People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Echahid Cheikh Larbi Tebessi University - Tebessa

Faculty of Exact Sciences and Natural and Life Sciences
Department: Mathematics

Final thesis
for obtaining of a MASTER's degree
Domain: Mathematics and Computer Science
Field: Mathematics
Option: Partial Differential Equations and Applications

Topic:

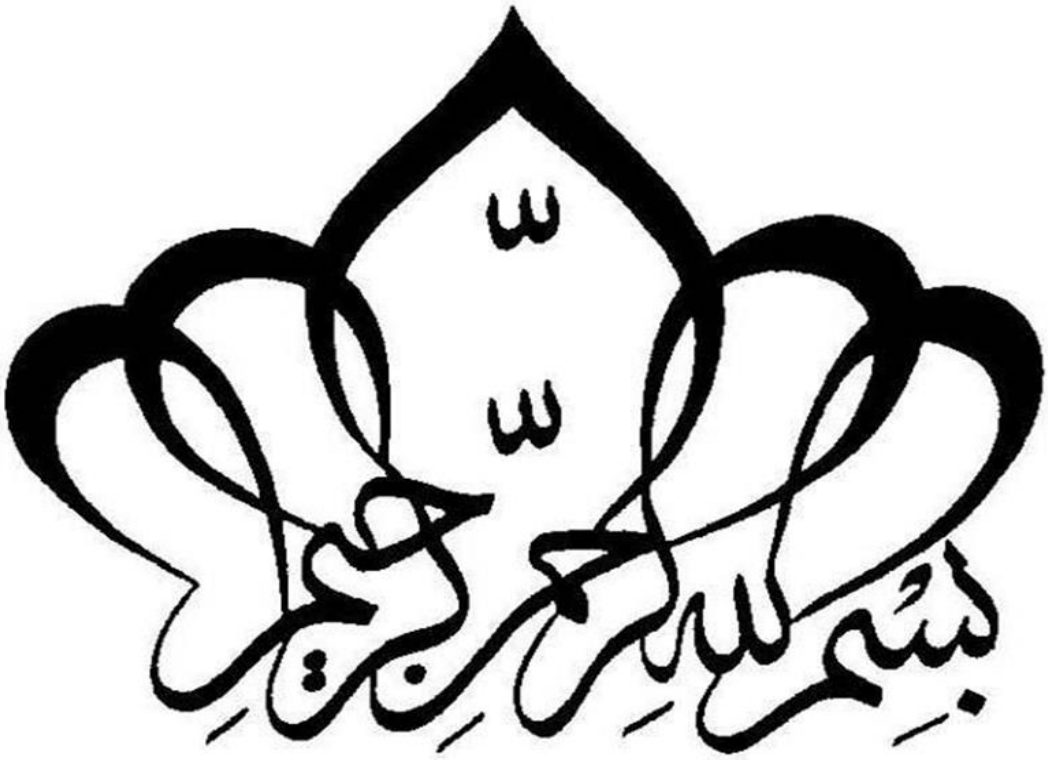
A common fixed point theorem and its application in dynamic programming

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Date of defense: 03 june 2025


بسم الله الرحمن الرحيم

Dedication

This study is dedicated to my beloved parents, who have been my source of inspiration and provided me with strength when I considered giving up. Their unwavering moral, spiritual, and emotional support has been invaluable.

I extend my gratitude for their encouragement and continual prayers.

I also want to thank my husband for his support and encouragement throughout the completion of this work, as well as my dear sisters and brothers.

To my family and all my friends, thank you for your love and support.

Acknowledgments

Foremost, I would like to thank ”ALLAH” almighty who provided me the power to complete this thesis.
I would like to express my sincere gratitude to my supervisor Dr. BARREH KHALED for the continuous support of my study and research, for his patience, human qualities, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I thank him for his great interest in this research.

Besides my supervisor, I would like to thank the rest of my thesis committee:

Mr. HAFDELLAH ABDELHAK , Professor at the University of Tebessa, who honored me by accepting to be president of the jury. I wish to express to him my respect- ful gratitude and consideration.

Mr. LAADJEL AMIROUCHE , Professor at the University of Tebessa, to whom I express my gratitude for having accepted to examine this work.

My sincere thanks also goes to all person at department of mathematics and computer sciences and at post-graduation at Larbi Tebessi university

Finally, I wish to express my thanks to all my teachers throughout my study journey and to all those who have helped me to the realization of this work.

ملخص

قمنا في هذه المذكرة بدراسة مبرهنة هامة في نظرية النقطة الثابتة، حيث نثبت وجود نقطة ثابتة مشتركة لأربع تطبيقات ذاتية المعرفة على فضاء متري تام، وذلك تحت شروط التوافق الضعيف والانكماش المعمم من نوع .Φ- contraction تعزز هذه المبرهنة النتائج السابقة وتوسع نطاقها، خاصة من حيث تطبيقها على حل معادلات دالية تظهر في سياق البرمجة الديناميكية، وهي معادلات نمذجية تصف القيم المثلى في عمليات اتخاذ القرار متعددة المراحل.

يقدم الباحثان تطبيقًا مباشرًا لهذه النظرية على نماذج رياضية مستمدة من البرمجة الديناميكية، مع إثبات وجود وتفرد الحلول لتلك النماذج. كما يتضمن العمل مثالًا عدديًا يوضح كيفية تحقق الشروط النظرية. يُعد هذا البحث مساهمة مهمة في ربط النظرية المجردة بنماذج عملية في التحسين والسيطرة، ويشكل مرجعًا في الدراسات الحديثة حول التثبيت في الفضاءات المترية.

الكلمات المفتاحية :نقطة ثابتة مشتركة، تطبيقات متوافقة بشكل ضعيف، انكماش Φ، فضاء متري، البرمجة الديناميكية، معادلة دالية.

Résumé

Cette thèse de master établit un théorème significatif de point fixe commun pour quatre auto-applications définis sur des espaces métriques complets, sous les hypothèses de compatibilité faible et d'une condition de Φ-contraction généralisée. Cette contribution étend et généralise des résultats précédents en théorie des points fixes, démontrant son applicabilité en résolvant des équations fonctionnelles qui apparaissent en programmation dynamique. Ces équations représentent généralement des formulations récursives des fonctions de valeur optimales dans des processus décisionnels multistages.

Les auteurs construisent un cadre complet en utilisant des outils de point fixe métriques, garantissant l'existence et l'unicité des solutions à ces systèmes fonctionnels. Un exemple illustratif est fourni pour confirmer les hypothèses théoriques. Cette contribution établit efficacement un lien entre la théorie abstraite des points fixes et les problèmes d'optimisation et de contrôle appliqués, enrichissant la base mathématique des modèles de programmation dynamique et ouvrant des avenues pour des recherches ultérieures en optimisation.

Mots-clés : Point fixe commun, Applications faiblement compatibles, Φ-Contraction, Espace métrique, Programmation dynamique, Équation fonctionnelle.

Abstract

This master's thesis establishes a significant common fixed point theorem for four self-mappings defined on complete metric spaces, under the assumptions of weak compatibility and a generalized Φ-contraction condition. This contribution extends and generalizes previous results in fixed point theory, demonstrating its applicability by solving functional equations that arise in dynamic programming. These equations typically represent recursive formulations of optimal value functions in multistage decision processes.

The authors construct a comprehensive framework using metric fixed-point tools, ensuring the existence and uniqueness of solutions to these functional systems. An illustrative example to confirm the theoretical assumptions. This contribution effectively bridges abstract fixed-point theory with applied optimization and control problems, enriching the mathematical foundation of dynamic programming models and opening avenues for further research in optimization.

Keywords: Common fixed point, Weakly compatible mappings, Φ-contraction, Metric space, Dynamic programming, Functional equations.

Contents

Introduction 4

1 Notions and Preliminaries **6**

1.1 Metric Spaces 6

1.1.1 Sequences and Cauchy Sequences 7

1.2 Partial metric spaces 8

1.3 Contraction Concepts 9

1.3.1 Definition and Examples 9

1.3.2 Some Types of Contractions 10

1.4 Fixed Point Theorem 14

1.5 proprieties of mappings 15

1.6 Picard-Jungck Operators 17

1.6.1 Jungck Iterative Scheme 17

1.6.2 Picard-Jungck Operators Types 17

2 COMMON FIXED POINT THEOREMS **18**

2.1 Common fixed point theorem 18

2.2 Common fixed point theorem for four mappings 25

3 APPLICATIONS IN DYNAMIC PROGRAMMING **33**

3.1 Structure of dynamic programming problems 34

3.2 Abstract dynamic programming models 34

Conclusion 43

Bibliography **45**

Introduction

The equation $fx = x$ plays a crucial role in fixed point theory, particularly when f is considered as a mapping (or operator). It is straightforward to demonstrate the solution to the equation $fx - x = 0$, where x represents the fixed point. A fixed point is defined as a point that remains unchanged by a given mapping or transformation. This concept can be observed in various phenomena, such as reproducible invariant temperatures like boiling and freezing points, or the triple point of a substance, which serves as a calibration standard for thermometers.

The significance of fixed points has led to extensive research on their existence. Under specific conditions pertaining to the mapping f and the space E, it is asserted that a mapping of E into itself can possess one or more fixed points. This area of study has yielded numerous results across two primary frameworks: the first involves topological spaces, illustrated by Brouwer's and Schauder's fixed point theorems, while the second focuses on metric spaces, prominently featuring the Contraction Mapping Theorem.

A pivotal concept within this framework is that of a **contraction** in complete metric spaces, which was introduced by Stefan Banach in 1922. This breakthrough established a new field of research centered on contraction mappings, commonly known as the Banach Contraction Principle (BCP). The exploration of common fixed points for self-mappings in metric spaces has gained significant attention due to its wide-ranging applications in nonlinear analysis, optimization theory, and especially in dynamic programming.

In this context, the notion of weakly compatible mappings, an extension of commuting mappings first introduced by Jungck and later expanded by Jungck and Rhoades has played a crucial role in developing and generalizing fixed point results for multiple mappings. Pathak and Tiwari, in their seminal paper, established a common fixed point theorem for four self-mappings defined on a complete metric space, under the assumptions of weak compatibility and a generalized *Phi*-contraction condition. The class of *Phi*-contractions encompasses a family of real-valued functions that are upper semi-continuous from the right, non-decreasing, and satisfy specific contractive conditions. This generalization includes many previously known contraction mappings, such as Banach-type, Kannan-type, and Chatterjea-type contractions.

The main result of their work not only provides a framework for ensuring the existence and uniqueness of a common fixed point but also establishes the resolution of systems of functional equations arising in dynamic programming. Specifically, their findings apply to equations of the form:

$$f(x) = \sup_{y \in D} H(x, y, f(T(x, y)))$$

which model optimal value functions in multistage decision processes. By leveraging fixed-point theory in this setting, it becomes possible to guarantee the solvability of such systems under abstract metric conditions.

In this master's thesis, we aim to extend these ideas further by investigating generalizations of common fixed point theorems for generalized contractions in metric spaces. Additionally, we will explore their implications for solving functional equations in dynamic programming, particularly those involving transformations and recursive formulations of value functions. Our work aligns with the mathematical foundations laid by Pathak and Tiwari and contributes to the broader theory of metric fixed point analysis and its applications in both computational and theoretical optimization. In this memoir, we present a structured exploration of these concepts across three chapters:

Chapter One: introduces essential concepts and preliminaries that lay the groundwork for the subsequent discussions. This foundational knowledge is crucial for understanding the more complex ideas presented later.

Chapter Two: delves into the core of our study, where we rigorously examine the common fixed point theorem. Here, we highlight its significance, applications, and the various conditions under which it holds.

Chapter Three: shifts focus to dynamic programming, where we introduce key concepts and demonstrate practical applications of the main results discussed earlier. This chapter illustrates how the theoretical framework established in the previous chapters can be applied to solve real-world problems.

Chapter 1

Notions and Preliminaries

1.1 Metric Spaces

In this chapter, we introduce essential concepts and foundational elements that will be crucial for the discussions in the following chapter. We outline the key principles of fixed point theory, including definitions and properties of fixed points. Additionally, we establish the necessary mathematical frameworks, such as metric spaces and topological concepts. By doing so, we prepare the reader for a deeper exploration of the common fixed point theorem that follows.

Definition 1.1.1 . [18] A *metric* on a set X is a function $d : X \times X \to \mathbb{R}$ such that:

(M1) If $x = y$ then $d(x, y) = 0$ (equality implies indistancy),

(M2) If $d(x, y) = 0$ then $x = y$ (indistancy implies equality),

(M3) $d(x, y) = d(y, x)$ (symmetry),

(M4) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$ (triangle inequality).

Example 1.1.1 . [18] Let $d(x, y) = |x - y|$; then $(\mathbb{R}, d)$ is a metric space. The first two conditions are obviously satisfied, and the third follows from the ordinary triangle inequality for real numbers:

$$d(x, y) = |x - y| = |(x - z) + (z - y)| \leq |x - z| + |z - y| = d(x, z) + d(z, y)$$

1.1.1 Sequences and Cauchy Sequences

A sequence in a metric space X is an infinite list

$$x_1, x_2, x_3, \ldots$$

of points in X. (More formally, a sequence in X can be defined as a function $f : \mathbb{N} \to X$, but we will not need this point of view here.) Note that the terms in the sequence do not have to be distinct; in particular, for any $x \in X$ we can talk about the constant sequence

$$x, x, x, \ldots$$

We will use the notation (x_n) for the sequence whose n-th term is x_n.

Definition 1.1.1.1 . A sequence (x_n) in a metric space (X, d) is said to *converge* to $x \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(x_n, x) < \varepsilon \quad \text{for all} \quad n \geq N.$$

Example 1.1.1.2 . In any metric space, a constant sequence $x, x, x, \ldots$ converges to x.

Example 1.1.1.3 . This is a standard elementary example. Consider the sequence $\left(\frac{1}{n}\right)$ in $\mathbb{R}$ with the standard metric. Intuitively, the terms of this sequence are getting closer and closer to 0. Let us prove that the sequence indeed converges to 0.

Let $\varepsilon > 0$. We want to find an index N such that for all $n \geq N$ we have

$$\frac{1}{n} < \varepsilon.$$

It is the Archimedean Property that tells us what N to choose: pick $N \in \mathbb{N}$ such that

$$\frac{1}{N} < \varepsilon.$$

Then, if $n \geq N$, we have

$$\frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

We conclude that the sequence $\left(\frac{1}{n}\right)$ converges to 0.

Definition 1.1.1.4 . A sequence (p_n) in a metric space (X, d) is said to be a *Cauchy*

sequence if for every $\varepsilon > 0$ there exists an integer N such that

$$d(p_n, p_m) < \varepsilon \quad \text{for all} \quad n, m \geq N.$$

Let E be a nonempty subset of a metric space (X, d), and let

$$S = \{d(p, q) : p, q \in E\}.$$

The *diameter* of E is defined as

$$\operatorname{diam}(E) = \sup S.$$

If (p_n) is a sequence in X and if E_N consists of the points $p_N, p_{N+1}, \ldots$, it is clear that (p_n) is a Cauchy sequence if and only if

$$\lim_{N \to \infty} \operatorname{diam}(E_N) = 0.$$

There are many types of metric spaces, for example:

- Cone metric spaces,
- Pseudo-metric spaces,
- B-metric spaces,
- Fuzzy metric spaces,
- Partial metric spaces, etc.

1.2 Partial metric spaces

Partial metric spaces, introduced by Matthews , are a generalization of the notion of the metric space in which the condition $d(x, x) = 0$ is replaced by the condition $d(x, x) \leq d(x, y)$.

Definition 1.2.1 [20] A *partial metric* or *p-metric* is a function $p : U^2 \to \mathbb{R}$ such that:

- For all $x, y \in U$, if $x = y$ then $p(x, x) = p(x, y) = p(y, y)$.
- For all $x, y \in U$, $p(x, x) \leq p(x, y)$.

- For all $x, y \in U$, $p(x, y) = p(y, x)$.
- For all $x, y, z \in U$, $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.

1.3 Contraction Concepts

1.3.1 Definition and Examples

Definition 1.2.1.1 . [21]Let (X, d) be a complete metric space. A function $f : X \to X$ is called a *contraction* if there exists $k < 1$ such that for any $x, y \in X$,

$$d(f(x), f(y)) \leq kd(x, y).$$

example 1.2.1.2 [21] Consider the metric space $(\mathbb{R}, d)$ where d is the Euclidean distance, i.e., $d(x, y) = |x - y|$. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \frac{x}{a} + b$$

is a contraction if $a > 1$. In this specific case, we can find a fixed point. Since a fixed point means that $f(x) = x$, we solve

$$x = \frac{x}{a} + b,$$

which leads to

$$x = \frac{ab}{a - 1}.$$

example 1.2.1.3 [21] We can create a similar contraction in the metric space $(\mathbb{R}^2, d)$ with the Euclidean distance. The function $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f(x, y) = \left(\frac{x}{a} + b, \frac{y}{c} + b\right)$$

is a contraction if $a, c > 1$. For a fixed point, we solve $f(x, y) = (x, y)$, which gives

$$x = \frac{ab}{a - 1}, \quad y = \frac{cb}{c - 1}.$$

1.3.2 Some Types of Contractions

1.3.2.1 Banach Contraction Principle

Several authors have defined contractual type mappings on a complete metric space X, which are generalizations of Banach's contraction, and have the property that each such mapping has a unique fixed point. We now present several definitions of different types of contractions.

Theorem 1.1 (Banach's Fixed Point Theorem) *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction on X. Then T has a unique fixed point $x \in X$ such that $T(x) = x$.*

Proof Let us choose any $x_0 \in X$, and define the sequence (x_n) by

$$x_{n+1} = T(x_n), \quad n = 0, 1, 2, \ldots$$

Step 1: Show that (x_n) is a Cauchy sequence.

By the contraction condition and recurrence relation, we have

$$d(x_{m+1}, x_m) = d(T(x_m), T(x_{m-1})) \leq Kd(x_m, x_{m-1}) \leq K^2 d(x_{m-1}, x_{m-2}) \leq \cdots \leq K^m d(x_1, x_0).$$

Hence, using the triangle inequality for $n > m$, we get:

$$d(x_m, x_n) \leq \sum_{k=m}^{n-1} d(x_k, x_{k+1}) \leq \sum_{k=m}^{n-1} K^k d(x_1, x_0) = K^m \frac{1 - K^{n-m}}{1 - K} d(x_1, x_0).$$

Since $0 < K < 1$, then $1 - K^{n-m} < 1$, and thus

$$d(x_m, x_n) \leq \frac{K^m}{1 - K} d(x_1, x_0).$$

As $m \to \infty$, the right-hand side tends to zero, so (x_n) is a Cauchy sequence. Since (X, d) is complete, there exists $x \in X$ such that

$$x_n \to x.$$

Step 2: Show that x is a fixed point.

We consider:

$$d(x, T(x)) \leq d(x, x_m) + d(x_m, T(x)) = d(x, x_m) + d(T(x_{m-1}), T(x)) \leq d(x, x_m) + Kd(x_{m-1}, x).$$

Since $x_n \to x$, both terms on the right tend to zero as $m \to \infty$. Therefore,

$$d(x, T(x)) = 0 \Rightarrow T(x) = x.$$

Step 3: Uniqueness.

Suppose x and $\tilde{x}$ are both fixed points of T, i.e., $T(x) = x$ and $T(\tilde{x}) = \tilde{x}$. Then

$$d(x, \tilde{x}) = d(T(x), T(\tilde{x})) \leq Kd(x, \tilde{x}).$$

Since $0 < K < 1$, this implies

$$(1 - K)d(x, \tilde{x}) \leq 0 \Rightarrow d(x, \tilde{x}) = 0 \Rightarrow x = \tilde{x}.$$

Thus, the fixed point is unique.

Remark 1.1 *It is essential that T is a contraction (i.e., $K < 1$) for Banach's Fixed Point Theorem to hold. If $d(T(x), T(y)) \leq d(x, y)$ for all $x, y \in X$ (i.e., $K = 1$), then a fixed point may not exist or may not be unique.*

For instance:

- *Let $T_1(x) = x + 1$ on $\mathbb{R}$, then $d(T_1(x), T_1(y)) = |x - y|$, but T_1 has no fixed point.*
- *Let $T_2(x) = x$ on $\mathbb{R}$, then every point is a fixed point, so fixed points are not unique.*

Let (X, d) be a complete metric space, and suppose $T : X \to X$ satisfies

$$d(Tx, Ty) \leq c \cdot d(x, y) \quad \text{for all } x, y \in X,$$

for some constant $c \in (0, 1)$. Then T has a unique fixed point $x^* \in X$ such that

$$T(x^*) = x^*.$$

Moreover, for any $x_0 \in X$, the sequence defined by $x_{n+1} = T(x_n)$ converges to x^*:

$$\lim_{n\to\infty} T^n(x_0) = x^*.$$

1.3.2.2 Rakotch Contraction

Definition 1.1 (Rakotch Contraction [22]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Rakotch contraction** if there is a monotonic and decreasing function $\alpha :]0, +\infty[\to [0, 1[$ such that for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) \leq \alpha(d(x, y)).$$

1.3.2.3 Kannan Contraction

Definition 1.2 (Kannan Contraction [16]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Kannan contraction** if there exists $a \in (0, \frac{1}{2})$ such that for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)].$$

1.3.2.4 Bianchini Contraction

Definition 1.3 (Bianchini Contraction [5]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Bianchini contraction** if there exists $h \in (0, 1)$ such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq h \max\{d(x, Tx), d(y, Ty)\}.$$

1.3.2.5 Reich Contraction

Definition 1.4 (Reich Contraction [22]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Reich contraction** if there exist $a, b, c > 0$ such that $a + b + c < 1$ and for all $x, y \in X$,*

$$d(Tx, Ty) \leq ad(x, Tx) + bd(y, Ty) + cd(x, y).$$

1.3.2.6 Sehgal Contraction

Definition 1.5 (Sehgal Contraction [23]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Sehgal contraction** if for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) < \max\{d(x, Tx), d(y, Ty), d(x, y)\}.$$

1.3.2.7 Zamfirescu Contraction

Definition 1.6 (Zamfirescu Contraction [26]) *Let (X, d) be a complete metric space and $T : X \to X$ a mapping. We say that T is a **Zamfirescu contraction** if for all $x, y \in X$, $x \neq y$,*

$$d(Tx, Ty) < \max\left\{d(x, Ty), \frac{d(x, Tx) + d(y, Ty)}{2}, \frac{d(x, Ty) + d(y, Tx)}{2}\right\}.$$

1.3.2.8 Wong Type Contraction

Definition 1.7 (Wong Type Contraction [11]) *Let A be a self-mapping on a metric space (M, d) and $\{f_i\}_{i=1}^{5}$ be a set of Wong (auxiliary) functions. We say that A is a **Wong type contraction** if the following inequality holds:*

$$d(Ap, Aq) \leq a_1 d(p, q) + a_2 d(p, Ap) + a_3 d(q, Aq) + a_4 d(p, Aq) + a_5 d(Ap, q),$$

for any $p, q \in M$ with $p \neq q$, where $a_i = \frac{f_i(d(p,q))}{d(p,q)}$.

1.3.2.9 Caristi contraction

Definition 1.8 (Caristi contraction [6]) *is a generalization of the Banach contraction principle. It was introduced by James Caristi in 1976. The idea is to replace the strict contraction condition with one involving a lower semicontinuous function.*

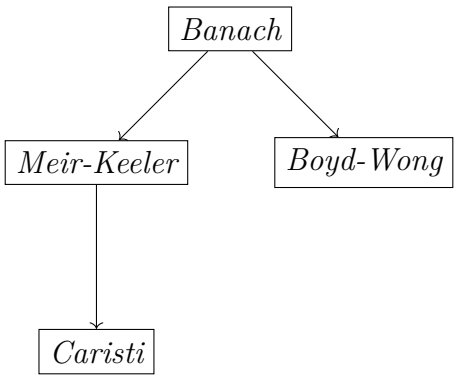
Definition (Caristi Fixed Point Theorem):
Let (X, d) be a complete metric space, and let $\varphi : X \to \mathbb{R}$ be a lower semicontinuous function that is bounded below. Suppose that $T : X \to X$ is a mapping such that for all $x \in X$,

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx).$$

Then T has a fixed point in X; that is, there exists $x^* \in X$ such that

$$Tx^* = x^*.$$

1.3.2.10 Diagram of Generalization



Each theorem generalizes or weakens the assumptions of the previous one in some way.

1.4 Fixed Point Theorem

We now examine how a familiar theorem from the theory of metric spaces can be transferred to partial metric spaces: complete spaces, Cauchy sequences, and the contraction fixed point theorem.

definition 1.3.1 *[13] For each partial metric space (X, p), a contraction is a function $f : X \to X$ for which there exists a $c \in [0, 1)$ such that for all $x, y \in X$,*

$$p(f(x), f(y)) \leq c\, p(x, y).$$

theorem 1.3.2 *[19]*

For each contraction f over a complete partial metric space (X, p), there exists a unique $x \in X$ such that $x = f(x)$. Also, $p(x, x) = 0$. Thus, the contraction fixed point theorem is extended to partial metric spaces.

1.5 proprieties of mappings

The aim of this paragraph is to recall the concept of commutative mappings, weakly commutative mappings, compatible mappings, partially ordered set, fixed point, coincidence point, common fixed point, and weakly compatibility of mappings.

definition 1.3.1.1 *[14] Let A and S be two self-mappings of a metric space (X, d). A and S are said to be commutative if*

$$ASx = SAx \quad \text{for all } x \in X.$$

definition 1.3.1.2 *[24]*

Let A and S be two self-mappings of a metric space (X, d). A and S are said to be weakly commutative if for all $x \in X$,

$$d(ASx, SAx) \leq d(Ax, Sx).$$

Obviously, commutativity implies weak commutativity, but the converse is not generally true, as shown in the following example.

exemple 1.3.1.3 *Let $(X, d) = ([0, 1], |\cdot|)$ where d is the usual metric. Define two mappings $A, S : X \to X$ as follows:*

$$A(x) = \frac{x}{2}, \quad S(x) = \frac{x}{2} + x = \frac{3x}{2}, \quad \text{for all } x \in X.$$

Then we compute:

$$|ASx - SAx| = \left|\frac{A(x)}{2} + x - \left(\frac{3x}{2}\right)\right| = \left|\frac{x}{4} + x - \frac{3x}{2}\right| = \left|\frac{5x}{4} - \frac{3x}{2}\right| = \left|\frac{-x}{4}\right| = \frac{x}{4},$$

and

$$|Ax - Sx| = \left|\frac{x}{2} - \frac{3x}{2}\right| = |-x| = x.$$

So, $d(ASx, SAx) = \frac{x}{4} \leq x = d(Ax, Sx)$, hence A and S are weakly commutative. However,

$$ASx = A(Sx) = A\left(\frac{3x}{2}\right) = \frac{3x}{4}, \quad SAx = S(Ax) = S\left(\frac{x}{2}\right) = \frac{3x}{4},$$

which shows $ASx = SAx$, so in this case they are actually commutative. But if you define slightly different S, the weak commutativity may hold while commutativity does not.

definition 1.3.1.4 *[15] Mappings A and S are said to be compatible if*

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0,$$

for every sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

definition 1.3.2.1 *[12] A pair (A, S) of self-mappings defined on a metric space (X, d) is said to be:*

1. ***Compatible:*** *if*

   $$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$$

   whenever $\{x_n\}$ is a sequence in X such that

   $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

   for some $t \in X$.

2. ***Non-compatible:*** *if there exists at least one sequence $\{x_n\}$ in X such that*

   $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

   for some $t \in X$, but

   $$\lim_{n\to\infty} d(ASx_n, SAx_n)$$

   is either non-zero or does not exist.

3. ***Weakly compatible:*** *if the mappings commute at their coincidence points, that is, $Ax = Sx$ for some $x \in X$ implies*

   $$ASx = SAx.$$

4. ***Tangential (or satisfying the property (E.A)):*** *if there exists a sequence $\{x_n\}$ in X such that*

   $$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$$

   for some $t \in X$.

1.6 Picard-Jungck Operators

1.6.1 Jungck Iterative Scheme

definition 1.3.3.1.1 *[25] Let X be a Banach space, $Y \subseteq X$ and $A, S : Y \to Y$ such that $A(Y) \subseteq S(Y)$. For $x_0 \in Y$, consider the following iterative scheme:*

$$Sx_{n+1} = Ax_n, \quad n \in \mathbb{N}.$$

For $Y = X$, this scheme is called the Jungck iterative scheme. *It was introduced by Jungck [14] in 1976 and reduces to the Picard iterative scheme when $S = I_X$, where I_X is the identity mapping on X.*

1.6.2 Picard-Jungck Operators Types

Inspired by Chandok et al. [8], we state the following definitions. In the sequel, let (X, d) be a metric space and $A, S : X \to X$ be two mappings such that $A(X) \subseteq S(X)$.

Definition 1.9 *The mappings A and S are called **Picard–Jungck operators** (briefly, PJO) if:*

> *[i)]A and S have a unique common fixed point z; The sequence $\{Sx_n\}$ converges to z for each $x \in X$.*

If $S = I_E$, where I_E is the identity mapping on X, we obtain the definition of a Picard operator (PO). Most of the operators such as those of Abbas and Khan [1], Berinde's (Theorems 3.3, 3.4) [4], Jungck's [14], and Sessa's [24] are PJO.

Furthermore, if the operators T and f in Theorem 2.1 of Chandok and Karapinar [7] are weakly compatible and the set of common fixed points of T and f is well ordered, then T and f are PJO (see Theorem 2.2 in [7]).

Chapter 2

COMMON FIXED POINT THEOREMS

In this chapter, we focus on the common fixed point theorem, a fundamental result in fixed point theory. We will explore its formulation, significance, and various applications across different mathematical contexts. By examining the conditions under which the theorem holds, we aim to highlight its relevance in both theoretical and practical scenarios. This foundational understanding will serve as a basis for further discussions in subsequent chapters.

2.1 Common fixed point theorem

2. ***Definition 2.1*** *A* ***common fixed point theorem*** *[10] concerns conditions under which two or more self-maps on a set (typically a metric space) have a shared fixed point. That is, for mappings $T_1, T_2, \ldots, T_n : X \to X$, a point $x \in X$ is a* ***common fixed point*** *if*

$$T_1(x) = T_2(x) = \cdots = T_n(x) = x.$$

These theorems generalize classical fixed point theorems (e.g., Banach's Fixed Point Theorem) to the case of multiple operators.

: Common Fixed Point Theorem for Commuting Maps

Let (X, d) be a complete metric space, and let $T_1, T_2 : X \to X$ be commuting and continuous mappings such that both are contractions; that is, there exists $0 < k < 1$ such that

$$d(T_i(x), T_i(y)) \leq k\, d(x, y) \quad \text{for all } x, y \in X,\ i = 1, 2.$$

Then, there exists a unique point $x^* \in X$ *such that*

$$T_1(x^*) = T_2(x^*) = x^*.$$

This point x^* *is called a* ***common fixed point*** *of* T_1 *and* T_2.

In what follows, we denote by Φ *the collection of all functions*

$$\varphi : [0, \infty) \to [0, \infty)$$

which are upper semicontinuous from the right, non-decreasing, and satisfy

$$\limsup_{s \to t^+} \varphi(s) < t \quad \text{and} \quad \varphi(t) < t \quad \text{for all} \quad t > 0.$$

Let X *denote a metric space endowed with metric* d *and let* $\mathbb{N}$ *denote the set of natural numbers. Now, let* A, B, S *and* T *be self-mappings of* X *such that*

$$A(X) \subset T(X) \quad \text{and} \quad B(X) \subset S(X), \tag{2.1}$$

$$\begin{aligned}[d^p(Ax, By) + a\, d^p(Sx, Ty)]\, d^p(Ax, By) \leq a \max \Big\{ & d^p(Ax, Sx) d^p(By, Ty), d^q(Ax, Ty) d^{q'}(By, Sx) \Big\} \\ & + \max \Bigg\{ \varphi_1 \left(d^{2p}(Sx, Ty) \right), \varphi_2 \left(d^r(Ax, Sx) d^{r'}(By, Ty) \right), \\ & \varphi_3 \left(d^s(Ax, Ty) d^{s'}(By, Sx) \right), \\ & \varphi_4 \left(\frac{1}{2} \left[d^l(Ax, Ty) d^{l'}(Ax, Sx) + d^l(By, Sx) d^{l'}(By, Ty) \right] \right) \Bigg\},\end{aligned} \tag{2.2}$$

for all $x, y \in X$, *where* $\varphi_i \in \Phi$ *for* $i = 1, 2, 3, 4$, *and* $a, p, q, q', r, r', s, s', l, l' \geq 0$ *with*

$$2p = q + q' = r + r' = s + s' = l + l' \leq 1.$$

Condition (2.2) *is commonly called a generalized* Φ*-contraction.*

Now, we pick $x_0 \in X$. *Since* $A(X) \subset T(X)$, *we can choose a point* $x_1 \in X$ *such that* $Ax_0 = Tx_1$. *Again, since* $B(X) \subset S(X)$, *for* $x_1 \in X$, *we can choose a point* $x_2 \in X$ *such that*

$Bx_1 = Sx_2$. Continuing in this way, we can construct a sequence $\{y_n\}$ in X such that

$$y_{2n} = Tx_{2n+1} = Ax_{2n}, \quad y_{2n+1} = Sx_{2n+2} = Bx_{2n+1} \quad (n \in \mathbb{N} \cup \{0\}). \tag{2.3}$$

First, we prove the following lemma:

Lemma 2.1. *Let us suppose $d_n = d(y_n, y_{n-1}), n \in \mathbb{N}$. Then, $\lim_{n\to\infty} d_n = 0$.*

Proof. *In (2.2), putting $x = x_{2n}$ and $y = x_{2n+1}$ and using (2.3), we get*

$$\begin{aligned}
&[d^p(Ax_{2n}, Bx_{2n+1}) + a\, d^p(Sx_{2n}, Tx_{2n+1})]d^p(Ax_{2n}, Bx_{2n+1}) \\
&\leq a\, \max\left\{d^p(Ax_{2n}, Sx_{2n})d^p(Bx_{2n+1}, Tx_{2n+1}), d^q(Ax_{2n}, Tx_{2n+1})d^r(Bx_{2n+1}, Sx_{2n})\right\} \\
&\quad + \max\Big\{\varphi_1(d'(Ax_{2n}, Sx_{2n}), d''(Bx_{2n+1}, Tx_{2n+1})), \\
&\qquad \varphi_2(d'(Ax_{2n}, Sx_{2n})d''(Bx_{2n+1}, Tx_{2n+1})), \\
&\qquad \varphi_3(d^s(Ax_{2n}, Tx_{2n+1})d^s(Bx_{2n+1}, Sx_{2n})), \\
&\qquad \varphi_4\left(\frac{1}{2}\left[d'(Ax_{2n}, Tx_{2n+1})d'(Ax_{2n}, Sx_{2n}) + d'(Bx_{2n+1}, Sx_{2n})d'(Bx_{2n+1}, Tx_{2n+1})\right]\right)\Big\}
\end{aligned}$$

$$[d^p(y_{2n}, y_{2n-1}) + a\, d^p(y_{2n-1}, y_{2n})]\, d^p(y_{2n}, y_{2n+1}) =$$

$$\left[d^p_{2n+1} + ad^p_{2n}\right] d^p_{2n+1} \leq a \max\{d^p(y_{2n}, y_{2n})d^p(y_{2n-1}, y_{2n+1}), d^q(y_{2n}, y_{2n+1}) + \max\{$$

$$\varphi_1\left(d^{2p}(y_{2n-1}, y_{2n})\right), \varphi_2\left(d^r(y_{2n+1}, y_{2n})\right), \varphi_3\left(d^s(d^s_{2n}, y_{2n})d^s(y_{2n+1}, y_{2n})\right),$$

$$\varphi_4\left(\frac{1}{2}\left[d^e(y_{2n}, y_{2n+1}) + d^e(y_{2n+1}, y_{2n-1}) + d^e(y_{2n+1}, y_{2n-1})\right]\right)\Big\}\Big\}$$

$$d^\ell(y_{2n+1}, y_{2n}) \leq d^\ell(y_{2n+1}, y_{2n-1}) + d^\ell(y_{2n-1}, y_{2n})$$

$$= d^\ell_{2n+1} + d^\ell_{2n}$$

$$d^{\ell}(y_{2n+1}, y_{2n-1})\, d^{\ell}(y_{2n+1}, y_{2n}) = (d^{\ell}_{2n+1} + d^{\ell}_{2n}) d^{\ell}_{2n+1}$$

$$= d^{\ell}_{2n+1} d^{\ell}_{2n+1} + d^{\ell}_{2n} d^{\ell}_{2n+1}$$

$$\left[d^{p}_{2n+1} + a d^{p}_{2n}\right] d^{p}_{2n+1} \leq a \max\left\{ d^{p}_{2n+1} > 0, \max\left\{ \varphi_1(d^{r}_{2n+1}), \varphi_3(\theta), \varphi_4\left(\frac{1}{2}\left[0 + d^{\ell'}_{2n+1} + d^{\ell}_{2n} d^{\ell'}_{2n+1}\right]\right)\right\}\right\}$$

or

$$\begin{aligned}
[d^{p}_{2n+1} + a d^{p}_{2n}] d^{p}_{2n+1} &\leq a \max\{d^{p}_{2n}, d^{p}_{2n}, 0\} + \max\left\{ \varphi_1(d'_{2n} d''_{2n+1}), \varphi_2(d'_{2n} d''_{2n+1}),\right. \\
&\qquad \left. \varphi_3(0), \varphi_4\left(\frac{1}{2}(d'_{2n} d''_{2n+1} + d'_{2n} d''_{2n+1})\right)\right\} \\
&\leq a d^{p}_{2n+1} d^{p}_{2n} + \max\left\{ \varphi_1(d'_{2n} d''_{2n+1}), \varphi_2(d'_{2n} d''_{2n+1}),\right. \\
&\qquad \left. \varphi_3(0), \varphi_4\left(\frac{1}{2}(d'_{2n} d''_{2n+1} + d'_{2n} d''_{2n+1})\right)\right\}
\end{aligned}$$

which implies

$$d^{2p}_{2n+1} \leq \max\left\{ \varphi_1(d^{2p}_{2n}), \varphi_2(d^{r}_{2n} d^{r'}_{2n+1}), \varphi_3(0), \varphi_4\left(\frac{1}{2}[d^{l+l'}_{2n+1} + d^{l}_{2n} d^{l'}_{2n+1}]\right)\right\}. \tag{2.4}$$

If $d_{2n+1} > d_{2n}$ then, we have

$$d^{2p}_{2n+1} \leq \max\left\{ \varphi_1(d^{2p}_{2n+1}), \varphi_2(d^{r+r'}_{2n+1}), \varphi_3(0), \varphi_4\left(\frac{1}{2}[d^{l+l'}_{2n+1} + d^{l+l'}_{2n+1}]\right)\right\} \leq \varphi_i(d^{2p}_{2n+1}) \quad (i = 1, 2, 4).$$

This, together with a well known result of Chang [9], which states that, if $\varphi_i \in \Phi$, where $i \in I$ (some indexing set), then there exists a $\varphi \in \Phi$ such that

$$\max\{\varphi_i, i \in I\} \leq \varphi(t), \quad \textit{for all } t > 0,$$

imply $d^{2p}_{2n+1} < d^{2p}_{2n+1}$, a contradiction.

Consequently, we have $d_{2n+1} \leq d_{2n}$ *for all* $n \in \mathbb{N}$, *and*

$$d_{2n+1} \leq \varphi(d_{2n}) \quad \text{for all } n \in \mathbb{N} \text{ and some } \varphi \in \Phi. \tag{2.5}$$

Similarly, for $x = x_{2n+2}$ *and* $y = x_{2n+1}$, *we have*

$$d_{2n+2}^{2p} \leq \max\left\{\varphi_1(d_{2n+1}^{2p}), \varphi_2(0), \varphi_3(0), \varphi_4\left(\frac{1}{2}[d_{2n+2}^{l+l'} + d_{2n+1}^{l} d_{2n+2}^{l'}]\right)\right\}. \tag{2.6}$$

A similar argument applied to (2.6) *will give*

$$d_{2n+2} \leq \varphi(d_{2n+1}) \quad \text{for all } n \in \mathbb{N}, \tag{2.7}$$

where $\varphi \in \Phi$ *is assumed to be the same as in the previous case.*

Therefore, for all $n \in \mathbb{N}$, *we have* $d_{n+1} \leq \varphi(d_n)$, *and by Lemma 2 of [9], we have*

$$\lim_{n\to\infty} d_n = 0.$$

Lemma 2.2. *The sequence* $\{y_n\}$ *defined in (2.3) is a Cauchy sequence.*

Proof. *We prove that the subsequence* $\{y_{2n}\}$ *of the sequence* $\{y_n\}$ *is a Cauchy sequence. On the contrary, let us suppose that* $\{y_{2n}\}$ *is not Cauchy. Then, there exists an* $\epsilon > 0$ *such that for each even integer* $2k$ *there exist even integers* $2m(k)$, $2n(k)$ $(n \in \mathbb{N})$ *with* $2m(k) > 2n(k) \geq 2k$, *such that*

$$d(y_{2n(k)}, y_{2m(k)}) > \epsilon \quad \text{and} \quad d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon, \tag{2.8}$$

that is, $2m(k)$ *is the least positive even integer such that* $2m(k) > 2n(k)$ *and* $d(y_{2n(k)}, y_{2m(k)-2}) \leq \epsilon$.

Hence, for each even integer $2k$, *we have*

$$\begin{aligned}
\epsilon &< d(y_{2n(k)}, y_{2m(k)}) \\
&\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\
&< \epsilon + d_{2m(k)-1} + d_{2m(k)}.
\end{aligned}$$

Hence, by Lemma 2.1 and (2.8) it follows that

$$\lim_{n\to\infty} d(y_{2n(k)}, y_{2m(k)}) = \epsilon. \tag{2.9}$$

By making use of the triangle inequality, for $\rho \in [0,1]$*, we have*

$$\begin{aligned} d_\rho(y_{2m(k)+2}, y_{2n(k)+1}) \leq\ & d_\rho(y_{2m(k)+2}, y_{2m(k)+1}) + d_\rho(y_{2m(k)+1}, y_{2m(k)}) \\ & + d_\rho(y_{2m(k)}, y_{2n(k)}) + d_\rho(y_{2n(k)}, y_{2n(k)+1}) \end{aligned}$$

or

$$d_\rho(y_{2m(k)+2}, y_{2n(k)+1}) - d_\rho(y_{2m(k)}, y_{2n(k)}) \leq d_{2m(k)+2} + d_{2m(k)+1} + d_{2n(k)+1}.$$

And

$$\begin{aligned} d_\rho(y_{2m(k)}, y_{2n(k)}) \leq\ & d_\rho(y_{2m(k)}, y_{2m(k)+1}) + d_\rho(y_{2m(k)+1}, y_{2m(k)+2}) \\ & + d_\rho(y_{2m(k)+2}, y_{2n(k)+1}) + d_\rho(y_{2n(k)+1}, y_{2n(k)}). \end{aligned}$$

or

$$d_\rho(y_{2m(k)}, y_{2n(k)}) - d_\rho(y_{2m(k)+2}, y_{2n(k)+1}) \leq d_{\rho 2m(k)+1} + d_{\rho 2m(k)+2} + d_{\rho 2n(k)+1}.$$

Thus, we obtain

$$|d_\rho(y_{2m(k)}, y_{2n(k)}) - d_\rho(y_{2m(k)+2}, y_{2n(k)+1})| \leq d_{\rho 2m(k)+1} + d_{\rho 2m(k)+2} + d_{\rho 2n(k)+1}. \tag{2.10}$$

Similarly, we have

$$|d_\rho(y_{2m(k)+1}, y_{2n(k)}) - d_\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{\rho 2m(k)+1}, \tag{2.11}$$

$$|d_\rho(y_{2m(k)+1}, y_{2n(k)+1}) - d_\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{\rho 2n(k)+1} + d_{\rho 2m(k)+1}, \tag{2.12}$$

$$|d_\rho(y_{2m(k)+2}, y_{2n(k)}) - d_\rho(y_{2n(k)}, y_{2m(k)})| \leq d_{\rho 2m(k)+1} + d_{\rho 2m(k)+2}. \tag{2.13}$$

By Lemma 2.2 and inequalities (2.10)–(2.13)*, we have*

$$\lim_{k\to\infty} d_\rho(y_{2m(k)+2}, y_{2n(k)+1}) = \lim_{k\to\infty} d_\rho(y_{2m(k)+1}, y_{2n(k)}) = \lim_{k\to\infty} d_\rho(y_{2m(k)+1}, y_{2n(k)+1}) = \tag{2.14}$$

$$\lim_{k\to\infty} d_\rho(y_{2m(k)+2}, y_{2n(k)}) = \epsilon. \tag{2.15}$$

Now, using (2.2) with $x = x_{2m(k)+2}$ *and* $y = x_{2n(k)+1}$ *along with (2.3) and a rearrangement, we obtain*

$$\begin{aligned}
&\big[d_p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) + a\, d_p(Sx_{2m(k)+2}, Tx_{2n(k)+1})\big]\, d_p(Ax_{2m(k)+2}, Bx_{2n(k)+1}) \\
&\le a \max\Big\{ d_p(Ax_{2m(k)+2}, Sx_{2m(k)+2})\, d_p(Bx_{2n(k)+1}, Tx_{2n(k)+1}), \\
&\qquad d_q(Ax_{2m(k)+2}, Tx_{2n(k)+1})\, d'_q(Bx_{2n(k)+1}, Sx_{2m(k)+2}) \Big\} \\
&\quad + \max\Big\{ \varphi_1\big(d_{2p}(Sx_{2m(k)+2}, Tx_{2n(k)+1})\big), \\
&\qquad \varphi_2\big(d_r(Ax_{2m(k)+2}, Sx_{2m(k)+2})\, d'_r(Bx_{2n(k)+1}, Tx_{2n(k)+1})\big), \\
&\qquad \varphi_3\big(d_s(Ax_{2m(k)+2}, Tx_{2n(k)+1})\, d'_s(Bx_{2n(k)+1}, Sx_{2m(k)+2})\big), \\
&\qquad \varphi_4\Big(\frac{1}{2}\big[d_l(Ax_{2m(k)+2}, Tx_{2n(k)+1})\, d'_l(Ax_{2m(k)+2}, Sx_{2m(k)+2}) \\
&\qquad\qquad + d_l(Bx_{2n(k)+1}, Sx_{2m(k)+2})\, d'_l(Bx_{2n(k)+1}, Tx_{2n(k)+1})\big]\Big)\Big\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&[d_p(y_{2m(k)+2}, y_{2n(k)+1}) + a\, d_p(y_{2m(k)+1}, y_{2n(k)})]d_p(y_{2m(k)+2}, y_{2n(k)+1}) \\
&\le a \max\Big\{ d_p(y_{2m(k)+2}, y_{2m(k)+1}) d_p(y_{2n(k)+1}, y_{2n(k)}), d_q(y_{2m(k)+2}, y_{2n(k)+1}) d'_q(y_{2n(k)+1}, y_{2m(k)+1}) \Big\} \\
&\quad + \max\Big\{ \varphi_1\big(d_{2p}(y_{2m(k)+1}, y_{2n(k)})\big), \varphi_2\big(d_r(y_{2m(k)+2}, y_{2m(k)+1}) d'_r(y_{2n(k)+1}, y_{2n(k)})\big), \\
&\qquad \varphi_3\big(d_s(y_{2m(k)+2}, y_{2n(k)}) d'_s(y_{2n(k)+1}, y_{2m(k)+1})\big), \\
&\qquad \varphi_4\Big(\frac{1}{2}\big[d_l(y_{2m(k)+2}, y_{2n(k)}) d'_l(y_{2m(k)+2}, y_{2m(k)+1}) + d_l(y_{2n(k)+1}, y_{2m(k)+1}) d'_l(y_{2n(k)+1}, y_{2n(k)})\big]\Big)\Big\}.
\end{aligned}$$

Letting $k \to \infty$ *and using Lemma 2.1, (2.9), (2.14) and the fact that* $\varphi_i \in \Phi$ *for* $i = 1, 2, 3, 4$, *we have:*

$$\epsilon^{2p} + a\epsilon^{2p} \le a\epsilon^{q+q'} + \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\},$$

or

$$\epsilon^{2p} \le \max\{\varphi_1(\epsilon^{2p}), \varphi_2(0), \varphi_3(\epsilon^{s+s'}), \varphi_4(0)\},$$

or

$$\epsilon^{2p} \leq \varphi(\epsilon^{2p}) < \epsilon^{2p},$$

a contradiction. Hence, $\{y_{2n}\}$ *is a Cauchy sequence in* X*. This proves that* $\{y_n\}$ *is Cauchy in* X*.*

2.2 Common fixed point theorem for four mappings

The following theorems are the essential results of our master thesis.

Theorem 2.1 *Let* A*,* B*,* S *and* T *be self-mappings of a complete metric space* X *satisfying conditions* (2.1) *and* (2.2)*. If the pairs* (A, S) *and* (B, T) *are weakly compatible and* $T(X)$ *or* $S(X)$ *is closed, then* A*,* B*,* S *and* T *have a unique common fixed point in* X*.*

Proof *Since* X *is complete, it follows from Lemma 2.2 that the sequence* $\{y_n\}$ *converges to a point* $z \in X$*. Consequently, the subsequences* $\{Ax_{2n}\}$*,* $\{Bx_{2n-1}\}$*,* $\{Sx_{2n}\}$*, and* $\{Tx_{2n+1}\}$ *of* $\{y_n\}$ *also converge to the same limit* z*.*

Now, suppose that $T(X)$ *is closed. Then, since* $\{Tx_{2n+1}\} \subset T(X)$*, there exists a point* $u \in X$ *such that* $z = Tu$*.*

Using (2.2) *with* $x = x_{2n}$ *and* $y = u$*, we have:*

$$\begin{aligned}
&[d^p(Ax_{2n}, Bu) + ad^p(Sx_{2n}, Tu)]\, d^p(Ax_{2n}, Bu) \\
&\leq a \max\left\{d^p(Ax_{2n}, Sx_{2n})d^p(Bu, Tu),\ d^q(Ax_{2n}, Tu)d^{q'}(Bu, Sx_{2n})\right\} \\
&\quad + \max\Big\{\varphi_1(d^{2p}(Sx_{2n}, Tu)),\ \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bu, Tu)), \\
&\qquad \varphi_3(d^s(Ax_{2n}, Tu)d^{s'}(Bu, Sx_{2n})), \\
&\qquad \varphi_4\left(\tfrac{1}{2}\left[d^l(Ax_{2n}, Tu)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bu, Sx_{2n})d^{l'}(Bu, Tu)\right]\right)\Big\}.
\end{aligned}$$

Letting $n \to \infty$, and using the fact that all sequences converge to z, we obtain:

$$[d^p(z,Bu) + ad^p(z,z)]\, d^p(z,Bu) \le a \max\{d^p(z,z)d^p(Bu,z),\ d^q(z,z)d^{q'}(Bu,z)\} + \max\left\{\varphi_1(d^{2p}(z,z)),\ \varphi_2(d^r(z,z)d^{r'}(Bu,z)),\ \varphi_3(d^s(z,z)d^{s'}(Bu,z)),\ \varphi_4\left(\tfrac{1}{2}d^{l+l'}(Bu,z)\right)\right\}.$$

This implies:

$$d^{2p}(z,Bu) \le \max\left\{\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0),\ \varphi_4\left(\tfrac{1}{2}d^{l+l'}(Bu,z)\right)\right\}.$$

$$d^{2p}(z,Bu) \le \varphi(d^{2p}(z,Bu)) < d^{2p}(z,Bu),$$

Which leads to a contradiction. Therefore, we must have $z = Bu$. Hence, $Tu = z = Bu$.

Now, since the pair (B,T) is weakly compatible, we have:

$$BTu = TBu \Rightarrow Bz = Tz.$$

now we show that z is a common fixed point of B and T.

Putting $x = x_{2n}$ and $y = z$ in (2.2), we obtain:

$$\begin{aligned}
&[d^p(Ax_{2n},Bz) + ad^p(Sx_{2n},Tz)]\, d^p(Ax_{2n},Bz)\\
&\le a\max\left\{d^p(Ax_{2n},Sx_{2n})d^p(Bz,Tz),\ d^q(Ax_{2n},Tz)d^{q'}(Bz,Sx_{2n})\right\}\\
&\quad + \max\left\{\varphi_1(d^{2p}(Sx_{2n},Tz)),\ \varphi_2(d^r(Ax_{2n},Sx_{2n})d^{r'}(Bz,Tz)),\right.\\
&\qquad \left.\varphi_3(d^s(Ax_{2n},Tz)d^{s'}(Bz,Sx_{2n})),\ \varphi_4\left(\tfrac{1}{2}\left[d^l(Ax_{2n},Tz)d^{l'}(Ax_{2n},Sx_{2n}) + d^l(Bz,Sx_{2n})d^{l'}(Bz,Tz)\right]\right)\right\}.
\end{aligned}$$

Letting $n \to \infty$, we get:

$$\begin{aligned}
&[d^p(z,Bz) + ad^p(z,Tz)]\, d^p(z,Bz) \\
&\leq a \max\left\{ d^p(z,z)d^p(Bz,Tz),\ d^q(z,Tz)d^{q'}(Bz,z) \right\} \\
&\quad + \max\left\{ \varphi_1(d^{2p}(z,Tz)),\ \varphi_2(d^r(z,z)d^{r'}(Bz,Tz)),\ \varphi_3(d^s(z,Tz)d^{s'}(Bz,z)), \right. \\
&\qquad \left. \varphi_4\left(\tfrac{1}{2}\left[d^l(z,Tz)d^{l'}(z,z) + d^l(Bz,z)d^{l'}(Bz,Tz)\right]\right) \right\}.
\end{aligned}$$

Now, using $d(z,z) = 0$, the inequality becomes:

$$d^{2p}(z,Bz) + ad^{2p}(z,Bz) \leq ad^{q+q'}(Bz,z) + \max\left\{ \varphi_1(d^{2p}(z,Bz)),\ \varphi_2(0),\ \varphi_3(d^{s+s'}(z,Bz)),\ \varphi_4(0) \right\},$$

or

$$(1+a)d^{2p}(z,Bz) \leq ad^{q+q'}(Bz,z) + \max\left\{ \varphi_1(d^{2p}(z,Bz)),\ \varphi_2(0),\ \varphi_3(d^{s+s'}(z,Bz)),\ \varphi_4(0) \right\}.$$

Hence,

$$\begin{aligned}
d^{2p}(z,Bz) &\leq \frac{a}{1+a}\, d^{q+q'}(Bz,z) \\
&\quad + \frac{1}{1+a} \max\left\{ \varphi_1\left(d^{2p}(z,Bz)\right),\ \varphi_2(0),\ \varphi_3\left(d^{s+s'}(z,Bz)\right),\ \varphi_4(0) \right\} \\
&< d^{2p}(z,Bz).
\end{aligned}$$

which is a contradiction. Thus, $z = Bz = Tz$. Therefore, z is a common fixed point of B and T.

Now we prove that z is common fixed point of A and S.
suppose that $S(X)$ is closed. Then, since $\{Sx_{2n}\} \subset S(X)$, there exists a point $u \in X$ such that $z = Su$.

Using (2.2) *with* $x = u$ *and* $y = x_{2n}$ *, we have:*

$$\begin{aligned}
&[d^p(Au, Bx_{2n}) + ad^p(Su, Sx_{2n})]\, d^p(Au, Bx_{2n}) \\
&\leq a \max\left\{d^p(Au, Su)d^p(Bx_{2n}, Sx_{2n}),\ d^q(Au, Sx_{2n})d^{q'}(Bx_{2n}, Su)\right\} \\
&\quad + \max\left\{\varphi_1(d^{2p}(Su, Sx_{2n})),\ \varphi_2(d^r(Au, Su)d^{r'}(Bx_{2n}, Sx_{2n})),\right. \\
&\qquad \varphi_3(d^s(Au, Sx_{2n})d^{s'}(Bx_{2n}, Su)), \\
&\qquad \left.\varphi_4\left(\tfrac{1}{2}\left[d^l(Au, Sx_{2n})d^{l'}(Au, Su) + d^l(Bx_{2n}, Su)d^{l'}(Bx_{2n}, Sx_{2n})\right]\right)\right\}.
\end{aligned}$$

Letting $n \to \infty$*, and using the fact that all sequences converge to* z*, we obtain:*

$$\begin{aligned}
[d^p(Au, z) + ad^p(z, z)]\, d^p(Au, z) \leq a \max\{d^p(Au, z)d^p(z, z),\ d^q(Au, z)d^{q'}(z, z)\} \\
+ \max\left\{\varphi_1(d^{2p}(z, z)),\ \varphi_2(d^r(Au, z)d^{r'}(z, z)),\right. \\
\left.\varphi_3(d^s(Au, z)d^{s'}(z, z)),\ \varphi_4\left(\tfrac{1}{2}d^{l+l'}(Au, z)\right)\right\}.
\end{aligned}$$

This implies:

$$d^{2p}(z, Au) \leq \max\left\{\varphi_1(0),\ \varphi_2(0),\ \varphi_3(0),\ \varphi_4\left(\tfrac{1}{2}d^{l+l'}(Au, z)\right)\right\}.$$

Which leads to:

$$d^{2p}(z, Au) \leq \varphi(d^{2p}(z, Au)) < d^{2p}(z, Au),$$

a contradiction. Therefore, we must have $z = Au$*. Hence,* $Su = z = Au$*.*

Now, since the pair (A, S) *is weakly compatible, we have:*

$$ASu = SAu \Rightarrow Bz = Sz.$$

Now we show that z *is a common fixed point of* A *and* S*.*

Putting $x = x_{2n}$ and $y = z$ in (2.2), we obtain:

$$
\begin{aligned}
&[d^p(Ax_{2n}, Bz) + ad^p(Sx_{2n}, Sz)]\, d^p(Ax_{2n}, Bz) \\
&\leq a \max\left\{d^p(Ax_{2n}, Sx_{2n})d^p(Bz, Sz),\ d^q(Ax_{2n}, Sz)d^{q'}(Bz, Sx_{2n})\right\} \\
&\quad + \max\Bigg\{\varphi_1(d^{2p}(Sx_{2n}, Sz)),\ \varphi_2(d^r(Ax_{2n}, Sx_{2n})d^{r'}(Bz, Sz)), \\
&\qquad \varphi_3(d^s(Ax_{2n}, Sz)d^{s'}(Bz, Sx_{2n})),\ \varphi_4\left(\tfrac{1}{2}\left[d^l(Ax_{2n}, Sz)d^{l'}(Ax_{2n}, Sx_{2n}) + d^l(Bz, Sx_{2n})d^{l'}(Bz, Sz)\right]\right)\Bigg\}.
\end{aligned}
$$

Letting $n \to \infty$, we get:

$$
\begin{aligned}
&[d^p(z, Bz) + ad^p(z, Sz)]\, d^p(z, Bz) \\
&\leq a \max\left\{d^p(z, z)d^p(Bz, Sz),\ d^q(z, Sz)d^{q'}(Bz, z)\right\} \\
&\quad + \max\Bigg\{\varphi_1(d^{2p}(z, Sz)),\ \varphi_2(d^r(z, z)d^{r'}(Bz, Sz)),\ \varphi_3(d^s(z, Sz)d^{s'}(Bz, z)), \\
&\qquad \varphi_4\left(\tfrac{1}{2}\left[d^l(z, Sz)d^{l'}(z, z) + d^l(Bz, z)d^{l'}(Bz, Sz)\right]\right)\Bigg\}.
\end{aligned}
$$

Now, using $d(z, z) = 0$, the inequality becomes:

$$
d^{2p}(z, Bz) + ad^{2p}(z, Bz) \leq ad^{q+q'}(Bz, z) + \max\left\{\varphi_1(d^{2p}(z, Bz)),\ \varphi_2(0),\ \varphi_3(d^{s+s'}(z, Bz)),\ \varphi_4(0)\right\},
$$

or

$$
(1 + a)d^{2p}(z, Bz) \leq ad^{q+q'}(Bz, z) + \max\left\{\varphi_1(d^{2p}(z, Bz)),\ \varphi_2(0),\ \varphi_3(d^{s+s'}(z, Bz)),\ \varphi_4(0)\right\}.
$$

Hence,

$$
\begin{aligned}
d^{2p}(z, Bz) &\leq \frac{a}{1 + a}\, d^{q+q'}(Bz, z) \\
&\quad + \frac{1}{1 + a} \max\left\{\varphi_1\big(d^{2p}(z, Bz)\big),\ \varphi_2(0),\ \varphi_3\big(d^{s+s'}(z, Bz)\big),\ \varphi_4(0)\right\} \\
&< d^{2p}(z, Bz).
\end{aligned}
$$

which is a contradiction. Thus, $z = Bz = Sz$. Therefore, z is a common fixed point of B and S. Consequently, z is a common fixed point of A, B, S, and T.

The uniqueness of z as a common fixed point of A, B, S, and T can be verified easily by assuming the existence of another fixed point and applying a similar contraction argument.

Remark 2.1 *If we assume $S(X)$ to be closed, then the above theorem also remains valid. We find the same result if $A(X)$ or $B(X)$ is assumed to be closed by condition (2.1).*

Remark 2.2 *the auther in Theorem 2.1 extends Theorem 3.5 of Pathak et al. [17].*

[Theorem 3.5] Let A, B, S and T be self-maps of a complete metric space (X, d) and suppose any one of the maps A or B is continuous. If the pairs $\{A, S\}$ and $\{B, T\}$ are weakly compatible, then A, B, S and T have a unique common fixed point $y' \in X$. Further y' is the common fixed point of A and S and of B and T.

Corollary 2.1 *Let A, B, S and T be self-mappings of a complete metric space X satisfying the following condition:*

$$d^{2p}(Ax, By) \leq h \max \Bigg\{ d^{2p}(Sx, Ty),\ d^{r}(Ax, Sx) d^{r'}(By, Ty),\ d^{s}(Ax, Ty) d^{s'}(By, Sx),$$
$$\frac{1}{2}\left[d^{l}(Ax, Ty) d^{l'}(Ax, Sx) + d^{l}(By, Sx) d^{l'}(By, Ty) \right] \Bigg\},$$

for all $x, y \in X$, where $0 < h < 1$, $2p = q + q' = r + r' = s + s' = l + l' \leq 1$, and the pairs (A, S) and (B, T) are weakly compatible.

If $T(X)$ or $S(X)$ is closed, then A, B, S, and T have a unique common fixed point in X.

Especially, when

$$\max \Big\{ d^{2p}(Sx, Ty),\ d^{r}(Ax, Sx)\, d^{r'}(By, Ty),\ d^{s}(Ax, Ty)\, d^{s'}(By, Sx),$$
$$\tfrac{1}{2}\left[d^{l}(Ax, Ty)\, d^{l'}(Ax, Sx) + d^{l}(By, Sx)\, d^{l'}(By, Ty) \right] \Big\} = d^{2p}(Sx, Ty).$$

we get Corollary 3.9 of Pathak et al. [17].

In Theorem 2.1, if we take $S = T = I_X$ (the identity mapping on X), then we obtain the following result:

Corollary 2.2 *Let A and B be self-mappings of a complete metric space X satisfying:*

$$\begin{aligned}
&[d^p(Ax,By)+a\,d^p(x,y)]\;d^p(Ax,By)\\
&\le a\max\Big\{d^p(Ax,x)\,d^p(By,y),\;\; d^q(Ax,y)\,d^{q'}(By,x)\Big\}\\
&\quad+\max\Bigg\{\phi_1\big(d^{2p}(x,y)\big),\;\;\phi_2\big(d^r(Ax,x)\,d^{r'}(By,y)\big),\\
&\qquad\phi_3\big(d^s(Ax,y)\,d^{s'}(By,x)\big),\;\;\phi_4\left(\tfrac{1}{2}\Big[d^l(Ax,y)\,d^{l'}(Ax,x)+d^l(By,x)\,d^{l'}(By,y)\Big]\right)\Bigg\}
\end{aligned}$$

for all $x,y\in X$, where $\phi_i\in\Phi$ for $i=1,2,3,4$, and the constants $a,p,q,q',r,r',s,s',l,l'\ge 0$ satisfy

$$2p=q+q'=r+r'=s+s'=l+l'\le 1.$$

Then A and B have a unique common fixed point in X.

As an immediate consequence of Theorem2.1 with $S=T$, we have the following:

Corollary 2.3 *Let A, B, and S be self-mappings of a complete metric space X such that*

$$A(X)\cup B(X)\subseteq S(X),$$

and for all $x,y\in X$,

$$\begin{aligned}
&\big[d^p(Ax,By)+a\,d^p(Sx,Sy)\big]\;d^p(Ax,By)\\
&\quad\le a\max\{d^p(Ax,Sx)d^p(By,Sy),\;\; d^q(Ax,Sy)d^{q'}(By,Sx)\}\\
&\qquad+\max\Bigg\{\phi_1\big(d^{2p}(Sx,Sy)\big),\;\;\phi_2\big(d^r(Ax,Sx)d^{r'}(By,Sy)\big),\\
&\qquad\phi_3\big(d^s(Ax,Sy)d^{s'}(By,Sx)\big),\;\;\phi_4\left(\frac{1}{2}\big[d^l(Ax,Sy)d^{l'}(Ax,Sx)+d^l(By,Sx)d^{l'}(By,Sy)\big]\right)\Bigg\}
\end{aligned}$$

where $\phi_i\in\Phi$ for $i=1,2,3,4$, and $a,p,q,q',r,r',s,s',l,l'\ge 0$ satisfy

$$0\le 2p=q+q'=r+r'=s+s'=l+l'\le 1.$$

If the pairs (A,S) and (B,S) are weakly compatible and $S(X)$ is closed, then A, B, and S have a unique common fixed point in X.

Theorem 2.2 *Let S, T, and $(A_n)_{n\in\mathbb{N}}$ be self-mappings of a complete metric space X. Suppose further that:*

- *For every $n \in \mathbb{N}$, the pairs (A_{2n-1}, S) and (A_{2n}, T) are weakly compatible.*
- *$A_{2n-1}(X) \subseteq T(X)$ and $A_{2n}(X) \subseteq S(X)$ for all $n \in \mathbb{N}$.*

if $S(X)$ or $T(X)$ is closed, and that for each $i \in \mathbb{N}$, the following condition holds for all $x, y \in X$:

$$
\begin{aligned}
&\big[d^p(A_ix, A_{i+1}y) + a\, d^p(Sx, Ty)\big] d^p(A_ix, A_{i+1}y) \\
&\quad \le a \max\big\{d^p(A_ix, Sx) d^p(A_{i+1}y, Ty),\ d^q(A_ix, Ty) d^{q'}(A_{i+1}y, Sx)\big\} \\
&\qquad + \max\Bigg\{\phi_1\big(d^{2p}(Sx, Ty)\big),\ \phi_2\big(d^r(A_ix, Sx) d^{r'}(A_{i+1}y, Ty)\big), \\
&\qquad\quad \phi_3\big(d^s(A_ix, Ty) d^{s'}(A_{i+1}y, Sx)\big),\ \phi_4\left(\frac{1}{2}\left[d^l(A_ix, Ty) d^{l'}(A_ix, Sx) + d^l(A_{i+1}y, Sx) d^{l'}(A_{i+1}y, Ty)\right]\right)\Bigg\},
\end{aligned}
$$

where $\phi_i \in \Phi$ for $i = 1, 2, 3, 4$, and $a, p, q, q', r, r', s, s', l, l' \ge 0$ satisfy

$$0 \le 2p = q + q' = r + r' = s + s' = l + l' \le 1.$$

Then S, T, and (A_n) have a common fixed point in X.

Chapter 3

APPLICATIONS IN DYNAMIC PROGRAMMING

In this chapter, we delve into the concepts of dynamic programming, emphasizing its role in optimization and decision-making processes. We will introduce key principles and methodologies that underpin dynamic programming approaches. Additionally, we will demonstrate how the common fixed point theorem can be applied to solve functional equations in this context. This exploration will illustrate the practical implications of our earlier theoretical findings.

Dynamic programming is a set of techniques for resolving sequential decision problems [2]. The techniques work by breaking down a multistage problem into a series of connected one-stage challenges. The optimality concept, created by Richard Bellman in the 1950s, is essential to this decomposition. It is significant because a functional equation connecting the ideal value for a $(t+1)$-stage problem to the ideal value for a t-stage problem can be solved to find the best solution for a multistage problem.

3.1 Structure of dynamic programming problems

The primary technique for analyzing a broad and varied class of sequential choice problems is dynamic programming, or DP for short [26] . Minimax problems, sequential zero-sum games, Markov and semi-Markov decision issues with discrete state space, and deterministic and stochastic optimum control problems with continuous state space are a few examples. These issues may differ greatly in nature, but they have many fundamental structures. Every situation has an underlying mapping that is based on a dynamic system and the related costs for every step. The DP operator, which is this mapping, offers a condensed "mathematical signature" of the issue. It establishes the ideal cost function as well as the cost function of policies. offers a handy abbreviation for describing and analyzing algorithms.

In short, dynamic programming helps us compute solutions step by step, using partial solutions to save effort and computation.

3.2 Abstract dynamic programming models

Throughout this section, we assume that X and Y are Banach spaces, $S \subset X$ is the state space, and $D \subset Y$ is the decision space. Let $R = (-\infty, \infty)$ and $B(S)$ denote the set of all bounded real-valued functions on S.

The basic form of the functional equation of dynamic programming is given by Bellman and Lee [3] as follows:

$$f(x) = \text{opt}_y \, H(x, y, f(T(x, y))),$$

where x and y represent the state and decision vectors respectively, T represents the transformation of the process, and $f(x)$ represents the optimal return with initial state x (where opt *denotes* max *or* min*).*

In this section, we study the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y \in D} H_i(x, y, f_i(T(x, y))), \quad x \in S, \tag{4.1}$$

$$g_i(x) = \sup_{y \in D} F_i(x, y, g_i(T(x, y))), \quad x \in S, \tag{4.2}$$

where $T : S \times D \to S$ and $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2$.

Suppose the mappings A_i and T_i $(i = 1, 2)$ are defined by:

$$A_i h(x) = \sup_{y \in D} H_i(x, y, h(T(x, y))), \; T_i k(x) = \sup_{y \in D} F_i(x, y, k(T(x, y))), \tag{4.3}$$

for all $x \in S$ and $h, k \in B(S)$, $i = 1, 2$.

Now, we present our main theorems of this section.

Theorem 3.1. *Suppose that the following conditions are satisfied:*

Assume the following conditions:

(i) H_i and F_i are bounded for $i = 1, 2$.

(ii) For all $(x, y) \in S \times D$, $k \in B(S)$, $t \in S$, $a \geq 0$, we have:

$$\begin{aligned} |H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq M^{-1} \Bigg\{ & \\ & \max\Big(|T_1 h(t) - A_1 h(t)| \cdot |T_2 k(t) - A_2 k(t)|, \; |T_1 h(t) - A_2 k(t)| \cdot |T_2 k(t) - A_1 h(t)|\Big) \\ & + \max\Big(\varphi_1\big(|T_1 h(t) - T_2 k(t)|\big), \; \varphi_2\big(|T_1 h(t) - A_1 h(t)|\big), \\ & \quad \varphi_3\big(|T_2 k(t) - A_2 k(t)|\big), \; \varphi_4\left(\frac{1}{2}\big(|T_1 h(t) - A_2 k(t)| + |T_2 k(t) - A_1 h(t)|\big)\right)\Big)\Bigg\}, \end{aligned}$$

where

$$M = \left[1 + a \sup_{t \in S} |T_1 k(t) - T_2 h(t)|\right],$$

and $\varphi_i \in \Phi$ for $i = 1, 2, 3, 4$. The mappings A_i and T_i $(i = 1, 2)$ are defined as in (4.3).

(iii) For sequences $\{h_n\}$, $\{k_n\} \subset B(S)$ and $h, k \in B(S)$ with

$$\lim_{n \to \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0, \quad \lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0,$$

there exist $h_i, k_i \in B(S)$ such that $k = T_2 h_i$ and $h = T_1 k_i$ for $i = 1$ or 2.

(iv) For any $h \in B(S)$, there exist $k_1, k_2 \in B(S)$ such that

$$A_1 h(x) = T_2 k_2(x), \quad A_2 h(x) = T_1 k_1(x), \quad \forall x \in S.$$

(v) For any $h, k \in B(S)$, with $A_1 h = T_1 h$, we have $T_1 A_1 h = A_1 T_1 h$, and with $A_2 k = T_2 k$, we have $T_2 A_2 k = A_2 T_2 k$.

Then, the system of functional equations (4.1) and (4.2) has a unique common solution in $B(S)$.

Proof. *Obviously, $B(S)$ endowed with the metric*

$$d(h,k) = \sup_{x \in D} |h(x) - k(x)|, \quad \forall h, k \in B(S),$$

is a complete metric space. Moreover, by condition (i), A_i and T_i are self-mappings of $B(S)$, and by condition (iv), it is clear that

$$A_1(B(S)) \subset T_2(B(S)) \quad \textit{and} \quad A_2(B(S)) \subset T_1(B(S)).$$

Also, by condition (v), the pairs (A_i, T_i) are weakly compatible for $i = 1, 2$.

Moreover, by (4.3) and (i), we have for any $\eta > 0$ there exist $y_i \in D$ $(i = 1, 2)$ such that

$$A_i h_i(x) < H_i(x_i, y_i, h_i(x)) + \eta, \tag{3.1}$$

Also,

$$A_1 h_1(x) \geq H_1(x, y_2, h_1(x_2)), \tag{3.2}$$

$$A_2 h_2(x) \geq H_2(x, y_2, h_2(x_1)). \tag{3.3}$$

Then, from (3.1), (3.2), (3.3) and (ii), we have

$$\begin{aligned}
A_1h_1(x) - A_2h_2(x) &\le H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1)) + \eta \\
&\le |H_1(x, y_1, h_1(x_1)) - H_2(x, y_1, h_2(x_1))| + \eta \\
&\le M^{-1}\Bigg(a \max\Big\{ |T_1h_1(x_1) - A_1h_1(x_1)| \cdot |T_2h_2(x_1) - A_2h_2(x_1)|, \\
&\quad |T_1h_1(x_1) - A_2h_2(x_1)| \cdot |T_2h_2(x_1) - A_1h_1(x_1)| \Big\} \\
&\quad + \max\Big\{ \varphi_1\big(|T_1h_1(x_1) - T_2h_2(x_1)|\big), \varphi_2\big(|T_1h_1(x_1) - A_1h_1(x_1)|\big), \\
&\quad \varphi_3\big(|T_2h_2(x_1) - A_2h_2(x_1)|\big), \\
&\quad \varphi_4\left(\frac{1}{2}\left(|T_1h_1(x_1) - A_2h_2(x_1)| + |T_2h_2(x_1) - A_1h_1(x_1)|\right)\right) \Big\} \Bigg) \\
&\le M^{-1}\Bigg(a \max\Big\{ d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1) \Big\} \\
&\quad + \max\Big\{ \varphi_1\big(d(T_1h_1, T_2h_2)\big), \varphi_2\big(d(T_1h_1, A_1h_1)\big), \\
&\quad \varphi_3\big(d(T_2h_2, A_2h_2)\big), \varphi_4\left(\frac{1}{2}\big(d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)\big)\right) \Big\} \Bigg) + \eta.
\end{aligned} \tag{3.4}$$

From (3.1), (3.2) and (ii), we also have

$$\begin{aligned}
A_1h_1(x) - A_2h_2(x) \ge -M^{-1}\Bigg\{ & a \max\Big(d(T_1h_1, A_1h_1) \cdot d(T_2h_2, A_2h_2),\ d(T_1h_1, A_2h_2) \cdot d(T_2h_2, A_1h_1) \Big) \\
& + \max\Big(\varphi_1\big(d(T_1h_1, T_2h_2)\big),\ \varphi_2\big(d(T_1h_1, A_1h_1)\big), \\
& \quad \varphi_3\big(d(T_2h_2, A_2h_2)\big),\ \varphi_4\left(\frac{1}{2}\left(d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)\right)\right) \Big) \Bigg\} - \eta
\end{aligned} \tag{3.5}$$

Using (3.4) and (3.5), we obtain:

$$\begin{aligned}
|A_1h_1(x) - A_2h_2(x)| \le M^{-1}\Bigg\{ & a \max\Big(d(T_1h_1, A_1h_1) \cdot d(T_2h_2, A_2h_2),\ d(T_1h_1, A_2h_2) \cdot d(T_2h_2, A_1h_1) \Big) \\
& + \max\Big(\varphi_1\big(d(T_1h_1, T_2h_2)\big),\ \varphi_2\big(d(T_1h_1, A_1h_1)\big), \\
& \quad \varphi_3\big(d(T_2h_2, A_2h_2)\big),\ \varphi_4\left(\frac{1}{2}\left(d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)\right)\right) \Big) \Bigg\} + \eta
\end{aligned} \tag{3.6}$$

Since (3.6) holds for any $x \in S$ and $\eta > 0$ is arbitrary, by taking the supremum over all $x \in S$, we have

$$\big(1+a\,d(T_1h_1, T_2h_2)\big)d(A_1h_1, A_2h_2) \leq a \max\Big\{d(T_1h_1, A_1h_1)d(T_2h_2, A_2h_2), d(T_1h_1, A_2h_2)d(T_2h_2, A_1h_1)\Big\}$$

$$+ \max\Big\{\varphi_1\big(d(T_1h_1, T_2h_2)\big), \varphi_2\big(d(T_1h_1, A_1h_1)\big), \varphi_3\big(d(T_2h_2, A_2h_2)\big), \varphi_4\left(\frac{1}{2}\big(d(T_1h_1, A_2h_2) + d(T_2h_2, A_1h_1)\big)\right)$$

Therefore, condition (2.2) is satisfied by mappings A_1, A_2, T_1 and T_2 and hence by Theorem 3.1, they have a common fixed point $h^* \in B(S)$, i.e. $h^*(x)$ is a unique common solution of the functional equations (4.1) and (4.2).

As an immediate consequence of Theorem 3.1 and Corollary 2.2, we have the following:

theorem3.2

Suppose the following conditions are satisfied:

(i) H_i is bounded for $i = 1, 2$.

(ii) For all $(x, y) \in S \times D$, $h, k \in B(S)$, $t \in S$, and $a \geq 0$, we have:

$$\begin{aligned}|H_1(x, y, h(t)) - H_2(x, y, k(t))| \leq N^{-1}\Bigg\{ & \\ a\,\max\Big(|h(t) - A_1h(t)| \cdot |k(t) - A_2k(t)|,\ |h(t) - A_2k(t)| \cdot |k(t) - A_1h(t)|\Big) & \\ + \max\Big(\varphi_1\big(|h(t) - k(t)|\big),\ \varphi_2\big(|h(t) - A_1h(t)|\big), & \\ \varphi_3\big(|k(t) - A_2k(t)|\big),\ \varphi_4\left(\frac{1}{2}\left(|h(t) - A_2k(t)| + |k(t) - A_1h(t)|\right)\right)\Big)\Bigg\}, &\end{aligned}$$

where

$$N = \left[1 + a \sup_{t \in S} |h(t) - k(t)|\right],$$

and $\varphi_i \in \Phi$ for $i = 1, 2, 3, 4$, and the mappings A_i are defined as in (4.3).

Then, the functional equations (4.1) and (4.2) have a unique common solution in $B(S)$.

Now, we furnish an example to validate Theorem 4.1.

Example 3.1 *Let $X = Y = \mathbb{R}$ be two Banach spaces endowed with the standard norm $\|\cdot\|$ defined by $\|x\| = |x|$ for all $x \in \mathbb{R}$. Let $S = [0,1] \subset X$ be the state space, $D = [1,\infty) \subset Y$ the decision space and T represents the transformation of the process. Define $T : S \times D \to S$ by*

$$T(x,y) = \frac{x}{y^2+1} \quad \text{for all} \quad x \in S, y \in D.$$

For any $h, k \in B(S)$, define $f_i, g_i : S \to \mathbb{R}$ $(i = 1,2)$ by

$$f_i(x) = g_i(x) = \frac{1}{4}\left(\frac{x}{x+1} + 1\right).$$

Define $H_i, F_i : S \times D \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ by

$$H_i(x,y,t) = \frac{1}{4}\left[\frac{x}{x+1}\sin\left(t \cdot \frac{y}{y+1}\right) + 1\right] \quad \text{and} \quad F_i(x,y,t) = \frac{1}{4}\left[\frac{x}{x+1}\cos\left(t \cdot \frac{y}{y+1}\right) + 1\right].$$

Clearly, $\|H_i\| \leq \frac{1}{2}$ and $\|F_i\| \leq \frac{1}{2}$. By varying y over D and taking supremum, we see that H_i yield f_i and H_i yield g_i, respectively $(i = 1,2)$, as defined above.

Define mappings A_i and T_i $(i = 1,2)$ by

$$A_1h(x) = \sup_{y\in D} H_1(x,y,h(T(x,y))), \quad A_2k(x) = \sup_{y\in D} H_2(x,y,k(T(x,y))),$$

$$T_1h(x) = \sup_{y\in D} F_1(x,y,h(T(x,y))), \quad T_2k(x) = \sup_{y\in D} F_2(x,y,k(T(x,y))),$$

for all $x \in S; h, k \in B(S)$.

Then, we see that

$$\begin{aligned}
A_1h(x) &= \sup_{y\in D} H_1(x,y,h(T(x,y))) \\
&= \sup_{y\in D} H_1\left(x,y,h\left(\frac{x}{y^2+1}\right)\right) \\
&= \sup_{y\in D} \frac{1}{4}\left[\frac{x}{x+1}\sin\left(\frac{h\left(\frac{x}{y^2+1}\right)}{y+1}\right) + 1\right] \\
&= \frac{1}{4}\left(\frac{x}{x+1} + 1\right) \\
&= f_1(x),
\end{aligned}$$

$$\begin{aligned}
A_2k(x) &= \sup_{y\in D} H_2(x,y,k(T(x,y))) \\
&= \sup_{y\in D} H_2\left(x,y,k\left(\frac{x}{y^2+1}\right)\right) \\
&= \sup_{y\in D} \frac{1}{4}\left[\frac{x}{x+1}\sin\left(\frac{k\left(\frac{x}{y^2+1}\right)}{y+1}\right)+1\right] \\
&= \frac{1}{4}\left(\frac{x}{x+1}+1\right) \\
&= f_2(x),
\end{aligned}$$

$$\begin{aligned}
T_1h(x) &= \sup_{y\in D} F_1(x,y,h(T(x,y))) \\
&= \sup_{y\in D} F_1\left(x,y,h\left(\frac{x}{y^2+1}\right)\right) \\
&= \sup_{y\in D} \frac{1}{4}\left[\frac{x}{x+1}\cos\left(\frac{h\left(\frac{x}{y^2+1}\right)}{y+1}\right)+1\right] \\
&= \frac{1}{4}\left(\frac{x}{x+1}+1\right) \\
&= g_1(x),
\end{aligned}$$

$$\begin{aligned}
T_2k(x) &= \sup_{y\in D} F_2(x,y,k(T(x,y))) \\
&= \sup_{y\in D} F_2\left(x,y,k\left(\frac{x}{y^2+1}\right)\right) \\
&= \sup_{y\in D} \frac{1}{4}\left[\frac{x}{x+1}\cos\left(\frac{k\left(\frac{x}{y^2+1}\right)}{y+1}\right)+1\right] \\
&= \frac{1}{4}\left(\frac{x}{x+1}+1\right) \\
&= g_2(x).
\end{aligned}$$

for all $x \in S; h, k \in B(S)$. *Also, weseethat*

$$M = \left[1 + a \sup_{t \in S} |T_1 k(t) - T_2 h(t)|\right] = \left[1 + a \sup_{t \in S} |g_1(x) - g_2(x)|\right] = 1.$$

Further, for any $k \in B(S)$, *define*

$$h_n(x) = \left(1 - \frac{1}{n}\right) h(x) \quad \textit{and} \quad k_n(x) = \left(1 - \frac{1}{n+1}\right) k(x)$$

so that

$$\lim_{n \to \infty} \sup_{x \in S} |h_n(x) - h(x)| = 0 \quad \textit{and} \quad \lim_{n \to \infty} \sup_{x \in S} |k_n(x) - k(x)| = 0.$$

Now, we observe that, for $\varphi_1(t) = t^2$,

(i) H_i *and* F_i *are bounded for* $i = 1, 2$.

(ii)

$$|H_1(x, y, h(t)) - H_2(x, y, k(t))| = \frac{1}{4}\left[\frac{x}{x+1} \sin\left(\frac{h(t)}{y+1}\right) + 1\right] - \frac{1}{4}\left[\frac{x}{x+1} \sin\left(\frac{k(t)}{y+1}\right) + 1\right]$$

$$= \frac{1}{4}\left(\frac{x}{x+1}\right)\left[\sin\left(\frac{h(t)}{y+1}\right) - \sin\left(\frac{k(t)}{y+1}\right)\right]$$

$$= \frac{1}{4}\left(\frac{x}{x+1}\right) \cdot 2 \sin\left(\frac{h(t)}{y+1} - \frac{k(t)}{y+1}\right) \cos\left(\frac{h(t)}{y+1} + \frac{k(t)}{y+1}\right).$$

$$|h(t) - k(t)| \leq \frac{1}{4}(x+1)$$

$$|h(t) - k(t)| \leq \frac{1}{4}\phi_1(h(t) - k(t)).$$

Finally, for any $h \in B(S)$, *there exist* $k_1, k_2 \in B(S)$ *such that*

$$A_1 h(x) = T_2 k_2(x), \quad A_2 h(x) = T_1 k_1(x), \quad x \in S.$$

Also, for any $h, k \in B(S)$, with $A_1 h = T_1 h$, we have $T_1 A_1 h = A_1 T_1 h$ and, with $A_2 k = T_2 k$, we have $T_2 A_2 k = A_2 T_2 k$.

Thus, all the assumptions of Theorem 3.1 are satisfied. Therefore, the system of equations (4.1) and (4.2) has a unique common solution in $B(S)$.

Conclusion

In this memoir, we have explored the profound implications of fixed point theory, particularly through the lens of the common fixed point theorem and its applications in dynamic programming. Our investigation began with a thorough introduction to the foundational concepts necessary for understanding the complexities of fixed points and their significance across various mathematical frameworks.

In Chapter One, we established the essential preliminaries that underpin our study. By examining the theoretical constructs surrounding fixed points, we prepared the groundwork for the detailed exploration that followed. This chapter emphasized the importance of understanding the underlying principles, which are crucial for navigating the more intricate discussions in subsequent chapters.

Chapter Two served as the core of our research, where we provided an in-depth analysis of the common fixed point theorem. We discussed its various forms, the conditions under which it holds, and its implications for both theoretical and applied mathematics. Our findings reaffirmed the theorem's central role in ensuring the existence and uniqueness of fixed points in metric spaces, thus contributing to a richer understanding of its applications in nonlinear analysis and optimization.

In Chapter Three, we transitioned to the realm of dynamic programming, where we introduced key concepts and demonstrated the practical utility of the main results derived from fixed point theory. By applying these theoretical insights to real-world problems, we illustrated how fixed points can facilitate the resolution of complex functional equations, ultimately enhancing decision-making processes in various fields. The applications discussed not only validate the theoretical framework but also emphasize the relevance of fixed point theory in contemporary research and practice.

Overall, this memoir highlights the interconnectedness of fixed point theory and dynamic programming, showcasing how foundational mathematical concepts can lead to significant advancements in solving practical problems.

The results obtained from our study pave the way for future research, encouraging further exploration of generalized contractions and their applications in diverse mathematical and computational contexts.

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