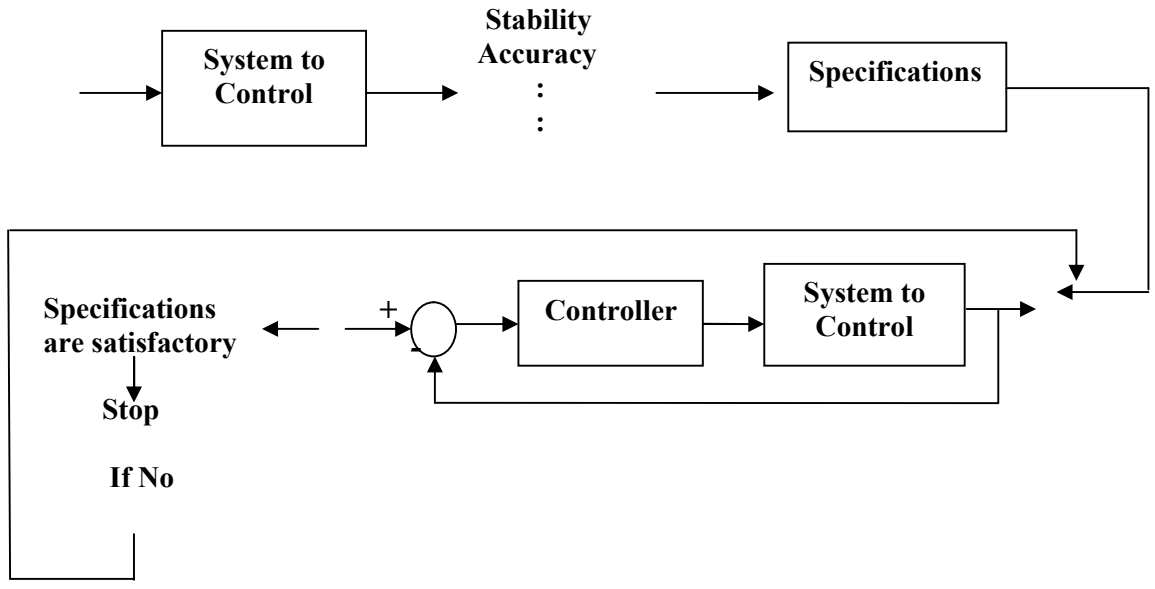


Figure 1.5: controller synthesis steps

Controller synthesis by classical méthods:

It consists of establishing the specifications:

1- Time response domain: rise time, settling time, peak overshoot and other step-response metrics.

2- Frequency response domain: phase margin, gain margin, bandwidth, … .

After defining the performance requirements, select the regulator type to include in the loop and then calculate its parameters to satisfy the performance requirements.

Classical Controllers:

- ***Proportional Action:*** $G_P(s) = K \rightarrow$ improves accuracy.
- ***Integral Action:*** $G_I(s) = 1/s \rightarrow$ eliminate steady-state error.
- ***Derivative Action:*** $G_p(s) = s \rightarrow$ improves stability.

4 STATE- SPACE REPRESENTATION:

State space representation is defined as a mathematical model of a physical system that expresses the system's behavior through input, output, and state variables, which are related by first-order differential or difference equations.

The State Space analysis applies to the non-linear and time-variant system. It helps in the analysis and design of linear, non-linear, multi-input, and multi-output systems. Earlier the transfer function applied to the linear time-invariant system but with the help of State Space analysis, it is possible to find the transfer function of the non-linear and time-variant systems.

The state representation is obtained from rewriting the higher-order differential equation into a system of first-order equations. By introducing new variables, a single nth-order differential equation is converted into "***n***" first-order differential equations. These first-order equations form the system's state equations, typically expressed in the form:

$$\begin{cases} \dot{x}_1 = f_1(x, u, t) \\ \dot{x}_2 = f_2(x, u, t) \\ \dot{x}_3 = f_3(x, u, t) \\ \quad\vdots \\ \dot{x}_n = f_n(x, u, t) \end{cases}$$

x(t) : state vector of dimension n (n : System order).

$x_1, x_2, \ldots, x_n$: state variable.

u(t) : the input or control signal

In the general case :

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{t}) \qquad \boldsymbol{x} \in \mathfrak{R}^{\boldsymbol{n}}, \boldsymbol{f} \in \mathfrak{R}^{\boldsymbol{n}}, \boldsymbol{u} \in \mathfrak{R}$$

Specefic cases:

- For linear system, the state equations are written:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

The relationship between outputs and states is:

$$\boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}(\boldsymbol{t})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}(\boldsymbol{t})\boldsymbol{u}(\boldsymbol{t})$$

- For linear time-invariant systems:

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t}) \end{cases}$$

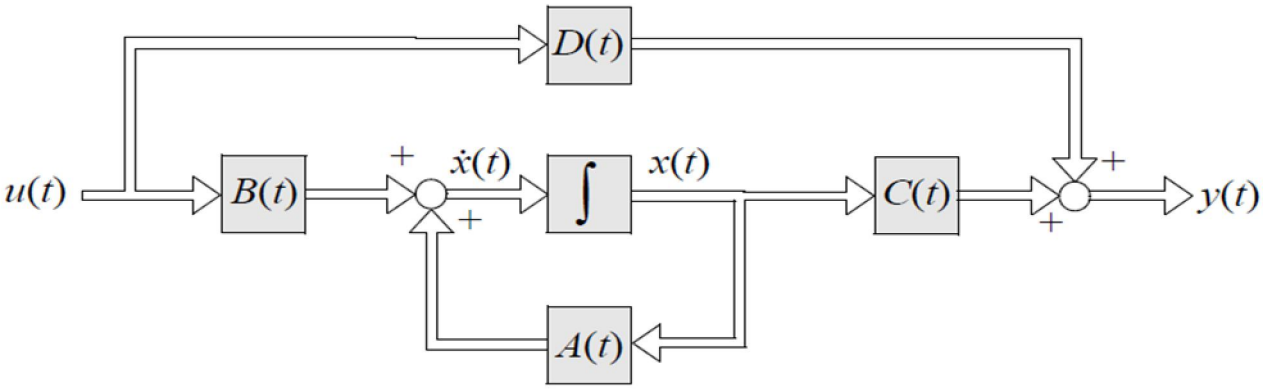


Figure 1.5: Block diagram representation of the linear state-space equations

Example 1:

Consider a DC motor coupled to a load; this motor is represented by figure 1.6:

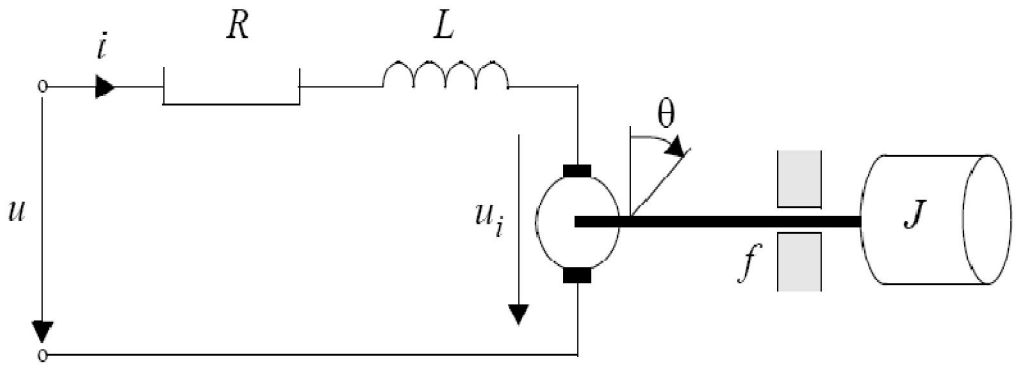


Figure 1.6: DC Moteur

$\boldsymbol{R}$ et $\boldsymbol{L}$: armature resistance and inductance.

$\boldsymbol{J}$: inertia

$\boldsymbol{f}$: load friction coefficient.

$\boldsymbol{u}$: motor variable supply voltage.

The modelisation of this system using a state-space representation, we will first write the differential equations governing its operation; that is, the electrical and mechanical equations of the motor..

The electrical equation: $\boldsymbol{u = Ri + L\frac{di}{dt} + u_i}$,

with: the induced voltage: $\boldsymbol{u_i = k\dot{\theta}}$ ($\boldsymbol{k}$**:** constant**)**

Mechanical equation: $\boldsymbol{J\ddot{\theta} = ki - f\dot{\theta}}$ **(**the torque supplied by DC motor is **"**$\boldsymbol{ki}$**")**

The input: $\boldsymbol{u}$ **,** The output: $\boldsymbol{\theta}$

We note that the two differential equations are functions of two quantities: $\boldsymbol{\theta}$ and ***i.*** The first quantitie is differentiated twice and the second is differentiated only once.

Therefore: state variables number is: **2 + 1 = 3** state variables.

We set: $\boldsymbol{x_1 = \theta, x_2 = \dot{x}_1 = \dot{\theta}, \dot{x}_2 = \ddot{\theta}, x_3 = i.}$

We obtain the state equations:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \dfrac{k}{J}x_3 - \dfrac{f}{J}x_2 \\ \dot{x}_3 = \dfrac{u}{L} - \dfrac{R}{L}x_3 - \dfrac{k}{L}x_2 \\ y = x_1 \end{cases}$$

Therefore, we can write equations in matrix form as follows:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & k/J \\ 0 & -k/L & -R/L \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} u \\ y = [1 \quad 0 \quad 0] . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \end{cases}$$

Example 2:

Consider the following series RLC circuit:

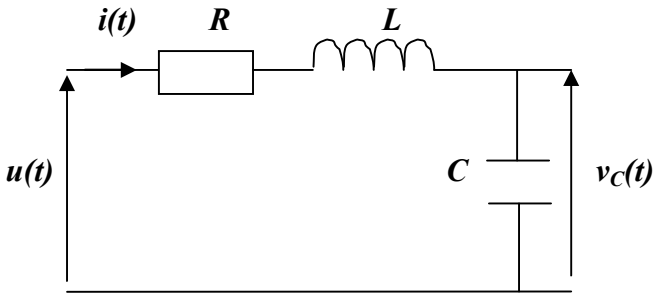


Figure 1.7: RLC Circuit

Input: ***u(t)***

Output: $y(t) = v_C(t)$

We will take as the state vector: $X = [\, v_C \;\; i \,]^T$, that's to say; $x_1(t) = v_C(t)$, $x_2(t) = i(t)$.

We have:

$$u(t) = Ri(t) + L\frac{di}{dt} + v_C(t) \;\; et \; i(t) = C\frac{dv_C}{dt}$$

The resulting state representation is:

$$\begin{cases} \dot{x}_1(t) = \dfrac{1}{C}x_2(t) \\ \dot{x}_2(t) = -\dfrac{R}{L}x_2(t) - \dfrac{1}{L}x_1(t) + \dfrac{u(t)}{L} \\ y(t) = x_1(t) \end{cases}$$

Therefore:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} u(t) \\ y(t) = [1 \quad 0] . \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

5 *SOLUTION OF STATE EQUATION :*

Recall the state equations for a linear, time-invariant (LTI) state space system:

$$\begin{cases} \dot{x} = Ax + Bu & x(t_0) = x_0 \\ y = Cx + Du \end{cases}$$

The difficulty in solving this system is the first equation, $\dot{x} = Ax + Bu$, because it is a differential equation. Once ***x(t)*** is found, substituting it into the second equation makes computing y(t) straightforward.

Because the system is linear, the superposition principle applies. Therefore, the general solution of the state equation is the sum of its homogeneous solution and a particular solution.

$$\boldsymbol{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}\, Bu(\tau)d\tau}$$

The transition matrix is: $\boldsymbol{\phi(t, t_0) = e^{A(t-t_0)}}$

6 TRANSITION FROM A STATE REPRESENTATION TO A TRANSFER FUNCTION:

The transition from a state representation to a transfer function is easily obtained by applying the Laplace transform to the state equation.

We have:

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{Ax(t) + Bu(t)} \\ \boldsymbol{y(t) = Cx(t) + Du(t)} \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} \boldsymbol{sX(s) = AX(s) + BU(s)} \cdots \boldsymbol{(1)} \\ \boldsymbol{Y(s) = CX(s) + DU(s)} \ldots \quad \boldsymbol{(2)} \end{cases}$$

$$\boldsymbol{(1) \Rightarrow (sI - A)X(s) = BU(s)}$$

$$\boldsymbol{\Rightarrow X(s) = (sI - A)^{-1}BU(s)}$$

Substituting into equation (2), we find:

$$\boldsymbol{Y(s) = C(sI - A)^{-1}BU(s) + DU(s) \Rightarrow G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D}$$

7 STABILITY, CONTROLLABILITY AND OBSERVABILITY :

7.1 Stability:

A system is stable if and only if the poles of its transfer function have negative real parts ***($\Re e(p) < 0$).***

If the system is described by state equations: $\boldsymbol{\dot{x}(t) = Ax(t) + Bu(t)}$, a system is stable when all eigenvalues of matrix A have negative real parts. ($\boldsymbol{\Re e(\lambda) < 0}$).

7.2 *Controlability:*

A system is completely controllable if and only if, given an initial time t_0 and a final time t_f, we can transfer the state $x(t)$ of the system from $x(t_0)$ to $x(t_f)$ by applying a command $u(t)$ between t_0 and t_f.

We can also say: A linear continuous-time time-invariant system: is controllable iff: for any initial value of the state at the initial time and for a given final time, there exists a control signal trajectory $\boldsymbol{u}([\mathbf{0} \quad \boldsymbol{t_f}])$ such that $\boldsymbol{x(t_f) = 0}$.

Kalman Condition:

A system is fully controllable iff :

$$\boldsymbol{rank}\underbrace{[\boldsymbol{B} \vdots \boldsymbol{AB} \vdots \boldsymbol{A^2B} \vdots \cdots \vdots \boldsymbol{A^{n-1}B}]}_{\text{Controllability matrix: } \phi_c} = \boldsymbol{rank(A) = n}$$

7.3 *Observability:*

A system is completely observable if and only if all its state variables are determined from the measurement of its outputs between the initial time t_0 and the final time t_f.

We can also say: The system is observable on $[\, t_0, t_f \,]$ if any initial state $x(t_0) = x_0$ is uniquely determined by the corresponding response $y(t)$ for $t \in [\, t_0, t_f \,]$.

Kalman Condition:

A system is fully observable iff :

$$\boldsymbol{rank}\underbrace{\left[\boldsymbol{C^T} \vdots \boldsymbol{A^TC^T} \vdots \boldsymbol{A^{T^2}C^T} \vdots \cdots \vdots \boldsymbol{A^{T^{(n-1)}}C^T}\right]}_{\text{Observability Matrix: } \phi_{obs}} = \boldsymbol{rank(A) = n}$$

Or :

$$\boldsymbol{rank}\left(\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{CA^2} \\ \vdots \\ \boldsymbol{CA^{(n-1)}} \end{bmatrix}\right) = \boldsymbol{n}$$

7.4 *Stabilizability:*

A state representation (the pair **(A, B)**) is called stabilizable if every unstable mode can be controlled. Equivalently, any mode that is not controllable must be stable.

7.5 *Detectability:*

A state representation (the pair **(A, C)** is detectable if all its unstable modes are observable. Equivalently, any mode that cannot be observed must be stable.

Example:

Consider the system described by the state equations:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = [0 \quad 3 \quad 5] x(t)$$

1. Verify controllability and observability.
2. Verify if the system is stabilizeable and detectable?

Solution:

1. Controllability:

$$\emptyset_{com} = [B \quad AB \quad A^2B] = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow det(\emptyset_{com}) = 0 \Rightarrow rank(\emptyset_{com}) < n \Rightarrow \textbf{the system is not fully controllable}$$

Observability:

$$\emptyset_{obs} = [C \quad CA \quad CA^2]^T = \begin{bmatrix} 0 & 3 & 5 \\ 0 & 3 & -10 \\ 0 & 3 & 20 \end{bmatrix}$$

$$\Rightarrow det(\emptyset_{obs}) = 0 \Rightarrow rank(\emptyset_{obs}) < n \Rightarrow \textbf{the system is not fully observable}$$

2. Stabilisability and detectability:

The eigenvalues:

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda + 1 & 0 & 0 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 2 \end{vmatrix} = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = 1 \\ \lambda_2 = -2 \end{cases}$$

The system is stabilizeable because the unstable mode **(1)** is controllable (the non-controllable mode **(-2)** is stable).

The system is detectable because the unstable mode **(1)** is observable (the non-observable mode **(-1)** is stable).

8 STATE FEEDBACK CONTROL:

For systems described by state equations, you can use either output feedback or state feedback.

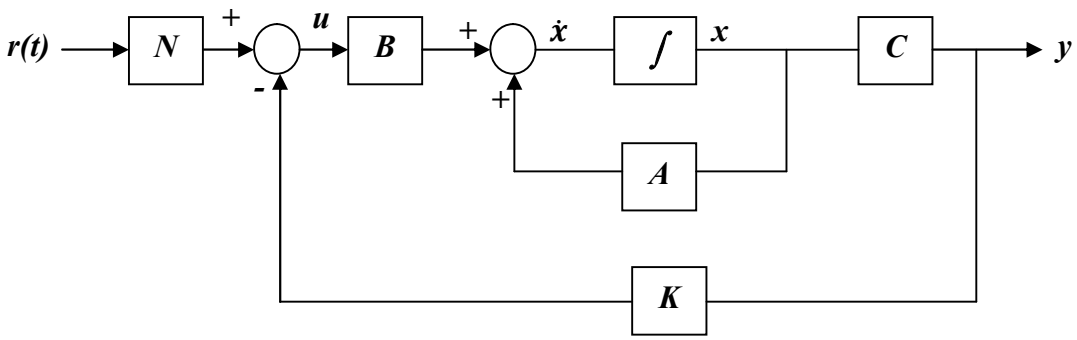


Figure 1.8: Output feedback control

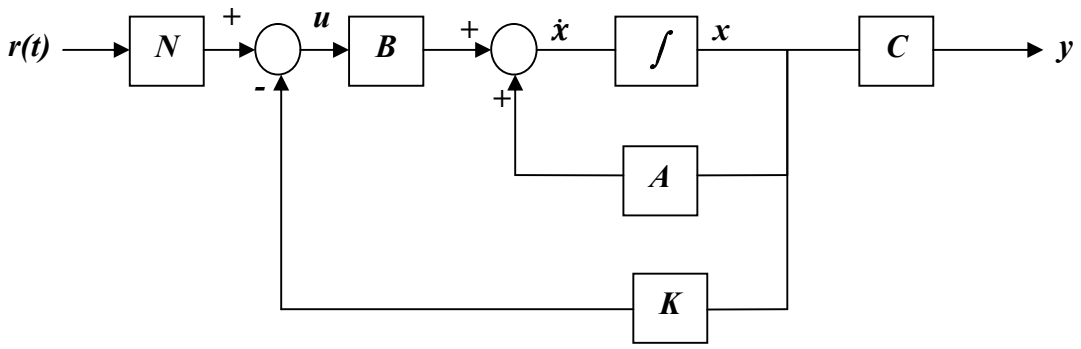


Figure 1.9: State feedback control

r(t) **:** the setpoint or refernce

N **:** the shaping or pre-filter matrix (also called the pre-compensation gain) — used to optimize steady-state behavior

K : the controller — responsible for optimizing transient response.

Consider the system: $\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t}) \end{cases}$

State feedback control: ***u(t) = N.r(t) – K.x(t)***

Therefore: $\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{N}\boldsymbol{r}(\boldsymbol{t}) - \boldsymbol{B}\boldsymbol{K}\boldsymbol{x}(\boldsymbol{t}) = (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{N}\boldsymbol{r}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{K})\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{N}\boldsymbol{r}(\boldsymbol{t}) \end{cases}$

To calculate the gain matrix K of state-feedback control ($u(t) = -K.x(t)$), we can use the state-feedback pole placement method. This method is a modern control theory technique that forces a system's closed-loop poles (eigenvalues) to exact, desired locations in the complex plane.

ALGORITHM : The linear time-invariant system (LTI) is fully controllable..

We want to calculate $K = ?$ so that the system with state feedback (that is the matrix $(A-KB)$ has the eigenvalues $\{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_n}\}$).

1- calculation of polynomial: $\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = det\,(sI - A)$

2- calculation of polynomial: $\bar{\Delta}(s) = det\big(sI - (A - KB)\big) = \big(s - \overline{\lambda_1}\big)\big(s - \overline{\lambda_2}\big) \dots \big(s - \overline{\lambda_n}\big)$

$$= s^n + \overline{a_{n-1}}s^{n-1} + \cdots + \overline{a_0}$$

3- calculation of: $\bar{K} = [a_0 - \overline{a_0}, \dots, a_{n-1} - \overline{a_{n-1}}]$

4- calculation of: q_i: $\begin{cases} q_n = B \\ q_{n-i} = Aq_{n-i+1} + a_{n-i}q_n \end{cases} \quad i = 1, \dots, (n-1)$

5- construction of matrix Q: $Q = [q_1, \dots, q_n]$

6- calculation of matrix P: $P = Q^{-1}$

7- The gain matrix: $K = -\bar{K}.P$

Example:

We consider the system described by the state equation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Calculate the gain matrix K to place the poles of the closed-loop (**C.L**) system at: $\overline{\lambda_1} = -3$, $\overline{\lambda_2} = -4$ by state feedback.

Solution:

The first step is the verification of the system controllability:

$$\emptyset_{com} = [B \quad AB] = \begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \Rightarrow det(\emptyset_{com}) = -2 \neq 0$$

Therfore: $rang(\emptyset_{com}) = 2 = n \Rightarrow$ *syetème is fully controllable*

1st Method:

$$\Delta(s) = det(sI - A) = \begin{vmatrix} s & -2 \\ 0 & s-3 \end{vmatrix} = s^2 - 3s \Rightarrow \lambda_1 = 0 \; et \; \lambda_2 = 3$$

$$\overline{\Delta}(s) = det(sI - (A - BK)) = (s - \overline{\lambda_1})(s - \overline{\lambda_2})$$

$$= (s+3)(s+4) = s^2 + 7s + 12$$

The matrix in **C.L** is:

$$(A - BK) = \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 2 \\ -k_1 & 3 - k_2 \end{bmatrix}$$

$$det(sI - (A - BK)) = \begin{vmatrix} s & -2 \\ k_1 & s - 3 + k_2 \end{vmatrix} = s^2 + (k_2 - 3)s + 2k_1$$

$$= s^2 + 7s + 12$$

Therefore, by identification, we obtain:

$$\begin{cases} k_2 - 3 = 7 \\ 2k_1 = 12 \end{cases} \Rightarrow \begin{cases} k_2 = 10 \\ k_1 = 6 \end{cases}$$

The gain matrix K is: $K = [6 \quad 10]$

2sd Method: The algorithm

1) $\Delta(s) = det(sI - A) = s^2 - 3s \Rightarrow a_1 = -3, a_0 = 0$

2) $\overline{\Delta}(s) = (s - \overline{\lambda}_1)(s - \overline{\lambda}_2) = s^2 + 7s + 12 \Rightarrow \overline{a}_1 = 7, \overline{a}_0 = 12$

3) $\overline{K} = [a_0 - \overline{a}_0 \quad a_1 - \overline{a}_1] = [-12 \quad -10]$

4) Calculation of q_i :

$$q_2 = B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$i = 1 \Rightarrow q_1 = Aq_2 + a_1 q_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

5) The matrix Q: $\quad Q = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

6) $P = Q^{-1} \Rightarrow P = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

7) $K = -[-12 \quad -10].\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow K = [6 \quad 10]$

9 REMINDER ABOUT MATRIX CALCULUS:

In mathematics, matrix calculus is a specialized notation for performing calculations with multiple variables, particularly on matrix spaces.

For our multivariable linear systems course, reviewing matrix calculus will help students manipulate state vectors and state matrices more confidently when describing MIMO systems. It also makes it easier to compute the ranks of controllability and observability matrices and to design state feedback controllers.

1. *The matrix trace:*

The sum of the diagonal elements of a square matrix is called the trace of the matrix. We use the notation "$tr(\boldsymbol{A})$" to denote the trace of the matrix $\boldsymbol{A}$: $tr(\boldsymbol{A}) = \sum_i a_{ii}$.

Definition: If $\boldsymbol{A}$ is a square matrix of size $\boldsymbol{(n \times n)}$, The trace of $\boldsymbol{A}$, noted $\boldsymbol{tr(A)}$, is the sum of its diagonal elements.

$$\boldsymbol{A = (a_{ij}) \Rightarrow tr(A) = \sum_{i=1}^{n} a_{ii}}$$

Properties:

- $tr(\boldsymbol{A}) = \sum_i a_{ii}$
- $tr(\boldsymbol{A}) = \sum_i \lambda_i$, $\lambda_i = \text{eig}(\boldsymbol{A})$
- $tr(\boldsymbol{A}) = tr(\boldsymbol{A}^T)$
- $tr(\boldsymbol{AB}) = tr(\boldsymbol{BA})$
- $tr(\boldsymbol{A} + \boldsymbol{B}) = tr(\boldsymbol{A}) + tr(\boldsymbol{B})$
- $tr(\boldsymbol{ABC}) = tr(\boldsymbol{BCA}) = tr(\boldsymbol{CAB})$
- $tr(\mathbf{uv}^T) = \mathbf{u}^T\mathbf{v}$
- $tr(\mathbf{Xvu}^T) = \mathbf{u}^T\mathbf{Xv}$

Example: Consider the following matix $\boldsymbol{A}$ of size **4×4:**

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2} & \mathbf{8} & \mathbf{0} & \mathbf{-3} \\ \mathbf{-1} & \mathbf{7} & \mathbf{4} & \mathbf{1} \\ \mathbf{6} & \mathbf{3} & \mathbf{-5} & \mathbf{9} \\ \mathbf{11} & \mathbf{-2} & \mathbf{5} & \mathbf{0} \end{pmatrix}$$

Calculate the matrix trace $\boldsymbol{tr(A)}$**.**

Solution:

$$\boldsymbol{tr(A) = \sum_{i=1}^{4} a_{ii} = (2 + 7 + (-5) + 0) = 4}$$

2. *Transpose of a matrix:*

The transpose of a matrix is the matrix whose $\boldsymbol{i^{th}}$ row is the $\boldsymbol{i^{th}}$ column of the original matrix and whose $\boldsymbol{j^{th}}$ column is the $\boldsymbol{j^{th}}$ row of the original matrix. We use a superscript "$\boldsymbol{T}$" to denote the transpose of a matrix; thus, if $\boldsymbol{A = (a_{ij})}$ then $\boldsymbol{A^T = (a_{ji})}$**.**

If the elements of the matrix are from the field of complex numbers, the conjugate transpose, also called the adjoint, is more useful than the transpose. We use a superscript "H" to denote the conjugate transpose of a matrix; thus, if $A = (a_{ij})$ then $A^H = (\overline{a_{ji}})$.

Properties:

- $(A + B)^T = B^T + A^T$.
- $(AB)^T = B^T A^T$.
- $(ABC \dots)^T = \cdots C^T B^T A^T$.
- $(A + B)^H = B^H + A^H$.
- $(AB)^H = B^H A^H$
- $(ABC \dots)^H = \cdots C^H B^H A^H$.

Example:

Consider the following matix A of size **4×4:**

$$A = \begin{pmatrix} 2 & 8 & 0 & -3 \\ -1 & 7 & 4 & 1 \\ 6 & 3 & -5 & 9 \\ 11 & -2 & 5 & 0 \end{pmatrix}$$

Calculate the transpose of matrix A.

Solution:

$$A^T = (a_{ji}) = \begin{pmatrix} 2 & -1 & 6 & 11 \\ 8 & 7 & 3 & -2 \\ 0 & 4 & -5 & 5 \\ -3 & 1 & 9 & 0 \end{pmatrix}$$

You can see that the elements of the main diagonal ***(2, 7, -5, 0)*** haven't moved.

3. *Determinants:*

If A is a square matrix of order **3**, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, we will define :

$$|A| = \sum_{j=1}^{3} (-1)^{1+j} a_{1j} |A_{1j}|$$

The determinant $|A|$ of a matrix A of order n is the expansion along the elements of the first row:

$$|A| = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} |A_{1j}|$$

The submatrix A_{ij} of A, is a new matrix, of order $(n-1)$, obtained by removing row i and column j.

Example1:

Evaluate the determinant of each of the following matrices:

$$A=\begin{bmatrix}3 & 0 & 2\\ 0 & -1 & 0\\ 1 & 1 & 2\end{bmatrix};\ B=\begin{bmatrix}3 & 0 & 1\\ 2 & 2 & 5\\ 3 & 0 & 2\end{bmatrix}$$

Solution:

- $\det(A)=(-1)^{1+1}\times 3\times\begin{vmatrix}-1 & 0\\ 1 & 2\end{vmatrix}+(-1)^{1+2}\times 0\times\begin{vmatrix}0 & 0\\ 1 & 2\end{vmatrix}+(-1)^{1+3}\times 2\times\begin{vmatrix}0 & -1\\ 1 & 1\end{vmatrix}$
  $=1\times 3\times(-2-0)+(-1)\times 0\times(0-0)+1\times 2\times(0+1)=-6+0+2=-4$
- $\det(B)=(-1)^{1+1}\times 3\times\begin{vmatrix}2 & 5\\ 0 & 2\end{vmatrix}+(-1)^{1+2}\times 0\times\begin{vmatrix}2 & 5\\ 3 & 2\end{vmatrix}+(-1)^{1+3}\times 1\times\begin{vmatrix}2 & 2\\ 3 & 0\end{vmatrix}$
  $=1\times 3\times(4-0)+(-1)\times 0\times(4-15)+1\times 1\times(0-6)$
  $=3\times 4+(-1)\times 0\times(-11)+1\times(-6)=12+0-6=6$

Properties:

- $det(I+A^TB)=det(I+AB^T)=det(I+B^TA)=det(I+BA^T)$.
- $det(A+uv^T)=(1+v^TA^{-1}u).det(A)$
- $det(A)=\prod_i\lambda_i\ ,\lambda_i=eig(A)$
- $det(AB)=det(A).det(B)$.
- $det(A^{-1})={}^{1}/_{det(A^{-1})}$
- $det(I+uv^T)=(1+v^Tu)$

Example 2:

$$A=\begin{bmatrix}1 & -3 & 3\\ 4 & 2 & 0\\ -2 & -7 & 5\end{bmatrix}$$

$$|A|=(+1)^{1+1}.\begin{vmatrix}2 & 0\\ -7 & 5\end{vmatrix}+(-3)^{1+2}.\begin{vmatrix}4 & 0\\ -2 & 5\end{vmatrix}+(3)^{1+3}.\begin{vmatrix}4 & 2\\ -2 & -7\end{vmatrix}$$
$$=10+(-27).(20)+81.(-24)=-2474$$

Example 3:

$$A=\begin{bmatrix}1 & 2 & 1 & 0\\ 0 & 3 & 1 & 1\\ -1 & 0 & 3 & 1\\ 3 & 1 & 2 & 0\end{bmatrix}$$

To minimize calculations, pick the row or column with the most zeros. Here, the fourth column. So we'll expand the determinant by cofactors along the fourth column, which means that:

$$|A| = a_{14}C_{14} + a_{24}C_{24} + a_{34}C_{34} + a_{44}C_{44}$$

Since $\mathbf{a_{14}}$ and a_{44} are zero, it would be pointless to try to find $\mathbf{C_{14}}$ et C_{44}. However, we will need the cofactors $\mathbf{C_{24}}$ et C_{34} …

$$C_{24} = (-1)^{2+4}M_{24} = 1\begin{vmatrix} 1 & 2 & 1 \\ -1 & 0 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 18$$

$$C_{34} = (-1)^{3+4}M_{24} = 1\begin{vmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -2$$

Therefore: $|A| = 1.18 + 1.(-2) = 16.$

4. ***The inverse of a matrix:***

The inverse of a square matrix A, denoted as A^{-1}, is a unique matrix that, when multiplied by the original matrix, results in an identity matrix I. This relationship is defined as $A^{-1}A = A^{-1}A = I$. A matrix is only invertible if it is a **square matrix** (equal number of rows and columns) and its **determinant is not zero** $det(A) \neq 0$. The inverse of a matrix A is calculated by:

$$A^{-1} = \frac{1}{\det(A)} adj(A), avec: \ (adj(A))_{i,j} = (-1)^{i+j}\det(A_{ji})$$

where: A_{ij} is the matrix obtained by removing the i^{th} raw and the j^{th} column of A. If A^{-1} exist, A is said to be invertible or regular, and $\det(A^{-1}) = 1/\det(A)$.

$adj(A)$ is **adjoint** (or **adjugate**) of a square matrix is the transpose of its cofactor matrix.

The adjoint is computed through a three-step process:

1. ***Find the Minors*****:** For every element in the matrix, eliminate its row and column, then calculate the determinant of the remaining submatrix.

2. ***Find the Cofactors*****:** Apply an alternating checkerboard of signs, $(-1)^{i+j}$, to the minors.

3. ***Take the Transpose*****:** Write the resulting cofactor matrix and then swap its rows with its columns.

Example:

By resuming: $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix} \Rightarrow A^{-1} = \begin{bmatrix} 2 & 2 & -5 \\ -2 & -1 & 4 \\ 1 & 0 & -1 \end{bmatrix}$ $avec: |A| = 1$

5. *Matrix rank:*

The linear dependence or independence of the vectors forming the rows or columns of a matrix is an important characteristic of the matrix. The maximum number of linearly independent vectors (those forming either the rows or the columns) is called the rank of the matrix. We use the notation rank($\boldsymbol{B}$) to denote the rank of the matrix $\boldsymbol{B}$. Because multiplication by a nonzero scalar does not change the linear independence of vectors, for the scalar $\boldsymbol{a}$ with $\boldsymbol{a \neq 0}$, we have $\boldsymbol{rank(aB) = rank(B)}$. If $\boldsymbol{A}$ is an $\boldsymbol{n \times m}$ matrix then $\boldsymbol{rank(A) < min(n, m)}$.

The rank $\boldsymbol{r}$ of a matrix $\boldsymbol{A \in \mathbb{C}^{n \times m}}$ is defined as the maximum number of linearly independent rows or columns in that matrix. It has full rank if and only if $\boldsymbol{min(m, n) = r}$. If this is not the case, the matrix is said to be rank deficient.

A square matrix $\boldsymbol{A}$ with full ranks is said to be regular. In this case, it follows $\boldsymbol{det\ (A) \neq 0}$.

Theorem: $\boldsymbol{rank(A) = rank(A^T) = rank(A^TA) = rank(AA^T)}$.

The results obtained according to this theorem are:

- $\boldsymbol{rank(A) = rank(A^T)}$,
- $\boldsymbol{rank(A^TA) = rank(AA^T)}$
- $\boldsymbol{rank(A) = rank(A^TA)}$
- $\boldsymbol{rank(A) = dim(\mathcal{V}(A))}$,
- $\boldsymbol{dim(\mathcal{V}(A)) = dim(\mathcal{V}(A^T))}$
- $\boldsymbol{rank(A^T) = rank(AA^T)}$

Example 1:

Let the matrix:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{0} & \mathbf{4} & \mathbf{6} \\ \mathbf{0} & \mathbf{2} & \mathbf{2} & \mathbf{0} \\ \mathbf{1} & \mathbf{2} & \mathbf{4} & \mathbf{3} \end{pmatrix}$$

We can see that the $\boldsymbol{2^{nd}}$ line is twice as long as the first and that the fourth $\boldsymbol{4^{th}}$ line can be formed by adding lines **1** and **3**.

Lines ***1*** and ***3*** are linearly independent (i.e., not proportional).

Finally, the rank of $\boldsymbol{A}$ is ***2***.

To confirm, we can calculate the determinant of the matrix:

$$det\ (A) = 0 \Rightarrow rank(A) < 4$$

Example 2:

Calculate the rank of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{0} & \mathbf{16} & \mathbf{8} & \mathbf{4} \\ \mathbf{2} & \mathbf{4} & \mathbf{8} & \mathbf{16} \\ \mathbf{16} & \mathbf{8} & \mathbf{4} & \mathbf{2} \\ \mathbf{4} & \mathbf{8} & \mathbf{16} & \mathbf{2} \end{pmatrix}$$

$$\boldsymbol{det(A) = (-1)^{1+1} a_{11} |A_{11}| + (-1)^{1+2} a_{12} |A_{12}| + (-1)^{1+3} a_{13} |A_{13}| + (-1)^{1+4} a_{14} |A_{14}|}$$
$$\boldsymbol{= 1 \times 0 \times |A_{11}| + (-1) \times 16 \times |A_{12}| + 1 \times 8 \times |A_{13}| + (-1) \times 4 \times |A_{14}|}$$

With:
$$\boldsymbol{|A_{11}| = \begin{vmatrix} 4 & 8 & 16 \\ 8 & 4 & 2 \\ 8 & 16 & 2 \end{vmatrix} = 1440}$$

$$\boldsymbol{|A_{12}| = \begin{vmatrix} 2 & 8 & 16 \\ 16 & 4 & 2 \\ 4 & 16 & 2 \end{vmatrix} = 3600}$$

$$\boldsymbol{|A_{13}| = \begin{vmatrix} 2 & 4 & 16 \\ 16 & 8 & 2 \\ 4 & 8 & 2 \end{vmatrix} = 1440}$$

$$\boldsymbol{|A_{14}| = \begin{vmatrix} 2 & 8 & 18 \\ 16 & 4 & 4 \\ 4 & 16 & 16 \end{vmatrix} = 2400}$$

Therefore**:**

$$\boldsymbol{|A| = 1 \times 0 \times 1440 + (-1) \times 16 \times 3600 + 1 \times 8 \times 1440 + (-1) \times 4 \times 2400 = -46080}$$

$$\boldsymbol{\Rightarrow |A| \neq 0 \Rightarrow rank(A) = 4}$$

As the matrix $\boldsymbol{A}$ has **4** rows or **4** columns, linearly independent. We conclude that the matrix has rank **4**.

CHAPITRE II : STATE REPRESENTATION OF MULTIVARIABLE SYSTEMS

1. INTRODUCTION:

Analyzing linear dynamical systems in the time domain traditionally uses differential-equation-based techniques. Because these time-domain methods become trying for more complex systems, engineers developed the Laplace transform to represent systems with transfer functions. Transfer functions have proven valuable for studying frequency response (Bode plots, Nyquist diagrams, etc.) and assessing stability. However, despite their usefulness, transfer-function representations have several limitations:

- The assumption of zero initial conditions simplifies analysis, but those initial states matter when studying systems in the time domain. In such cases, the solution often depends strongly on the system's past behavior, so neglecting initial conditions can lead to incorrect conclusions.
- Another drawback of this approach is that it does not adapt well to the multi-variable case (multiple inputs, multiple outputs: MIMO systems).
- The Laplace variable "s" prohibits the use of numerical methods.
- Non-linear systems are difficult to describe using transfer functions.
- Compensation for an unstable pole by a zero.

For such reasons, advanced control theory often uses a type of modern modeling called state-space representation (or internal representation). The concept of state is used whenever information about internal variables is needed to make a decision concerning a system. State vector representation helps overcome these difficulties and unify the framework for both continuous and discrete dynamical systems.

We can therefore list some advantages of state representation:

- A very large class of physical processes can be represented by a state-space model.
- State-space models take the initial state into account.
- State-space models are more easily adaptable to multivariable cases.

Models of continuous-time linear systems (reminder of the single-variable case):

Models of continuous-time linear systems (multivariable case):

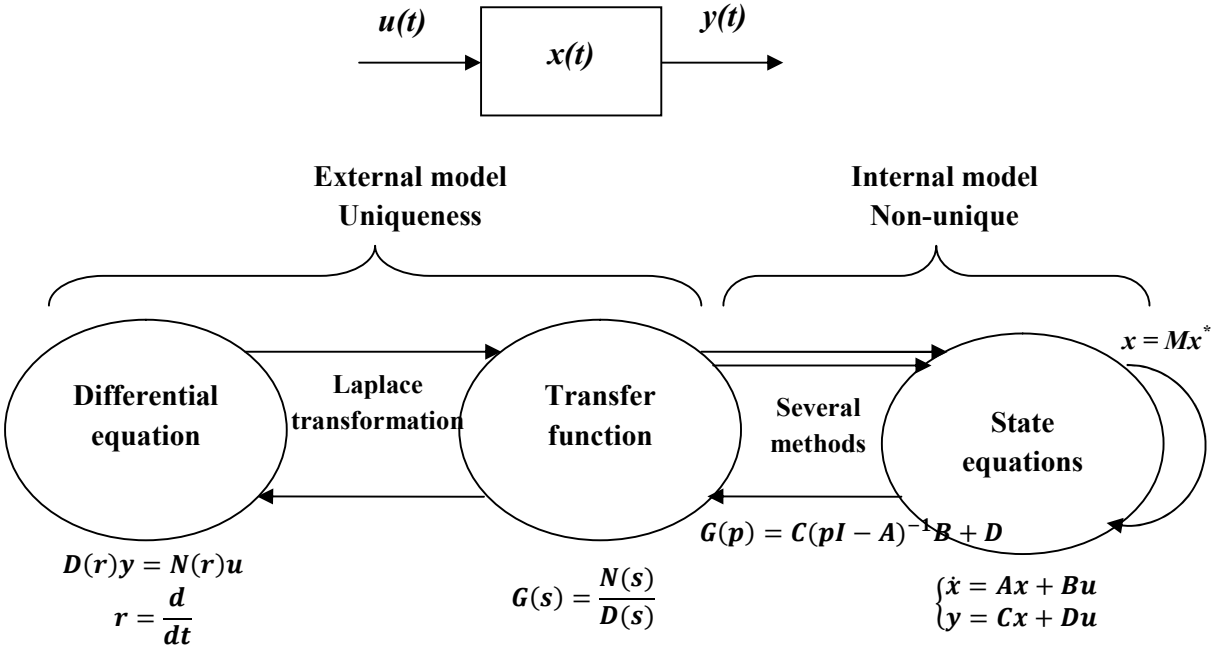


Figure 2.1: Different representations of SISO system

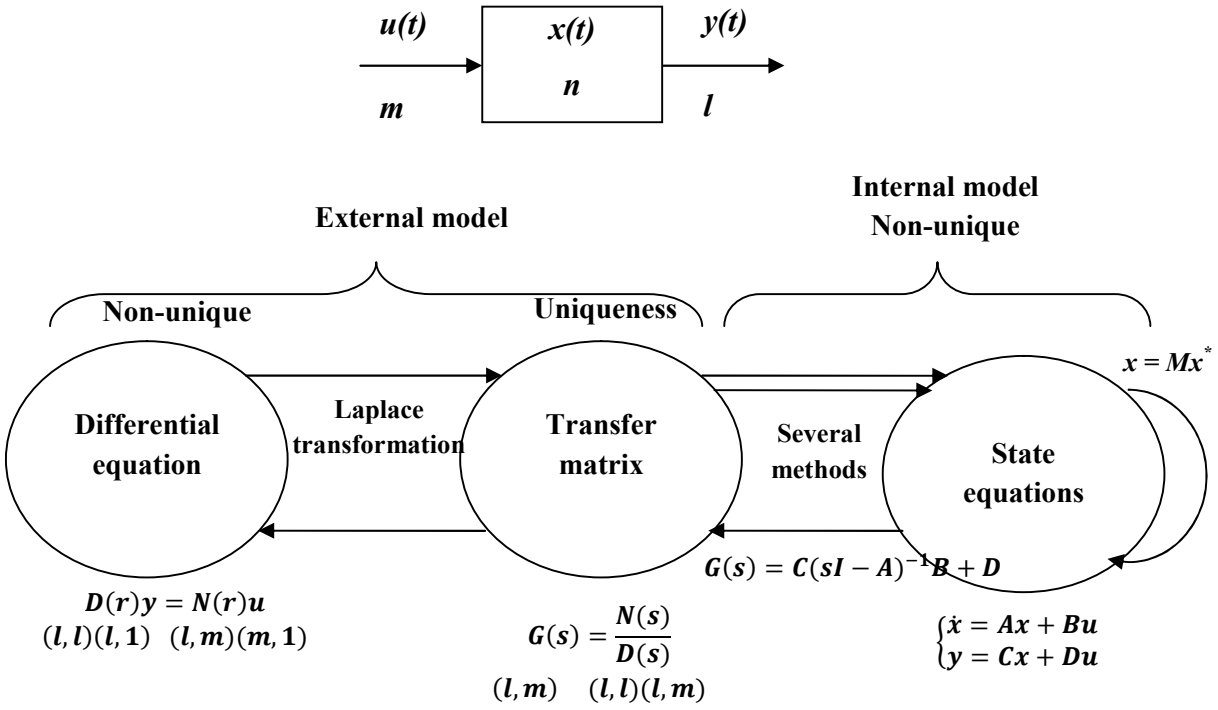


Figure 2.2: Different representations of MIMO system

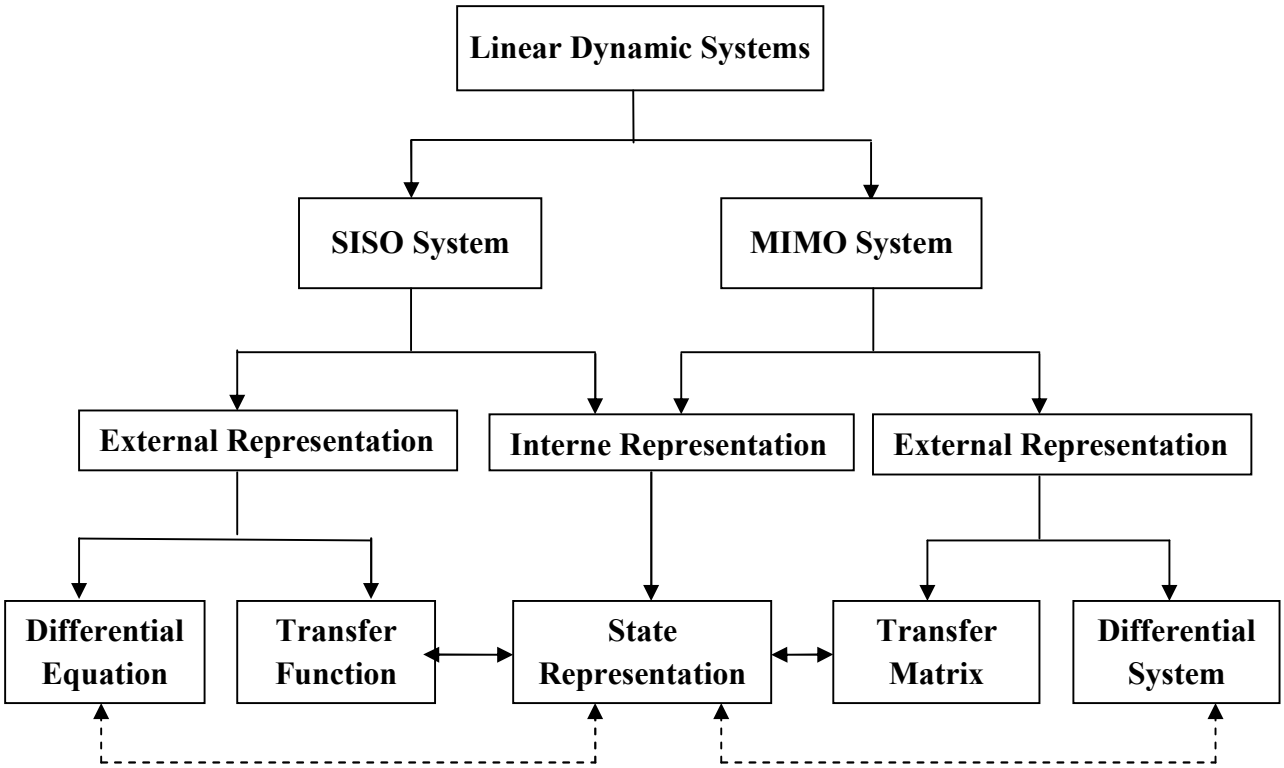


Figure 2.3: Flowchart of Different types of system

2. ANALOGUE STATE MODEL :

1. *Definition:*

After applying the physical laws governing them, many physical processes can be described by differential and algebraic equations with the following structure:

$$\begin{cases} \dot{x_1}(t) = f_1[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \\ \dot{x_2}(t) = f_2[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \\ \vdots \\ \dot{x_n}(t) = f_n[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \end{cases}$$

$$\begin{cases} y_1(t) = g_1[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \\ y_2(t) = g_2[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \\ \vdots \\ y_l(t) = g_l[x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t), t\ ] \end{cases}$$

The variable t represents time. We now introduce vector notation:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}; u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix} \text{ et } y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_l(t) \end{bmatrix}$$

The functions f et g are given by:

$$f[x(t),u(t),t]=\begin{bmatrix} f_1[x(t),u(t),t] \\ f_2[x(t),u(t),t] \\ \vdots \\ f_n[x(t),u(t),t] \end{bmatrix}; g[x(t),u(t),t]=\begin{bmatrix} g_1[x(t),u(t),t] \\ g_2[x(t),u(t),t] \\ \vdots \\ g_l[x(t),u(t),t] \end{bmatrix}$$

By definition :

$$\dot{x}(t)=\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix}$$

Therefore, the previous relationships can be written in the form:

$$\begin{cases} \dot{x}(t)=f[x(t),u(t),t] & \textbf{State equation} \\ y(t)=g[x(t),u(t),t] & \textbf{Output equation} \end{cases}$$

2. *State représentation of linear systems*

An extremely important special case of a state model is that in which the functions f and g have the following form:

$$f[x(t),u(t),t]=A(t)x(t)+B(t)u(t)$$

$$g[x(t),u(t),t]=C(t)x(t)+D(t)u(t)$$

A(t), B(t), C(t)* et *D(t) are respectively matrices of dimensions ***(n×n), (n×m), (l×n)*** et ***(l×m).***

The state representation then becomes:

$$\begin{cases} \dot{x}(t)=A(t)x(t)+B(t)u(t) \\ y(t)=C(t)x(t)+D(t)u(t) \end{cases}$$

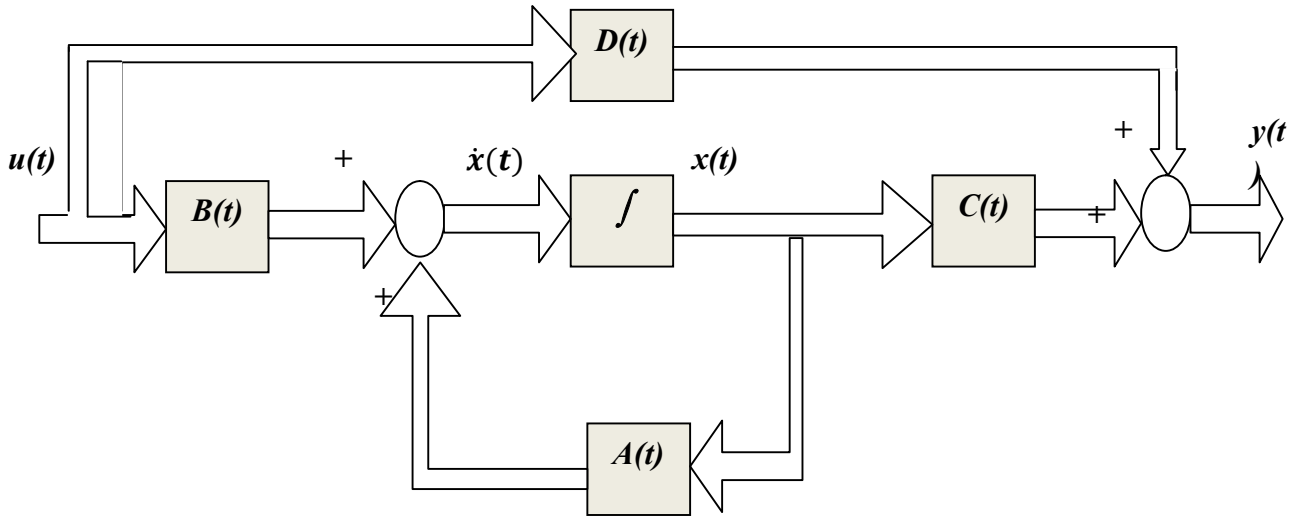


Figure 2.4: Functional diagram of state representation of an analog system

An analog system governed by the relationships above is called **linear**. If the matrices in those equations change over time, the system is **non-stationary**; if they do not, it is **stationary**.

3. DISCRETE STATE MODEL:

Intrinsically discrete systems, discretized versions of continuous systems, and numerical simulation algorithms can all be described using difference and algebraic equations. Any difference equation, regardless of its form, can be rewritten as a first-order difference equation system by choosing appropriate state variables. In the discrete setting, the role of the derivative is taken by a shift over the sampling period. Consequently, a discrete state model is represented by the following structure:

$$\begin{cases} \boldsymbol{x_1(k+1) = f_1[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \\ \boldsymbol{x_2(k+1) = f_2[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \\ \vdots \\ \boldsymbol{x_n(k+1) = f_n[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \end{cases}$$

$$\begin{cases} \boldsymbol{y_1(k) = g_1[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \\ \boldsymbol{y_2(k) = g_2[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \\ \vdots \\ \boldsymbol{y_l(tk) = g_l[x_1(k), \dots, x_n(k), u_1(k), \dots, u_m(k), k\ ]} \end{cases}$$

The preceding relationships can be written in the form:

$$\begin{cases} \boldsymbol{x(k+1) = f[x(k), u(k), k)} & \boldsymbol{(1)} \\ \boldsymbol{y(k) = g[x(k), u(k), k]} & \boldsymbol{(2)} \end{cases}$$

Equation ***(1)*** governing the dynamic behavior of the process is called the **state equation**; relation ***(2)*** giving its output is the **output equation**. The **state model** is said to be discrete or internal.

State representation of linear discrete systems:

A useful special case of the discrete state-space model is that in which the functions $\boldsymbol{f}$ and $\boldsymbol{g}$ have the following form:

$$\boldsymbol{f[x(k), u(k), k] = \Phi(k)x(k) + \Gamma(k)u(k)}$$
$$\boldsymbol{g[x(k), u(k), k] = C(k)x(k) + D(k)u(k)}$$

The state representation then becomes:

$$x(k+1) = \Phi(k)x(k) + \Gamma(k)u(k)$$
$$y(k) = C(k)x(k) + D(k)u(k)$$

The following figure provides the functional diagram of such a system; q^{-1} denotes the delay operator:

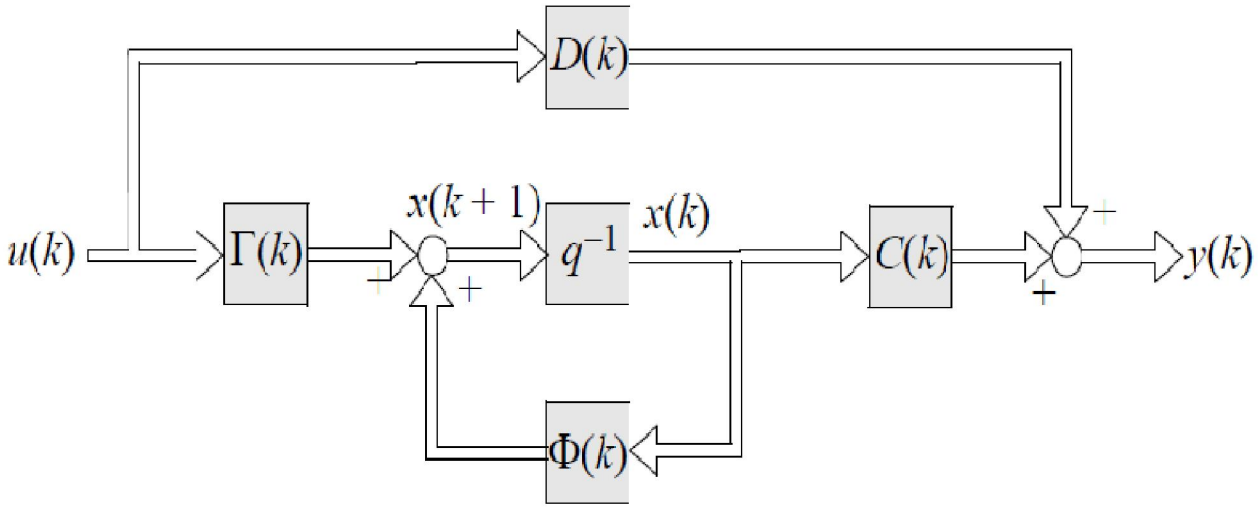


Figure 2.5: Functional diagram of the state representation of linear discrete systems

4. SOLUTION OF THE LINEAR STATE EQUATION:

The state equation solution: $\dot{x}(t) = Ax(t) + Bu(t)$ is : $x(t) = x_h(t) + x_p(t)$, with :

$x_h(t)$: the homogeneous solution

$x_p(t)$ **:** the particular solution

Homogeneous state equation:

To solve the state equation, start by considering the system without external input and with only nonzero initial conditions. This gives the homogeneous equation:

$$\dot{x}(t) = Ax(t),\ x(t_0) = x_0 \qquad (3)$$

To achieve this, its solution is assumed to be sufficiently smooth so that a serial expansion is possible:

$$x(t) = A_0 + A_1(t - t_0) + A_2(t - t_0)^2 + \cdots \qquad (4)$$

By setting $t = t_0$, we immediately find that $A_0 = x_0$. If we differentiate **(4)** and substitute the resulting expression into **(3)**, we get:

$$A_1 + 2A_2(t - t_0) + 3A_3(t - t_0)^2 + \cdots = Ax(t)$$

and, in : $t = t_0$: $A_1 = Ax_0$. By continuing the successive differentiations of the series and the homogeneous equation evaluated at $t = t_0$, the following expression is obtained:

$$x(t) = \left[I + A(t - t_0) + \frac{A^2(t - t_0)^2}{2} + \frac{A^3(t - t_0)^3}{6} + \cdots \right] x_0$$

By analogy with the scalar case, the term in brackets is defined as the exponential of matrix and denoted:

$$e^{A(t-t_0)} = I + A(t - t_0) + \frac{A^2(t - t_0)^2}{2} + \frac{A^3(t - t_0)^3}{6} + \cdots$$

$$= \sum_{k=0}^{\infty} A^k \frac{(t - t_0)^k}{k!}$$

Particular solution:

The particular solution of state equation when the control is non-zero is obtained by the method of parameter variation. We assume that the solution has the form:

$$x(t) = e^{A(t-t_0)} v(t)$$

Where $v(t)$ is a vector of variable parameters to be determined, as opposed to constant parameters $x(t_0)$.

By substituting this expression into the state equation: $\dot{x}(t) = Ax(t) + Bu(t)$, we obtain:

$$Ae^{A(t-t_0)} v(t) + e^{A(t-t_0)} \dot{v}(t) = Ae^{A(t-t_0)} v(t) + Bu(t)$$

and, by the fact that the inverse of the matrix exponential is obtained by changing the sign of its exponent:

$$\dot{v}(t) = e^{-A(t-t_0)} Bu(t)$$

We obtain :

$$v(t) = \int_{t_0}^{t} e^{-A(\tau - t_0)} Bu(\tau) d\tau$$

Therefore, the particular solution is:

$$x(t) = \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau$$

General Solution:

Because the system is linear, the superposition principle applies; therefore the full solution or general solution of the state equation equals the sum of the homogeneous solution and a particular solution.

$$\boldsymbol{x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau}$$

Calculating the matrix exponential:

1) Serial development:

$$\boldsymbol{e^{At} = I + At + \frac{A^2t^2}{2} + \cdots + \frac{A^kt^k}{k!} + \cdots = \sum_{k=0}^{\infty} A^k\frac{t^k}{k!}}$$

2) Inverse Laplace Transform:

$$\boldsymbol{e^{At} = \mathcal{L}^{-1}((pI - A)^{-1})}$$

3) Base change (diagonalization or Jordan form):

$$\boldsymbol{Av_i = \lambda_i v_i}$$

If A has n independent eigenvectors, then A is diagonalizable. Let T be the matrix formed by the eigenvectors and Λ the diagonal matrix having the eigenvalues of A on the diagonal.

Therefore:

$$\boldsymbol{e^{At} = Ve^{\Lambda t}V^{-1}}$$

With: V is the matrix that contains the eigenvectors

$$\boldsymbol{V = [v_1\ v_2 \cdots v_n], \qquad e^{\Lambda t} = \begin{bmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_2 t} \end{bmatrix}}$$

Example 1:

A system is represented by state equations whose state matrix A is:

$$\boldsymbol{A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$$

Calculate the transition matrix: $\boldsymbol{\phi = e^{At}}$

Solution:

$$(sI_n - A)^{-1} = \begin{bmatrix} s+1 & -1 & -2 \\ 0 & s+2 & -1 \\ 0 & 0 & s+2 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & \frac{1}{(s+1)(s+2)} & \frac{2s+5}{(s+1)(s+2)^2} \\ 0 & \frac{1}{s+2} & \frac{1}{(s+2)^2} \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}$$

By decomposition into simple elements, we find:

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$\frac{2s+5}{(s+1)(s+2)^2} = \frac{-1}{(s+2)^2} - \frac{3}{s+2} + \frac{3}{s+1}$$

Therefore:

$$e^{At} = \mathcal{L}^{-1}\{(pI - A)^{-1}\} = \begin{bmatrix} e^{-t} & -e^{-2t} + e^{-t} & -(t+3)e^{-2t} + 3e^{-t} \\ 0 & e^{-2t} & te^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

Example 2:

If $A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *and* $u(t)$: *u nitstep*

Calculate $\phi(t)$ by the diagonalization method and ***x(t).***

Solution:

- Calculation of eigenvalues:

$$|\lambda I - A| = 0 \Rightarrow \begin{vmatrix} \lambda+2 & 2 \\ 1 & \lambda+3 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 5\lambda + 4 = 0 \Rightarrow \begin{cases} \lambda_1 = -1 \\ \lambda_2 = -4 \end{cases}$$

- Calculation of des eigenvectors: $Av_i = \lambda_i v_i$

$$Av_1 = \lambda_1 v_1 \Rightarrow \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \cdot \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = -\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix}$$

$$\Rightarrow \begin{cases} -v_{11} - 2v_{12} = 0 \\ -v_{11} - 2v_{12} = 0 \end{cases}$$

We take: $\mathbf{v_{12}} = \mathbf{1} \Rightarrow \mathbf{v_{11}} = \mathbf{-2}$, therfore: $\boldsymbol{v_1} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{1} \end{bmatrix}$

The same rule applies to calculating the vector $\boldsymbol{v_2}$, we find: $\boldsymbol{v_2} = \begin{bmatrix} \mathbf{1} \\ \mathbf{1} \end{bmatrix}$

Therefore, the matrix: $T = [\boldsymbol{v_1}\ \boldsymbol{v_2}] = \begin{bmatrix} \mathbf{-2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} \Rightarrow \boldsymbol{T^{-1}} = \begin{bmatrix} \mathbf{-1/3} & \mathbf{1/3} \\ \mathbf{1/3} & \mathbf{2/3} \end{bmatrix}$

$$\phi = \boldsymbol{e^{At} = Te^{\Omega t}T^{-1}} = \begin{bmatrix} \mathbf{-2} & \mathbf{1} \\ \mathbf{1} & \mathbf{1} \end{bmatrix} . \begin{bmatrix} \boldsymbol{e^{-t}} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{e^{-4t}} \end{bmatrix} . \begin{bmatrix} \mathbf{-1/3} & \mathbf{1/3} \\ \mathbf{1/3} & \mathbf{2/3} \end{bmatrix}$$

$$\phi = \begin{bmatrix} \frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{e^{-t}} + \frac{\boldsymbol{e^{-4t}}}{\mathbf{3}} & -\frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{e^{-t}} + \frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{e^{-4t}} \\ -\frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{e^{-t}} + \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{e^{-4t}} & \frac{\mathbf{1}}{\mathbf{3}}\boldsymbol{e^{-t}} + \frac{\mathbf{2}}{\mathbf{3}}\boldsymbol{e^{-4t}} \end{bmatrix}$$

$$\boldsymbol{x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau} \text{ with: } \boldsymbol{e^{At}x(0) = 0}$$

Therefore:

$$\boldsymbol{x(t)} = \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{4}}(\mathbf{7} - \mathbf{8}\boldsymbol{e^{-t}} + \boldsymbol{e^{-4t}} \\ \frac{\mathbf{-5}}{\mathbf{4}} + \boldsymbol{e^{-t}} + \frac{\boldsymbol{e^{-4t}}}{\mathbf{4}} \end{bmatrix}$$

CHAPTRE III : CONTROLLABILITY AND OBSERVABILITY

1 INTRODUCTION:

Controllability and observability are intrinsic properties of systems described by state-space models and do not depend on the specific form of representation. These fundamental concepts describe, respectively, the ability of a control input to influence the system's state and the ability to determine a state from the system's measured outputs.

Furthermore, these concepts (controllability and observability) are also the criteria used to convert a state-space model into a transfer function representation, since transfer function modeling is validated only if it fulfills the conditions of controllability and observability of systems in state-space form.

2 CONTROLLABILITY:

2.1 System controllability:

Consider the system described by the state equations:

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{A}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{B}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{C}\boldsymbol{x}(\boldsymbol{t}) + \boldsymbol{D}\boldsymbol{u}(\boldsymbol{t}) \end{cases} \quad (*)$$

The system described by (*) is called controllable if, for any initial state $x_0 \in \mathbb{R}^n$ at $t = 0$ and any final state $x_f \in \mathbb{R}^n$ at $t = t_f$, there exist piecewise-continuous control input $u(t) \in \mathbb{R}^m$ on the interval $[0, t_f]$ which steers the system from the state x_0 to the state x_f.

2.2 Theorem:

The system (*) is controllable if and only if:

$$\boldsymbol{rank}(\boldsymbol{\phi_{com}}) = \boldsymbol{n}$$

with: $\boldsymbol{\phi_{com}} = [\boldsymbol{B}\ \boldsymbol{AB}\ \ldots \boldsymbol{A^{n-1}B}]$ $(dim(\phi_{com}) = (n, n \times m))$

We then say that the pair (A, B) is controllable.

3 OBSERVABILITY:

3.1 Definition:

The system described by (*) is called observable if, for any initial state x_0 à $t = 0$, that state can be uniquely determined from the output $y(t)$ and the control input $u(t)$ for $t \geq 0$.

3.2 Theorem:

The system (*) is observable if and only if:

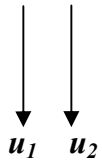
$$rank(\phi_{obs}) = n$$

with: $\phi_{com} = [B\ AB\ \ldots A^{n-1}B]$ $(dim(\phi_{obs}) = (n \times l, n))$

We then say that the pair (A, C) is observable.

4 CONTROLLABILITY OF A MIMO SYSTEM:

Example: A two-input system: $B = [b_1\ b_2]$



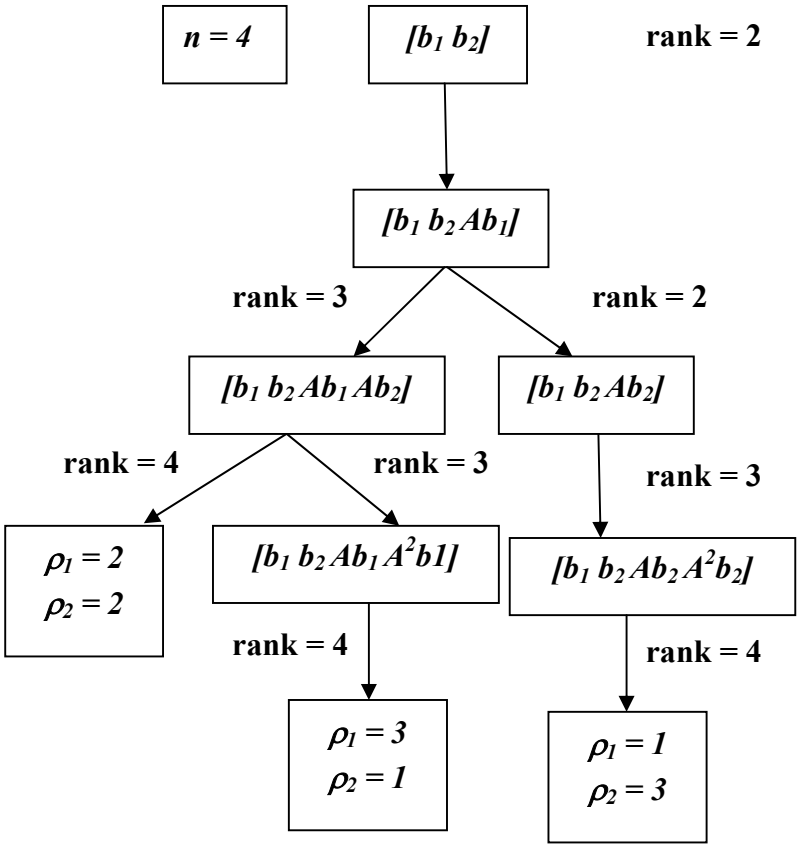
$$\phi_{com} = [B\ AB\ \ldots A^{n-1}B] = [b_1\ b_2\ Ab_1\ Ab_2 \cdots A^{n-1}b_1\ A^{n-1}b_2]$$

if $rank(\phi_{obs}) = n \Rightarrow$ system is controllable.

1) $rank([b_1\ b_2]) = 2$ (B full rank)

2) We add the column Ab_1 and we check the rank, if the rank has changed from **2** to **3,** we keep this column, otherwise we remove it. Then, we add the following column (Ab_2) and we start again.

Controllability indices ρ_i (for each u_i): $\sum_{i=1}^{m} \rho_i = rank(\emptyset_{com})$



- The columns$[\boldsymbol{b_i \;\; Ab_i \;\cdots A^{\rho_i-1}b_i}]$
- When $\boldsymbol{A^k b_i}$ does not increase the rank, the same is true for $\boldsymbol{A^{k+1}b, A^{k+2}b, \ldots .}$
- As soon as we observe that the addition of $\boldsymbol{Ab_1}$ does not increase the rank, we directly conclude that:

$$\boldsymbol{\rho_1 = 2 \text{ et } \rho_2 = n - \rho_1 = 3.}$$

Example:

Consider the system

$$\dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{-3} \\ \mathbf{4} & \mathbf{5} & \mathbf{-6} \\ \mathbf{7} & \mathbf{7} & \mathbf{-9} \end{bmatrix} \boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

Calculate the controllability indices and verify the controllability of the system.

Solution:

$$\emptyset_{com} = [\boldsymbol{B\ AB\ A^2B}] = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{-3} & \mathbf{1} & \mathbf{24} \\ \mathbf{-1} & \mathbf{0} & \mathbf{-1} & \mathbf{-6} & \mathbf{-1} & \mathbf{12} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{-9} & \mathbf{0} & \mathbf{18} \end{bmatrix}$$

Controllability indices: $\boldsymbol{\rho_1 = ?,\ \rho_2 = ?}$

$$rang[b_1\ b_2] = 2$$

$$rank[B\ Ab_1] = rank\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = 2 \Rightarrow \rho_1 = 1\ ,\ \rho_2 = 2?$$

$$rank[b_1\ b_2\ Ab_2] = rank\begin{bmatrix} 1 & 0 & -3 \\ -1 & 0 & -6 \\ 0 & 1 & -9 \end{bmatrix} = 3\ (det = -3) \Rightarrow \rho_2 = 2$$

Therefore: $rank(\emptyset_{com}) = \rho_1 + \rho_2 = 1 + 2 = 3 \Rightarrow$ Fully controllable system.

5 CONCEPT OF DUALITY:

Observability and controllability are dual notions. To illustrate this, we examine the two systems ***(S)*** and ***(S*)*** defined by:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \quad (S) \qquad \begin{cases} \dot{x}^*(t) = A^T x^*(t) + C^T u^*(t) \\ y^*(t) = B^T x^*(t) \end{cases} \quad (S^*)$$

(S^*) is called a dual or auxiliary system **(S).**

It is possible to demonstrate that ***(S)*** is controllable if and only if ***(S*)*** is observable and that ***(S)*** is observable if and only if ***(S*)*** is controllable.

Indeed, ***(S*)*** is observable if and only if:

$$\left[B^T\ B^T A^T\ B^T A^{T^2}\ \ldots B^T (A^T)^{n-1}\right]^T \ is\ of\ rank\ n$$

That's to say $[B\ AB\ A^2B\ \ldots A^{n-1}B]$ or even if and only if ***(S)*** is controllable.

Or: ***(S^*)*** contollable:

$$\phi_{com} = \left[C^T\ A^T C^T\ A^{T^2} C^T\ \ldots (A^T)^{n-1} C^T\right]$$

$$= [C\ CA\ CA^2\ \ldots CA^{n-1}]^T$$

$$= \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \equiv (S)\ observable$$

6 CONTROLLABILITE AND OBSERVABILITY OF MIMO SYSTEM:

6.1 Controllability:

$$\emptyset_{com} = [b_1\ b_2\ \ldots\ b_m\ Ab_1 Ab_2\ \ldots\ Ab_m\ A^2 b_1\ A^2 b_2 \cdots\ A^{n-1} b_m]$$

6.2 Controllability Indice (ρ):

Controllability indice ρ_i relating to input u_i is the number of states that can be controlled by the single input u_i.

6.2.1 Choice or selection by lines:

1- The following table is constructed:

b_1	b_2	b_3	….	b_m	
×	×	×		×	A^0
×	0	×		×	A^1
		0		×	A^2
					:
					:
			….		A^{n-1}

2- We fill in the table line by line

3- Linearly independent vectors are indicated by placing a cross *(×)* in the corresponding cells.

4- When you detect a vector that is linearly dependent on the previous vectors, place a zero *(0)* in its cell. Leave all cells below it blank, since those corresponding vectors will also be linearly dependent.

5- We stop the procedure as soon as we find " n " linearly independent vectors (n crosses × in the table).

6- Controllability indice ρ_i at the input u_i is equal to the number of crosses (×) in column b_i.

- ✓ We can make a selection per column, using the same procedure.

6.3 Criteria for the MS controllability:

- We calculate the controllability indices ρ_i ***(i = 1, …., m).***

- The system is said to be controllable if $\sum_{i=1}^{m} \rho_i = n$.

- In this case, the controllability matrix ϕ_{com} reduces to the square matrix:

$$\phi_{com} = [|b_1\ Ab_1\ \dots\ A^{\rho_1-1}b_1||b_2\ Ab_2\ \dots\ A^{\rho_2-1}b_2|\dots|b_m\ Ab_m\ \dots\ A^{\rho_m-1}b_m|]$$

- We can decompose the system into " m " subsystems, each of order ρ_i controllable by u_i.

7 STUDY OF SOME CANONICAL FORMS:

7.1 Notion of state representation similarity:

When you model a dynamic system with state equations, the resulting state representation isn't unique. You can derive alternative or "**similar**" representations that describe the same dynamics.

Consider a state representation of a system, given by the equations:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

We can choose a transformation matrix T called a similar transformation matrix; this matrix is singular (invertible), that is its determinant is non-zero; it allows us to transform the state vector X into another state vector $\hat{X}$:

$$\widehat{X} = T.X \quad (ou\ X = T\widehat{X})$$

Therfore: $X = T^{-1}.\widehat{X} \Rightarrow \dot{X} = T^{-1}.\dot{\widehat{X}}$

The state equation becomes: $\begin{cases} T^{-1}.\dot{\widehat{X}} = A.T^{-1}.\widehat{X} + BU \\ y = C.T^{-1}.\widehat{X} + DU \end{cases} \Rightarrow \begin{cases} \dot{\widehat{X}} = T.A.T^{-1}.\widehat{X} + T.BU \\ y = C.T^{-1}.\widehat{X} + DU \end{cases}$

We put: $\begin{cases} \widehat{A} = T.A.T^{-1} \\ \widehat{B} = T.B \\ \widehat{C} = C.T^{-1} \\ \widehat{D} = D \end{cases} \Rightarrow \begin{cases} \dot{\widehat{X}} = \widehat{A}\widehat{X} + \widehat{B}U \\ y = \widehat{C}\widehat{X} + \widehat{D}U \end{cases}$

7.2 Transition from transfer function to state representation:

As mentioned earlier, a linear dynamical system has a single transfer function, but it can be represented by different state-space forms, known as "**realizations**". In the following sections, we will see how to derive several realization forms from a given transfer function.

In general, the transfer function is written:

$$G(p) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

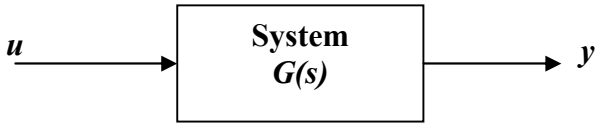
Initially, it is assumed that $m < n$, meaning that there is no direct transmission ($D = 0$). In this case the transfer function is "**strictly proper**". From this starting point, we can derive realizations in diagonal, Jordan, or companion form.

Next, the case $m = n$ is examined (the "proper" transfer function). It is then shows how the results derived for the $m < n$ case can still be useful in this proper case.

7.2.1 *Controllable canonical form:*

Example 1 (m< n):

Consider the system with the transfer function ***G(s):***



$$G(p) = \frac{5s^2 + 7s + 9}{s^3 + 8s^2 + 6s + 2} = \frac{N(s)}{D(s)}$$

Let's decompose the transfer function as follows:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} . \frac{X(s)}{U(s)} = \frac{N(s)}{D(s)}$$

We put: $\begin{cases} N(s) = \frac{Y(s)}{X(s)} = 5s^2 + 7s + 9 \\ \frac{1}{D(s)} = \frac{X(s)}{U(s)} = \frac{1}{s^3+8s^2+6s+2} \end{cases}$

Therefore, we obtain: $\begin{cases} Y(s) = (5s^2 + 7s + 9).X(s) & (1) \\ U(s) = (s^3 + 8s^2 + 6s + 2).X(s) & (2) \end{cases}$

$$(2) \Rightarrow U(s) = (s^3 + 8s^2 + 6s + 2).X(s) \Rightarrow \dddot{x}(t) + 8\ddot{x}(t) + 6\dot{x}(t) + 2x(t) = u(t)$$

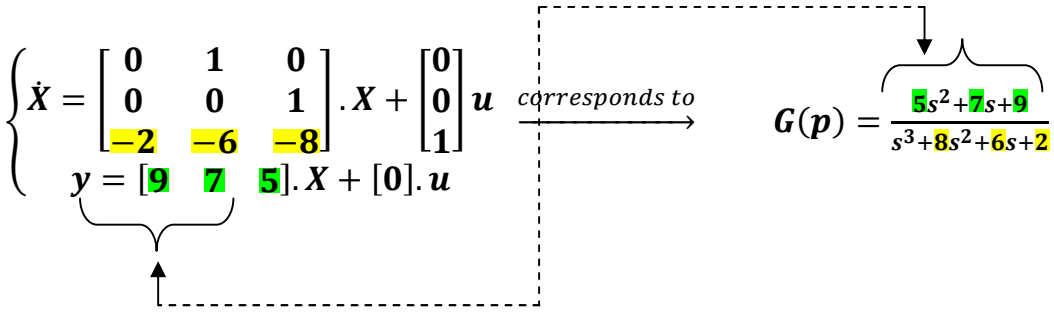
We put: $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -2x_1 - 6x_2 - 8x_3 + u(t) \end{cases}$

That is to say: $\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -6 & -8 \end{bmatrix} . X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$

$$(1) \Rightarrow Y(s) = (5s^2 + 7s + 9).X(s) \Rightarrow y(t) = 5\ddot{x}(t) + 7\dot{x}(t) + 9x(t)$$

Therefore: $y(t) = 5x_3 + 7x_2 + 9x_1 \Rightarrow y = [9 \quad 7 \quad 5].X + [0].u$

The desired state representation is:



Generalization:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0} \quad avec: \ m < n$$

$$\begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} U \\ y = [b_0 \quad b_1 \quad b_2 \quad \cdots \quad b_m \quad \vdots \quad 0^* \quad \ldots \quad 0^*]X + [D]U \end{cases}$$

0^*: Extra zeros in the case ($m \leq n-2$).

To obtain the controllable canonical form, you need:

- Write ***D(s)*** so that the coefficient of the highest degree (of s^n) is equal to **1**.
- Fill in the coefficients of the denominator of ***G(s)*** preceded by a **(-)** sign in the last line of ***A***.
- Insert ***(1)*** into the upper diagonal of the matrix ***A***.
- Fill in the coefficients of the numerator of ***G(s)*** in the last row of matrix ***C***.
- The matrix ***D*** is zero (***[D] = 0***).

Example 2 (m = n):

Consider the system whose transfer function "proper" ($m = n$) is:

$$G(s) = \frac{5s^2 + 7s + 9}{s^2 + 2s + 15} = \frac{N(s)}{D(s)}$$

We can write ***G(s)*** in the form:

$$G(s) = \frac{5s^2 + 7s + 9}{s^2 + 2s + 15} = 5 + \frac{-3s - 66}{s^2 + 2s + 15} = 5 + \widehat{G}(s)$$

$$\widehat{G}(s) = \frac{-3s - 66}{s^2 + 2s + 15} \ (is \ \textbf{strictly proper})$$

Therefore:

$$y_1 = [5]U$$

$$and \begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 \\ -15 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \\ y_2 = [-66 \quad -3] + [0]U \end{cases} \quad with: \quad y = y_1 + y_2$$

We obtain:

$$\begin{cases} \dot{X} = \begin{bmatrix} 0 & 1 \\ -15 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix} U \\ y_2 = [-66 \quad -3] + [5]U \end{cases}$$

7.2.2 Observable canonical form:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + s^{n-1} + \cdots + a_1 s + a_0}$$

Dividing this transfer function by s^n, we find:

$$G(s) = \frac{N(s)}{D(s)} = \frac{b_m s^{m-n} + s^{m-n-1} + \cdots + b_1 s^{-n} + b_0 s^{-n}}{1 + a_{n-1} s^{-1} + \cdots + a_1 s^{1-n} + a_0 s^{-n}}$$

The following state representation is found:

$$\begin{cases} \dot{X} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \ldots & 1 & -a_{n-1} \end{bmatrix} X + \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \\ b_m \\ 0^* \\ \vdots \\ 0^* \end{bmatrix} U \\ y = [0 \quad 0 \quad 0 \quad \cdots \quad 1] X + [0] U \end{cases}$$

7.2.3 Generalization of canonical forms:

$$G(s) = \frac{N(s)}{D(s)}$$

With: ***N(s)*** of degree ***m***
D(s) of degree ***n*** } $n > m$

- ***Obtaining modal forms (or diagonals):***

Decompose the T.F into simple elements:

$$G(s) = \frac{N(s)}{D(s)} = \frac{\alpha}{s - \lambda_1} + \cdots + \frac{\alpha_n}{s - \lambda_n}$$

The associated modal forms are such that the diagonal elements or eigenvalues of A are the poles of the T.F and: $c_i b_i = \alpha_i \quad \forall i = 1, 2, \ldots, n.$

Let:

$$A = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \ldots & \vdots & \ddots & 0 \\ 0 & \ldots & 0 & \lambda_n \end{bmatrix}; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, C = [c_1 \quad c_2 \quad \ldots \quad c_n]$$

Case of a multiple mode:

$$G(s) = \frac{N(s)}{(s-\lambda)^n}$$

1) Decomposition into simple elements:

$$G(s) = \frac{\alpha_1}{s-\lambda} + \frac{\alpha_2}{(s-\lambda)^2} + \cdots + \frac{\alpha_n}{(s-\lambda)^n}$$

Hence the state model:

$$A = \begin{bmatrix} \lambda & 1 & \ldots & 0 \\ 0 & \lambda & \cdots & \vdots \\ \ldots & \vdots & \ddots & 1 \\ 0 & \ldots & 0 & \lambda \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, C = [\alpha_n \quad \alpha_{n-1} \quad \ldots \quad \alpha_1]$$

2) ***General Method:***

We put: $N(s) = N_1(s).N_2(s)$

We obtain:

$$A = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix}, B = \begin{bmatrix} \vdots \\ \ddot{N}_2(\lambda)/2! \\ \dot{N}_2(\lambda)/1! \\ N_2(\lambda) \end{bmatrix}, C = [N_1(\lambda) \quad \dot{N}_1(\lambda)/1! \quad \ddot{N}_1(\lambda)/2! \quad \ldots]$$

- ***General case: obtaining companion forms:***

$$G(s) = \frac{N(s)}{D(s)} = \frac{N_1(s)N_2(s)}{D(s)}$$

Horizontal companion form:

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & & & 1 \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = A^{-1} \cdot \begin{bmatrix} \vdots \\ \ddot{N}_2(0)/2! \\ \dot{N}_2(0)/1! \\ N_2(0) \end{bmatrix}$$

$$C = [N_1(0) \quad \dot{N}_1(0)/1! \quad \ddot{N}_1(0)/2! \quad \ldots]$$

with $\mathcal{A}$ is the ***Toeplitz*** matrix:

$$\mathcal{A} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ a_{n-1} & \mathbf{1} & \cdots & \vdots & \mathbf{0} \\ \vdots & & & \mathbf{1} & \mathbf{0} \\ a_1 & a_2 & \dots & a_{n-1} & \mathbf{1} \end{bmatrix}$$

Vertical companion form:

$$A = \begin{bmatrix} -a_{n-1} & \mathbf{1} & \cdots & \mathbf{0} \\ -a_{n-2} & \mathbf{0} & \cdots & \vdots \\ \vdots & & & \mathbf{1} \\ -a_0 & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, B = \begin{bmatrix} \vdots \\ \ddot{N}_2(\mathbf{0})/2! \\ \dot{N}_2(\mathbf{0})/1! \\ N_2(\mathbf{0}) \end{bmatrix},$$

$$C = [N_1(\mathbf{0}) \quad \dot{N}_1(\mathbf{0})/\mathbf{1}! \quad \ddot{N}_1(\mathbf{0})/\mathbf{2}! \quad \dots] \cdot \mathcal{A}^{-1}$$

Particular case:

If we choose: $N_1(s) = N(s)$ *et* $N_2(s) = \mathbf{1}$

We find the classic forms known in monovariable form:

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}; B = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{1} \end{bmatrix}; C = [b_0 \quad \cdots \quad b_m \quad \mathbf{0} \quad \cdots]$$

CHAPTER IV : REPRESENTATION OF MULTIVARIABLE SYSTEMS BY TRANSFER MATRIX

1 INTRODUCTION:

We define the transfer matrix for a multivariable system, a generalization of the single-variable transfer function. This matrix relates the output vector ***Y(s)*** to the input vector ***U(s)*** by:

$$\boldsymbol{Y(s) = H(s).U(s).}$$

The multivariable system described by the state representation:

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \boldsymbol{Ax}(\boldsymbol{t}) + \boldsymbol{Bu}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{Cx}(\boldsymbol{t}) + \boldsymbol{Du}(\boldsymbol{t}) \end{cases} \quad \boldsymbol{avec}:\ \boldsymbol{A}: (\boldsymbol{n} \times \boldsymbol{n}), \boldsymbol{B}: (\boldsymbol{n} \times \boldsymbol{m}), \boldsymbol{C}: (\boldsymbol{l} \times \boldsymbol{n}) \boldsymbol{et}\ \boldsymbol{D}: (\boldsymbol{l} \times \boldsymbol{m})$$

has a transfer matrix:

$$\boldsymbol{H(p) = C(pI - A)^{-1}B + D}$$

of dimension $(\boldsymbol{l} \times \boldsymbol{m})$**.**

Matrix Structure:

The matrix is filled by individual scalar transfer functions $\boldsymbol{G_{ij}(s)}$ representing the dynamic response from the $\boldsymbol{j^{th}}$ input to the $\boldsymbol{i^{th}}$ output:

$$\boldsymbol{G(s)} = \begin{bmatrix} \boldsymbol{G_{11}(s)} & \boldsymbol{G_{12}(s)} & \cdots & \boldsymbol{G_{1m}(s)} \\ \boldsymbol{G_{21}(s)} & \boldsymbol{G_{22}(s)} & \cdots & \boldsymbol{G_{2m}(s)} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{G_{l1}(s)} & \boldsymbol{G_{l2}(s)} & \cdots & \boldsymbol{G_{lm}(s)} \end{bmatrix}$$

For example:

$$\boldsymbol{G_{21}(s)} = \left.\frac{\boldsymbol{Y_2(s)}}{\boldsymbol{U_1(s)}}\right|_{\boldsymbol{U_2,\ldots U_m=0}}$$

represents how the first input affects the second output when all other inputs are held at zero.

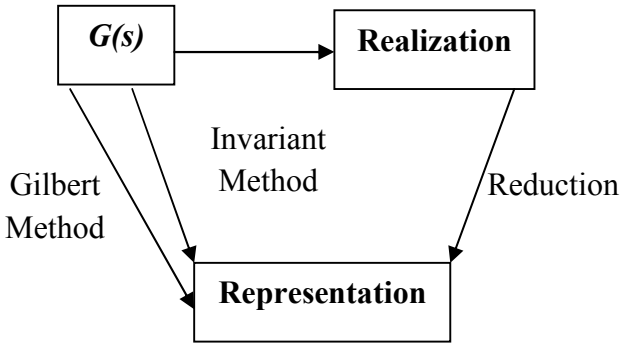
2 TRANSITION FROM A TRANSFER MATRIX TO A STATE REPRESENTATION:

Consider a system described by its transfer matrix ***G(s)***. We call:

Realization: a triplet (A, B, C) such that $\boldsymbol{G(s) = C(sI - A)^{-1}B}$ with ***A*** of non-minimal dimension.

Representation: a triplet (A, B, C) such that $\boldsymbol{G(s) = C(pI - A)^{-1}B}$ with ***A*** of minimal dimension.

We can then distinguish several approaches to obtain a state representation shown in the following figure:



2.1 Method by reducing a realization:

The findings from the single-variable case can be extended to the multi-variable setting, but this should be done cautiously.

Consider the transfer matrix:

$$\boldsymbol{G(s) = \begin{bmatrix} \dfrac{1}{(s-1)^2} & \dfrac{1}{(s-1)(s+3)} \\ \dfrac{-2}{(s-1)(s+3)^2} & \dfrac{s-2}{(s+3)^2} \end{bmatrix}}$$

Inputs number: ***m =2,*** Outputs number: ***l =2***

We use the controllable canonical form:

$$\boldsymbol{G_{11}(s) = \frac{1}{(s-1)^2} = \frac{1}{s^2 - 2s + 1};\ G_{12}(s) = \frac{1}{s^2 + 2s - 3}}$$

$$\boldsymbol{G_{21}(s) = \frac{-2}{(s-1)(s^2 + 6s + 9)^2} = \frac{-2}{s^3 + 5s + 3s - 9};\ G_{22}(s) = \frac{s-2}{(s+3)^2} = \frac{s-2}{s^2 + 6s + 9}}$$

Therefore:

$$A = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & +3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & -3 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & -9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad C = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 1 & -2 \end{bmatrix}$$

This realization has no reason to be minimal (in the sense of the dimension of the state space).

Consider: $\boldsymbol{D(s) = (s-1)^2.(s+3)^2}$**, i**t is the smallest common multiple or least common multiple (***LCM***).

$$G(s) = \frac{1}{(s-1)^2(s+3)^2}\begin{bmatrix} (s+3)^2 & (s-1)(s+3) \\ -2(s-1) & (s-2)(s-1)^2 \end{bmatrix}$$

$$= \frac{1}{(s-1)^2(s+3)^2}\left\{s^3\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + s^2\begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix} + s\begin{bmatrix} 6 & 2 \\ -2 & 5 \end{bmatrix} + \begin{bmatrix} 9 & -3 \\ 2 & -2 \end{bmatrix}\right\}$$

with: $\boldsymbol{D(s) = s^4 + 4s^3 - 2s - 12s + 1}$

We can confirm that a representation analogous to the previous companion form, can be:

$$A = \begin{bmatrix} -4 \times I_2 & 2 \times I_2 & 12 \times I_2 & -I_2 \\ I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 \\ 0 & 0 & I_2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \textit{and} \quad C = \left[\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} 9 & -3 \\ 2 & -2 \end{bmatrix}\right]$$

2×2

dim(A) is not the same as the previous one: ***dim(A) = 8 = (r×m)×(r×m),*** with: ***r = 4*** et ***m = 2***.

It's a ***controller*** form. The controller form suggests that there is an ***observer*** form:

$$A = \begin{bmatrix} -4 \times I_2 & I_2 & 0 & 0 \\ 2 \times I_2 & 0 & I_2 & 0 \\ 12 \times I_2 & 0 & 0 & I_2 \\ -I_2 & 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 1 \\ 0 & -4 \end{bmatrix} \\ \begin{bmatrix} 6 & 2 \\ -2 & 5 \end{bmatrix} \\ \begin{bmatrix} 9 & -3 \\ 2 & -2 \end{bmatrix} \end{bmatrix} \qquad et\ C = [I_2 \quad 0 \quad 0 \quad 0]$$

Based on the results obtained in the single-variable case, we can assume that the dimension of the state space is linked to the observability and controllability of the system (a system is minimal precisely when it is both completely controllable and completely observable.).

2.2 Gilbert method:

We assume that the roots of the denominator of ***G(s)*** are real and distinct. We can write:

$$G(s) = \frac{M(s)}{\psi(s)}$$

M(s): polynomial matrix ***l×m*** et ***ψ(s):*** common denominator.

Then we have:

$$G(p) = \frac{M(s)}{\psi(s)} = \sum_{i=1}^{n} \frac{M_i}{s - \lambda_i}$$

The poles or modes are λ_i and their multiplicity order is equal to the ***rank(M_i).*** We have:

$$system\ order = n = \sum_{i=1}^{n} rank(M_i)$$

Example:

Consider the multivariable system described by the transfer matrix:

$$G(s) = \frac{\begin{bmatrix} 2(2s^2+4s+1) & s(s+2) \\ 2s(3s+5) & (s+2)(3s+1) \end{bmatrix}}{s(s+1)(s+2)}$$

$$= \frac{M_1}{s} + \frac{M_2}{s+1} + \frac{M_3}{s+2}$$

$$\frac{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}{s} + \frac{\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}}{s+2}$$

The system order is:

$$n = rang(M_1) + rang(M_2) + rang(M_3)$$

$$= 2 + 1 + 1 = 4$$

We can write:

$$\begin{cases} Y_1(s) = \overbrace{\frac{[1 \quad 0]U(s)}{s}}^{X_1} + \overbrace{\frac{[2 \quad 1]U(s)}{s+1}}^{X_2} + \overbrace{\frac{[1 \quad 0]U(s)}{s+2}}^{X_3} \\ Y_2(s) = \underbrace{\frac{[0 \quad 1]U(s)}{s}}_{X_4} + \underbrace{\frac{[4 \quad 2]U(s)}{s+1}}_{X_5} + \underbrace{\frac{[2 \quad 0]U(p)}{s+2}}_{X_6} \end{cases}$$

Therefore:

$$\begin{cases} X_1(s) = \frac{[1 \quad 0]U(s)}{s} \Rightarrow \dot{x}_1(t) = u_1(t) \\ X_2(s) = \frac{[2 \quad 1]U(s)}{s+1} \Rightarrow \dot{x}_2(t) = -x_2 + 2u_1 + u_2 \\ X_3(s) = \frac{[1 \quad 0]U(s)}{s+2} \Rightarrow \dot{x}_3(t) = -2x_3 + u_1 \\ X_4(s) = \frac{[0 \quad 1]U(p)}{s} \Rightarrow \dot{x}_4(t) = u_2(t) \\ X_5(s) = \frac{[4 \quad 2]U(s)}{s+1} \Rightarrow \dot{x}_5(t) = -x_5 + 4u_1 + 2u_2 \\ X_6(s) = \frac{[2 \quad 0]U(s)}{s+2} \Rightarrow \dot{x}_6(t) = -2x_6 + 2u_1 \end{cases}$$

Therefore:

$$A=\begin{bmatrix}0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&-2&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-2\end{bmatrix};B=\begin{bmatrix}1&0\\2&1\\1&0\\0&1\\4&2\\2&0\end{bmatrix};C=\begin{bmatrix}1&1&1&0&0&0\\0&0&0&1&1&1\end{bmatrix}$$

We can also write:

$$\begin{cases}Y_1(s)=\overbrace{\frac{[1\quad 0]U(s)}{s}}^{X_1}+\overbrace{\frac{[2\quad 1]U(s)}{p+1}}^{X_2}+\overbrace{\frac{[1\quad 0]U(s)}{p+2}}^{X_3}\\ Y_2(s)=\underbrace{\frac{[0\quad 1]U(s)}{s}}_{X_4}+\underbrace{\frac{2[2\quad 1]U(s)}{s+1}}_{2X_2}+\underbrace{\frac{2[1\quad 0]U(s)}{s+2}}_{2X_3}\end{cases}$$

Therefore:

$$A=\begin{bmatrix}0&0&0&0\\0&-1&0&0\\0&0&-2&0\\0&0&0&0\end{bmatrix};B=\begin{bmatrix}1&0\\2&1\\1&0\\0&1\end{bmatrix};C=\begin{bmatrix}1&1&1&0\\0&2&2&1\end{bmatrix}$$

Another possibility is:

$$\begin{cases}Y_1(s)=\overbrace{\frac{[1\quad 0]U(s)}{p}}^{X_1}+\overbrace{\frac{[2\quad 1]U(s)}{p+1}}^{\frac{1}{2}X_3}+\overbrace{\frac{[1\quad 0]U(s)}{p+2}}^{\frac{1}{2}X_4}\\ Y_2(s)=\underbrace{\frac{[0\quad 1]U(s)}{s}}_{X_2}+\underbrace{\frac{2[2\quad 1]U(s)}{s+1}}_{X_3}+\underbrace{\frac{2[1\quad 0]U(s)}{s+2}}_{X_4}\end{cases}$$

Therefore:

$$A=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&-1&0\\0&0&0&-2\end{bmatrix};B=\begin{bmatrix}1&0\\0&1\\4&2\\2&0\end{bmatrix};C=\begin{bmatrix}1&0&1/2&1/2\\0&1&1&1\end{bmatrix}$$

2.3 *Invariant Method:*

Smith-McMillan Method:

Consider the system described by:

$$\boldsymbol{G(s) = \frac{M(s)}{\psi(s)}}$$

M(s): polynomial matrix ***l×m*** and ***ψ(s):*** common denominator.

Let us consider specifically the polynomial matrix ***M(s)***. We define the notion of equivalent matrices. ***S(s)*** and ***M(s)*** are equivalent if there exist two unimodular matrices (whose determinants are nonzero constants independent of (***s***)) such that:

$$\underbrace{\boldsymbol{M(s)}}_{\boldsymbol{l\times m}} = \underbrace{\boldsymbol{V(s)}}_{\boldsymbol{l\times l}}.\underbrace{\boldsymbol{S(s)}}_{\boldsymbol{l\times m}}.\underbrace{\boldsymbol{W(s)}}_{\boldsymbol{m\times m}}$$

An important special case is where ***S(s)*** is pseudo-diagonal:

$$\boldsymbol{S(s)} = \begin{bmatrix} \boldsymbol{\gamma_1(s)} & & & & \\ & \ddots & & & \\ & & \boldsymbol{\gamma_r(s)} & & \\ & & & \mathbf{0} & \\ & & & & \ddots \\ & & & & \mathbf{0} \end{bmatrix}$$

This is the form of ***Smith*** and ***rank(M(s)) = r***. The $\gamma_i(p)$ are the invariants of ***S(s).***

There are two methods to find ***S(s)*** :

- ***Pseudo-diagonalization*** algorithm that allows simultaneous finding ***V(p), S(p)*** and ***W(p).***
- Direct method for finding ***S(s),*** but not ***V(s)*** and ***Wsp).***

The direct method is as follows:

1. Let $\Delta_0 = 1$ and define $\Delta_i(s)$ as the greatest common divisors (**GCD**) of of ***S(s)*** minor's of order ***i*** .

2. We have :

$$\boldsymbol{\gamma_1(p) = \frac{\Delta_1(p)}{\Delta_0(p)}, \gamma_2(p) = \frac{\Delta_2(p)}{\Delta_1(p)}, \ldots, \gamma_i(p) = \frac{\Delta_i(p)}{\Delta_{i-1}(p)}, \ldots}$$

Note that $\gamma_i(s)$ divides $\gamma_{i+1}(s)$.

Example:

Consider the polynomial matrix:

$$M(s) = \begin{bmatrix} 1 & -1 \\ s^2 + s - 4 & 2s^2 - s - 8 \\ s^2 - 4 & 2(s^2 - 4) \end{bmatrix}$$

The structure of ***S(s)*** is a priori as follows:

$$S(s) = \begin{bmatrix} \gamma_1(s) & 0 \\ 0 & \gamma_2(s) \\ 0 & 0 \end{bmatrix}$$

Let's apply the algorithm:

1. $\Delta_0 = 1$

2. The **first-order** miners are the elements of the matrix. Their **GCD** is equal to $\Delta_1 = 1$. Second-order miners (order ***2***) are :

$$3s^2\text{-}12$$

$$3s^2\text{-}12$$

$$3s^3\text{-}12s$$

The **GCD** is $\Delta_2(s) = 3s^2 - 12$. So we have :

$$\gamma_1(s) = \frac{\Delta_1}{\Delta_0} = 1, \gamma_2(s) = \frac{\Delta_2(s)}{\Delta_1(s)} = 3(s^2 - 4)$$

$$\text{and } S(s) = \begin{bmatrix} 1 & 0 \\ 0 & 3(s^2 - 4) \\ 0 & 0 \end{bmatrix}$$

Given the above, we can express this as:

$$G(p) = \frac{V(p)S(p)W(p)}{\psi(p)} = V(p)\frac{S(p)}{\psi(p)}W(p)$$

The matrix $\frac{S(s)}{\psi(s)}$ is a diagonal matrix of rational fractions. The elements on the diagonal can be simplified between the numerator and denominator.

By noting:

$$V(s) = [V_1(s) \quad \dots \quad V_l(s)], W(s) = \begin{bmatrix} W_1(s) \\ \vdots \\ W_m(s) \end{bmatrix}$$

$$\frac{S(s)}{\psi(s)} = S(s) = \begin{bmatrix} \frac{E_1(s)}{\psi_1(s)} & & & & \\ & \ddots & & & \\ & & \frac{E_r(s)}{\psi_r(s)} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

We arrive at:

$$G(s) = \frac{V(s)S(s)W(s)}{\psi(s)} = V(s)\frac{S(s)}{\psi(s)}W(s)$$

$$= \sum_{i=1}^{r} V_i(s)\frac{E_i(s)}{\psi_i(s)}W_i(s)$$

We define: The poles of $G(s)$ are the roots of $\psi_i(s)$

The zeros of $G(s)$ are the roots of $E_i(s)$

The system order $n = \sum_{i=1}^{r} deg\ (\psi_i(s))$

Example:

$$G(s) = \frac{\begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2(s^2-4) \end{bmatrix}}{s(s+1)(s+2)}$$

We have: $S(s) = \begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}$

Therfore:

$$\frac{E_1(s)}{\psi_1(s)} = \frac{\gamma_1(s)}{\psi(s)} = \frac{1}{s(s+1)(s+2)}$$

$$\frac{E_2(s)}{\psi_2(s)} = \frac{\gamma_2(s)}{\psi(s)} = \frac{3(s^2-4)}{s(s+1)(s+2)} = \frac{3(s-2)}{s(s+1)}$$

and $$G(s) = V(s).\begin{bmatrix} \frac{1}{s(s+1)(s+2)} & 0 \\ 0 & \frac{3(s-2)}{s(s+1)} \\ 0 & 0 \end{bmatrix}.W(p)$$

(Smith-McMillan form**)**

The order: $n = 3+2 = 5$

The poles: 0 is a double pole

-1 is a double pole

-2 is a simple pole

The zeros: ***2*** is a simple zero

Example:

Consider the system described by the transfer matrix:

$$\boldsymbol{G(s)=\frac{\begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2(s^2-4) \end{bmatrix}}{s(s+1)(s+2)}}$$

Provide the state representation of this system.

Solution:

$$\boldsymbol{M(s)=V(s).S(s).W(s)}$$

The matrix ***S(s)*** is already calculated in the previous example.

$$\boldsymbol{S(s)=\begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}} \text{ and we take } \boldsymbol{W(s)=\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$$

Therefore: $\boldsymbol{M(s)=V(s).\begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$,

with: $\boldsymbol{V(s)=\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}}$

Therefore: $\boldsymbol{\begin{bmatrix} 1 & -1 \\ s^2+s-4 & 2s^2-s-8 \\ s^2-4 & 2(s^2-4) \end{bmatrix}=\begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix}.\begin{bmatrix} 1 & 0 \\ 0 & 3(s^2-4) \\ 0 & 0 \end{bmatrix}.\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}$

$$\Rightarrow \begin{cases} v_{11}=1, \quad -1=-v_{11}+3v_{12}(p^2-4), & v_{13}=0 \\ p^2+p-4=v_{21}, \ 2p^2-p-8=-v_{21}+3v_{22}(p^2-4), & v_{23}=0 \\ v_{31}=p^2-4, \quad 2p^2-8=-v_{31}+3v_{32}(p^2-4), & v_{33}=1 \end{cases}$$

We find:

$$\boldsymbol{V(s)=\begin{bmatrix} 1 & 0 & 0 \\ s^2+s-4 & 1 & 0 \\ s^2-4 & 1 & -1 \end{bmatrix}}$$

The state representation:

$$G(s) = \sum_{i=1}^{2} G_i(s) = \sum_{i=1}^{2} V_i(s)\frac{E_i(s)}{\psi_i(s)}W_i(s)$$

$$\Rightarrow \begin{cases} G_1(s) = \dfrac{V_1(s)E_1(s)W_1(s)}{\psi_1(p)} \\ G_2(s) = \dfrac{V_2(s)E_2(s)W_2(s)}{\psi_2(s)} \end{cases}$$

Therefore:

$$G_1(s) = \frac{\overbrace{\begin{bmatrix} 1 \\ s^2+p-4 \\ s^2-4 \end{bmatrix}}^{N_1}.1.\overbrace{[1 \quad -1]}^{N_2}}{s(s+1)(s+2)}$$

$$\rightarrow \begin{cases} \dot{x}_1 = A_1x_1 + B_1U \\ y_1 = C_1x_1 \end{cases}$$

With: $s(s+1)(s+2) = s^3 + 3s^2 + 2s \Rightarrow a_2 = 3;\ a_1 = 2;\ a_0 = 0$

$$\mathcal{A}_1 = \begin{bmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_1 & a_2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \Rightarrow \mathcal{A}_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 7 & -3 & 1 \end{bmatrix}$$

Therefore:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix};\ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \end{bmatrix};\ C_1 = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 1 \\ -4 & 0 & 1 \end{bmatrix}$$

$$G_2(p) = \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.3(s-2).[0 \quad 1]}{s(s+1)} = \frac{\overbrace{\begin{bmatrix} 0 \\ 3(s-2) \\ 3(s-2) \end{bmatrix}}^{N_1}.\overbrace{[0 \quad 1]}^{N_2}}{s(s+1)}$$

$$\rightarrow \begin{cases} \dot{x}_1 = A_1x_1 + B_1U \\ y_1 = C_1x_1 \end{cases}$$

$$\mathcal{A}_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \mathcal{A}_2^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\boldsymbol{A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix};\ B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix};\ C_2 = \begin{bmatrix} 0 & 0 \\ -6 & 3 \\ -6 & 3 \end{bmatrix}}$$

The state representation is:

$$\boldsymbol{A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -2 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix};\ B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix};\ C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -4 & 1 & 1 & -6 & 3 \\ -4 & 0 & 1 & -6 & 3 \end{bmatrix}}$$

$$\boldsymbol{D = zeros(3,2) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}$$

CHAPTER V : STATE FEEDBACK CONTROL OF MULTIVARIABLE LINEAR SYSTEMS

1 INTRODUCTION:

State feedback is currently the fundamental tool of automatic control; it provides a systematic procedure that ensures at least the stabilization of any linear time-invariant system. In practical applications, implementing state feedback requires that all state variables be available (measurable).

State feedback control (or pole placement control) is a method used for correcting linear systems. Its principle is to determine a state feedback control law (a controller) that allows the system's poles to be placed in the closed-loop configuration at values specified by the operator. This control method ensures the stability of the closed-loop system and achieves the desired dynamic behavior.

The goal of this state-feedback control design is to determine the line vector ***K*** (gain matrix) in such a way as to satisfy specifications based on either closed-loop eigenvalue placement (dynamic performance) or structural placement (closed-loop transfer function or state equation). These feedback matrix gains can also be obtained using optimal control principles. This state-feedback control requires that the system ***(A, B)*** be controllable.

2 GENERAL PRINCIPLE OF STATE FEEDBACK CONTROL:

State feedback control for systems described by state-space models plays the same role as closed-loop control does for systems represented by transfer functions. The idea is still to control the system with a reference signal and automatically generate the control signal by continuously comparing the reference setpoint value with the system's actual behavior. The difference between the reference and the actual behavior forms the basis of the system's control signal. In state feedback control, we don't measure the output signal and feed it back to the input; instead, we use the complete state vector to understand the system's behavior.

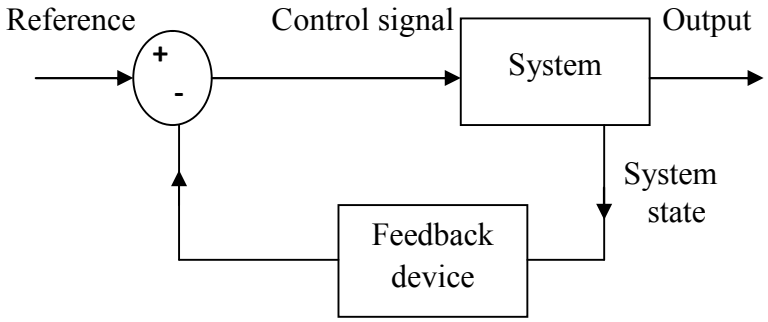


Figure 5.1: State feedback principle

3 *CASE OF MONOVARIABLE SYSTEMS:*

Consider the system described by a state representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

The system is assumed to be controllable.

The principle of the method consists of using a state feedback control law of the form:

$$u(t) = Ny_c(t) - Kx(t)$$

with: $y_c(t)$: the reference (the desired output)

N: is a pre-filter scalar

$K = [k_1\ k_2 \ldots k_n]$ is a line vector of $"n"$ components called the state feedback gain vector.

$x \in \mathcal{R}^n$: state vector (column vector)

K: modifies the system dynamics

N: modifies the system static gain

The **T.F** in **O.L** is : $G_{OL}(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$

The characteristic equation in **O.L** is:

$$det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0$$

In C**.L**, we have: $\begin{cases} \dot{x}(t) = Ax(t) + B(Ny_c - Kx(t)) \\ y(t) = Cx(t) \end{cases}$

Therefore: $\dot{x}(t) = (A - BK)x(t) + BNy_c$

The **T.F** in **C.L** is:

$$G_{BF}(s) = \frac{Y(s)}{Y_c(s)} = C(sI - (A - BK))^{-1}BN$$

The characteristic equation in **C.L** is:

$$det(sI - (A - BK)) = s^n + \beta_{n-1}(K)s^{n-1} + \cdots + \beta_1(K)s + \beta_0(K) = 0$$

The coefficients are in function of K

Dynamics in **O.L**: A —Application of state feedback control→ Dynamics in **C.L**: $(A\text{-}BK)$

3.1 *Calculation of the control:*

Since the system is controllable from the single input, there exists a transformation matrix $\boldsymbol{T}$ such that: $\tilde{\boldsymbol{x}} = \boldsymbol{Tx}$ (or $\boldsymbol{x} = \boldsymbol{T}\tilde{\boldsymbol{x}}$) ($\boldsymbol{T}$ invertible) which allows the system to be put into controllability canonical form:

$$\begin{cases} \dot{\tilde{\boldsymbol{x}}} = \boldsymbol{TAT^{-1}}\tilde{\boldsymbol{x}} + \boldsymbol{TBu}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = \boldsymbol{CT}\tilde{\boldsymbol{x}}(\boldsymbol{t}) \end{cases}$$

with:

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \boldsymbol{-a_0} & \boldsymbol{-a_1} & \boldsymbol{-a_2} & \cdots & \boldsymbol{-a_{n-1}} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

The characteristic polynomial in **O.L** is:

$$\boldsymbol{D_{BO}(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

By applying the state feedback control:

$\boldsymbol{u(t) = Ny_c(t) - Kx(t)}$ with: $\boldsymbol{K = [k_0\ k_1\ \cdots k_{n-1}]}$

In controllability companion canonical base, the command becomes:

$$\tilde{\boldsymbol{x}} = \boldsymbol{Tx} \Rightarrow \boldsymbol{x = T^{-1}}.\tilde{\boldsymbol{x}} \Rightarrow \boldsymbol{u(t) = Ny_c(t) - KT^{-1}\tilde{x}(t)}$$

With: $\tilde{\boldsymbol{K}} = \boldsymbol{KT^{-1}} = \left[\tilde{\boldsymbol{k}}_{\boldsymbol{0}}\ \tilde{\boldsymbol{k}}_{\boldsymbol{1}}\ \cdots \tilde{\boldsymbol{k}}_{\boldsymbol{n-1}}\right]$

Consider the system in controllability companion canonical base:

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \boldsymbol{-a_0} & \boldsymbol{-a_1} & \boldsymbol{-a_2} & \cdots & \boldsymbol{-a_{n-1}} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix} \boldsymbol{u}(\boldsymbol{t})$$

In **C.L**:

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \boldsymbol{-a_0} & \boldsymbol{-a_1} & \boldsymbol{-a_2} & \cdots & \boldsymbol{-a_{n-1}} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix} (\boldsymbol{Ny_c - \tilde{K}\tilde{x}})$$

$$\dot{\tilde{\boldsymbol{x}}} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{1} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{1} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{1} \\ \boldsymbol{-(a_0 + \tilde{k}_0)} & \boldsymbol{-(a_1 + \tilde{k}_1)} & \boldsymbol{-(a_2 + \tilde{k}_2)} & \cdots & \boldsymbol{-(a_{n-1} + \tilde{k}_{n-1})} \end{bmatrix} \tilde{\boldsymbol{x}} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \boldsymbol{1} \end{bmatrix} . \boldsymbol{Ny_c}$$

The system in **C.L** is under controllability companion form, therefore, the characteristic polynomial in **C.L** is:

$$\boldsymbol{D_{CL}(s) = s^n + (a_{n-1} + \widetilde{k}_{n-1})s^{n-1} + \cdots + (a_1 + \widetilde{k}_1)s + (a_0 + \widetilde{k}_0)}$$

By imposing an "n" pole in **C.L**: $p_1, \ldots, p_n$.

The desired characteristic polynomial in **C.L** is written:

$$\boldsymbol{D^d_{CL}(s) = (s - s_1)(s - s_2) \ldots (s - s_n}$$

$$\boldsymbol{D^d_{CL}(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

By identification:

$$\begin{cases} \alpha_0 = a_0 + \widetilde{k}_0 \\ \alpha_1 = a_1 + \widetilde{k}_1 \\ \vdots \\ \alpha_{n-1} = a_{n-1} + \widetilde{k}_{n-1} \end{cases} \Rightarrow \begin{cases} \widetilde{k}_0 = \alpha_0 - a_0 \\ \widetilde{k}_1 = \alpha_1 - a_1 \\ \vdots \\ \widetilde{k}_{n-1} = \alpha_{n-1} - a_{n-1} \end{cases} \Rightarrow \widetilde{k}_i = \alpha_i - a_i, i = \overline{0:n-1}$$

where: a_i: the coefficients of the characteristic polynomial in **O.L**.

α_i: the coefficients of the characteristic polynomial in **C.L**.

Finally, we return to the initial basis: $\boldsymbol{K = \widetilde{K}.T}$

3.2 ***Pole placement algorithm - state feedback:***

Consider the system to be controlled:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases}$$

We have "n" desired poles in **C.L**.

The state feedback control to be applied to the system is: $\boldsymbol{u(t) = Ny_c(t) - Kx(t)}$

Step1: Controllability verification (calculation of the Q_c matrix)

Step2: Put the system into controllability companion form:

$$\boldsymbol{\widetilde{x} = Tx \Rightarrow x = T^{-1}\widetilde{x}\ (calcul\ de\ T\ et\ T^{-1})}$$

$$\begin{cases} \dot{\widetilde{x}} = \widetilde{A}\widetilde{x} + \widetilde{B}u \\ y = \widetilde{C}\widetilde{x} \end{cases}$$

$$\boldsymbol{with: \widetilde{A} = TAT^{-1}, \widetilde{B} = TB, \widetilde{C} = CT^{-1}}$$

Step3: Determination of the characteristic polynomial in **O.L**:

$$\boldsymbol{D_{OL}(p) = p^n + a_{n-1}p^{n-1} + \cdots + a_1 p + a_0}$$

Step4: Determination of the characteristic polynomial in **C.L**:

$$\boldsymbol{D^d_{CL}(s) = s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0}$$

Step5: Calculating the state feedback: $\widetilde{K} = [\widetilde{k}_0\ \widetilde{k}_1\ \cdots \widetilde{k}_{n-1}]$ in the canonical controllability base:

$$\widetilde{k}_i = \alpha_i - a_i, i = \overline{0:n-1}$$

Step6: Calculating the state feedback: $K = [k_0\ k_1\ \cdots k_{n-1}]$ in the initial basis: $K = \widetilde{K}.T$

Example: Consider the system described by the state equations:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & -4\\ 2 & 5\end{bmatrix}.\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}-1\\ 1\end{bmatrix}u$$

$$y(t) = [1 \quad 1].\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Design a state-feedback controller so that the closed-loop (C.L) system has a double pole at $s = -1$.

Solution:

1) $Q_c = [B\ AB] = \begin{bmatrix}-1 & -2\\ 1 & 3\end{bmatrix}; det(Q_c) = -1$

2) ${Q_c}^{-1} = \frac{1}{-1}\begin{bmatrix}3 & -1\\ 2 & -1\end{bmatrix}^T = \begin{bmatrix}-3 & -2\\ 1 & 1\end{bmatrix}$

Therfore: $T = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$

$$\widetilde{A} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}-2 & -4\\ 2 & 5\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}$$

$$\widetilde{B} = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}\begin{bmatrix}-1\\ 1\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\widetilde{C} = [1 \quad 1]\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix} = [1 \quad 0]$$

3) $D_{OL}(p) = \begin{vmatrix}s+2 & +4\\ -2 & s-5\end{vmatrix} = (s+2)(s-5) + 8 = s^2 - 3s - 2 \Rightarrow a_0 = -2, a_1 = -3$

4) $D_{CL}^d(s) = (s+1)^2 = s^2 + 2s + 1 \Rightarrow \alpha_0 = 1, \alpha_1 = 2$

5) $\widetilde{K} = [\alpha_0 - a_0\ \alpha_1 - a_1] = [3 \quad 5]$

6) $K = [k_0\ k_1] = \widetilde{K}.T = [3 \quad 5].\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix} = [3 \quad 8]$

3.3 *Determination of the state vector:*

State feedback control assumes a thorough understanding of the system, but above all, access to the state vector. Three cases are possible:

- All internal signals composing the state vector are accessible for measurement; in this case, the state variables are measurable and judiciously placed sensors allow access to the information necessary for state feedback;
- Not all state variables are measurable, but the system is completely observable; it is then possible to reconstruct the state vector at a given time from knowledge of the output signal and

the input signal of the system over a previous time interval; a state observer is used for this purpose;

- The system is not completely observable; it is therefore necessary to estimate the state vector using a state estimator.

3.4 Static performance and state feedback - the pre-filtering:

In the case where the system is not subject to any external disturbance, the objective of the control is to bring the system to a new equilibrium point.

The placement of the poles → to satisfy the dynamic constraints imposed on the system (speed and stability).

A control law must be determined such that the static gain of the closed loop (**C.L**) model is unity. (output = reference) (that's to say **error = 0**). To obtain this static gain, the pre-filtering scalar ***N*** must be carefully chosen.

$$\boldsymbol{G_{CL}(s) = \frac{Y(s)}{Y_c(s)} = C(sI - (A - BK))^{-1}B.N}$$

Static Gain: $\boldsymbol{G_{CL}(0) = \frac{Y(0)}{Y_c(0)} = -C(A - BK)^{-1}B.N}$

$$\boldsymbol{G_{CL}(0) = 1 \Rightarrow \frac{Y(0)}{Y_c(0)} = -C(A - BK)^{-1}B.N = 1}$$

$$\boldsymbol{\Rightarrow = (-C(A - BK)^{-1}B.N)^{-1}}$$

4 CASE OF MULTIVARIABLE SYSTEMS:

Consider the system: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$

State feedback control law: $\boldsymbol{u = Ny_c - Kx}$

with: y_c**:** desired outputs (***l×1***)

N: is pre-filtering matrix (***m×l***)

K : state feedback matrix (***m×n***)

4.1 System fully controllable by (r < m) inputs:

The system is decomposed into "***r***" controllable subsystems each by a single input.alors, we put $\boldsymbol{\tilde{x} = Tx \; ou \; x = T^{-1}\tilde{x}}$**,** we obtain the new system:

$$\begin{cases} \dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u \\ y = \tilde{C}\tilde{x} \end{cases}$$

Where: $\boldsymbol{\tilde{A} = TAT^{-1}, \tilde{B} = TB, \tilde{C} = CT^{-1}}$

The control: $\boldsymbol{u = Ny_c - \tilde{K}\tilde{x} \; with: K = \tilde{K}.T \Rightarrow \tilde{K} = T^{-1}K}$

$$\widetilde{A}=\begin{bmatrix}\begin{bmatrix}0 & 1 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1\\ -a_0^1 & -a_1^1 & \dots & -a_{n_1-1}^1\end{bmatrix} & \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ * & * & \dots & *\end{bmatrix} & \dots & \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ * & * & \dots & *\end{bmatrix}\\ \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ * & * & \dots & *\end{bmatrix} & \begin{bmatrix}0 & 1 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1\\ -a_0^2 & -a_1^2 & \dots & -a_{n_2-1}^2\end{bmatrix} & \dots & \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ * & * & \dots & *\end{bmatrix}\\ \vdots & \vdots & & \vdots\\ \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\end{bmatrix} & \begin{bmatrix}0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 0\\ 0 & 0 & \dots & 0\end{bmatrix} & \dots & \begin{bmatrix}0 & 1 & \dots & 0\\ 0 & 0 & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \dots & 1\\ -a_0^r & -a_1^r & \dots & -a_{n_r-1}^r\end{bmatrix}\end{bmatrix}$$

with: the matrices in the diagonal are noted: $\widetilde{A}_{11}, \widetilde{A}_{22}, \dots, \widetilde{A}_{rr}$, each of these matrices represents the controllable canonical form of the corresponding subsystem.

$$\widetilde{B}=\begin{bmatrix}0 & 0 & 0\\ \vdots & \vdots & \vdots\\ 1 & * & *\\ 0 & 0 & 0\\ \vdots & \vdots & \vdots\\ 0 & 1 & *\\ \vdots & \vdots & \vdots\\ 0 & 0 & 0\\ \vdots & \vdots & \vdots\\ 0 & 0 & 1\end{bmatrix}\begin{matrix}\overset{r}{\longleftrightarrow}\\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{matrix}\begin{bmatrix}\overset{m-r}{\longleftrightarrow}\\ \\ \\ B_1^*\\ \\ \\ B_2^*\\ \\ \vdots\\ B_r^*\\ \\ \end{bmatrix}\begin{matrix}\\ \\ \\ \updownarrow n_1\\ \\ \\ \updownarrow n_2\\ \\ \\ \updownarrow n_r\\ \\ \end{matrix}$$

The characteristic equation of the global system in **O.L** is written: $D_{OL}(s)=|pI-\widetilde{A}|=$

$$\prod_{i=1}^{r}|sI-\widetilde{A}_{ii}|=\prod_{i=1}^{r}(s^{n_i}+a_{n_i-1}^{i}s^{n_i-1}+\cdots+a_0^i)$$

n_i**:** controllability indice

Therefore, each input u_i controls a single-variable subsystem composed of n_i states

$$u=Ny_c-\widetilde{K}\widetilde{x}$$

$\widetilde{K}$ **:** is the matrix ($m\times n$) whose first " r " lines are not zero and the remaining "m-r" lines are zero.

$$\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \\ \vdots \\ u_m \end{bmatrix} = -\begin{bmatrix} \widetilde{k}_0^1\widetilde{k}_1^1 \cdots \widetilde{k}_{n_1-1}^1 & 0 & 0 & \dots & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \widetilde{k}_0^2\widetilde{k}_1^2 \cdots \widetilde{k}_{n_2-1}^2 & \dots & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots & \ddots & & \vdots & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & \widetilde{k}_0^r\widetilde{k}_1^r \cdots \widetilde{k}_{n_r-1}^r & & \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} \widetilde{x}_1 \\ \vdots \\ \widetilde{x}_{n_1} \\ \widetilde{x}_{n_1+1} \\ \vdots \\ \widetilde{x}_{n_1+n_2} \\ \vdots \\ \widetilde{x}_{n_1+n_2+1} \\ \vdots \\ \widetilde{x}_{n_1+n_2+\cdots+n_r} \end{bmatrix} + N y_c$$

By substituting the control law into the state equation:

$$\dot{\widetilde{x}} = \left(\widetilde{A} - \widetilde{B}\widetilde{K}\right)\widetilde{x} + N y_c \quad o\grave{u} \quad \widetilde{A}_{BF} = \left(\widetilde{A} - \widetilde{B}\widetilde{K}\right)$$

$$\widetilde{A}_{BF} = \begin{bmatrix} \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ -(a_0^1 + \widetilde{k}_0^1) & -(a_1^1 + \widetilde{k}_1^1) & \cdots & -(a_{n_1-1}^1 + \widetilde{k}_{n_1-1}^1) \end{bmatrix} & \dots & \dots & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ * & * & \cdots & * \end{bmatrix} \\ \vdots & \ddots & & \vdots \\ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} & \dots & \dots & \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \\ -(a_0^r + \widetilde{k}_0^r) & -(a_1^r + \widetilde{k}_1^r) & \cdots & -(a_{n_1-1}^r + \widetilde{k}_{n_1-1}^r) \end{bmatrix} \end{bmatrix}$$

The characteristic polynomial of the global system in **C.L** is:

$$D_{CL}(s) = \left|sI - \widetilde{A}_{BF}\right| = \prod_{i=1}^{r}\left|sI - \widetilde{A}_{ii_{CL}}\right| = \prod_{i=1}^{r}\left(s^{n_i} + (a_{n_i-1}^i + \widetilde{k}_{n_i-1}^i)s^{n_i-1} + \cdots + (a_0^i + \widetilde{k}_0^i)\right)$$

The desired characteristic polynomial in **C.L** is:

$$D_{CL}^d(s) = \left|sI - \widetilde{A}_{BF}^d\right| = \prod_{i=1}^{r}\left|sI - \widetilde{A}_{ii_{CL}}^{\ d}\right| = \prod_{i=1}^{r}(s^{n_i} + \alpha_{n_i-1}^i s^{n_i-1} + \cdots + \alpha_0^i)$$

By identification:

$$\begin{cases} \alpha_0^i = a_0^i + \widetilde{k}_0^i \\ \alpha_1^i = a_1^i + \widetilde{k}_1^i \\ \vdots \\ \alpha_{n_i-1}^i = a_{n_i-1}^i + \widetilde{k}_{n_i-1}^i \end{cases} \Rightarrow \begin{cases} \widetilde{k}_0^i = \alpha_0^i - a_0^i \\ \widetilde{k}_1^i = \alpha_1^i - a_1^i \\ \vdots \\ \widetilde{k}_{n_i-1}^i = \alpha_{n_i-1}^i - a_{n_i-1}^i \end{cases}$$

with: a_j^i **:** coefficient of the characteristic equation in **O.L** du sous système **"i".**

α_j^i **:** coefficient of the desired characteristic equation in **C.L** of sub-system **"i".**

Finally: $K = \widetilde{K}.T$

Pole placement algorithm (decomposition into r (r < m) controllable sub-systems):

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

$\downarrow$

n poles in **C.L**

(desired poles) $\longrightarrow$ $u(t) = Ny_c(t) - Kx(t)$

$K = ?$

Step1: Decompose the system into ***r (r < m)*** controllable subsystems using the similarity matrix $\boldsymbol{T}$.

Step2: Determining the characteristic polynomial of the global system in **O.L**: $\boldsymbol{D_{OL}(s)}$

Step3: Determining the desired characteristic polynomial in **C.L**: $\boldsymbol{D_{CL}^{d}(s)}$

Step4: Calculating the state feedback $\widetilde{\boldsymbol{K}}$ in the canonical base companion of controllability for: $\boldsymbol{i = 1 \ldots r}$.

with: $\widetilde{k}_j^i = \alpha_j^i - a_j^i \ \ with: j = 0 \ldots (n_i - 1)$

Step5: Calculating the state feedback matrix $\boldsymbol{K}$ in the initial base: $\boldsymbol{K = \widetilde{K}.T}$.

Example: Consider the system:

$$\dot{x} = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} -1 & -1 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} x$$

Calculate the feedback gain $\boldsymbol{K}$ that places the closed-loop (**C.L**) poles at the specified locations:

$$p_1 = 1;\ p_2 = -2;\ p_3 = -3;\ p_4 = -4.$$

Solution:

1) Decompose the system into ***r (r < 3)*** sub-systems controllables:

$$Q_c = [B\ AB\ A^2B\ A^3B] = [b_1\ b_2\ b_3\ Ab_1\ Ab_2\ Ab_3\ A^2b_1\ A^2b_2 \ldots]$$

$$or\ \ Q_c = [b_1\ Ab_1\ A^2b_1\ A^3b_1\ b_2\ \ Ab_2\ A\ A^2b_2\ A^3b_2\ b_3 \ldots]$$

$$Q_c = \begin{array}{c} \begin{matrix} b_1 & Ab_1 & A^2b_1 & A^3b_1 & b_2 & Ab_2 & A^2b_2 & \ldots \end{matrix} \\ \begin{bmatrix} -1 & 0 & -2 & -2 & -1 & \vdots & \vdots & \ldots \\ 0 & -1 & -1 & 0 & 1 & \vdots & \vdots & \ldots \\ 0 & 1 & -2 & -3 & 2 & \vdots & \vdots & \ldots \\ 1 & -1 & -2 & -1 & 0 & \vdots & \vdots & \ldots \end{bmatrix} \end{array}$$

The controllability indices:

b_1	b_2	b_3	
×	×		A^0
×			A^1
×			A^2
0			A^3

$$det\begin{bmatrix}-1 & 0 & -2 & -2\\ 0 & -1 & -1 & 0\\ 0 & 1 & -2 & -3\\ 1 & -1 & -2 & -1\end{bmatrix} = 0 \Rightarrow \textbf{The column } (A^3b_1) \textbf{ rejected}$$

$$\textbf{The column } (A^3b_1) \textbf{ is replaced by } b_2 \Rightarrow det\begin{bmatrix}-1 & 0 & -2 & -2\\ 0 & -1 & -1 & 0\\ 0 & 1 & -2 & -3\\ 1 & -1 & -2 & -1\end{bmatrix} = 15$$

Therefore: $n_1 = 3;\ n_2 = 1;\ n_3 = 0 \Rightarrow \sum_i n_i = 4\ et\ \ r = 2$

The system is fully controllable ***(rang(Q_c) = 4).***

$$Q_c = [b_1\ Ab_1\ A^2b_1\ \ b_2] = \begin{bmatrix}-1 & 0 & -2 & -2\\ 0 & -1 & -1 & 0\\ 0 & 1 & -2 & -3\\ 1 & -1 & -2 & -1\end{bmatrix} \Rightarrow {Q_c}^{-1} = \begin{bmatrix}-2/5 & -8/15 & 1/15 & 3/5\\ 0 & -2/3 & 1/3 & 0\\ -1/5 & 1/15 & -2/15 & -1/5\\ -1/5 & 2/5 & 1/5 & -1/5\end{bmatrix}$$

Calculation of T and T^1:

$$\begin{cases} n_1 = 3 \rightarrow T_1\ (3\times 3) \rightarrow q_3\\ n_2 = 1 \rightarrow T_2\ (1\times 1) \rightarrow q_4 \end{cases} \Rightarrow T = \begin{bmatrix} q_3\\ q_3A\\ q_3A^2\\ q_4 \end{bmatrix}$$

Therefore: $T = \begin{bmatrix}-1/5 & 1/15 & -2/15 & -1/5\\ -1/3 & -1/3 & 1/3 & -1/3\\ -1 & 0 & 1 & 0\\ -1/5 & 2/5 & 1/5 & -1/5\end{bmatrix} \Rightarrow T^{-1} = \begin{bmatrix}-3 & 1 & -1 & 20/15\\ 0 & -1 & 0 & 5/3\\ -3 & 1 & 0 & 20/15\\ 0 & -2 & 1 & -5/3\end{bmatrix}$

4) $\tilde{A} = TAT^{-1} = \left[\begin{array}{ccc:c} 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & -1 & 1 & 5/3\\ \hdashline 0 & 0 & 0 & 0 \end{array}\right]$

$$\tilde{B} = TB = \left[\begin{array}{c:cc} 0 & 0 & -2/15\\ 0 & 2/3 & -2/3\\ 1 & 3 & -2\\ \hdashline 0 & 1 & 1/5 \end{array}\right]$$

2) The characteristic polynomial in **O.L:**

$$\boldsymbol{D_{OL}(s) = \left|sI - \widetilde{A}\right| = \prod_{i=1}^{2}\left|sI - \widetilde{A}_{ii}\right| = \left|sI - \widetilde{A}_{11}\right|.\left|sI - \widetilde{A}_{22}\right|}$$

$$\boldsymbol{= \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 1 & s-1 \end{vmatrix}.s = (s^3 - s^2 + s).s \Rightarrow a_2^1 = -1;\ a_1^1 = 1;\ a_0^1 = 0; a_0^2 = 0}$$

3) The desired characteristic polynomial in **C.L:**

$$\boldsymbol{D_{CL}^d(s) = \left|sI - \widetilde{A}_{CL}^d\right| = \prod_{i=1}^{2}\left|sI - \widetilde{A}_{ii}{}_{CL}^{\ d}\right| = \prod_{i=1}^{2}(s^{n_i} + \alpha_{n_i-1}^i s^{n_i-1} + \cdots + \alpha_0^i)}$$

$$\boldsymbol{= (s+1)(s+2)(s+3)(s+4)}$$

$$\boldsymbol{= \left(s^3 + \alpha_2^1 s^2 + \alpha_1^1 s + \alpha_0^1\right)\left(s + \alpha_0^2\right) = (s^3 + 6s^2 + 11s + 6)(s+4)}$$

Therefore:

$$\boldsymbol{\alpha_2^1 = 6;\ \alpha_1^1 = 11;\ \alpha_0^1 = 6;\ \alpha_0^2 = 4}$$

4) $\boldsymbol{\widetilde{K} = \begin{bmatrix} \widetilde{k}_0^1 & \widetilde{k}_1^1 & \widetilde{k}_2^1 & 0 \\ 0 & 0 & 0 & \widetilde{k}_0^2 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$

$$\boldsymbol{\begin{cases} \widetilde{k}_0^1 = \alpha_0^1 - a_0^1 = 6 \\ \widetilde{k}_1^1 = \alpha_1^1 - a_1^1 = 10 \\ \widetilde{k}_2^1 = \alpha_2^1 - a_2^1 = 7 \\ \widetilde{k}_0^2 = \alpha_0^2 - a_0^2 = 4 \end{cases} \Rightarrow \widetilde{K} = \begin{bmatrix} 6 & 10 & 7 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

5) $\boldsymbol{K = \widetilde{K}.T = \begin{bmatrix} 6 & 10 & 7 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.\begin{bmatrix} -1/5 & 1/15 & -2/15 & -1/5 \\ -1/3 & -1/3 & 1/3 & -1/3 \\ -1 & 0 & 1 & 0 \\ -1/5 & 2/5 & 1/5 & -1/5 \end{bmatrix}}$

$$\boldsymbol{= \begin{bmatrix} -11.53 & -2.93 & 9.53 & -4.53 \\ -0.8 & 1.6 & 0.8 & -0.8 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

4.2 Young and Willems Method:

The ***Young-Willems*** method is a pole placement method for multi-input systems. The proposed procedure is an elegant alternative based on the characteristic polynomial and therefore applicable to all cases (single poles and multiple poles).

Principle:

Let ***p(λ)*** and ***d(λ)*** be the polynomials of the uncorrected and corrected system:

$$p(\lambda) = det(\lambda I - A) = a_0 + a_1\lambda + \cdots + a_{n-1}\lambda^{n-1} + \lambda^n$$

$$d(\lambda) = det(\lambda I - A + BK) = d_0 + d_1\lambda + \cdots + d_{n-1}\lambda^{n-1} + \lambda^n$$

In this method, the gain ***K (m×n)*** is of the form:

$$K = fg^T$$

where: ***f*** et ***g*** are vectors with ***r*** and ***n*** components. If there exists a vector ***f*** such that the pair ***(A, B.f)*** is controllable, that is; such that the following square controllability matrix is invertible:

$$C_f = (Bf, ABf, \ldots, A^{n-1}Bf)$$

C_f: is the controllability matrix of the single-input pseudo-system ***{A, Bf}***

Therefore ***g*** is obtained by: $g = \left(C_f^T\right)^{-1}.X^{-1}.\delta$

With:

$$X = \begin{pmatrix} 1 & & & \\ a_{n-1} & 1 & & \\ \vdots & \ddots & \ddots & \\ a_1 & \ldots & a_{n-1} & 1 \end{pmatrix}; \ \delta = \begin{pmatrix} d_{n-1} - a_{n-1} \\ d_{n-2} - a_{n-2} \\ \vdots \\ d_0 - a_0 \end{pmatrix}$$

<u>Algorithm for pole placement using the Young and Willems method:</u>

To calculate the gain value, we will follow the following algorithm:

<u>Step1:</u> calculate the polynomial $p(\lambda)$

<u>Step2:</u> calculate the polynomial $d(\lambda)$

<u>Step3:</u> deduce the value of $\delta(\lambda)$

<u>Step4:</u> choose the matrix value ***f***

<u>Step5:</u> give the values of C_f et ***X***

<u>Step6:</u> calculate ***g***

<u>Step7:</u> give the gain value $K = fg^T$

<u>Step8:</u> Calculate the control $u(t) = Nr(t) - K.x(t)$

Example:

$$A=\begin{pmatrix}0&1&0\\0&0&1\\4&4&-1\end{pmatrix};B=\begin{pmatrix}1&0\\0&0\\0&1\end{pmatrix},p=\{-1,-2,-3\}\ et\ f=\begin{pmatrix}1\\1\end{pmatrix}$$

$$C_f=(Bf\ ABf\ A^2Bf)\ \ et\ f=\begin{pmatrix}f_1\\f_2\end{pmatrix}\ \ (m=2)$$

$$C_f=\begin{pmatrix}f_1&0&f_2\\0&f_2&4f_1-f_2\\f_2&4f_1-f_2&2f_2-4f_1\end{pmatrix}\Rightarrow det(C_f)=f_1(4f_1f_2+{f_2}^2-16{f_1}^2)-{f_2}^3\neq 0$$

We take: $f=\begin{pmatrix}1\\1\end{pmatrix}$

$$\Rightarrow C_f=\begin{pmatrix}1&0&1\\0&1&3\\1&3&-2\end{pmatrix}\Rightarrow det(C_f)=-12$$

$$p(\lambda)=|\lambda I-A|=\begin{vmatrix}\lambda&-1&0\\0&\lambda&-1\\-4&-4&\lambda+1\end{vmatrix}=\lambda^3+\lambda^2-4\lambda-4\Rightarrow a_0=-4,a_1=-4;a_2=1$$

$$d(\lambda)=(\lambda+1)(\lambda+2)(\lambda+3)=\lambda^3+6\lambda^2+11\lambda+6\Rightarrow d_0=6,d_1=11;d_2=6$$

$$\delta=\begin{pmatrix}d_2-a_2\\d_1-a_1\\d_0-a_0\end{pmatrix}=\begin{pmatrix}5\\15\\10\end{pmatrix}$$

$$X=\begin{pmatrix}1&0&0\\1&1&0\\-4&1&1\end{pmatrix}$$

We have: $g=\left(C_f^T\right)^{-1}.X^{-1}.\delta$

$$\left(C_f^T\right)^{-1}=\frac{1}{12}\begin{pmatrix}11&-3&1\\-3&3&3\\1&3&-1\end{pmatrix}\ et\ X^{-1}=\begin{pmatrix}1&0&0\\-1&1&0\\5&-1&1\end{pmatrix}$$

Therefore:
$$g=\frac{1}{12}\begin{pmatrix}11&-3&1\\-3&3&3\\1&3&-1\end{pmatrix}\begin{pmatrix}1&0&0\\-1&1&0\\5&-1&1\end{pmatrix}\begin{pmatrix}5\\15\\10\end{pmatrix}=\frac{1}{12}\begin{pmatrix}45\\75\\15\end{pmatrix}$$

and $\boldsymbol{K = \frac{1}{12}\begin{pmatrix} 45 & 75 & 15 \\ 45 & 75 & 15 \end{pmatrix}}$.

5 OBSERVER AND OUTPUT FEEDBACK CONTROL:

5.1 Review observer for monovariable systems:

State estimation works by creating a virtual copy of the system dynamics that uses the control input ***U*** and incorporates system outputs (measurements) ***Y*** to correct any discrepancies between the model and the real system.

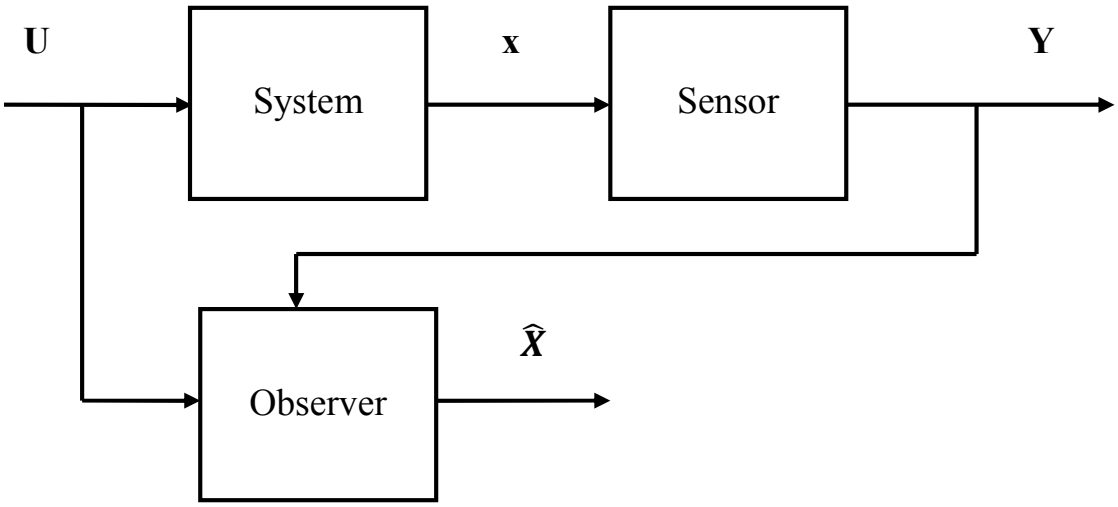


Figure 5.2: System with observer

Consider the following linear system:

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX \end{cases}$$

The implementation of state feedback control ***u(x)*** requires sensors to provide an approximate value of the state ***x(t)*** at each time ***t***. We use software sensors, also called Observers. These are algorithms based on a model system.

These software sensors provide an online estimate of the unmeasured state variables of the system at each time ***t***.

A dynamic observer (**Luenberger observer**) has the following form:

$$\begin{cases} \dot{\widehat{X}} = A\widehat{X} + BU + L(Y - \widehat{Y}) \\ \widehat{Y} = C\widehat{X} \end{cases}$$

The state evolution is corrected using the model based on the difference between the observed output and the output reconstructed by the observer: ***(Y - Ŷ)***

The observer can be rewritten as follows:

$$\dot{\widehat{\boldsymbol{X}}} = (\boldsymbol{A} - \boldsymbol{LC})\widehat{\boldsymbol{X}} + \boldsymbol{BU} + \boldsymbol{LY}$$

The matrix $\boldsymbol{L}$ is called the gain matrix and must be chosen so that the error on the state converges exponentially to $\boldsymbol{0}$, i.e. $\widetilde{\boldsymbol{X}} = \left(\widehat{\boldsymbol{X}} - \boldsymbol{X}\right) \to \boldsymbol{0}$. For this, it suffices to choose $\boldsymbol{L}$ such that the matrix ***(A - LC)*** is a **Hurwitz** matrix, i.e. that its eigenvalues have negative real parts in the continuous case (or a **Schur** matrix, i.e. that its eigenvalues have a modulus less than **1** in the discrete case).

Remark:

- For a single-variable system (or (monovariable system), the gain $\boldsymbol{L}$ is a vector of dimension ***(n×1).***
- For a multivariable system, the gain $\boldsymbol{L}$ is a matrix of dimension ***(n× l).***

$$\widetilde{\boldsymbol{X}} = \left(\widehat{\boldsymbol{X}} - \boldsymbol{X}\right) \to \boldsymbol{0}$$
$$= \boldsymbol{e(t)} \ \ (\boldsymbol{Error})$$

$$\boldsymbol{e(t)} \to \boldsymbol{0} \Rightarrow \boldsymbol{\dot{e}(t)} \to \boldsymbol{0}$$

$$\boldsymbol{\dot{e}(t)} = \dot{\widehat{\boldsymbol{X}}} - \dot{\boldsymbol{X}}$$
$$= (\boldsymbol{A} - \boldsymbol{LC})\widehat{\boldsymbol{X}} - (\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{X}$$
$$= (\boldsymbol{A} - \boldsymbol{LC})\left(\widehat{\boldsymbol{X}} - \boldsymbol{X}\right)$$
$$= (\boldsymbol{A} - \boldsymbol{LC})\boldsymbol{e(t)}$$

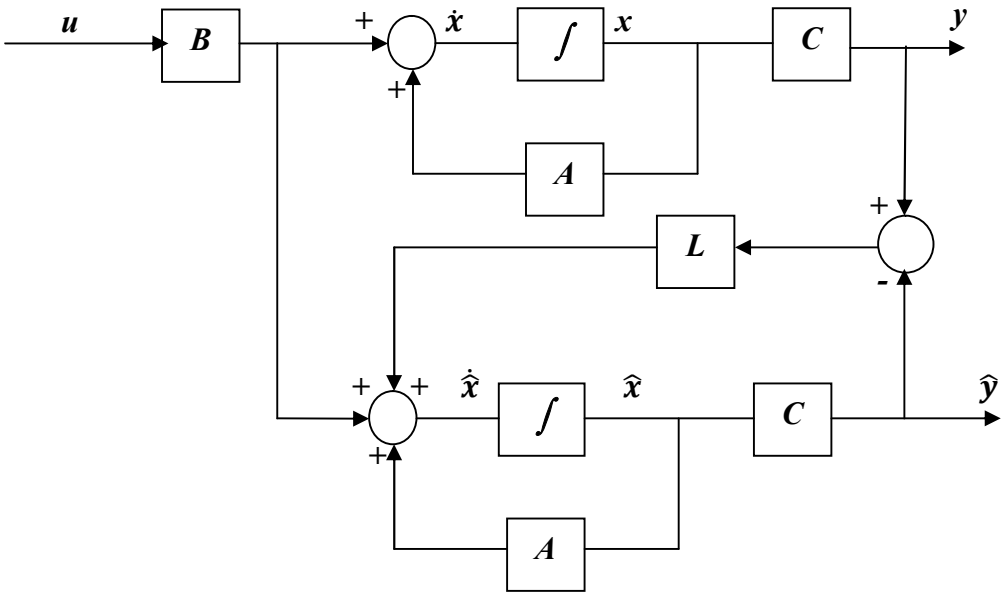


Figure 5.3: Block diagram of the system with observer

To calculate the gain matrix $\boldsymbol{L}$ of the full-order observer, we can use the following algorithm:

ALGORITHM : The LIT system is observable.

We want to calculate $\boldsymbol{L = ?}$ So that the system with observer (i.e., the matrix ***(A-LC)***) has the eigenvalues: $\{\overline{\lambda_1}, \overline{\lambda_2}, \dots, \overline{\lambda_n}\}$

1- Calculate the polynomial $\Delta(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0 = det\,(sI - A)$

2- Calculate the polynomial $\bar{\Delta}(s) = det\big(sI - (A - LC)\big) = \big(s - \overline{\lambda_1}\big)\big(s - \overline{\lambda_2}\big) \dots \big(s - \overline{\lambda_n}\big)$

$$= s^n + \overline{a_{n-1}}s^{n-1} + \cdots + \overline{a_0}$$

3- Calcul of: $\bar{L} = [a_0 - \overline{a_0}, \dots, a_{n-1} - \overline{a_{n-1}}]$

4- Calcul of q_i: $\begin{cases} q_n = C^T \\ q_{n-i} = A^T q_{n-i+1} + a_{n-i} q_n \qquad i = 1, \dots, (n-1) \end{cases}$

5- $Q = [q_1, \dots, q_n]$

6- $P = Q^{-1}$

7- $L = -P^T . \bar{L}^T$

5.2 Reconstructed feedback state control by an observer:

The Luenberger linear observer has a notable property called the separation principle: in the case of linear state-feedback control, the control synthesis and observer synthesis processes can be performed independently. Indeed, if the controlled system is stable, and if the observer thus designed is stable (that is, the matrices $A - BK$ and $A - LC$ are **Hurtwitz**), then the system controlled by the reconstructed state-feedback is stable.

Consider the following linear time-invariant system, observable and controllable, equipped with a Luenberger observer:

$$\begin{cases} \dot{X} = AX + BU \\ Y = CX \end{cases} \quad and \quad \begin{cases} \dot{\hat{X}} = A\hat{X} + BU + L(Y - \hat{Y}) \\ \hat{Y} = C\hat{X} \end{cases}$$

By implementing a feedback loop with state feedback $u = -K\hat{X}$, the dynamics of the closed-loop system can then be written as:

$$\begin{cases} \dot{X} = AX - BK\hat{X} \\ \dot{\hat{X}} = (A - BK)\hat{X} + L(Y - C\hat{X}) \end{cases}$$

We can make the following variable change to write the reconstruction error:

$\tilde{X} = X - \hat{X}$ Hence, by replacing, $\dot{\tilde{X}} = (A - LC)\tilde{X}$

By writing a new augmented system, consisting of the state and the reconstruction error, we obtain:

$$\begin{pmatrix} \dot{X} \\ \dot{\tilde{X}} \end{pmatrix} = \begin{pmatrix} A - BK & BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} X \\ \tilde{X} \end{pmatrix}$$

The matrix is block triangular, so the closed-loop system's spectrum is just the union of the spectra of its diagonal blocks — namely, the spectra of the original controlled subsystem and the original observed subsystem. Therefore, for linear time-invariant systems, designing state feedback together with an observer is straightforward: the controller and the observer can be synthesized independently.

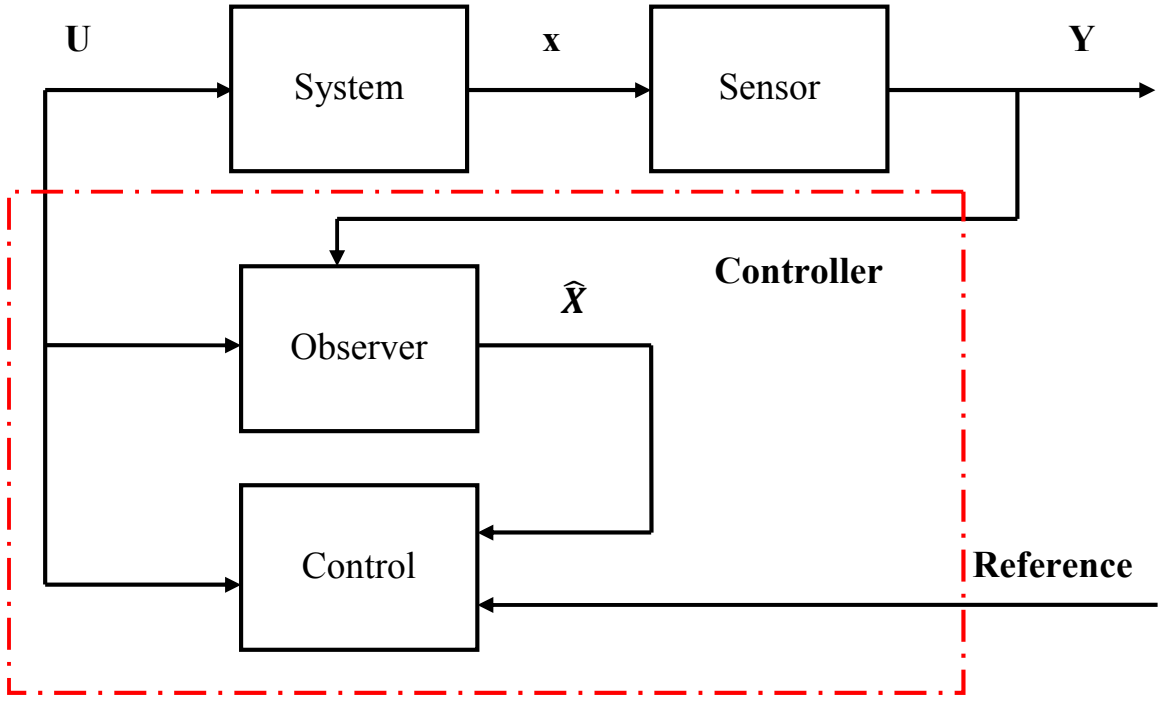
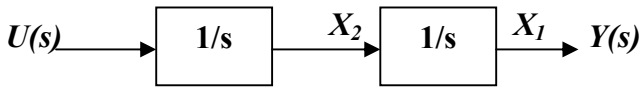


Figure 5.4: Reconstructed feedback state control by an observer

Example:

Consider the system composed of a double integrator represented by the diagram below:



1. Write the system state model. Give the control that would allow obtaining the following poles in **C.L**: $\boldsymbol{s_{1,2} = -1 \pm j.}$
2. Give the gain of a full-order observer (Luenberger observer) to obtain the following poles: $\boldsymbol{s_{1,2} = -2 \pm 2j.}$ Write the state equations for this observer.
3. Draw the blockl diagram of the system with the observer.

Solution:

1. State Model:

$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{cases}$$

Controllability: $\emptyset_{com} = [B\ AB] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det(\emptyset_{com}) = -1 \neq 0$

$$\Rightarrow rank(\emptyset_{com}) = rang(A) = 2 \Rightarrow The\ system\ is\ fully\ controllable$$

The control: $det(\lambda I - (A - BK)) = (\lambda + 1 + j)(\lambda + 1 - j)$

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$$

$$\Rightarrow |\lambda I - (A - BK)| = \begin{vmatrix} \lambda & -1 \\ k_1 & \lambda + k_2 \end{vmatrix} \Rightarrow \lambda(\lambda + k_2) + k_1 = \lambda^2 + k_2\lambda + k_1 = \lambda^2 + 2\lambda + 2$$

$$\Rightarrow K = [2 \quad 2]$$

2. The observer:

Observability: $\emptyset_{obs} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \det(\emptyset_{obs}) = 1 \neq 0$

$$\Rightarrow rank(\emptyset_{obs}) = rank(A) = 2 \Rightarrow sThe\ system\ is\ fully\ observable$$

Calcul of observer gain: $det(\lambda I - (A - LC)) = (\lambda + 2 + 2j)(\lambda + 2 - 2j)$

$$A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} [1 \quad 0] = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}$$

$$\Rightarrow |\lambda I - (A - LC)| = \begin{vmatrix} \lambda + l_1 & -1 \\ l_2 & \lambda \end{vmatrix} \Rightarrow \lambda(\lambda + l_1) + l_2 = \lambda^2 + l_1\lambda + l_2 = \lambda^2 + 4\lambda + 8$$

$$\Rightarrow L = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

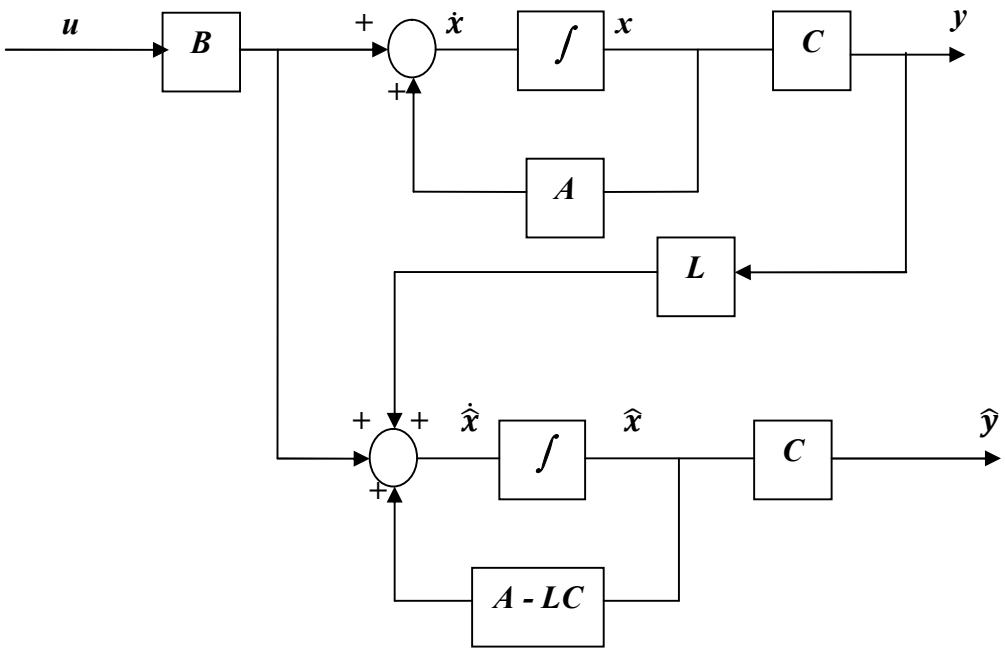
Observer Equation:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y - C\hat{x}) = (A - LC)\hat{x} + Bu + Ly$$

$$\Rightarrow \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} . \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 4 \\ 8 \end{bmatrix} (y - \hat{x}_1)$$

$$\Rightarrow \begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ -8 & 0 \end{bmatrix} . \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 4 \\ 8 \end{bmatrix} y$$

3. Block diagram:



5.3 Reconstructed feedback state control by an observer for multivariable system:

As with the monovariable case, using the **separation theorem**, the matrix K (control gain) and the matrix L (observer gain) are calculated independently (separately).

Calculation of K (already done)

Calculation of L:

In the initial base, we have

$$\text{System} : \begin{cases} \dot{X} = AX + BU \\ Y = CX \end{cases}$$

$$\text{Observer: } \begin{cases} \dot{\widehat{X}} = A\widehat{X} + BU + L(Y - \widehat{Y}) \\ \widehat{Y} = C\widehat{X} \end{cases} \Rightarrow \dot{\widehat{X}} = (A - LC)\widehat{X} + BU + LY \quad \widehat{Y} = C\widehat{X}$$

$A_{obs} = A - LC$ is the observer's evolution matrix, where $L(nx\ l)$

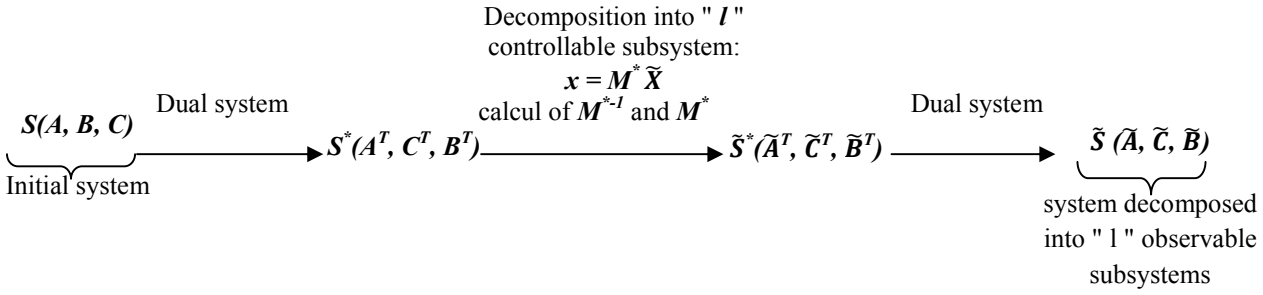
To facilitate calculations, the system is decomposed into l observable subsystems.

$$\text{System: } \begin{cases} \dot{\widetilde{X}} = \widetilde{A}\widetilde{X} + \widetilde{B}U \\ Y = \widetilde{C}\widetilde{X} \end{cases}$$

$$\text{Observer: } \begin{cases} \dot{\widehat{X}} = (\widetilde{A} - \widetilde{L}\widetilde{C})\widehat{X} + \widetilde{B}U + \widetilde{L}Y \\ \widehat{Y} = \widetilde{C}\widehat{X} \end{cases}$$

The observer's evolution matrix: $\widetilde{A}_{obs} = \widetilde{A} - \widetilde{L}\widetilde{C}$ where: $\widetilde{L}(nx\ l)$

Then, following the algorithm, we decompose the system into l observable subsystems:



The system in observable companion form: $\begin{cases} \dot{\widetilde{X}} = \widetilde{A}\widetilde{X} + \widetilde{B}U \\ Y = \widetilde{C}\widetilde{X} \end{cases}$

$$\widetilde{A}^T = M^{*-1}A^T M^*, \widetilde{C}^T = M^{*-1}C^T, \widetilde{B}^T = B^T M^*$$

$$\tilde{A} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \dots 0 & -a_0^1 \\ 1 & 0 & & -a_1^1 \\ 0 & \ddots & & \vdots \\ 0 & \dots & 1 & -a_{n_1-1}^1 \end{bmatrix} & \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} & \dots & \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} \\ \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} & \begin{bmatrix} 0 & 0 & \dots 0 & -a_0^2 \\ 1 & 0 & & -a_1^2 \\ 0 & \ddots & & \vdots \\ 0 & \dots & 1 & -a_{n_2-1}^2 \end{bmatrix} & \dots & \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} \\ \vdots & \vdots & & \vdots \\ \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} & \begin{bmatrix} 0 & \dots & 0 & * \\ \vdots & & & \vdots \\ 0 & \dots & 0 & \\ 0 & \dots & 0 & * \end{bmatrix} & \dots & \begin{bmatrix} 0 & 0 & \dots 0 & -a_0^l \\ 1 & 0 & & -a_1^l \\ 0 & \ddots & & \vdots \\ 0 & \dots & 1 & -a_{n_l-1}^l \end{bmatrix} \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} [0 & \dots & 1] & [0 & \dots & 0] & \dots & [0 & \dots & 0] \\ [0 & \dots & r_{12}] & [0 & \dots & 1] & \dots & [0 & \dots & 0] \\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ [0 & \dots & r_{1_l}] & [0 & \dots & r_{1_l}] & \dots & [0 & \dots & 1] \end{bmatrix}$$

This time, the matrix $\widetilde{\boldsymbol{A}}$ is not block triangular, which will complicate the calculation of the open-loop characteristic equation. However, we can choose a gain matrix $\boldsymbol{L}$ such that the matrix:

$\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}} = \widetilde{\boldsymbol{A}} - \widetilde{\boldsymbol{L}}\widetilde{\boldsymbol{C}}$ is triangular.

.

For reasons of simplicity and readability of the expressions, we limit to the case where the system is decomposed into two observable subsystems ***(l = 2)***.

$$\tilde{A} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \begin{bmatrix} a_{11} \\ \dots \\ a_{21} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & \begin{bmatrix} a_{12} \\ \dots \\ a_{22} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \end{bmatrix};$$

$$\tilde{C} = \begin{bmatrix} [0 & \dots & 1] & [0 & \dots & 0] \\ [0 & \dots & r_{21}] & [0 & \dots & 1] \end{bmatrix}; \qquad \tilde{L} = \begin{bmatrix} \tilde{l}_{11} & \tilde{l}_{12} \\ 0 & \tilde{l}_{22} \end{bmatrix}$$

$$\tilde{A}_{obs} = \tilde{A} - \tilde{L}\tilde{C} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \begin{bmatrix} \dots \\ a_{11} - \tilde{l}_{11} - r_{21}\tilde{l}_{12} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & \begin{bmatrix} \dots \\ a_{12} - \tilde{l}_{12} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & \begin{bmatrix} \dots \\ a_{11} - r_{21}\tilde{l}_{12} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \begin{bmatrix} \dots \\ a_{22} - \tilde{l}_{22} \end{bmatrix} \end{bmatrix}$$

$$A_{obs}^{d} = \begin{bmatrix} \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \begin{bmatrix} \alpha_{11} \\ \dots \\ \alpha_{21} \end{bmatrix} & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & \begin{bmatrix} \alpha_{12} \\ \dots \\ \alpha_{22} \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{bmatrix} & & \begin{bmatrix} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \dots \\ \dots & \dots & \ddots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix} & \end{bmatrix}$$

To obtain a triangular matrix $\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}}$, we take: $\boldsymbol{a_{12}} = [\boldsymbol{0} \quad \cdots \quad \boldsymbol{0}]^{\boldsymbol{T}}$.

By identifying $\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}}$ and $\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}}{}^{\boldsymbol{d}}$ term by term, he comes:

$$\begin{array}{cc} \alpha_{22} = a_{22} - \tilde{l}_{22} & \tilde{l}_{22} = a_{22} - \alpha_{22} \\ a_{21} - \tilde{l}_{21} = 0 \quad \Rightarrow & \tilde{l}_{21} = a_{21} \\ \alpha_{11} = a_{11} - \tilde{l}_{11} - r_{12}\tilde{l}_{21} & \tilde{l}_{11} = a_{11} - r_{12}\tilde{l}_{21} - \alpha_{11} \end{array}$$

In the general case:

$$\tilde{l}_{ll} = a_{ll} - \alpha_{ll}$$

$$\tilde{l}_{ij} = a_{ij} \quad j > i$$

$$\tilde{l}_{ii} = a_{ii} - \alpha_{ii} - \sum_{j=i+1}^{l} r_{ji}\tilde{l}_{ij}$$

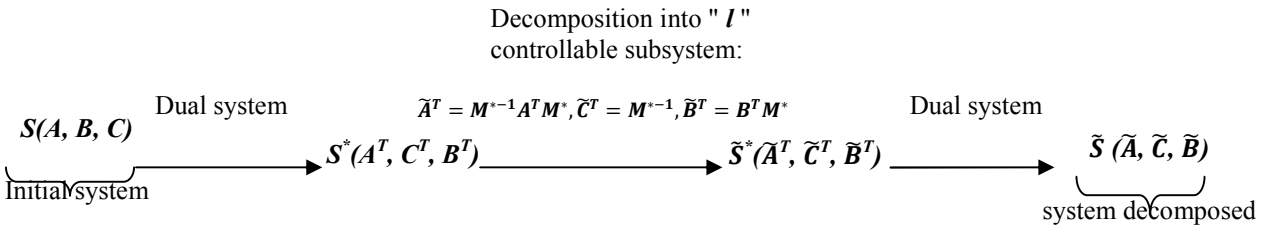
Finally: $\boldsymbol{L} = (\boldsymbol{M}^{*-1})^{\boldsymbol{T}}\tilde{\boldsymbol{L}}$

Algorithm for calculating observer L - Decomposition into l observable subsystems:

Let a given system: $\begin{cases} \dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{BU} \\ \boldsymbol{Y} = \boldsymbol{CX} \end{cases}$

We have n desired poles for the state observer: $\dot{\widehat{\boldsymbol{X}}}(\boldsymbol{t}) = \boldsymbol{A}\widehat{\boldsymbol{X}}(\boldsymbol{t}) + \boldsymbol{Bu}(\boldsymbol{t}) + \boldsymbol{L}(\boldsymbol{Y} - \widehat{\boldsymbol{Y}})$

Step1: Decompose the system into ***l*** observable subsystems: (Calculate of $\boldsymbol{M}^{*-1}$ ***and*** $\boldsymbol{M}^{*}$)



$$\begin{cases} \dot{\widetilde{\boldsymbol{X}}} = \widetilde{\boldsymbol{A}}\widetilde{\boldsymbol{X}} + \widetilde{\boldsymbol{B}}\boldsymbol{U} \\ \boldsymbol{Y} = \widetilde{\boldsymbol{C}}\widetilde{\boldsymbol{X}} \end{cases}$$

Step2: Determination of vectors $\boldsymbol{a_{ij}}$ and scalars $\boldsymbol{r_{ij}}$.

$$\widetilde{\boldsymbol{A}}=\begin{bmatrix}[\widetilde{\boldsymbol{A}}_{11}] & \cdots & [\widetilde{\boldsymbol{A}}_{1l}]\\ \vdots & \ddots & \vdots\\ [\widetilde{\boldsymbol{A}}_{l1}] & \cdots & [\widetilde{\boldsymbol{A}}_{ll}]\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}0 & 0 & \cdots 0 & -a_0^1\\ 1 & 0 & \ddots & -a_1^1\\ 0 & \ddots & 0 & \vdots\\ 0 & 0 & 1 & -a_{n_1-1}^1\end{bmatrix} & \cdots & \begin{bmatrix}0 & 0 & \cdots 0 & *\\ \vdots & & & *\\ 0 & 0 & 0 & \vdots\\ 0 & 0 & 0 & *\end{bmatrix}\\ \vdots & \ddots & \vdots\\ \begin{bmatrix}0 & 0 & \dots 0 & *\\ \vdots & & & *\\ 0 & 0 & 0 & \vdots\\ 0 & 0 & 0 & *\end{bmatrix} & \cdots & \begin{bmatrix}0 & 0 & \dots 0 & -a_0^l\\ 1 & 0 & \ddots & -a_1^l\\ 0 & \ddots & 0 & \vdots\\ 0 & 0 & 1 & -a_{n_l-1}^1\end{bmatrix}\end{bmatrix}$$

$$\widetilde{\boldsymbol{C}}=\begin{bmatrix}[0 & \cdots & 1] & [0 & \cdots & 0] & \dots & [0 & \cdots & 0]\\ [0 & \cdots & r_{21}] & [0 & \cdots & 1] & \dots & [0 & \cdots & 0]\\ & \vdots & & & \vdots & & \ddots & & \vdots & \\ [0 & \cdots & r_{l1}] & [0 & \cdots & r_{l2}] & \cdots & [0 & \cdots & 1]\end{bmatrix}$$

Step3: Determination of vectors $\boldsymbol{\alpha_{ij}}$, We want a triangular matrix $\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}}{}^{\boldsymbol{d}}$, therefore:

$$\boldsymbol{D_{obs}^d(s)}=\prod_{i=1}^{n}(\boldsymbol{sI}-\boldsymbol{p_i})=\prod_{i=1}^{l}\left(\boldsymbol{s}+\boldsymbol{\alpha}_{n_i-1}^{i}\boldsymbol{s}^{n_i-1}+\cdots+\boldsymbol{\alpha}_1^i\boldsymbol{s}+\boldsymbol{\alpha}_0^i\right)$$

$$=\underbrace{\left(\boldsymbol{s}^{n_1}+\boldsymbol{\alpha}_{n_i-1}^{1}\boldsymbol{s}^{n_1-1}+\cdots+\boldsymbol{\alpha}_0^1\right)}_{\boldsymbol{A_{obs\ 11}^d}}\times\ldots.\underbrace{\left(\boldsymbol{s}^{n_l}+\boldsymbol{\alpha}_{n_l-1}^{l}\boldsymbol{s}^{n_l-1}+\cdots+\boldsymbol{\alpha}_0^l\right)}_{\boldsymbol{A_{obs\ ll}^d}}$$

$$\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}}^{\boldsymbol{d}}=\begin{bmatrix}[\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}\,11}^{\boldsymbol{d}}] & \cdots & [\boldsymbol{0}]\\ \vdots & \ddots & \vdots\\ [*] & \cdots & [\widetilde{\boldsymbol{A}}_{\boldsymbol{obs}\,ll}^{\boldsymbol{d}}]\end{bmatrix}=\begin{bmatrix}\begin{bmatrix}0 & 0 & \cdots 0 & -\alpha_0^1\\ 1 & 0 & \ddots & -\alpha_1^1\\ 0 & \ddots & 0 & \vdots\\ 0 & 0 & 1 & -\alpha_{n_1-1}^1\end{bmatrix} & \cdots & \begin{bmatrix}0 & 0 & \cdots 0 & 0\\ \vdots & & & \\ 0 & 0 & 0 & \vdots\\ 0 & 0 & 0 & 0\end{bmatrix}\\ \vdots & \ddots & \vdots\\ \begin{bmatrix}0 & 0 & \dots 0 & *\\ \vdots & & & *\\ 0 & 0 & 0 & \vdots\\ 0 & 0 & 0 & *\end{bmatrix} & \cdots & \begin{bmatrix}0 & 0 & \dots 0 & -\alpha_0^l\\ 1 & 0 & \ddots & -\alpha_1^l\\ 0 & \ddots & 0 & \vdots\\ 0 & 0 & 1 & -\alpha_{n_l-1}^1\end{bmatrix}\end{bmatrix}$$

Step4: calculation of the state feedback $\tilde{L}$ in the canonical observability companion basis

$$\tilde{\boldsymbol{L}} = \begin{bmatrix} \tilde{l}_{11} & \tilde{l}_{12} & \dots & \tilde{l}_{1l} \\ 0 & \tilde{l}_{22} & \dots & \tilde{l}_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{l}_{ll} \end{bmatrix} \quad \boldsymbol{avec}: \quad \begin{matrix} \tilde{l}_{ll} = a_{ll} - \alpha_{ll} \\ \tilde{l}_{ij} = a_{ij} \quad j > i \\ \tilde{l}_{ii} = a_{ii} - \alpha_{ii} - \sum_{j=i+1}^{l} r_{ji}\, \tilde{l}_{ij} \end{matrix}$$

Step5: calcul de la matrice des gains de l'observateur $\boldsymbol{L}$ dans la base initiale:

$$\boldsymbol{L} = (\boldsymbol{M}^{*-1})^{T} \tilde{\boldsymbol{L}}$$

Example:

We have a MIMO system defined by:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix};\ \boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

The objective is to find the **2×2** dimension gain matrix $\boldsymbol{L}$ such that the poles of the matrix:

$\boldsymbol{A_c = A - LC}$, are $\lambda_1 = -3;\ \lambda_2 = -4$.

Step 1: Matrix expression

Let us define the unknown gain matrix: $\boldsymbol{L} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}$

Let's first calculate the $\boldsymbol{LC}$ product:

$$\boldsymbol{LC} = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} . \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} l_{11} + l_{12} & l_{12} \\ l_{21} + l_{22} & l_{22} \end{pmatrix}$$

Let's now calculate the observer's state matrix $\boldsymbol{A - LC:}$

$$\mathbf{A_c} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} - \begin{pmatrix} l_{11} + l_{12} & l_{12} \\ l_{21} + l_{22} & l_{22} \end{pmatrix} = \begin{pmatrix} 1 - l_{11} - l_{12} & 1 - l_{12} \\ -l_{21} - l_{22} & 2 - l_{22} \end{pmatrix}$$

Step 2: Calculation of the characteristic polynomial of $\boldsymbol{A - LC}$

The characteristic polynomial is given by: $\boldsymbol{P(s) = det\ (sI - A_c)}$

$$\boldsymbol{sI - A_c} = \begin{pmatrix} s - 1 + l_{11} + l_{12} & -1 + l_{12} \\ l_{21} + l_{22} & s - 2 + l_{22} \end{pmatrix}$$

$$\Rightarrow \boldsymbol{P(s) = (s - 1 + l_{11} + l_{12})(s - 2 + l_{22}) - (l_{21} + l_{22})l_{12} - 1)}$$

$$= \boldsymbol{s^2 + (-1 + l_{11} + l_{12})s + (-2 + l_{22})s + (-1 + l_{11} + l_{12})(-2 + l_{22}) - (l_{21} + l_{22})l_{12} - 1)}$$

$$= \boldsymbol{s^2 + (-3 + l_{11} + l_{12} + l_{22})s + (3 + l_{21} + l_{22} - l_{21}l_{12} + l_{11}l_{22})}$$

Expanding this, we obtain an expression of the form: $\boldsymbol{s^2 + \alpha_1 s + \alpha_0}$

By identification, we obtain:

$$\begin{cases} \boldsymbol{\alpha_1 = (-3 + l_{11} + l_{12} + l_{22})} \\ \boldsymbol{\alpha_0 = (2 + l_{21} - 2l_{12} - 2l_{11} - l_{21}l_{12} + l_{22}l_{11})} \end{cases}$$

***Step 3*:** Identification with the desired polynomial

The desired poles are ***-3*** and ***-4***. The desired characteristic polynomial is:

$$\boldsymbol{P_{des}(s) = (s + 3)(s + 4) = s^2 + 7s + 12}$$

By identifying the coefficients, we obtain the following system of equations:

$$\begin{cases} \boldsymbol{-3 + l_{11} + l_{12} + l_{22} = 7} \\ \boldsymbol{2 + l_{21} - 2l_{12} - 2l_{11} - l_{21}l_{12} + l_{22}l_{11} = 12} \end{cases} \Rightarrow \begin{cases} \boldsymbol{l_{11} + l_{12} + l_{22} = 10} \\ \boldsymbol{l_{21} - 2l_{12} - 2l_{11} - l_{21}l_{12} + l_{22}l_{11} = 10} \end{cases}$$

We have **2** equations for **4** unknowns. This is a fundamental property of MIMO systems: the placement of poles does not define a unique solution. We have additional degrees of freedom.

To find a simple solution, let's arbitrarily fix two values that simplify the calculations.

Let's say, for example: $\boldsymbol{l_{12} = 1}$ *and* $\boldsymbol{l_{21} = 0}$

Substituting into the previous equations, we find:

$$\begin{cases} \boldsymbol{l_{11} + l_{22} = 9} \\ \boldsymbol{-2l_{11} + l_{22}l_{11} = 12} \end{cases} \Rightarrow \begin{cases} \boldsymbol{l_{11} = 4} \\ \boldsymbol{l_{22} = 5} \end{cases}$$

Therefore, the gain matrix $\boldsymbol{L}$ for this system is:

$$\boldsymbol{L = \begin{pmatrix} 4 & 1 \\ 0 & 5 \end{pmatrix}}$$

EXERCISES

State representation – State feedback control (monovariable systems)

EXERCISE N°1 :

Consider the system described by the state equations: $\dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{Bu}$,

1) $\boldsymbol{A} = \begin{bmatrix} -4 & 1 \\ -2 & -2 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$;

Verify controllability?

2) $\boldsymbol{A} = \begin{bmatrix} -2 & 1 \\ \alpha & -4 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

For what value of « α » is the system controllable?

EXERCISE N°2 :

Consider the system : $\begin{cases} \dot{\boldsymbol{X}} = \boldsymbol{AX} + \boldsymbol{Bu} \\ \boldsymbol{y} = \boldsymbol{CX} \end{cases}$

Avec : $\boldsymbol{A} = \begin{bmatrix} 0 & 9 \\ 1 & 0 \end{bmatrix}; \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \boldsymbol{C} = [0 \quad 9]$;

1- Is the system controllable?

2- Is the system observable ?

EXERCISE N°3 :

$$\begin{cases} \dot{\boldsymbol{x}}(\boldsymbol{t}) = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}.\boldsymbol{x}(\boldsymbol{t}) + \begin{bmatrix} 0 \\ 4 \end{bmatrix}\boldsymbol{u}(\boldsymbol{t}) \\ \boldsymbol{y}(\boldsymbol{t}) = [1 \quad 0].\boldsymbol{x}(\boldsymbol{t}) \end{cases}$$

1- Is the system controllable?

2- Compute the gain matrix $\boldsymbol{K}$ that places the **C.L** poles at the desired locations:

$\bar{\lambda}_1 = -2 - j2\sqrt{3}$ and $\bar{\lambda}_2 = -2 + j2\sqrt{3}$ using the algorithm and the direct method.

EXERCISE N°4 :

Consider the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ -1 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}.\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.u \; ; y = [0 \quad 0 \quad 1].\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1- Is the system controllable?

2- Compute the gain matrix K that places the **C.L** poles at the desired locations at: -2 ; $-2 \pm 2j$.

EXERCISE N°5 :

We consider the system to be controlled, represented by the following matrices:

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -2 \end{bmatrix}; B = \begin{bmatrix} 1 \\ b \end{bmatrix}; C = [1 \quad 1]; D = 0.$$

1- Determine the system poles to be adjusted.

2- Determine, by calculation, whether the system is controllable, expressed as a function of the parameter *« b »*.

3- For $b = 0$, determine if the poles in the **C.L** can be placed using state feedback at the location: $\lambda_1 = -4$; $\lambda_2 = -5$.

4- For $b = 2$, determine the state feedback gain to place the poles in **C.L** at: $\lambda_1 = -4$; $\lambda_2 = -5$.

EXERCISE N°6 :

We consider the analog system given in the following controllable form:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}; B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; C = [1 \quad -2 \quad 3]; D = 0.$$

1- Determine the system stability, with : $p_1 = -0.5437$.

2- Determine the system controllability.

3- Determine the state feedback gain K, such that the poles in **C.L**. are located at: $p_1 = -10$; $p_{2,3} = -20$.

EXERCISE N°7 :

A continuous linear system is described by the state equations below, where $u = u(t)$ is the scalar control, and $y = y(t)$ is the scalar output.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} . \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} . u \; ; y = [1 \quad 1 \quad 1]. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Of the form: $\dot{X} = AX + Bu \; ; \; y = CX$

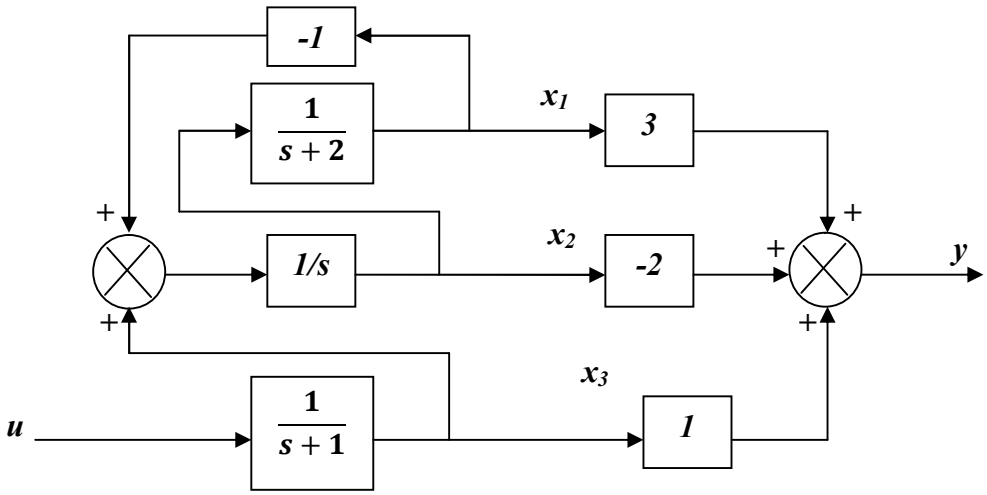
1- Using the Kalman criterion, calculate the dimensions of the controllable subspace and the observable subspace of the state space.

2- Calculate the transfer function ***F(s) = Y(s)/U(s).***

- Check the pole(s) which is (are) not completely controllable and observable.

EXERCISE N°8 :

Consider the linear system with three state variables represented by the figure below, ***u(t)*** is the input signal, ***y(t)*** the output signal.



1- Using Laplace transforms of state variables x_i, $i = 1, 2, 3$ and of u et y, Calculate the transfer function between u and y. Verify that it has a triple pole.

2- Write the matrix equations of state evolution in the form :

$$\begin{cases} \dot{X} = AX + Bu \\ y = CX \end{cases} \quad where \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

3- Using the Kalman criterion, show that the system is completely controllable.

EXERCISE N°9:

Consider the linear system represented by its transfer function:

$$G(p) = \frac{9}{s^2 - 9} \qquad with: \; C = [0 \quad 9]; D = 0.$$

1- Find the matrices ***A, B*** of the state representation.

2- Is the pair ***(A, B)*** controllable??

3- Calculate the gain matrix ***K*** ensuring pole placement in : $p_{1,2} = -3 \pm 3j$.

State representation of multivariable systems (1)

EXERCISE N°1:

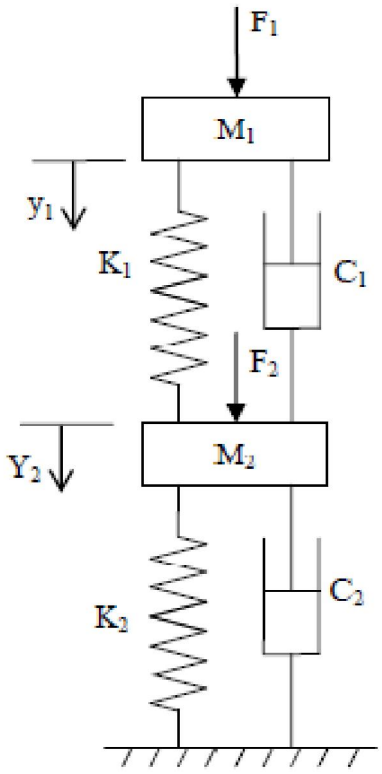
Consider the MIMO system represented by state equations:

$$\begin{cases} \dot{x} = \begin{bmatrix} -3 & -2 \\ 1 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 3 \\ -2 & -1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \\ 1 & -2 \end{bmatrix} x \end{cases}$$

1. What are the characteristics of this system (order, number of inputs and number of outputs)?
2. What are the system modes? Verify the stability.
3. Calculate the transfer matrix and what each element represents?

EXERCISE N°2:

Consider the **MIMO** mechanical system with two inputs (***F_1 and F_2***) and two outputs (***y_1 and y_2***):



1. Give the differential equations of the mechanical system
2. Give the state representation (***A, B, C, D***).

EXERCISE N°3:

Consider the system defined by the following state equations:

$$\begin{cases} \dot{X} = AX + Bu \\ y = CX \end{cases}$$

with:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} et\ C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

1. Determine if the system is controllable and observable
2. Determine the system modes, and indicate whether the system is completely stable.

EXERCISE N°4:

The state representation of an n^{th} order multivariable system with m independent scalar inputs and l distinct scalar outputs, can be expressed in the compact form as:

$$\begin{cases} \dot{X} = AX + Bu \\ y = CX \end{cases}, avec: B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{l \times n}$$

The system can also be written in the following way::

$$\begin{cases} \dot{X} = AX + \sum_{k=1}^{m} b_k u_k \\ y = c_i X, \quad i \in [1, l] \end{cases}$$

$$A = \begin{bmatrix} -11 & 3 & 0 \\ -18 & 4 & 0 \\ 1 & 0 & -1 \end{bmatrix}; B = \begin{bmatrix} 1 & -5 \\ 3 & -11 \\ -2 & 3 \end{bmatrix} and\ C = [2 \quad -1 \quad 1]$$

1. What are the characteristics of this system (order, number of inputs and number of outputs)? Determine the different matrices b_k and c_i.
2. What are the system modes?
3. Prove that the system is observable?
4. Prove that the system is not controllable by one of the inputs and specify which one, justifying your answer.

EXERCISE N°5:

We consider the linearized equations describing a satellite near a circular orbit traversed at a constant speed ω:

$$\ddot{r} = 3\omega^2 r + 2\omega\dot{\theta} + u_r$$
$$\ddot{\theta} = -2\omega\dot{r} + u_\theta$$

The satellite is controlled by two motors. The first provides a radial force u_r and the second a tangential force u_θ. The measured output y is the radial position r.

We ask:

1. Due to a technical fault, you need to shut down one of the motors. Which motor will you choose?
2. Determine if this system is observable when the measured output is the radial position r, if possible without calculating the rank of a matrix. If not, what output should be used?
3. To determine the initial conditions which cannot be distinguished from one another when only $y(t)$ and $u(t)$ are known.

State representation of multivariable systems (2)

EXERCISE N°1:

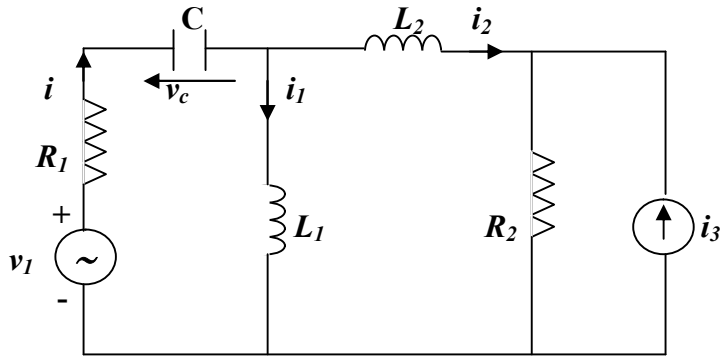
Consider the linear system described by:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -3 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t); \quad y(t) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 1 \end{pmatrix} x(t)$$

1. Calculate the transfer matrix and briefly explain what each element represents??
2. Verify the system's controllability.
3. Verify the system's observability.

EXERCISE N°2:

Consider the following electrical circuit:



The states are: - Voltage across the capacitor

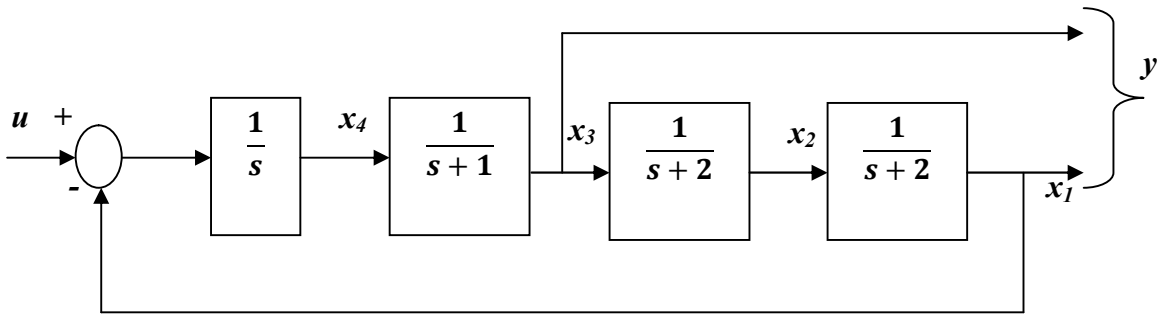
- Currents in the inductors

The output: $y(t) = v_c(t)$

1. Give the differential equations of the electrical system
2. Give the state-space representation (***A, B, C, D***).
3. What are the characteristics of this system?

EXERCISE N°3:

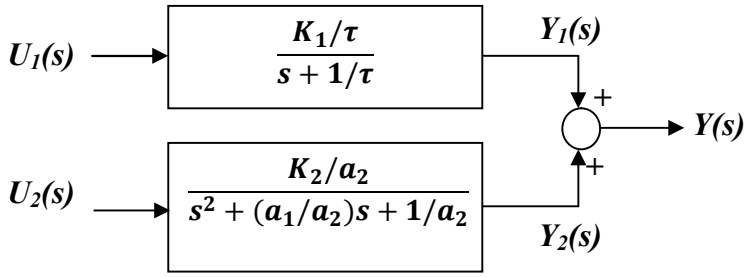
Consider the system below:



1. Give the state-space representation (***A, B, C, D***).
2. What are the characteristics of this system (order, number of inputs and number of outputs)?
3. What is the dimension of the transfer matrix (with explanation) and what does each element represent?

EXERCISE N°4:

Consider the functional block diagram of a multivariable system with two inputs and one output:



- Give the state equations for the two systems.
- By combining the two systems, give the state representation of the multivariable system.

EXERCISE N°5:

Consider a system representing an electromagnetic positioner, described by the following equations:

- Mechanical equation: $\boldsymbol{F = M\frac{d^2x}{dt^2} + f\frac{dx}{dt} + kx = Ai}$

- Electrical equation: $\boldsymbol{V = Ri + L\frac{di}{dt} + E}$

- The output ***y*** is the position ***x***

With: ***x*** is the displacement of the mobile, ***F*** is the force exerted on the mobile by the magnetic field, ***A*** is a constant, ***f*** is the friction coefficient, ***k*** represents the spring's stiffness, i.e., the coefficient of its restoring force, ***E*** is the electromotive force of the displacement; this force is proportional (coefficient β) to the displacement speed.

1- Determine the state variables number.

2- Write the state equations for this system (***A, B, C, D***).

N.A: $\mathbf{M = 30Kg, k = 15N/m, f = 15N/(m/s), A = 6N/A,}$

$\mathbf{R = 10\Omega, L = 10H, \beta = 0.2V/(m/s)}$

3- Calculate the static gain of the system.

Transition from a transfer matrix to a state representation

EXERCISE N°1:

Consider the multivariable system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-1 & -4 & 4\\ -3 & -4 & 6\\ -3 & -5 & 7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1 & 2\\ 1 & 0\\ 0 & 1\end{bmatrix}u$$

$$y = \begin{bmatrix}0 & 1 & 0\\ 1 & 3 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

- Find the observability indices of y_1 and y_2.
- Deduce whether the system is completely observable or not?

EXERCISE N°2:

Consider the multivariable system represented by its transfer matrix:

$$M(s) = \frac{1}{(s-1)(s+1)(s+2)}\begin{bmatrix}s^2+6 & s^2+s+4\\ 2s^2-7s-2 & s^2-5s-2\end{bmatrix}$$

Using Gilbert's method, find a state representation of the system.

EXERCISE N°3:

Consider the multivariable system represented by its transfer matrix:

$$H(s) = \begin{bmatrix}\frac{s}{s+1} & 0\\ \frac{1}{s+1} & \frac{s+1}{s^2}\end{bmatrix}$$

Using Smith-McMillan method:

1. Determine the system order.
2. Find the modes and zeros of the system.

EXERCISE N°4:

Consider the multivariable system represented by its transfer matrix:

$$\boldsymbol{G(s)} = \frac{\mathbf{1}}{\boldsymbol{(s+2)^2}}\begin{pmatrix} \boldsymbol{s+3} & \boldsymbol{3s+8} \\ \mathbf{1} & \boldsymbol{s+4} \end{pmatrix}$$

- Find the system order and modes.
- Find the state representation of the system using the invariant method, with:

$$\boldsymbol{W(s)} = \begin{pmatrix} \mathbf{1} & \boldsymbol{s+4} \\ \mathbf{0} & \mathbf{-1} \end{pmatrix}$$

EXERCISE N°5:

Consider the multivariable system represented by its transfer matrix:

:

$$\boldsymbol{G(s)} = \begin{pmatrix} \dfrac{\mathbf{1}}{\boldsymbol{(s+1)(s+2)}} & \dfrac{\mathbf{2}}{\boldsymbol{s+2}} \\ \dfrac{\mathbf{1}}{\boldsymbol{(s+1)}} & \dfrac{\mathbf{1}}{\boldsymbol{(s+1)}} \end{pmatrix}$$

- Determine the system order and modes.
- Find the state-space representation of the system using Gilbert's method..

State feedback Control

EXERCISE N°1:

Consider the system: $\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} u(t)$

Calculate the state feedback control that places the poles at $\{-4, -3, -5\}$ by putting the system in controllable canonical form.

EXERCISE N°2:

Consider the system: $\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 4 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u(t)$

- Calculate the gain of the state-feedback control that places the poles to $\{-1, -2, -3\}$ using the Young-Willems method.

- Provide the expression of the control.

EXERCISE N°3:

Consider the following linear MIMO system:

$$\dot{X} = \begin{bmatrix} 1 & 0 & 2 & -1 \\ -1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{bmatrix} X + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} U$$

1. Decompose the system into ***r*** (***r < m, m*** is the number of inputs) controllable subsystems (choice by column).

2. Calculate the state feedback imposing the following poles in closed loop: ***-1, -2, -3 ± j.***

EXERCISE N°4:

Consider the system:

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 1 & 2 & 3 \\ 2 & 1 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2 \\ 0 & 2 \end{bmatrix} U$$

Calculate the gain matrix ***K*** such that the matrix ***(A - B.K)*** has the eigenvalues:

-4 ± 3j et ***-5 ± 4j.***

GENERAL CONCLUSION

This course provides fundamental concepts on multivariable linear systems. It begins with a review of single-variable systems, their state representation, and their control using state feedback or pole placement. Then, it applies these basic concepts to multivariable systems, covering state representation and transfer matrices, controllability and observability, the transition from a transfer matrix to a state representation, and state feedback control.

By reading this course, students will be able to distinguish between a SISO system and a MIMO system. Furthermore, they will master the mathematical tools and fundamental concepts introduced for single-variable systems, thoroughly understanding the solved examples provided for each concept. The student will be able to understand the theories of multivariable systems: state representation and its solution, principles of controllability and observability, study of some canonical forms, transformation of the transfer matrix to a state representation using the Gilbert method and the invariant method (Smith-McMillan) and state feedback control.

This course contains several examples with solutions in each chapter to facilitate understanding of the theoretical concepts introduced and many exercices in the end of document.

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