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Thème

About some Zeraoulia ordinary differential equations

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وكان فضل الله عليك عظيماً

شكر و عرفان

لحمد لله الذي كان لنا نورًا في العتمة، وسندًا في لحظات الضعف

وملهمًا في طريقٍ لم يكن دائمًا سهلاً ... فله الحمد حتى يبلغ الحمد منتهاه

لم يكن هذا العمل مجرد صفحات تُكتب،

بل كان رحلةً من الصبر والتعب، من الأمل والتحدي، ومن السقوط والنهوض من جديد...

رحلة لم نكتصل إلى نهايتها لولا أيدٍ امتدت لنا بصدق، وقلوبٍ آمنت بنا

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، الذي كان أكثر من مجرد مؤطر، كان دليلاً في دربٍ مليء بالتساؤلات، ونورًا في لحظات التردد،

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.لكم منا أصدق الدعاء وأعمق الامتنان

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{وآخر دعواهم أن الحمد لله ربي العالمين }

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الى من احمل اسمه بكل فخر "**خموج محمد**"الى من حصد الاشواك عن دربي ليمهد لي

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Abstract

This thesis investigates three-dimensional nonlinear dynamical systems of the Lorenz-type, which exhibit chaotic behavior characterized by the presence of strange attractors and quasi-attractors.

The study focuses on the **Zeraoulia generalized system**, which is a generalized model capable of realizing several well-known chaotic systems, including the Lorenz, Chen, Lü, and Chua models. The analytical approach is based on determining the system's equilibrium points and evaluating their local stability through Jacobian matrix analysis and eigenvalues computation. Additionally, bifurcation analysis is performed to explore how parameters variations affect system dynamics and the transition to chaos.

The work provides a precise characterization of the studied system's dynamics and highlights new chaotic features

Key words:

- Zeraoulia generalized model
- Attractors
- Chaos

Résume

Ce mémoire est consacré à l'étude des systèmes dynamiques non linéaires tridimensionnels de type Lorenz, qui présentent un comportement chaotique caractérisé par l'existence d'attracteurs étranges et de quasi-attracteurs.
L'analyse porte sur le **système généralisée de Zeraoulia**, un modèle généralisé capable de reproduire les modèles chaotiques les plus célèbres tels que Lorenz, Chen, Lü et Chua.
La méthodologie repose sur la détermination des points d'équilibre du système et l'étude de leur stabilité locale via la matrice jacobienne et le calcul des valeurs propres associées.
De plus, une analyse de bifurcation est effectuée pour examiner l'effet de la variation des paramètres sur la dynamique du système et la transition vers le chaos.
Cette approche fournit une caractérisation précise de la dynamique du système étudié et met en évidence de nouvelles propriétés chaotiques.

Les mots clés :

- Le model généralisé de Zeraoulia
- Attractrurs
- Chaos

ملخص

تتناول هذه المذكرة دراسة الأنظمة الديناميكية غير الخطية ثلاثية الأبعاد من نوع لورينز، والتي تتميز بسلوك فوضوي يظهر من خلال وجود الجواذب الغريبة .تركز الدراسة بشكل أساسي على نموذج زراولية الموحدوهو نموذج قادر على توليد عدة أنواع من الجواذب المعروفة مثل أنظمة لورينز، وتشن، ولو، وتشوا .تعتمد المنهجية المتبعة على تحديد نقاط التوازن للنظام وتحليل استقرارها المحلي باستخدام مصفوفة جاكوبي وحساب القيم الذاتية.كما تتضمن الدراسة تحليلاً للتشعبلدراسة تأثير تغير الوسائط على ديناميكية النظام

وخلصت الدراسة إلى تقديم خصائص دقيقة للديناميكيات المدروسة، مما يساهم في فهم أعمق للأنظمة الفوضوية وتطبيقاتها

الكلمات المفتاحية :

- الجملة المعممة لزراولية
- فوضى
- جاذبات

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Introduction

This work aims to study a three-dimensional nonlinear dynamical system of the Lorenz-type, which exhibits chaotic behavior characterized by the presence of strange attractors and quasi-attractors. This is Zeraoulia unified system studied in [1]. The analysis is based on the determination of the system's equilibrium points and the study of their properties through the use of the Jacobian matrix and the computation of the associated eigenvalues, allowing the identification of local stability in the neighborhood of these points. In addition, bifurcation analysis is performed to investigate the effect of parameter variations on the system dynamics and the transition from regular to chaotic behavior. This analytical and numerical approach provides an accurate characterization of the dynamics of the studied system, establishes connections with well-known chaotic models, and highlights new chaotic features.
We focus on continuous 3-D time systems when studying chaos and bifurcation theory. We have divided this master thesis into three chapters as follows:

- **Chapter 1** is dedicated to presenting the generalized system describing the four systems: Lorenz, Chen, Lü and Chua models.
- **Chapter 2** is interested in the searching of bounded chaotic orbits in some 3-dimensional systems by using Lyapunouv theory.
- **Chapter 3** is focused only on the searching of chaos in a Zeraoulia 3-dimensional smooth piecewise Chua circuit**.**

Chapter 1

Zeraoulia generalized model of some Lorenz-type and quasi-attractors type strange attractors

In this chapter, a continuous-time three-dimensional autonomous system is presented from [1]. This system is capable to realise the almost well known Lorenz-type and quasi-attractors-type strange attractors; by special choices of a real function and eight bifurcations parameters; especially the Lorenz, the Chen, the lü and the Chua models are obtained; in addition to others new chaotic attractors.

1.1 Classification of Strange attractors

Several researchers have defined and studied strange attractors. The first was Lorenz [8] in 1963, where he proposed a simple mathematical model of a weather system which was made up of three linked nonlinear differential equations, showing rates of change in temperature and wind speed. Some surprising results clearly showed the complex behavior from supposedly simple equations. Also, it has been noted that the behavior of this system of equations was sensitively dependent on the initial conditions of the model. He spelled out the implication of his discovery, saying it implied that if there were any errors in observing the initial state of the system which is inevitable in any real system, therefore, the prediction of a future state of the system was impossible. Later, Ruelle in 1979, when computer simulation came along, the first fractal shape identified took the form of a butterfly; it arose from graphing the changes in weather systems modelled by Lorenz. The Lorenz's attractor shows just how and why weather prognostication is so involved and notoriously wrong because of the butterfly effect. This amusing name reflects the possibility

that a *butterfly in the Amazon might, in principal, ultimately alter the weather in Kansas*.
Observations of chaotic behavior in the field of electrical circuit date to van der pole in 1927, who reported *irregular noise* from his neon bulb circuit. A major breakthrough came with the discovery of chaotic attractors in a three-dimensional autonomous system of equation by Lorenz [8]. He purposed to create and analyse a model for the unpredictable behaviour of the weather. The first proposed real physical dynamical system, capable of generating chaotic phenomena in the laboratory, similar to those in the Lorenz system is invented by Chua [7], who synthesized a simple third-order autonomous circuit exhibiting a remarkable variety of dynamical phenomena i.e., strange attractors.
Strange attractors at the present time are classified into three principal classes: *Hyperbolic, Lorenz-type and quasi-attractors*.

Definition 1.1 *The hyperbolic attractors are the limit sets for which Smale's **Axiom A** is satisfied, and are structurally stable. Periodic orbits and homoclinic orbits are dense and are of the same saddle type. That is stable (resp. unstable) manifold of all the trajectories have the same dimension.*

Definition 1.2 *The Lorenz attractors are not structurally stable, thought their homoclinic and heteroclinic orbits are structurally stable (hyperbolic).*

Definition 1.3 *The quasi-attractors is the limit set enclosing periodic orbits of different topological types (for example stable and saddle periodic orbits), structurally unstable orbits.*

For example the attractors generated by Chua's circuit [7] associated with saddle-focus homoclinic loop are quasi-attractors. Noting that this type is more complex than the above two attractors.

1.2 The Zeraoulia generalized model of some types of strange attractors

A question arises: *It is possible to construct a mathematical model which permits to visualise some of these types of attractors?*. The answer is positive: it is possible to construct a model that can give some attractors of Lorenz-type and quasi-attractors-type. This new model is given by the following system of equations:

$$\begin{cases} \frac{dx}{dt} = a_1(y - f(x)) \\ \frac{dy}{dt} = a_2 x + by + a_3 z - a_4 xz, \\ \frac{dz}{dt} = -a_5 z + a_6 xy - a_7 y, \end{cases} \tag{1.1}$$

where $(a_i)_{1\leq i\leq 7}$ and b are the bifurcation parameters and $f(x)$ is some real function. It has to be noted that the model (1.1) is a smooth quadratic one with only two nonlinearities xy and xz if and only if $f(x)$ is a function with the variable x. This model is capable of realising the almost well known three-dimensional chaotic systems for example:

1. For $f(x) = x, a_3 = a_7 = 0, b = -1.$ one have the Lorenz,
2. For $f(x) = x, a_3 = a_7 = 0, a_2 = b - a_1$ one have the Chen model,
3. For $f(x) = x, a_2 = a_3 = a_7 = 0$ and $a_4 = 1.$one have the Lü model,
4. For $f(x) = v_1 x + \frac{1}{2}(v_0 - v_1)(|x+1| - |x-1|)$ and $a_2 = a_3 = 1, a_4 = a_5 = a_6 = 0$ and $b = -1$ one has the Chua model, where in each case the remaining constants are the set of bifurcation parameters. See Fig.1.1.

The new chaotic attractor has several important properties. It has natural symmetry under the coordinate transform $(x, y, z) \longrightarrow (-x, -y, z)$ if and only if $a_7 = 0$ and f is an odd function, and it is invariant under the coordinate transform $(x, y, z) \longrightarrow (-x, -y, -z)$ if and only if $a_4 = a_6 = 0$ and f is an odd function. These transformations persist for all values of the system parameters. Therefore for the system (1.1) the divergence of the flow is given by:

$$\nabla V = \frac{\partial x\prime}{\partial x} + \frac{\partial y\prime}{\partial y} + \frac{\partial z\prime}{\partial z} = a_1 f\prime(x) + b - a_5 \tag{1.2}$$

Proposition 1.1 *If $f'(x) > \frac{b-a_5}{a_1}$ and $a_1 > 0$ or $f'(x) < \frac{b-a_5}{a_1}$and $a_1 < 0$ or $a_1 = 0$ and $b < a_5$ for all values of x then the system (1.1) has a bounded globally attracting $\omega-$limit set.*

Then the system (1.1) is dissipative. Thus, all trajectories are ultimately confined to a specific subset having zero volume and the asymptotic motion settles onto an attractor, this result has been confirmed by some computer simulations.

1.3 Importance of dissipative systems

Dissipative systems are open, non-conservative systems that operating far from equilibrium, they continuously exchange energy with their environment and are characterized by irreversible energy losses. Such systems cannot retain all the energy supplied to them and therefore require a permanent energy input to sustain their dynamics.

- Dissipative systems function under non-equilibrium thermodynamic conditions.

- They require a continuous external energy supply to maintain their structure.
- Part of the supplied energy is irreversibly dissipated into the environment.
- These systems are unable to conserve all incoming energy.
- Dissipative behavior can be described using storage function in mathematical models.
- A system is dissipative if the stored energy does not exceed the supplied energy.
- Dissipativity theory provides a rigorous analytical framework for such systems.

1.4 Importance of equilibrium points

Equilibrium points constitute a fundamental elements in the dynamical systems as they determine the local and global behavior of system trajectories. In particular,

- An Equilibrium point represents a steady-state configuration of the system, where no evolution in time occurs.
- The number and location of equilibrium points depend on the nonlinear structure of the function $f(x)$ and a nonlinear system may admit multiple equilibria.
- The stability properties of these points (stable, unstable, saddle) determine the long-term behavior of nearby trajectories.
- The presence of multiple equilibrium points may lead to complex dynamics, including bifurcation and chaotic behavior.
- The global structure of attractors in nonlinear systems is often organized around equilibrium points.

Therefore, the study of equilibrium points is essential for understanding the qualitative and asymptotic behavior of nonlinear systems.

1.5 Stability and complex eigenvalue analysis

Complex eigenvalues $\lambda = \alpha \pm bi$ indicate oscillatory behavior in linear systems, where the real part α determines stability (damping) and the imaginary part b represents the frequency of oscillation. A system is stable if all eigenvalues have negative real parts $\alpha < 0$, unstable if there exist a real part $\alpha > 0$, and marginally stable if the real part is zero.

- **Stable Systems** $(Re(\boldsymbol{\lambda}) < \mathbf{0})$: If the real part of all complex eigenvalues is negative, the system's oscillations decay over time, leading back to a steady state.
- **Unstable Systems** $(Re(\lambda) > 0)$: If at least one eigenvalue has a positive real part, the system is unstable, with oscillations that grow exponentially.
- **Marginally Stable** $(Re(\lambda) = 0)$: If the real part is zero, the system oscillates with constant amplitude.
- **Role of Imaginary Part** $(Im(\lambda) = b)$: The imaginary part determines the frequency of rotation or oscillation in the system.
- **Complex Conjugate Pairs:** For real matrices, complex eigenvalues always occur in conjugate pairs $(\lambda = \alpha \pm bi)$.
- **Physical Interpretation:** In structural dynamics, the real part represents damping, while the imaginary part represents the frequency of oscillation. Positive real parts correspond to unstable modes, often appearing in pairs.
- **System Behavior**: Complex eigenvalues indicate that solutions involve both exponential decay/growth and oscillations.

For this purpose the Jacobian matrix must be computed and evaluated at these equilibria, as follow:

$$J(P) = \begin{pmatrix} -a_1\, f'(x) & a_1 & 0 \\ a_2 - a_4 z & b & a_3 - a_4 x \\ a_6 y & a_6 x - a7 & -a_5 \end{pmatrix} \tag{1.3}$$

where $P = P(x, y, z)$ is an equilibrium point.

1.6 Cardan method's for solving a cubic characteristic equation

The precise value of the eigenvalues of matrix (1.2) is obtained by the use of the Cardan method's for solving a cubic characteristic equation $P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$. By setting $\omega = \frac{-A}{3}$. These yields: $G(X) = X^3 + PX + Q$, where $P = -\frac{A^2}{3} + B$ and $Q = \frac{2A^3}{27} - \frac{AB}{3} + C$. We set $\Delta = 4P^3 + 27Q^2$, resulting the following:

(i) If $\Delta > 0$, there is a unique real eigenvalues: $\lambda_R = \frac{-A}{3} + X_R$ where:

$$X_R = \left(-\frac{Q}{2} + \sqrt[2]{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{Q}{2} - \sqrt[2]{\frac{Q^2}{4} + \frac{P^3}{27}}\right)^{\frac{1}{3}}$$

along with two complexes conjugate eigenvalues: $(\lambda_C)^{\pm} = \omega + (X_C)^{\pm}$, where:

$$(X_C)^{\pm} = -\frac{X_R}{2} \pm \frac{i}{2}\sqrt[2]{4P + 3(X_R)^2}$$

(ii) If $\Delta < 0$, the system has three real and distinct eigenvalues:

$$\lambda_1 = \frac{-A}{3} + 2\sqrt{-\frac{P}{3}}\sin(\frac{\theta}{3}), \lambda_2 = \frac{-A}{3} + 2\sqrt{-\frac{P}{3}}\sin(\frac{2\pi+\theta}{3}), \lambda_3 = \frac{-A}{3} + 2\sqrt{-\frac{P}{3}}\sin(\frac{4\pi+\theta}{3}).$$

where $\theta = \arcsin(\sqrt{\frac{-27Q^2}{4P^3}}) \in [0, \pi]$. The case $\Delta = 0$ corresponds to a measure-zero set of parameters. So, by a slight perturbation of parameters, without changing the behavior of the system, a system belonging to one of the two cases is obtained.

1.7 Stability of equilibrium points

The equilibrium points of the system (1.1) can be found by solving the following three equations:

$$a_1(y - f(x)) = 0, a_2x + by + a_3z - a_4xz = 0, -a_5z + a_6xy - a_7y = 0. \tag{1.4}$$

then two principal cases can be distinguished:
(i) If $a_5 \neq 0$ then after (1.4) one obtain that all equilibrium points of system (1.1) are given by:

$$P = \left(x, f(x), \frac{-a_7\ f(x) + a_6\ x\ f(x)}{a_5}\right) \tag{1.5}$$

where x is the root of the equation:

$$a_2x + f(x)\left(b + \frac{g(x)}{a_5}\right) = 0$$

where

$$g(x) = -a_1a_6x^2 + (a_3a_6 + a_1a_7)x - a_3a_7$$

(ii) If $a_5 = 0$, then by (1.4) one obtains that all equilibrium points of system (1.1) are given by:

$$P = \left(x, f(x), \frac{-a_2\ f(x) + b\ f(x)}{a_3 - a_4x}\right) \tag{1.6}$$

where x is the root of the equation

$$(a_6x - a_7) \times f(x) = 0$$

By impose some supplementary conditions on the coefficients $(a_i)_{1\leq i\leq 7}$ and b. It can be remarked that the origin is an equilibrium point for the system (1.1) if for example $f(0) = 0$. Noting also that the number of equilibrium points depends mainly on the shape of the function $f(x)$ and the set of coefficients $(a_i)_{1\leq i\leq 7}$ and b where for such equilibrium $P(x, y, z)$ one has:

$$\begin{aligned} A &= -a_1\ f'(x) + a_5 - b \\ B &= a_1((a_5 - b)\ f\prime(x) + a_4z - a_2) + g_1(x) \\ C &= a_1(g_1(x)\ f\prime(x) + g_2(x, y, z)) \end{aligned}$$

where

$$g_1(x) = a_4a_6x^2 - (a_3a_6 + a_4a_7)x - (a_3a_7 + a_5b),$$

$$g_2(x, y, z) = a_3a_6y - a_4a_5z - a_4a_6xy + a_2a_5.$$

Here, the values of x, y, z depends of $f(x)$ formula, and the set of coefficients $(a_i)_{1\leq i\leq 7}$ and b.
In this chapter, some functions are being used to generate chaotic attractors from model (1.1). For example:

$$\begin{aligned} f_1(x) &= x \\ f_2(x) &= sgn(x) \\ f_3(x) &= v_1x + \frac{1}{2}(v_0 - v_1)(|x + 1| - |x - 1|) \\ f_4(x) &= v_1x + v_0sgn(x) \\ f_5(x) &= (v_1x + v_0) \times sgn(x) \end{aligned}$$

Note that the function f_3 is the well known Chua's piecewise-linear function. On the other hand the model (1.1) is piecewise-linear with the conditions that: $a_4 = a_6 = 0$ and f itself is a piecewise-linear function.

1.8 The case where $f(x) = v_1x + \frac{1}{2} \times (v_0 - v_1)\,(|x + 1| - |x - 1|)$

In this case, we choose the values of the slopes of the inner and outer regions for the Chua's circuit as follow: $v_0 = -0.43, v_1 = 0.41$. We obtain the two chaotic attractors: the first is a *scroll-type attractor* and the second is a *double scroll-type attractor* obtained for nonlinear system i.e.,: $a_4 \neq 0, a_6 \neq 0$ and $f(x)$ is the same as in the original Chua's circuit [7].

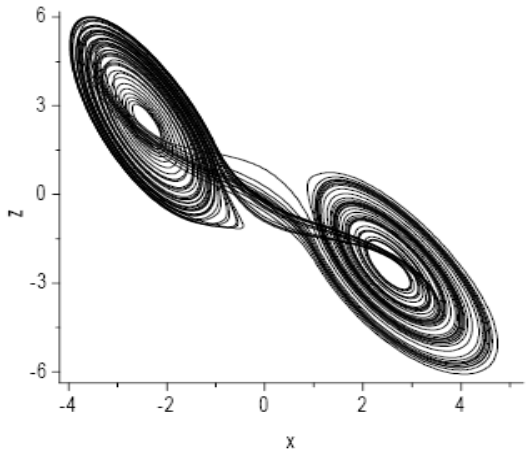


Figure 1.1: A double scroll type chaotic attractor obtainedfor: $a_1 = 13$, $a_2 = 1$, $a_3 = 1$, $a_4 = 0$, $a_5 = 1$, $a_6 = 1$,$a_7 = 14$, $b = -1$.

From these figures the double scroll can occurring in the two following systems:

$$\begin{cases} \frac{dx}{dt} = a_1(y - f(x)), \\ \frac{dy}{dt} = x - y + z - a_4 xz, \\ \frac{dz}{dt} = -a_5 z + xy - a_7 y, \end{cases} \text{ and } \begin{cases} \frac{dx}{dt} = a_1(y - f(x)), \\ \frac{dy}{dt} = x - y + z, \\ \frac{dz}{dt} = -a_5 z + xy - a_7 y, \end{cases} \tag{1.7}$$

which are the originals Chua's circuit in addition of the two nonlinearities xz and xy in the first system (1.1) and the same Chua's circuit in addition of a single quadratic nonlinearity xy in the second system (1.7). As a summary, the double scroll chaotic attractor can be occurring in special nonlinear quadratic systems with still the same three segment piecewise-linear function. If we take a_4 as the bifurcation parameter and fix the other parameters, it can be remarked that the double scroll-type attractors appear for small values of a_4, but for the big values of a_4 the system converges to a periodic solutions.

An important example of this case is when $f_6(x) = \alpha x + \beta \times sgn(x)$ where one gets a double scroll type attractor with only two equilibria having the same stability type for all bifurcation parameters (See Fig 1.2). Note that the function f_6 is a generalized form for f_1,f_2 and f_4 mentioned above.Now, let us now study the stability type for the equilibrium points. For this purpose the Jacobian matrix must be computed and evaluated at these equilibria, as follow:

$$J(P) = \begin{pmatrix} -a_1 f'(x) & a_1 & 0 \\ a_2 - a_4 z & b & a_3 - a_4 x \\ a_6 y & a_6 x - a7 & -a_5 \end{pmatrix} \tag{1.8}$$

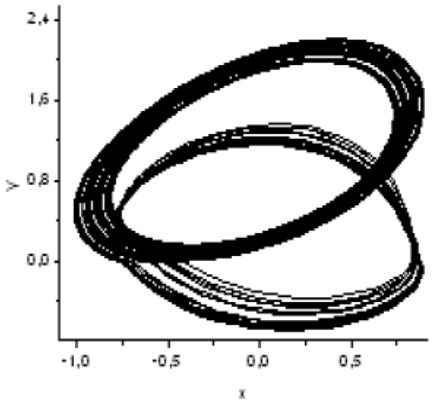


Figure 1.2: A two scroll chaotic attractor for: $a_1 = 10.5$, $a_2 = 1$, $a_3 = 6$, $a_4 = 0.8$, $a_5 = 0$, $a_6 = 0$, $a_7 = 14$, $b = -1$. Here $f(x) = v_1 x + v_0 \times sgn(x)$, where $v_0 = -0.43, v_1 = 0.41$.

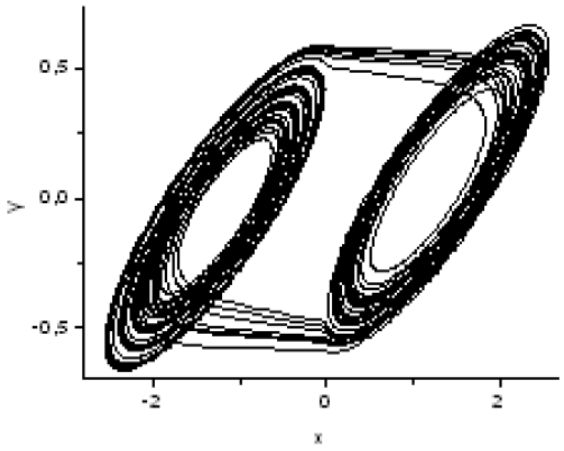


Figure 1.3: Projection of the plan $x - y$ of a double scroll type chaotic attractor obtained for $a_1 = 12$, $a_2 = a_3 = 1$, $a_4 = 0$, $a_5 = 1$, $a_6 = 0$, $a_7 = 14$, $b = -1$ and f_6 is given by: $f_6(x) = \alpha x + \beta \times sgn(x)$ where $\alpha = 0.41$ and $\beta = -0.42$.

where $P = P(x, y, z)$ is an equilibrium point. The precise value of the eigenvalues (1.8) is obtained by the use of the Cardan method's for solving a cubic characteristic equation $P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C$, where for such equilibrium $P(x, y, z)$ one has:

$$\begin{aligned} A &= -a_1\ f'(x) + a_5 - b \\ B &= a_1((a_5 - b)\ f'(x) + a_4 z - a_2) + g_1(x) \\ C &= a_1(g_1(x)\ f'(x) + g_2(x, y, z)) \end{aligned}$$

where

$$g_1(x) = a_4 a_6 x^2 - (a_3 a_6 + a_4 a_7)x - (a_3 a_7 + a_5 b),$$

$$g_2(x, y, z) = a_3 a_6 y - a_4 a_5 z - a_4 a_6 xy + a_2 a_5.$$

where the values of x, y, z depends of $f(x)$ formula, and the set of coefficients $(a_i)_{1\leq i\leq 7}$ and b. Assuming that $a_5 \neq 0$, then the equilibrium points (x, y, z) are the real solutions of the system

$$\begin{cases} a_1(y - f(x)) = 0 \\ a_2 x + bf + a_3 z - a_4 xz = 0 \\ -a_5 z + a_6 xf - a_7 f = 0 \end{cases} \tag{1.9}$$

that is, $y = f$ and $z = -\left(\frac{fa_7 - fxa_6}{a_5}\right)$ and the equilibria of (1.9) are $(x, y, z) = \left(x, f, -\left(\frac{fa_7 - fxa_6}{a_5}\right)\right)$ with the conditions

$$\begin{aligned} & a_2 x + bf + a_3\left(-\left(\frac{fa_7 - fxa_6}{a_5}\right)\right) - a_4 x\left(-\left(\frac{fa_7 - fxa_6}{a_5}\right)\right) \\ = & \frac{bfa_5 - fa_7 a_3 + xa_5 a_2 - fx^2 a_6 a_4 + fxa_6 a_3 + fxa_7 a_4}{a_5} = 0 \end{aligned}$$

i.e.,

$$bfa_5 - fa_7 a_3 + xa_5 a_2 - fx^2 a_6 a_4 + fxa_6 a_3 + fxa_7 a_4 = 0 \tag{1.10}$$

In this chapter we are interested by some particular shape of the function f and both parameters $(a_i)_{1\leq i\leq 7}$ and b are chosen in a way that the system (1.1) gives some others chaotic attractors. To see some chaotic behavior of the system (1.1) we present various numerical results to show its chaoticity. The nature and the shape of the attractors depend mainly on the $(a_i)_{1\leq i\leq 7}$ and b values as well as the shape of the function f.

1.9 The case where $f(x) = x$

In this case, the choice of the function f_1 gives a smooth autonomous system with three equilibria when $\Delta > 0$ and two equilibria when $\Delta = 0$, and a single equilibrium if $\Delta < 0$ where Δ is the

expression given by:

$$\Delta = (a_3a_6 - a_1a_7)^2 + 4a_1a_5a_6(a_2 + b)$$

then the system in this case gives at most two kinds of attractors: *one scroll chaotic attractor*. (See Fig.1.4) and *two-scroll chaotic attractors*. (See Fig.1.3).

A new chaotic attractor of three-dimensional system is coined by Chen and Ueta [6], in the pursuit of anti controlling chaos for Lorenz model. This new chaotic model reassembles the Lorenz system [8]. However the Chen model appears to be more complex and more sophisticated. They both are three-dimensional autonomous with only two quadratic terms, but it is topologically not equivalent to the Lorenz equation. Many other systems of three smooth autonomous ordinary differential equations with two quadratic nonlinear terms have been found in addition of the Chen system, aiming to generate chaotic attractors. Liu & Chen [9] found another relatively simple three-dimensional smooth autonomous chaotic system with only quadratic nonlinearities. These systems are not topologically equivalent, they have many similar properties, e.g., they are all dissipative systems, have at most three equilibria and two scroll chaotic attractors, possess Hopf bifurcation and pitch bifurcation, etc...

In the case where $f(x) = x$, equation (1.9) takes the form

$$x \times \left(ba_5 + a_2a_5 - a_3a_7 - x^2a_4a_6 + xa_3a_6 + xa_4a_7\right) = 0 \tag{1.11}$$

then $x = 0$ or

$$ba_5 + a_2a_5 - a_3a_7 - x^2a_4a_6 + xa_3a_6 + xa_4a_7 = 0$$

and the solutions are

$$x_1 = -\frac{(ba_5 + a_2a_5 - a_3a_7)}{a_3a_6 + a_4a_7}, \text{ if } -a_4a_6 = 0 \text{ and } a_3a_6 + a_4a_7 \neq 0 \tag{1.12}$$

$$\begin{cases} x_2 = \frac{1}{2a_4a_6}\left(-\sqrt{\Delta} + a_3a_6 + a_4a_7\right) \\ x_3 = \frac{1}{2a_4a_6}\left(\sqrt{\Delta} + a_3a_6 + a_4a_7\right) \end{cases}, \text{ if } -a_4a_6 \neq 0 \tag{1.13}$$

where if $\Delta > 0$ and

$$\Delta = a_3^2a_6^2 + a_4^2a_7^2 + 4a_2a_4a_5a_6 - 2a_3a_4a_6a_7 + 4ba_4a_5a_6 \tag{1.14}$$

Hence the equilibrium points of (1.8) are

$$(x, y, z) = (0, 0, 0), \tag{1.15}$$

$$P(x, y, z) = \left(x, x, -\left(\frac{xa_7 - x^2a_6}{a_5}\right)\right) \tag{1.16}$$

In this case of (1.16), the Jacobian matrix (1.8) at $P(x,y,z)$ is given by

$$J\left(x,x,-\left(\frac{xa_7-x^2a_6}{a_5}\right)\right)=\begin{pmatrix} a_1 & a_1 & 0 \\ a_2-a_4\left(-\left(\frac{xa_7-x^2a_6}{a_5}\right)\right) & b & a_3-a_4x \\ a_6x & a_6x-a_7 & -a_5 \end{pmatrix} \tag{1.17}$$

where in (1.17), $g=f'=1$.

1.9.1 The case where $f(x)=x, a_3=a_7=0, b=-1$

One have the Lorenz system and the equilibrium points $(x,y,z)=(0,0,0)$ exists without any conditions.

1. The equilibrium points $(x,y,z)=\left(x,f,-\left(\frac{fa_7-fxa_6}{a_5}\right)\right)$ cannot exists because $-a_4a_6=0\wedge a_3a_6+a_4a_7=0$.
2. The equilibrium points $(x,y,z)=\left(x_2,x_2,\frac{x_2^2a_6}{a_5}\right)$ exists if $\Delta=4a_4a_5a_6\left(a_2-1\right)>0$ and $a_5\neq 0$.
3. The equilibrium points $(x,y,z)=\left(x_3,x_3,\frac{x_3^2a_6}{a_5}\right)$ exists if $\Delta=4a_4a_5a_6\left(a_2-1\right)>0$ and $a_5\neq 0$.

The Jacobian matrix (1.8) evaluated at $\left(x_2,x_2,\frac{x_2^2a_6}{a_5}\right)$ is given by:

$$J\left(x_2,x_2,\frac{x_2^2a_6}{a_5}\right)=\begin{pmatrix} a_1 & a_1 & 0 \\ \frac{a_5a_2-a_6x_2^2a_4}{a_5} & -1 & -a_4x_2 \\ a_6x_2 & a_6x_2 & -a_5 \end{pmatrix} \tag{1.18}$$

The Jacobian matrix (1.8) evaluated at $\left(x_3,x_3,\frac{x_3^2a_6}{a_5}\right)$ is given by:

$$J\left(x_3,x_3,\frac{x_3^2a_6}{a_5}\right)=\begin{pmatrix} a_1 & a_1 & 0 \\ \frac{a_5a_2-a_6x_3^2a_4}{a_5} & -1 & -a_4x_3 \\ a_6x_3 & a_6x_3 & -a_5 \end{pmatrix} \tag{1.19}$$

The Lorenz system represents a special case of the preceding system and it is given by:

$$\begin{cases} x'=a(y-x) \\ y'=cx-xz-y \\ z'=xy-bz \end{cases} \tag{1.20}$$

and we have

$$a=a_1,a_2=c,a_4=a_6=1,a_5=b, \tag{1.21}$$

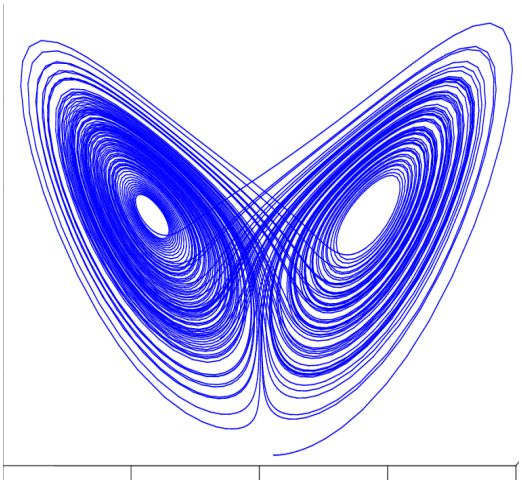


Figure 1.4: The Lorenz chaotic attractor obtained for $a = 10, b = \frac{8}{3}, c = 28$.

For the special case

$$a = 10, b = \frac{8}{3}, c = 28 \tag{1.22}$$

System (1.20) has the chaotic attractor shown in Fig. 1.4. We have

$$\left\{\begin{array}{c} a(y-x)=0 \\ cx-xz-y=0 \\ xy-bz=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} y=x \\ cx-xz-x=0 \\ xy-bz=0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{c} y=x \\ x=0 \text{ or } z=c-1 \\ y=\pm\sqrt{b(c-1)} \end{array}\right.$$

So, the equilibrium point exist if $b \times (c-1) > 0$. In this case the Jacobian matrix (1.20) is given by:

$$J(x,y,z) = \begin{pmatrix} -10 & 10 & 0 \\ 1 & -1 & \pm 6\sqrt{2} \\ \pm 6\sqrt{2} & \pm 6\sqrt{2} & -\frac{8}{3} \end{pmatrix} \tag{1.23}$$

and the eigenvalues of the matrix (1.23) are: $2.960\,5, 1.686\,4 + 3.895\,0i, 1.686\,4 - 3.895\,0i$. So, the two equilibria are unstable.

The Jacobian matrix (1.20) evaluated at $(0,0,0)$ is given by

$$J(0,0,0) = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.24}$$

and the eigenvalues of the matrix (1.24) are: $11.828, -22.828, 0$ and the equilibrium point is unstable.

1.9.2 The case where $f(x) = x, a_3 = a_7 = 0, a_2 = b - a_1$

One has the Chen model.

1. The equilibrium points(1.16) $(x, y, z) = \left(x, x, \frac{x^2 a_6}{a_5}\right)$ cannot exists because $-a_4 a_6 = 0 \wedge a_3 a_6 + a_4 a_7 = 0$.
2. The equilibrium points $(x, y, z) = \left(x_2, x_2, \frac{x^2 a_6}{a_5}\right)$ exists if $\Delta = 4a_4 a_5 a_6 (2b - a_1) > 0$ and $a_5 \neq 0$.
3. The equilibrium points $(x, y, z) = \left(x_2, x_2, \frac{x^2 a_6}{a_5}\right)$ exists if $\Delta = 4a_4 a_5 a_6 (2b - a_1) > 0$ and $a_5 \neq 0$.

The Jacobian matrix (1.3) evaluated at $\left(x_2, x_2, \frac{x_2^2 a_6}{a_5}\right)$ is given by

$$J\left(x_2, x_2, \frac{x_2^2 a_6}{a_5}\right) = \begin{pmatrix} a_1 & a_1 & 0 \\ \frac{a_5(b-a_1) - a_6 x_2^2 a_4}{a_5} & b & -a_4 x_2 \\ a_6 x_2 & a_6 x_2 & -a_5 \end{pmatrix}$$

The Jacobian matrix (1.3) evaluated at $\left(x_3, x_3, \frac{x_3^2 a_6}{a_5}\right)$ is given by

$$J\left(x_3, x_3, \frac{x_3^2 a_6}{a_5}\right) = \begin{pmatrix} a_1 & a_1 & 0 \\ \frac{a_5(b-a_1) - a_6 x_3^2 a_4}{a_5} & b & -a_4 x_3 \\ a_6 x_3 & a_6 x_3 & -a_5 \end{pmatrix}$$

This Chen system

$$\begin{cases} x' = a(y - x) \\ y' = (c - a)x - xz + cy \\ z' = xy - bz \end{cases} \tag{1.25}$$

represents a special case of the preceding system, where $a = a_1, c - a) = a_2, a_4 = a_6 = 1, a_5 = b$. For $a = 35, b = 3, c = 28$, the Chen system has the attractor shown in Fig.1.5.
In this case, we have

$$\begin{cases} a(y - x) = 0 \\ (c - a)x - xz + cy = 0 \\ xy - bz = 0 \end{cases} \Leftrightarrow \begin{cases} y = x \\ z = 2c - a \\ y = \pm\sqrt{b(2c - a)} \end{cases} \tag{1.26}$$

So, the equilibrium point exist if $b \times (2c - 1) > 0$ and the Jacobian matrix of (1.25) is

$$J(x, x, z) = \begin{pmatrix} -35 & 35 & 0 \\ -28 & 28 & \pm 3\sqrt{7} \\ \pm 3\sqrt{7} & \pm 3\sqrt{7} & -3 \end{pmatrix} \tag{1.27}$$

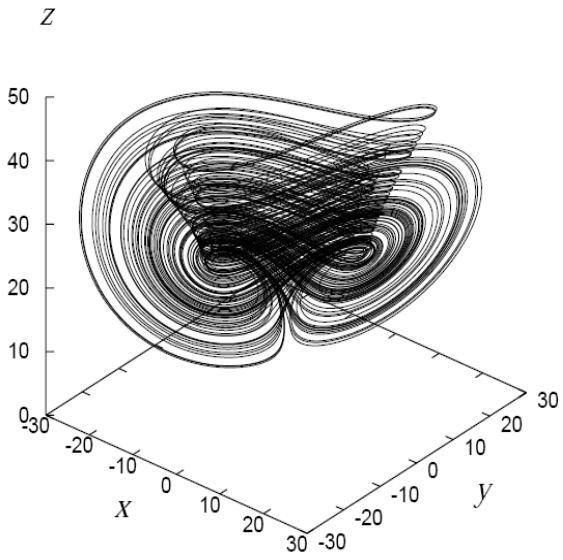


Figure 1.5: The Chen chaotic attractor for $a = 35, b = 3, c = 28$.

The eigenvalues of the matrix (1.27) are: $-18.428, 4.2140 + 29.769i, 4.2140 - 14.885i$ and the equilibrium point is unstable.
The Jacobian matrix (1.27) evaluated at $(0, 0, 0)$ is given by

$$J(0,0,0) = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{1.28}$$

and the eigenvalues of the matrix(1.28) are: $23.836, -30.836, -3$ and the equilbrium point is unstable.

1.9.3 The case where $f(x) = x, a = a_1, a_4 = 1, a_5 = b, a_6 = 1, b = c$

One has the Lü model.

1. The equilibrium points $(x, y, z) = \left(x, x, \frac{x^2 a_6}{a_5}\right)$ cannot exists because $-a_4 a_6 = 0 \wedge a_3 a_6 + a_4 a_7 = 0$.

2. The equilibrium points $(x, y, z) = \left(x_2, x_2, \frac{x^2 a_6}{a_5}\right)$ exists if $\Delta = 4ba_5 a_6 > 0$ and $a_5 \neq 0$.

3. The equilibrium points $(x, y, z) = \left(x_2, x_2, \frac{x^2 a_6}{a_5}\right)$exists if $\Delta = 4ba_5 a_6 > 0$ and $a_5 \neq 0$.

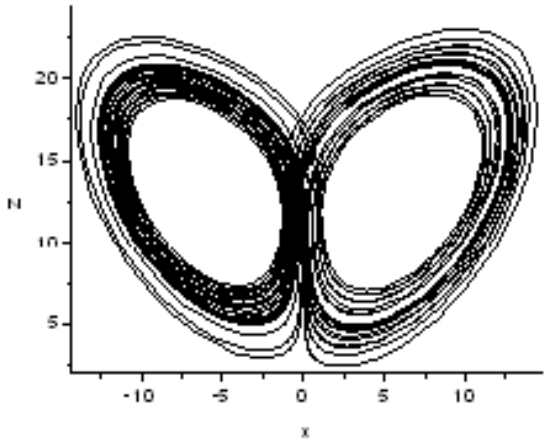


Figure 1.6: The Lü chaotic attractor for $a = 35, b = 3, c = 20$.

The Jacobian matrix (1.3) evaluated at $\left(x_2, x_2, \frac{x_2^2 a_6}{a_5}\right)$ is given by

$$J\left(x_2, x_2, \frac{x_2^2 a_6}{a_5}\right) = \begin{pmatrix} a_1 & a_1 & 0 \\ \frac{-a_6 x_2^2}{a_5} & b & -x_2 \\ a_6 x_2 & a_6 x_2 & -a_5 \end{pmatrix} \tag{1.29}$$

The Jacobian matrix (1.3) evaluated at $\left(x_3, x_3, \frac{x_3^2 a_6}{a_5}\right)$ is given by

$$J\left(x_3, x_3, \frac{x_3^2 a_6}{a_5}\right) = \begin{pmatrix} a_1 & a_1 & 0 \\ \frac{-a_6 x_3^2}{a_5} & b & -x_3 \\ a_6 x_3 & a_6 x_3 & -a_5 \end{pmatrix} \tag{1.30}$$

This example of Lü model represents a special case of the preceding system (1.1) and it is given by

$$\begin{cases} x' = a(y - x) \\ y' = cy - xz \\ z' = -bz + xy \end{cases} \tag{1.31}$$

For chaotic behaviour, parameters are selected as $a = 36, b = 3, c = 20$ and the chaotic attractor is shown in Fig. 1.6.

The equilibrium point exists if $bc > 0$ and the eigenvalues are:$-22.652, 1.8258 + 13.689i, 1.8258 - 13.689i$ and the equilbrium points are unstable.

The Jacobian matrix of (1.3) evaluated at $(0,0,0)$ is

$$J(0,0,0) = \begin{pmatrix} -36 & 36 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & -3 \end{pmatrix} \tag{1.32}$$

So the eigenvalues (1.32) are: $-3, 20, -36$ and the equilibrium point is unstable.

1.10 The case where $f_2(x) = sgn(x), a_1 = 20, a_2 = 5, a_3 = 2, a_4 = 34, a_5 = 5, a_6 = 2, a_7 = 6, b = 13$

In these two cases the model gives a *scroll-type chaotic attractor* similar to the one given in Fig. 1.3.

$$f_2(x) = sgn(x) = \begin{cases} 1, x \geq 0 \\ -1, x < 0 \end{cases} \tag{1.33}$$

(1) For $x \geq 0$, then $f_2(x) = 1$ and the system becomes

$$\begin{cases} \frac{dx}{dt} = 20(y-1) \\ \frac{dy}{dt} = 5x + 13y + 2z - 34xz, \\ \frac{dz}{dt} = 5z + 2xy - 6y, \end{cases} \tag{1.34}$$

The equilibrium point are the solutions of the system (1.34)

$$\begin{cases} 20(y-1) = 0 \rightarrow y = 1 \\ 5x + 13y + 2z - 34xz = 0 \\ 5z + 2xy - 6y = 0 \end{cases} \tag{1.35}$$

That is, by (1.35) $y = 1, x = \left(\frac{-13-2z}{5-34z}\right)$ and

$$-225z + 170z^2 + 56 = 0 \Leftrightarrow z = \begin{cases} \frac{1}{340}\sqrt{5}\sqrt{2509} + \frac{45}{68} = 0.991\,19 \\ \frac{45}{68} - \frac{1}{340}\sqrt{5}\sqrt{2509} = 0.332\,34 \end{cases} \tag{1.36}$$

Hence,

$$x = \begin{cases} 0.522\,03, \text{ if } z = 0.99119 \\ 2.\,169\,1, \text{ if } z = 0.33234 \end{cases} \tag{1.37}$$

From (1.36)-(1.37), the equilibrium points are

$$(x, y, z) = (0.52203, 1, 0.99119) \text{ and } (x, y, z) = (2.\,169\,1, 1, 0.33234)$$

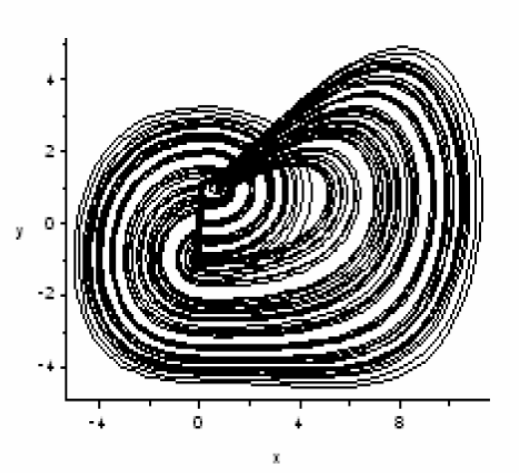


Figure 1.7: A chaotic attractor for $a_1 = 20, a_2 = 5, a_3 = 2, a_4 = 34, a_5 = 5, a_6 = 2, a_7 = 6, b = 13$.

In this case the Jacobian matrix is given by

$$J\left(0.52203, 1, 0.99119\right) = \begin{pmatrix} 0 & 20 & 0 \\ -28.70046 & 13 & -15.74902 \\ 2 & -4.95594 & 5 \end{pmatrix} \tag{1.38}$$

with eigenvalues: $6.761\,8 + 21.324i, 6.761\,8 - 21.324i$ and $4.476\,4$. By the same method, the eigenvalues of the Jacobian matrix (1.35) of $(x, y, z) = (2.1691, 1, 0.33234)$ are: $13.002 + 10.528\,i, 13.002 - 10.528i$ and $-8.003\,5$ and both points are unstable.

(2) For $x < 0$, we have $f_2(x) = 1$ and the equilibrium point are $(x, y, z) = (-0.26377, -1, 1.3055)$. In this case the Jacobian matrix of (1.35) is given by

$$J\left(-0.26377, -1, 1.3055\right) = \begin{pmatrix} 0 & 20 & 0 \\ -39.387 & 13 & 10.96818 \\ -2 & -6.52754 & 5 \end{pmatrix} \tag{1.39}$$

with eigenvalues: $6.983\,9 + 28.622i, 6.983\,9 - 28.622i$ and $4.032\,2$. Overally, the system has three equlibrium points and they all unstable. The chaotic attractor is shown in Fig.1.7

Chapter 2

On some bounded 4-D chaotic attractors

Bounded chaotic attractors and the estimate of their bounds is important in chaos control, chaos synchronization, and their applications. Such an estimation is quite difficult to achieve technically, however. Several works on this topic were realized for some 3-D quadratic continuous-time systems [2; and references therein].
In recent years, the study of hyperchaotic dynamical systems have attracted the attention of many engineers, physicists and mathematicians. A large amount of literature developed concerning the bounded of solutions of dynamical systems of three and four dimensions. In the present chapter, we study the boundedness of solution for a constructed generalized four-dimension hyperchaotic system using a technique combining the generalized Lyapunov function theory and optimization. The results obtained in this chapter are the generalizations of some results given in [2-4-5-10-12].

2.1 Estimating the bounds for the generalized 4-D hyperchaotic system

Let us consider the following system

$$X' = f(X) \tag{2.1}$$

where $X \in \mathbb{R}^4, f : \mathbb{R}^4 \to \mathbb{R}^4, X = (x, y, z, w), t_0 \geq 0$ is the initial time, and $X(t, t_0, X_0)$ is a solution to system (2.1) satisfying $X(t, t_0, X_0) = X_0$ which for simplicity is denoted as $X(t)$. Assume $\Omega \in \mathbb{R}^4$ is a compact set. Define the distance between the solution $X(t, t_0, X_0)$ and the set Ω by $\rho(X(t, t_0, X_0), \Omega) = \inf_{Y \in \Omega} \|X(t, t_0, X_0) - Y\|$, and denote

$$\Omega_\varepsilon = \{X - \rho(X, \Omega) < 0\}$$

Clearly, we have $\Omega \subset \Omega_\varepsilon$.

Definition 2.1 *Suppose that there is a compact set $\Omega \in \mathbb{R}^4$. If, for every $X_0 \in \mathbb{R}^4/X$, we have $\lim_{T\to\infty}\rho(X(t),\Omega) = 0$, that is, for any $\varepsilon > 0$, there is a $T > t_0$ such that for $t \geq T$, $X(t,t_0,X_0) \subset \Omega_\varepsilon$, then the set Ω is called an ultimate bound for system (2.1). If, for any $X_0 \in \Omega$ and all $t \geq t_0$, $X(t,t_0,X_0) \subset \Omega$, then the set Ω is called the positively invariant set for system (2.1).*

In this section, we consider the following *generalized 4-D hyperchaotic system*:

$$\begin{cases} x' = (5\alpha+1)y - (5\alpha+1)x + \frac{15}{10}(1-\alpha)w \\ y' = (65\alpha+26)x - y - \frac{25}{10}\alpha xz + 6\alpha\frac{z^2}{x} - \alpha xw + (\alpha-1)xz \\ z' = \frac{-15}{10}\alpha x + \frac{25}{10}\alpha y - (\frac{3\alpha+7}{10})z - 6\alpha\frac{yz}{x} + (1-\alpha)xy \\ w' = -(\frac{2\alpha}{10}+1)w + \alpha xy + (\alpha-1)x \end{cases} \tag{2.2}$$

This system is chaotic for all $0 \leq \alpha \leq 1$; with $\alpha = 0$ it reduces to the original Lorenz-Stenflo system [11] and with $\alpha = 1$ it is the original Lorenz-Haken system [10].
The bounds of system (2.2) are obtained based on *multivariate function analysis* concerned with locating maxima and minima. In particular, we find large regions in the bifurcation parameter space where system (2.2) is bounded. We have proved the following result:

Theorem 2.1 *When $0 < \alpha \leq 1$, the chaotic system (2.2) is contained in the sphere*

$$\Omega = \left\{(x,y,z,w) : \frac{10}{15}x^2 + y^2 + z^2 + (w - (\frac{205\alpha+80}{3\alpha}))^2 \leq l\right\} \tag{2.3}$$

Proof. Let

$$V = \frac{10}{15}x^2 + y^2 + z^2 + (w - (\frac{205\alpha+80}{3\alpha}))^2 \tag{2.4}$$

Then , the time derivative along the orbits of system (2.2) is given by

$$\begin{aligned} \frac{1}{2}\frac{dV}{dt} &= \frac{10}{15}xx' + yy' + zz' + (w - (\frac{205\alpha+80}{3\alpha}))w' \\ &= \frac{10}{15}(5\alpha+1)xy - \frac{10}{15}(5\alpha+1)x^2 + (1-\alpha)xw \\ &\quad +(65\alpha+26)xy - y^2 - \frac{25}{10}\alpha yz + 6\alpha\frac{yz^2}{x} \\ &\quad -\alpha xyw + (\alpha-1)xyz - \frac{15}{10}\alpha xz + \frac{25}{10}\alpha yz \\ &\quad -(\frac{3\alpha+7}{10})z^2 - 6\alpha\frac{yz^2}{x} + (1-\alpha)xyz - (\frac{2\alpha}{10}+1)w^2 \\ &\quad +\alpha xyw + (\alpha-1)xw + (\frac{205\alpha+80}{3\alpha})(\frac{2\alpha}{10}+1)w \\ &\quad -\alpha(\frac{205\alpha+80}{3\alpha})xy - (\frac{205\alpha+80}{3\alpha})(\alpha-1)x \end{aligned}$$

that is,

$$\begin{aligned}
\frac{1}{2}\frac{dV}{dt} &= -\frac{10}{15}(5\alpha+1)x^2 - y^2 - (\frac{3\alpha+7}{10})z^2 - \frac{15}{10}\alpha xz \\
&\quad -(\frac{3\alpha+7}{10})z^2 \ -(\frac{2\alpha}{10}+1)w^2 + (\frac{205\alpha+80}{3\alpha})(\frac{2\alpha}{10}+1)w \\
&\quad +(\frac{205\alpha+80}{3\alpha})(1-\alpha)x \\
&= -\left[\sqrt{\frac{10}{15}(5\alpha+1)}x - \frac{1}{2}\sqrt{\frac{15}{10(5\alpha+1)}}(\frac{205\alpha+80}{3\alpha})(1-\alpha)\right]^2 \\
&\quad +\frac{1}{4}\frac{15}{10(5\alpha+1)}(\frac{205\alpha+80}{3\alpha})^2(1-\alpha)^2 - y^2 - (\frac{3\alpha+7}{10})z^2 \\
&\quad -\frac{15}{10}\alpha xz - (\frac{2\alpha}{10}+1)\left[w - \frac{1}{2}(\frac{205\alpha+80}{3\alpha})\right]^2 + \\
&\quad \frac{1}{4}(\frac{2\alpha}{10}+1)(\frac{205\alpha+80}{3\alpha})^2 \\
&= -\left[\sqrt{\frac{10}{15}(5\alpha+1)}x - \frac{1}{2}\sqrt{\frac{15}{10(5\alpha+1)}}(\frac{205\alpha+80}{3\alpha})(1-\alpha)\right]^2 \\
&\quad -y^2 - (\frac{3\alpha+7}{10})z^2 - \frac{15}{10}\alpha xz - (\frac{2\alpha}{10}+1)\left[w - \frac{1}{2}(\frac{205\alpha+80}{3\alpha})\right]^2 \\
&\quad +\frac{1}{4}(\frac{205\alpha+80}{3\alpha})^2(\frac{25\alpha^2+22\alpha+25}{10(5\alpha+1)})
\end{aligned}$$

Put

$$\begin{cases} \check{s} = \sqrt{\frac{10}{15}(5\alpha+1)}x - \frac{1}{2}\sqrt{\frac{15}{10(5\alpha+1)}}(\frac{205\alpha+80}{3\alpha})(1-\alpha) \\ \hat{w} = w - \frac{1}{2}(\frac{205\alpha+80}{3\alpha}) \end{cases}$$

Then

$$\begin{aligned}
\frac{1}{2}\frac{dV}{dt} = -\check{s}^2 \quad & -y^2 - (\frac{3\alpha+7}{10})z^2 - \frac{15}{10}\alpha xz - (\frac{2\alpha}{10}+1)\hat{w}^2 \\
& +\frac{1}{4}(\frac{205\alpha+80}{3\alpha})^2(\frac{25\alpha^2+22\alpha+25}{10(5\alpha+1)})
\end{aligned}$$

For $V' = 0$, we have the following four-dimensional sphere Γ :

$$\check{s}^2 + y^2 + (\frac{3\alpha+7}{10})z^2 + \frac{15}{10}\alpha xz + (\frac{2\alpha}{10}+1)\hat{w}^2 = \frac{1}{4}(\frac{205\alpha+80}{3\alpha})^2(\frac{25\alpha^2+22\alpha+25}{10(5\alpha+1)}) \tag{2.5}$$

The corresponding matrix of the left-hand side quadratic from of Eq. (2.5) is given by

$$A = \begin{bmatrix} 1 & 0 & \frac{15}{10} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{15}{10} & 0 & (\frac{3\alpha+7}{10}) & 0 \\ 0 & 0 & 0 & (\frac{2\alpha}{10}+1) \end{bmatrix}$$

with positive eigenvalues given by:

$$\begin{cases} 1, \frac{17}{20} + \frac{3}{20}\alpha + \frac{3}{20}\sqrt{(26 - 2\alpha + \alpha^2)} \\ \frac{17}{20} + \frac{3}{20}\alpha - \frac{3}{20}\sqrt{(26 - 2\alpha + \alpha^2)}, \frac{1}{5}\alpha + 1. \end{cases}$$

since if $\frac{17}{20} + \frac{3}{20}\alpha - \frac{3}{20}\sqrt{(26 - 2\alpha + \alpha^2)} = 0$, only for $\alpha = -0.4583333$, hence it is positive for all $0 < \alpha \leq 1$. Hence, for all $0 < \alpha \leq 1$, A is positive definite matrix. Thus, Eq. (2.5) is a four-dimensional ellipsoidal surface. Outside Γ, that is, when

$$\begin{cases} \check{s}_1^2 + y^2 + (\frac{3\alpha+7}{10})z^2 + \frac{15}{10}\alpha xz + (\frac{2\alpha}{10} + 1)\hat{w}^2 > \frac{1}{4}(\frac{205\alpha+80}{3\alpha})^2(\frac{25\alpha^2+22\alpha+25}{10(5\alpha+1)}) \\ \check{s}_1 = \frac{10}{15}\sqrt{(5\alpha+1)}x + \frac{1}{2}\sqrt{\frac{15}{10(5\alpha+1)}}(\frac{205\alpha+80}{3\alpha})(1-\alpha) \end{cases}$$

we have $V' < 0$, which implies that the surface $V = const$ is monotonously decreasing. Since Γ is a closed set and V is continuous on Γ, the extreme values of V can be attained on Γ. Denote the maximum value of V as l, that is $l = \max V_{(x,y,z,w)\in\Gamma}$. Namely, if we denote $\sqrt{\frac{15}{10}}x = \check{s}$, Eq (2.4) becomes

$$V = \check{s}^2 + y^2 + z^2 + (w - (\frac{205\alpha + 80}{3\alpha}))^2 \tag{2.6}$$

and the ellipsoid surface Γ becomes Γ' as follows:

$$\check{s}_1^2 + y^2 + (\frac{3\alpha + 7}{10})z^2 + \frac{15}{10}\alpha xz + (\frac{2\alpha}{10} + 1)\hat{w}^2 = \frac{1}{4}(\frac{205\alpha + 80}{3\alpha})^2(\frac{25\alpha^2 + 22\alpha + 25}{10(5\alpha + 1)}) \tag{2.7}$$

Thus, l is the maximum value of the square of the distance from the points on Γ' to point $\left(0, 0, 0, \frac{205\alpha+80}{3\alpha}\right)$, which obviously exists. So, it is easy to compute it when the parameters in system (2.2) are given. For the set Ω given by (2.3), we have $\Gamma \subset \Omega$. Next, we will show that:

$$\lim_{t\to\infty} \rho(X(t), \Omega) = 0 \tag{2.8}$$

using the proof by absurd, where $X(t) = (x(t), y(t), z(t), w(t))$. If (2.8) does not hold, we can conclude that the orbits of system (2.2) are outside the set Ω, thus $V' < 0$. Therefore, the function $V(X(t))$ monotonously decreases outside the set Ω, which leads to the following result

$$\lim_{t\to\infty} V(X(t)) = V^* > l \tag{2.9}$$

■

2.2 Estimating the bounds for the chaotic Lorenz-Haken system

For $\alpha = 1$, we find the *chaotic Lorenz-Haken system* given by

$$\begin{cases} x' = -6x + 6y \\ y' = 91x - y - \frac{25}{10}z + 6\frac{z^2}{x} - xw \\ z' = -\frac{15}{10}x + \frac{25}{10}y - z - 6\frac{z}{x} \\ w' = -\frac{12}{10}w + xy \end{cases} \tag{2.10}$$

Define the following Lyapunov function

$$V = \frac{10}{15}x^2 + y^2 + z^2 + (w - 95)^2 \tag{2.11}$$

and the four-dimension ellipsoidal surface

$$\Gamma : (2x)^2 + y^2 + z^2 + \frac{15}{10}xz + \frac{12}{10}(w - \frac{95}{2})^2 = (\frac{95}{2})^2(\frac{6}{5}) = \frac{5415}{2} \tag{2.12}$$

to get

$$l = \max_{(x,y,z,w)\in\Gamma} \left[\frac{10}{15}x^2 + y^2 + z^2 + (w - 95)^2\right] \tag{2.13}$$

Let

$$\begin{cases} F = \frac{10x^2}{15} + y^2 + z^2 + (w - 95)^2 + \lambda g\,(x, y, z, w) \\ g\,(x, y, z, w) = (2x)^2 + y^2 + z^2 + \frac{15xz}{10} + \frac{12(w-\frac{95}{2})^2}{10} - \frac{5415}{2} \end{cases} \tag{2.14}$$

Thus, we have

$$\begin{cases} F_x = \frac{20}{15}x + 4\lambda x + \frac{15}{10}z = 0 \\ F_y = 2y + 2\lambda y = 0 \\ F_z = 2z + 2\lambda z + \frac{15}{10}x = 0 \\ F_w = 2(w - 95) + \frac{24}{10}\lambda(w - \frac{95}{2}) = 0 \\ (2x)^2 + y^2 + z^2 + \frac{15}{10}xz + \frac{12}{10}(w - \frac{95}{2})^2 = \frac{5415}{2} \end{cases} \tag{2.15}$$

We have

$$(x, y, z, w, \lambda) = (0, 0, 0, 95, 0) \text{ or } (x, y, z, w, \lambda) = \left(0, 0, 0, \frac{-190}{114}\right) \tag{2.16}$$

Then

$$l = \max_{(x,y,z,w)\in\Gamma} \left[\frac{10}{15}x^2 + y^2 + z^2 + (w - 95)^2\right] = 95 \tag{2.17}$$

Thus, the chaotic Lorenz-Haken system is contained in the the ellipsoid surface

$$\Omega = \left\{(x, y, z, w) : \frac{10}{15}x^2 + y^2 + z^2 + (w - 95)^2 \leq 95\right\} \tag{2.18}$$

Finally, we have proved the following result:

Theorem 2.2 *The Lorenz-Haken system (2.10) is contained in part of the 4-D ellipsoid defined by* $\Omega = \left\{(x, y, z, w)\ \frac{10}{15}x^2 + y^2 + z^2 + (w - 95)^2 \leq 95\right\}$.

2.3 Estimating the bounds for the Lorenz–Stenflo system

In this section, we find upper and lower bounds for the *Lorenz-Stenflo system* [12] given by

$$\begin{cases} x' = -\sigma(x - y) + sw \\ y' = -xz + rx - y \\ z' = xy - bz \\ w' = -x - \sigma w \end{cases} \tag{2.19}$$

In particular, we find large regions in the bifurcation parameter space $(\sigma, r, b, s) \in \mathbb{R}^4$ where system (2.19) is bounded. To estimate the bound for the Lorenz-Stenflo system (2.19), we consider the Lyapunov function $V(x, y, z, w)$ defined by

$$V(x, y, z, w) = \frac{\frac{1}{s}x^2 + y^2 + \left(z - \left(r + \frac{\sigma}{s}\right)\right)^2 + w^2}{2}$$

The derivative of V along the solutions of (2.19) is given by

$$\frac{dV}{dt} = -\frac{\sigma}{s}x^2 - y^2 - b\left(z - \frac{\sigma + rs}{2s}\right)^2 - \sigma w^2 + \frac{b\left(r + \frac{\sigma}{s}\right)^2}{4}$$

Let

$$H(x, y, z, w) = \frac{x^2}{\frac{b(\sigma+rs)^2}{4\sigma s}} + \frac{y^2}{\frac{b\left(r+\frac{\sigma}{s}\right)^2}{4}} + \frac{\left(z - \frac{\sigma+rs}{2s}\right)^2}{\frac{(\sigma+rs)^2}{4s^2}} + \frac{w^2}{\frac{b(\sigma+rs)^2}{4\sigma s^2}} - 1$$

Thus to prove the boundedness of system (2.19), we assume that it is bounded, and then we find its bound, i.e., assume that σ, s, and b are strictly positive and $r \geq 0$. Then if system (2.19) is bounded, the function $\frac{dV}{dt}(x, y, z, w)$ has a maximum value, and the maximum point (x_0, y_0, z_0, w_0) satisfies $H(x_0, y_0, z_0, w_0) = 0$. Now consider the 4-D ellipsoid defined by

$$\Gamma = \left\{(x, y, z, w) \in \mathbb{R}^4 : H(x, y, z, w) = 0, \sigma > 0, s > 0, b > 0, r \geq 0\right\}$$

and define the function

$$F(x, y, z, w) = G(x, y, z, w) + \lambda H(x, y, z, w)$$

where $G(x,y,z,w) = x^2+y^2+z^2+w^2$ and $\lambda \in \mathbb{R}$ is a finite parameter. We have $\max_{(x,y,z,w)\in\Gamma} G = \max_{(x,y,z,w)\in\Gamma} V$ and

$$\begin{cases} \frac{\partial F}{\partial x} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}x \\ \frac{\partial F}{\partial y} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2+b\sigma^2\right)}{b(\sigma+rs)^2}y \\ \frac{\partial F}{\partial z} = \frac{2\left(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2\right)}{(\sigma+rs)^2}z - \frac{2\left(2r\lambda s^2+2\sigma\lambda s\right)}{(\sigma+rs)^2} \\ \frac{\partial F}{\partial w} = \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}w \end{cases}$$

In this case, the Hessian matrix of the function F is diagonal with the elements (eigenvalues)

$$\begin{cases} \frac{2\left(br^2s^2+2brs\sigma+4\lambda s\sigma+b\sigma^2\right)}{b(\sigma+rs)^2}, \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2+b\sigma^2\right)}{b(\sigma+rs)^2} \\ \frac{2\left(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2\right)}{(\sigma+rs)^2}, \frac{2\left(br^2s^2+2brs\sigma+4\lambda s^2\sigma+b\sigma^2\right)}{b(\sigma+rs)^2} \end{cases}$$

Thus the scalar function F has a maximum point if all eigenvalues of the corresponding Hessian matrix are strictly negative, that is,

$$\lambda < \min\left(\frac{-b(\sigma+rs)^2}{4s\sigma}, \frac{-b(\sigma+rs)^2}{4s^2}, \frac{-(\sigma+rs)^2}{4s^2}, \frac{-b(\sigma+rs)^2}{4s^2\sigma}\right)$$

If $s \geq 1$, $0 < \sigma \leq s$, and $0 < \sigma < bs$, we have

$$\begin{cases} \frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-b(\sigma+rs)^2}{4s^2}\right) = \frac{-b(s-\sigma)(\sigma+rs)^2}{4s^2\sigma} \leq 0 \\ \frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-(\sigma+rs)^2}{4s^2}\right) = \frac{-(\sigma+rs)^2(-\sigma+bs)}{4s^2\sigma} \leq 0 \\ \frac{-b(\sigma+rs)^2}{4s\sigma} - \left(\frac{-b(\sigma+rs)^2}{4s^2\sigma}\right) = \frac{-b(s-1)(\sigma+rs)^2}{4s^2\sigma} \leq 0 \end{cases}$$

Thus $\lambda < \frac{-b(\sigma+rs)^2}{4s\sigma}$. Then the only critical point of F is $x_0 = 0, y_0 = 0, z_0 = \frac{2s(\sigma+rs)\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}$, and $w_0 = 0$, and hence

$$\max_{(x,y,z,w)\in\Gamma} G = \left(\frac{2s\sigma\lambda+2rs^2\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}\right)^2 = f(\lambda)$$

In this case, there exists a parameterized family (in λ) of bounds of system (2.19). We remark that for different values of λ, one can get different estimates for system (2.19). Some calculations lead to $f'(\lambda) = \frac{(\sigma+rs)^4 8s^2\lambda}{(r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2)^3}$. We have $r^2s^2+2rs\sigma+4\lambda s^2+\sigma^2 < 0$ for all $-\infty < \lambda < \frac{-b(\sigma+rs)^2}{4s\sigma}$, and hence $f'(\lambda) > 0$, which means that $f(\lambda)$ is an increasing function, that is,

$$\begin{aligned} \lim_{\lambda\to-\infty}\left(\frac{2s\sigma\lambda+2rs^2\lambda}{\sigma^2+4s^2\lambda+r^2s^2+2rs\sigma}\right)^2 &= Q^2 \\ &= \frac{(2rs^2+2\sigma s)^2}{16s^4} \\ &< f(\lambda) < \frac{1}{4}b^2\frac{(\sigma+rs)^2}{(-\sigma+bs)^2} = R^2 \end{aligned}$$

Finally, we have $\max_{(x,y,z,w)\in\Gamma}\left(x^2+y^2+z^2+w^2\right) < \frac{b^2(\sigma+rs)^2}{4(-\sigma+bs)^2} = R^2$, which is the upper bound for the Lorenz–Stenflo system (2.19). For the other values of $(\sigma, r, b, s) \in \mathbb{R}^4$, the same logic applies. Finally, we have proved the following result:

Theorem 2.3 *The Lorenz-Stenflo system (2.19) is contained in part of the 4-D ellipsoid defined by* $\Omega = \left\{(x,y,z,w)\in\mathbb{R}^4 : Q^2 < \frac{\frac{1}{s}x^2+y^2+\left(z-\left(r+\frac{\sigma}{s}\right)\right)^2+w^2}{2} \leq R^2\right\}$ *for all* $r \geq 0, b > 0,\ s \geq 1, 0 < \sigma \leq s, \sigma < bs$, *and all initial conditions, where* $Q^2 = \frac{\left(2rs^2+2\sigma s\right)^2}{16s^4}$ *and* $R^2 = \frac{b^2(\sigma+rs)^2}{4(bs-\sigma)^2}$.

We remark that if $\sigma \to bs$, then the upper bound converges to infinity. The volume of the resulting set in $\mathbb{R}^4$ is

$$\frac{1}{4}b^2\frac{(\sigma+rs)^2}{(-\sigma+bs)^2} - \left(\frac{\left(2rs^2+2\sigma s\right)^2}{16s^4}\right) = \frac{\sigma\left(\sigma+rs\right)^2(2bs-\sigma)}{4s^2\left(bs-\sigma\right)^2} > 0$$

since $bs > \sigma$.

Chapter 3

About three scroll chaotic attractors

In this chapter we demonstrated numerically the existence of a *3-scroll chaotic attractor* with three equilibria obtained from direct modification of Chua's equation. This system has some properties similar to the original Chua's system and other well-known modified Chua's systems. Some detailled analysis of the dynamics of this system was also presented.
Chaos is the idea that a system will produce very different long-term behaviors when the initial conditions are perturbed only slightly. Piecewise linear chaotic systems are used to construct simple electronic circuits with applications in electronic engineering, especially in secure communications. Circuits also provide a very useful vehicle for studying chaos as well as easy applications. Chua's circuit [7] is one of the most interesting examples of chaos in circuit theory. This circuit is structurally the simplest and dynamically the most complex member of the Chua's circuit family. Chua's circuit is given by the following closed-form dimensionless equations:

$$\begin{cases} x' = \alpha(y - f(x)), \\ y' = x - y + z, \\ z' = -\beta y, \end{cases} \tag{3.1}$$

where α and β are constant parameters and

$$f(x) = m_1 x + \frac{1}{2}(m_0 - m_1)\left(|x+1| - |x-1|\right) \tag{3.2}$$

is the characteristic function of system (1-2), with m_0, m_1 the slopes of the outer and the inner regions. System (3.1)-(3.2) gives a chaotic attractor called the *double-scroll attractor*. One of the basic properties of multi-scroll attractors is that they surround a large number of equilibrium points. In this chapter, we report the finding of a *three-scroll chaotic attractor* but with only three equilibria rather than more as in the usual case where this attractor requires at least five equilibria. In addition, it is shown numerically that the system (3.1)-(3.2) can generate the classical *one-scroll* and *two-scroll chaotic attractors*.

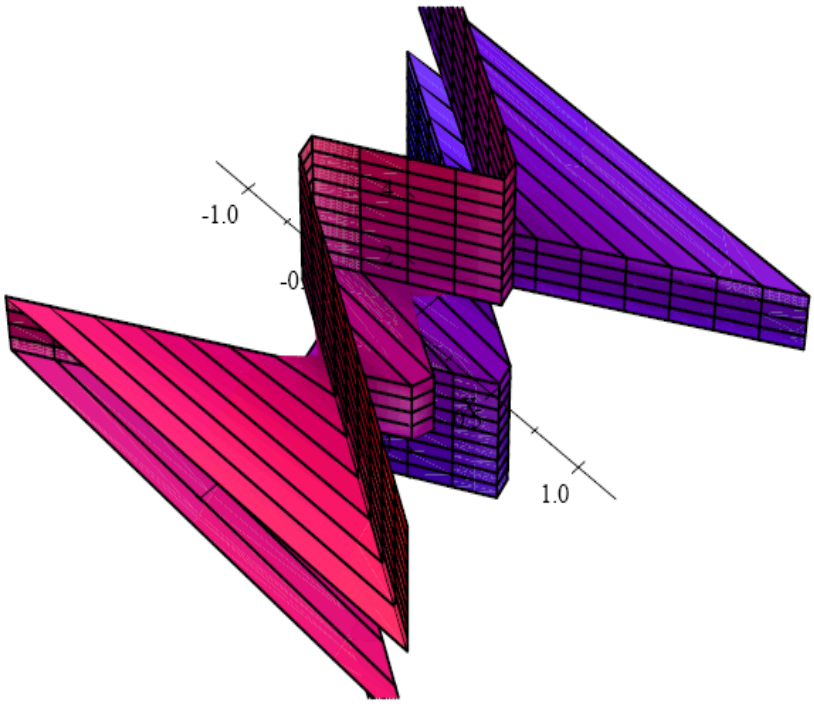


Figure 3.1: A 3-D graph of equation (3) for $m_0 = -0.43, m_1 = 0.41$ and $a = 1.0$.

3.1 The proposed model for three scroll chaotic attractors

In this chapter, a piecewise-linear version of Chua's circuit (3.1)-(3.2) is proposed in [3] where we replace the characteristic function (3.2) by a piecewise-linear function (in the x, z variables) given by:

$$h_a(x,z) = \begin{cases} f(x), |z| \geq a, \\ -f(x), |z| \leq a, \end{cases} \tag{3.3}$$

where a is a positive parameter. The proposed function has the advantage that is has two variables x and z and it has not been previously proposed or studied. Note that $h_a(x,z)$ is an *odd-symmetric discontinuous function* (except for $a = 0$ or $a = \frac{m_1 - m_0}{m_1}$). A 3-D graph of equation (3.3) is shown in Fig. 3.1. Hence, the system is given by:

$$\begin{cases} x' = \alpha\left(y - h_a\left(x,z\right)\right) \\ y' = x - y + z \\ z' = -\beta y. \end{cases} \tag{3.4}$$

Suppose that $m_0 < 0$ and $m_1 > 0$. Then one has a new continuous-time, three-dimensional, autonomous, piecewise-linear system. We have discovered that the model (3.3)-(3.4) generates

various chaotic attractors including the classical 1-scroll, double-scroll, and three-scroll chaotic attractors. See [14] for some examples about generation of n-double scrolls ($n = 1, 2, 3, 4$,...). The equation (3.3)-(3.4) is piecewise-linear, which simplifies its circuitry realization. This fact is very important because many works have been devoted to building simple electronic chaotic circuits with piecewise-linear functions. This situation is quite interesting both theoretically and practically in terms of new chaos generation techniques and possible engineering applications of chaos.

3.2 Some basic properties

We remark that $\lim_{a \longrightarrow 0} h_a(x, z) = f(x)$, and so it is possible to conclude that the dynamics of system (3.3)-(3.4) is very close to the original Chua's circuit (3.1)-(3.2) when the bifurcation parameter a is close to zero. This result is confirmed numerically in Section 3.4. For $a < 0$, one has $h_a(x, z) = f(x)$, and so systems (3.1)-(3.2) and (3.3)(-(3.4) are identical. Notably, the system (3.3)-(3.4) is not diffeomorphic with the Chua's system (3.1)-(3.2) (the two systems are not topologically equivalent) since the system (3.3)-(3.4) has nine linear regions with respect to x and z, but the latter has only three regions with respect to the x variable. Dynamically, it is clear that the two systems do not have the same sequence of bifurcations as will be shown in Section 3.4.

The chaotic system (3.3)-(3.4) has several important properties in common with Chua's attractor, but there are some differences between them. It has a natural symmetry under the coordinate transformation $(x, y, z) \longrightarrow (-x, -y, -z)$, which persists for all values of the system parameters.

Proposition 3.1 *If* $\alpha > \max\left(\frac{1}{m_0}, -\frac{1}{m_1}\right)$, *then the system (3.3)-(3.4) has a bounded, globally attracting* ω*-limit set.*

Proof. The divergence of the flow for system (3.3)-(3.4) is given by:

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = \begin{cases} -\alpha m - 1, |z| \geq a \\ \alpha m - 1, |z| \leq a, \end{cases} \tag{3.5}$$

where $m = \begin{cases} m_1, |x| \geq 1 \\ m_0, |x| \leq 1 \end{cases}$. The function $h(x)$ can be rewritten as

$$h(x) = \begin{cases} m_1 x + (m_0 - m_1) & x \geq 1 \\ m_0 x & |x| \leq 1 \\ m_1 x + (m_0 - m_1) & x \leq 1 \end{cases}$$

(1) If $|z| \geq a$, then system (3.5) takes the form

$$\begin{cases} x' = \alpha(y - h(x)) \\ y' = x - y + z \\ z' = -\beta y \end{cases}$$

If $|z| \geq a$ and $x \geq 1$, then $\frac{dx'}{dx} = -\alpha m_1, \frac{dy'}{dy} = -1, \frac{dz'}{dz} = 0$. If $|z| \geq a$ and $|x| \leq 1$, then $\frac{dx'}{dx} = -\alpha m_0, \frac{dy'}{dy} = -1, \frac{dz'}{dz} = 0$. If $|z| \geq a$ and $x \leq -1$, then $\frac{dx'}{dx} = -\alpha m_1, \frac{dy'}{dx} = -1, \frac{dz'}{dx} = 0$.
(2) If $|z| \leq a$, then system (3.5) takes the form

$$\begin{cases} x' = \alpha(y + h(x)) \\ y' = x - y + z \\ z' = -\beta y \end{cases}$$

If $|z| \leq a$ and $x \geq 1$, then $\nabla V = \alpha m_1 - 1$. If $|z| \leq a$ and $|x| \leq 1$, then $\nabla V = \alpha m_0 - 1$. If $|z| \leq a$ and $x \leq 1$, then $\nabla V = \alpha m_1 - 1$.
So, if $|z| \geq a$ and $x \geq 1$ or $|z| \geq a$ and $x \leq 1$, or $|z| \leq a$ and $x \geq 1$, or $|z| \leq a$ and $x \leq 1$ then we get

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = -\alpha m_1 - 1$$

If $|z| \geq a$ and $|x| \leq 1$ or $|z| \leq a$ and $|x| \leq 1$

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = -\alpha m_0 - 1$$

So, the final result

$$\nabla V = \frac{\partial x'}{\partial x} + \frac{\partial y'}{\partial y} + \frac{\partial z'}{\partial z} = \begin{cases} -\alpha m - 1 & |z| \geq a \\ \alpha m - 1 & |z| \leq a \end{cases}$$

where $m = \begin{cases} m_1, & |x| \geq 1 \\ m_0, & |x| \leq 1 \end{cases}$. Thus if $\alpha > \max\left(\frac{1}{m_0}, -\frac{1}{m_1}\right)$, then the system (3.3)-(3.4) has a bounded, globally attracting ω-limit set, and it is dissipative just like Chua's system (3.1)-(3.2). ■

Proposition 3.2 *The asymptotic motion converges onto an attractor.*

Proof. The variation of the volume $V(t)$ of a small element $\delta\Omega(t) = \delta x \delta y \delta z$ is determined by the divergence given in (3.5), and in this case the exponential contraction rate is

$$\delta\Omega(t) = \begin{cases} \exp(-\alpha m - 1), & |z| \geq a \\ \exp(\alpha m - 1), & |z| \leq a \end{cases}$$

Hence, a volume element V_0 is contracted in time t by the flow into a volume element $V_0 \exp((\nabla V)t)$. Then each volume containing the system trajectory shrinks to zero as $t \longrightarrow +\infty$. Thus all system orbits will be confined to a specific subset having zero volume, and the asymptotic motion converges onto an attractor. This result has been confirmed by numerical simulations in Section 3.4.

■

3.3 Equilibrium points and their stability

In this section, we suppose that $\alpha > 0$ and $\beta > 0$. Due to the shape of the vector field of system (3.3)-(3.4), the phase space can be divided into two principal linear regions denoted by R_1 and R_2:

$$R_1 = \{(x, y, z) \in \mathbb{R}^3 : |z| \leq a\} \tag{3.6}$$

$$R_2 = \{(x, y, z) \in \mathbb{R}^3 : |z| \geq a\}, \tag{3.7}$$

in which R_1 is itself divided into three linear regions:

$$\begin{cases} \Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 1, |z| \leq a\} \\ \Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, |z| \leq a\} \\ \Omega_3 = \{(x, y, z) \in \mathbb{R}^3 : x \leq -1, |z| \leq a\}, \end{cases} \tag{3.8}$$

and R_2 is divided into six linear regions:

$$\begin{cases} \Omega_4 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 1, z \geq a\} \\ \Omega_5 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, z \geq a\} \\ \Omega_6 = \{(x, y, z) \in \mathbb{R}^3 : x \leq -1, z \geq a\} \\ \Omega_7 = \{(x, y, z) \in \mathbb{R}^3 : x \geq 1, z \leq -a\} \\ \Omega_8 = \{(x, y, z) \in \mathbb{R}^3 : |x| \leq 1, z \leq -a\} \\ \Omega_9 = \{(x, y, z) \in \mathbb{R}^3 : x \leq -1, z \leq -a\}. \end{cases} \tag{3.9}$$

Thus, the system (3.3)-(3.4) has nine piecewise-linear regions with respect to the variables x and z, which confirm that this system is topologically different from the original Chua's circuit as mentioned above. In this case, all equilibrium points of system (3.3)-(3.4) are given by $X_{eq(x)} = (x, 0, -x)$ where x is the solution of the equation $h_a(x, -x) = 0$. Thus one has the following two cases:

Proposition 3.3 *If $a \leq 1 - \frac{m_0}{m_1}$, then there exist three equilibrium points for the system (3.3)-(3.4) as follows:*

$$P^{\pm} = \left(\pm \left(\frac{m_0 - m_1}{m_1} \right), 0, \mp \left(\frac{m_0 - m_1}{m_1} \right) \right) \text{ and } P^0 = (0, 0, 0),$$

Proof. We hawe

$$h(x) = \begin{cases} m_1 x + (m_0 - m_1) & x \geq 1 \\ m_0 x & |x| \leq 1 \\ m_1 x + (m_0 - m_1) & x \leq 1 \end{cases}$$

and

$$X_{eq(x)} = (x, 0, -x)$$

For $|x| \leq 1$

$$m_0 x = 0 \Leftrightarrow \begin{cases} x = 0 & \text{(exist)} \\ m_0 = 0 & \text{(not exist)} \end{cases}$$

so equilibrium point for the system (3.3)-(3.4) as follow:

$$P^0 = (0, 0, 0)$$

For $x \geq 1$, we have $m_1 x + (m_0 - m_1) = 0 \Leftrightarrow x = \left(-\frac{(m_0 - m_1)}{m_1}\right)$. So the equilibrium point for the system (3.3-)-(3.4) as follow:

$$P^+ = \left(-\left(\frac{m_0 - m_1}{m_1}\right), 0, \left(\frac{m_0 - m_1}{m_1}\right)\right)$$

For $x \leq -1$, we have $m_1 x - (m_0 - m_1) = 0 \Leftrightarrow x = \left(\frac{(m_0 - m_1)}{m_1}\right)$. So the equilibrium point for the system (3.3)-(3.4) as follow:

$$P^- = \left(\frac{m_0 - m_1}{m_1}, 0, -\frac{m_0 - m_1}{m_1}\right)$$

then there exist three equilibrium points for the system (3.3)-(3.4) as follows:

$$\begin{cases} P^{\pm} = \left(\pm\left(\frac{m_0 - m_1}{m_1}\right), 0, \mp\left(\frac{m_0 - m_1}{m_1}\right)\right) \\ \\ P^0 = (0, 0, 0) \end{cases}$$

such that $P^+ \in \Omega_6$, $P^- \in \Omega_7$, and $P^0 \in \Omega_2$ as shown in Fig.3.2, and the Jacobian matrix evaluated at these equilibria is given as follows: For $P^{\pm}$ one has:

$$J^{\pm} = \begin{pmatrix} \pm\alpha m_1 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}, \tag{3.10}$$

Second, if $a > 1 - \frac{m_0}{m_1}$, then only $P^0 \in \Omega_2$ exists (So no equilibrium is in R_1). The Jacobian matrix evaluated at P^0, is given by

$$J^0 = \begin{pmatrix} \alpha m_0 & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}. \tag{3.11}$$

The eigenvalues are the solutions of the following characteristic cubic equation:

$$P(\lambda) = \lambda^3 + A\lambda^2 + B\lambda + C = 0, \tag{3.12}$$

where the values of $A, B,$ and C are determined for each equilibrium point as follows: For $P^{\pm}$ one has in the first case that:

$$\begin{cases} A_1 = \alpha m_1 + 1 \\ B_1 = \beta + \alpha\,(m_1 - 1)) \\ C_1 = \alpha\beta m_1. \end{cases} \tag{3.13}$$

For P^0 one has in the first case

$$\begin{cases} A_0 = \alpha m_0 + 1 \\ B_0 = -\alpha + \beta + \alpha m_0 \\ C_0 = \alpha\beta m_0, \end{cases} \tag{3.14}$$

and in the second case one has

$$\begin{cases} A_0 = -\alpha m_0 + 1 \\ B_0 = \beta + \alpha\,(-m_0 - 1)) \\ C_0 = -\alpha\beta m_0. \end{cases} \tag{3.15}$$

It is easy to show that in the first case one has:

$$C_0 C_1 < 0. \tag{3.16}$$

Then P^0 and $P^{\pm}$ have different topological types: $P^{\pm}$ is always unstable since C_1 is negative, and therefore one eigenvalue of the Jacobian matrix $J^{\pm}$ is real and positive. Also, note that the two equilibrium points $P^{\pm}$ have the same Jacobian matrix, and thus the two equilibria have the same type of stability in the first case. The exact value of the eigenvalues is obtained by using the Cardan method for solving a cubic equation. On the other hand, the Routh-Hurwitz conditions lead to the conclusion that the real parts of the roots λ (solution of (3.14)) are negative if and only if $A > 0$, $C > 0$, and $AB - C > 0$. Thus, in the first case, $P^{\pm}$ are stable if and only if:

$$\beta > (1 - m_1)\alpha(1 + m_1\alpha) \tag{3.17}$$

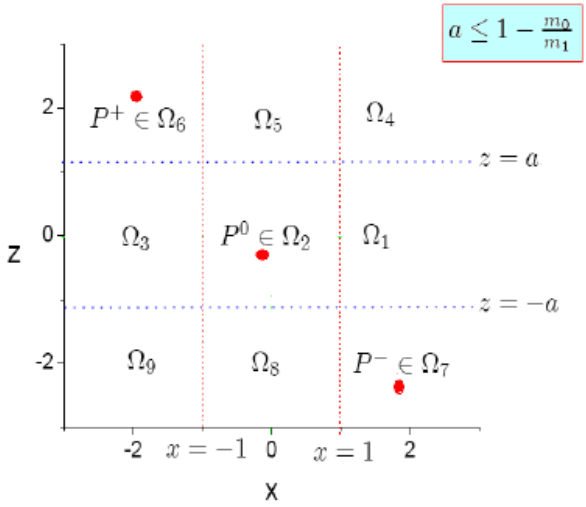


Figure 3.2: Equilibria of system (3-4).

since we assume that $m_1 > 0, \alpha > 0$, and $\beta > 0$, and in the second case P^0 is stable if and only if:

$$\beta > \alpha(1 + m_0)(1 - \alpha m_0).$$

■

An illustration of equilibria is shown in Fig. 3.2.

3.4 Numerical simulations

We are interested in the case $\alpha = 10.0, \beta = 14.0, m_0 = -0.43, m_1 = 0.41$, and $a = 1.0$, where system (3.1) display a three scroll chaotic attractor showxn in Fig.3.2.

1. For $|x| \leq 1, |z| \geq 0.1$, we define the parameters $\alpha = 10.0, \beta = 14.0, a = 1.0, m_0 = -0.43, , m_1 = 0.41$ then the equilibrium point is $P^0 = (0.0.0)$ and the Jacobian matrix evaluated at P^0 is given by

$$J^0 = \begin{pmatrix} -4.3 & 10 & 0 \\ 1 & -1 & 1 \\ 0 & -14.0 & 0 \end{pmatrix} \tag{3.18}$$

eigenvalues: $0.198\,83 + 3.\,244\,4i, 0.198\,83 - 3.\,244\,4i, -5.\,697\,7$. So the equilibrium point P^0 is unstable.

2. For $x \leq 1, |z| \geq 0.1$, we define the parameters $\alpha = 10.0, \beta = 14.0, a = 1.0, m_0 = -0.43, m_1 = 0.41$, then the equilibrium point is $P^- = \left(\left(+\left(\frac{m_0-m_1}{m_1}\right), 0, -\left(\frac{m_0-m_1}{m_1}\right)\right)\right)$ and the Jacobian matrix evaluated at P^- is given by

$$J^- = \begin{pmatrix} -4.1 & 10.0 & 0 \\ 1 & -1 & 1 \\ 0 & -14 & 0 \end{pmatrix} \tag{3.19}$$

and the eigenvalues are: $0.208\,73+3.\,218\,7i, 0.208\,73-3.\,218\,7i, -5.\,517\,5$. So the equilibrium point P^- is unstable.

3. For $x \geq 1, |z| \geq 0.1$, we define the parameters $\alpha = 10.0, \beta = 14.0, a = 1.0, m_0 = -0.43, m_1 = 0.41$, then the equilibrium point is $P^+ = \left(\left(-\left(\frac{m_0-m_1}{m_1}\right), 0, +\left(\frac{m_0-m_1}{m_1}\right)\right)\right)$ and the Jacobian matrix evaluated at P^- is given by

$$J^+ = \begin{pmatrix} 4.1 & 10.0 & 0 \\ 1 & -1 & 1 \\ 0 & -14.0 & 0 \end{pmatrix} \tag{3.20}$$

The eigenvalues: $5.\,223\,2, -1.\,061\,6+3.\,140\,5i, -1.\,061\,6-3.\,140\,5i$. So, the equilibrium points P^+ is unstable.

In all these numerical methods, the data are calculated in **double precision**. Thus for $\alpha = 10.0, \beta = 14.0, m_0 = -0.43, m_1 = 0.41$, and $a = 1.0$, the system (3.3)-(3.4) has the 3-scroll chaotic attractor shown in Fig.3.3. This attractor resembles the so-called *three-scroll chaotic attractor*, but with only three equilibria, contrary in the usual case where this attractor requires at least five equilibria [13].
Fig.3.4 show the bifurcation diagrams of the variable y_n plotted versus control parameters $a \in [0, 4.0]$, and with $\alpha = 10.0, \beta = 14.0$. In all these cases, the chaotic attractors are obtained via a *border collision bifurcation scenario* from a stable period-2 orbit. The dynamical behaviors of the system (3.3)-(3.4) are investigated numerically.

1. For $\alpha = 10.0, \beta = 14.0$, and $0 \leq a \leq 4.0$ varies, the system (3.3)-(3.4) is chaotic and displays both the 1-scroll and 3-scroll chaotic attractors until a is in a small neighborhood of $a = 3.0$ where the system converges to periodic orbit.

2. Note that all 3-scroll chaotic attractors appear when $a \leq 2.\,048\,8$ because the system (3.3)-(3.4) has three equilibria

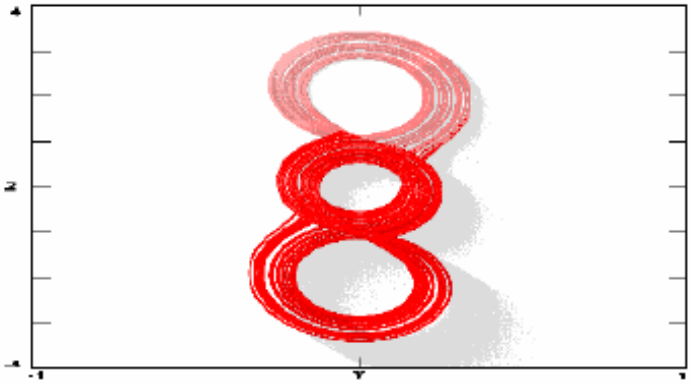


Figure 3.3: Projection onto the $y - z$ plane of the 3-scroll chaotic attractor obtained for: $\alpha = 10.0, \beta = 14, a = 1$.

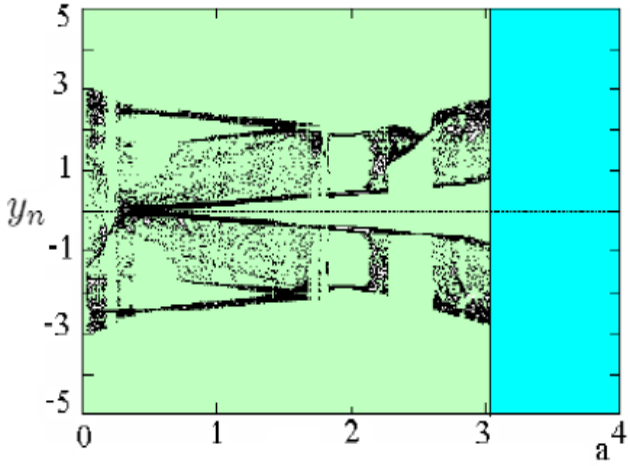


Figure 3.4: Bifurcation diagram of the variable yn plotted versus control parameter $a \in [0, 4]$ with $\alpha = 10.0, \beta = 14.0$.

3. For $a > 2.048\,8,$ the system (3.3)-(3.4) has at most one-scroll chaotic attractors because it has only one equilibrium point.

4. On the other hand, one can observe that the so-called *double-scroll* can occur for some small values of a, for example for $0 < a < 0.17$ as shown in Fig.3.3.

Conclusion

In this work, we have investigated a **generalized Zeraoulia system** including Lorenz-type dynamical systems with a particular focus on strange attractors and quasi-attractors. The study began with a classification of chaotic attractors and an introduction to the generalized model, highlighting the importance of dissipative systems and equilibrium points in understanding nonlinear dynamics. A detailled stability analysis was carried out using the Jacobian matrix and eigenvalues computation, allowing the characterization of equilibrium points and their local behavior. Various cases of the nonlinear functions were examined, providing deeper insight into the system's structure and the conditions under which stability or instability occurs. Furthermore, the study was extended to bounded 4-dimensional chaotic systems, where bounds were estimated for different models, including Lorenz-Haken and Lorenz-Stenflo systems. In addition, a new approach to three-scroll chaotic attractors was explored, and their fundamental properties, equilibrium points, and stability were analyzed. The numerical simulations played a crucial role in confirming the theoretical results by illustrating the presence of chaotic dynamics and complex attractor structures. These results demonstrate that the proposed models not only reproduce known chaotic behaviors but also reveal new dynamic features that enrich the theory of nonlinear systems. Overally, this work contributes to a better understanding of complex dynamical systems, and opens perspectives for future research in chaos theory, hyperchaotic systems, and their applications in areas such as secure communications and advanced modeling.

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