People's Democratic Republic of Algeria Ministry
of higher education and Scientific Research
Larbi Tebessi University – Tebessa
Faculty of Exact Sciences and Natural and Life Sciences
Departement : *Mathematics*
End of study dissertation for obtaining the master's degree
Domain : *Mathematics and Informatics*
Field : *Mathematics*
Option : *PDE and Applications*

Theme:

Mathematical Analysis of double-age structured model

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Academic year : 2024 - 2025

شكر وتقدير

بسم الله و الحمد لله ما أشرقت شمس طموح إلا بنوره، ولا تفتحت زهرة حلم إلا بسقياه. في هذه اللحظة المفصلية التي تختلط فيها دموع الفرح بنبضات الإنجاز، لا يسعني إلا أن أرفع أكف الحمد والشكر لمن بيده تحقيق الغايات، وبلوغ الأمنيات، الحمد لله أولاً وآخراً، ظاهرًا وباطنًا، سرًا وعلانية فهو الذي ألهم وسدّد، ووفّق وأعان.

شكرٌ خاص، بحجم الجهد المبذول، وعُمق الأثر المخلّد، للأستاذ الفاضل والمؤطّر الكريم : نابتي عبد الرزاق، الذي لم يكن مجرد موجّه أكاديمي، بل كان نبراسًا أضاء درب البحث، ومدادًا شحذ الهمة. ثم أزجي عبارات الشكر والتقدير الى لجنة المناقشة الموقرة: الاستاذ عكروت كمال والأستاذ بومعزة النوري، على ما بذلوه من جهدٍ في القراءة والتوجيه، وما تفضلوا به من ملاحظات ثمينة ونقد بنّاء سيكون زادًا في دروب العلم القادمة.

ولا أنسى، بكل الحب والتقدير، جميع الأساتذة الكرام الذين غرسوا فينا بذور العلم ، وصقلوا فينا شغف المعرفة، ورافقونا رحلة بعد رحلة حتى أزهرت عقولنا وأينعت أفكارنا. ونسأل الله في ختام هذا الجهد، أن يبارك فيه، ويجعله من العلم النافع وأن يسدد خطانا لما فيه الخير.

إهـــــداء

﴿وَآخِرُ دَعْوَاهُمْ أَنِ الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ﴾

الى من كانت دعواتها سرّ توفيقي إلى من علّمتني الصبر، وغرست في قلبي حبّ العطاء، إلى من سهرت لأرتاح، وضحّت لأكبر، واحتملت لأصل امي

إلى السند الذي لا يميل، والنبراس الذي أنار لي دروب الحياة. إلى من علّمني أن الخطوة الأولى تبدأ بالإيمان، وأن النجاح لا يُصنع إلا بالإرادة والعمل ابي

إلى من تقرأ عيناي ، إلى من شاركتني الضحكات، وخفّفت عني الآهات، إلى السند في الحياة اختي

إلى من كانوا وما زالوا الامتداد الجميل لروحي، إلى الذين نشأت معهم، فكبر في قلبي حبهم، وتعلّق في وجداني وجودهم اخوتي

إلى دفءِ البيت، وطمأنينة العمر، ومصدر الحب الأول جدي و جدتي

لرجالٍ كانوا العون عند الحاجة، والسند حين يُفتقد السند أخوالي و أعمامي

إلى من كان لهن في قلبي ركنٌ خاص عماتي

إلى رفقاء الدرب الذين جعلوا المشوار أخف، والقلوب أدفأ، فكان وجودهم نورا في دروب الحياة، وجسرًا لعبور كل صعاب الزمن.

ملخص

تُبرز هذه المذكرة أهمية النماذج الرياضية ذات البنية العمرية في فهم سلوك الأمراض المعدية المعقّدة. يُظهر النموذج المدروس كيف تؤثر العوامل الزمنية كالعمر وفترة الكمون على انتشار العدوى، كما يبرهن على إمكانية التحكم في الديناميكيات الوبائية من خلال استراتيجيات تدخّل فعالة. وقد بيّنت النتائج أن التلقيح الموجّه يمكن أن يحدّ بشكل ملحوظ من انتشار المرض، مما يعزّز دور النمذجة في دعم القرارات الصحية القائمة على أسس علمية.

الكلمات المفتاحية:نموذج وبائي ذو بنية عمرية مزدوجة؛ انتكاس؛ عدد التكاثر الاساسي؛ الاستقرار الشامل؛ دالة ليابونوف التابعية؛ التحكم الامثل؛ المشتقة على طريقة غاتو.

Résumé

Ce mémoire met en évidence l'importance des modèles mathématiques à structure d'âge dans la compréhension du comportement des maladies infectieuses complexes. Le modèle étudié montre comment des facteurs temporels tels que l'âge biologique et la période d'incubation influencent la propagation de l'infection, et démontre la possibilité de contrôler la dynamique épidémique par des stratégies d'intervention efficaces. Les résultats indiquent qu'une vaccination ciblée peut réduire considérablement la propagation de la maladie, ce qui renforce le rôle de la modélisation mathématique dans l'appui aux décisions de santé publique fondées sur des bases scientifiques.

Mots-clés: Modèle épidémique structuré par âge; Rechute; Nombre de reproduction de base; Stabilité globale; Fonctionnelle de Lyapunov; Contrôle optimal; Dérivée de Gâteaux.

Abstract

This thesis highlights the importance of age-structured mathematical models in understanding the behavior of complex infectious diseases. The studied model illustrates how temporal factors such as biological age and incubation period affect the spread of infection, and demonstrates the possibility of controlling epidemic dynamics through effective intervention strategies. The results show that targeted vaccination can significantly reduce the spread of the disease, reinforcing the role of mathematical modeling in supporting scientifically based public health decisions.
keywords: Age-structure epidemic model; relapse; basic reproduction number; global stability; Lyapunov functional; optimal control; gateaux derivative.

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Introduction

Infectious diseases have long represented a persistent threat to humanity, having caused major health crises throughout history that claimed millions of lives and deeply affected the social and economic fabric of societies. Despite significant advances in science and medicine that have made it possible to control many epidemics, infectious diseases continue to present new challenges due to complex factors such as globalization, climate change, urban expansion, and the increasing interaction between humans and nature. These conditions have led to more intricate epidemic patterns and highlighted the urgent need for precise analytical tools capable of understanding transmission mechanisms, predicting outbreak dynamics, and proposing effective containment strategies.

In this context, *mathematical modeling* has emerged as a central tool in epidemiology, enabling researchers to represent biological and behavioral phenomena through systems of equations that can be rigorously analyzed. These models have evolved from simple frameworks based on compartmentalization—dividing the population into susceptible, infected, and recovered groups—to more refined approaches that incorporate additional biological features such as *age structure*, *latency*, and the *possibility of reinfection*. Some infectious diseases, such as tuberculosis and certain viral infections, can remain dormant in the host for extended periods before reactivation, necessitating the use of models that capture such complexity [19].

This study proposes a *mathematical model with a double age structure*—based on both the biological age of individuals and the duration of the latent period. The model divides the population into four compartments: the **susceptible** $s(t, a)$, the **exposed** or latent individuals $e(t, \theta)$, the **infectious** $I(t)$, and the **recovered** $R(t)$. This structure enables a more accurate representation of diseases that involve a *variable latent period* and potential *loss of immunity or viral reactivation* [10].

The thesis is divided into two chapters. The **first chapter** provides a qualitative analysis of the model *without any intervention*, focusing on the *basic reproduction number* R_0, a fundamental

threshold used to evaluate the infectious potential of a disease. It further analyzes the *existence and stability of equilibrium points* and investigates the long-term behavior of the system using mathematical tools that ensure the model's well-posedness and consistency [35].

The **second chapter** explores the application of *optimal control theory* to reduce the disease burden through a *targeted treatment strategy* aimed at the susceptible population. Unlike classical approaches that use Pontryagin's Maximum Principle, this part employs *Gâteaux derivatives* from variational analysis to establish the existence of an optimal control that balances minimizing infection levels and reducing intervention costs [34]. Numerical simulations are presented in the impact of treatment of disease dynamics and demonstrating the potential of this framework to improve public health outcomes [23].

Preliminaries

Definition 0.1 (Cauchy Sequence) *Page 47, [27]*

Let (X, d) be a metric space. A sequence $(x_n) \subset X$ is called a *Cauchy sequence* if:

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall m, n \geq N,\ d(x_n, x_m) < \varepsilon.$$

Definition 0.2 (Banach Space) *Pages 58-59, [22]*

A **Banach space** is a normed vector space $(X, \|\cdot\|)$ such that every Cauchy sequence $(x_n) \subset X$ converges to a limit in X. In other words, X is complete with respect to the norm $\|\cdot\|$.

Definition 0.3 (Lipschitz Function) *Page 9, [5]*

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is called **Lipschitz** if there exists a constant $C \geq 0$ such that:

$$d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.$$

Any such C is referred to as a Lipschitz constant of f. The minimal Lipschitz constant is called the **dilatation** of f and denoted by $\operatorname{dil} f$.

Definition 0.4 (Strong Convergence) *Page 24, [22]*

A sequence (x_n) in a Hilbert space H is said to converge **strongly** to $x \in H$ if:

$$\lim_{n \to \infty} \|x_n - x\| = 0$$

Definition 0.5 (Weak Convergence) *Page 35, [22]*

A sequence (x_n) in a Hilbert space H is said to converge **weakly** to $x \in H$ if:

$$\lim_{n \to \infty} \langle x_n, y \rangle = \langle x, y \rangle \quad \text{for all } y \in H$$

Definition 0.6 (Uniformly bounded) *Page 205, [22]*

Let X and Y be normed vector spaces, and let $\{T_\alpha\}_{\alpha \in A}$ be a family of bounded linear operators from X to Y, i.e.,

$$T_\alpha : X \to Y, \quad \forall \alpha \in A.$$

The family $\{T_\alpha\}_{\alpha \in A}$ is said to be *uniformly bounded* if there exists a constant $M > 0$ such that

$$\|T_\alpha\| := \sup_{\|x\|_X = 1} \|T_\alpha(x)\|_Y \leq M, \quad \text{for all } \alpha \in A.$$

Equivalently, this means:

$$\sup_{\alpha \in A} \|T_\alpha\| < \infty.$$

Definition 0.7 (Equicontinuous) *Page 154, [28]*

Let X be a metric space, and let $\mathcal{F}$ be a family of functions $f : X \to \mathbb{R}$. The family $\mathcal{F}$ is ***equicontinuous at a point*** *$x_0 \in X$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$ and all $x \in X$ with $d(x, x_0) < \delta$, we have:*

$$|f(x) - f(x_0)| < \varepsilon.$$

If the condition holds for every $x_0 \in X$, then $\mathcal{F}$ is said to be ***equicontinuous on*** *X.*

Definition 0.8 (Uniform convergence) *Page 147 [28]*

Let (f_n) be a sequence of functions $f_n : X \to \mathbb{R}$, where X is a metric space. We say that f_n **converges uniformly** to a function $f : X \to \mathbb{R}$ if:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \forall x \in X, \ |f_n(x) - f(x)| < \varepsilon.$$

That is, the convergence is uniform over all $x \in X$.

Definition 0.9 (Closed set) *Page 32, [28]*

A subset C of a metric space (X, d) is called **closed** if it contains all its limit points. Equivalently, C is closed if every convergent sequence (x_n) in C converges to a limit x that is also in C.

Formally:

$$C \subseteq X \text{ is closed } \iff \forall (x_n) \subseteq C, \ x_n \to x \implies x \in C.$$

Definition 0.10 (Convex set) *Page 23, [4]*

A subset C of a vector space V over $\mathbb{R}$ is called **convex** if for any two points $x, y \in C$ and for every $t \in [0, 1]$, the point

$$(1 - t)x + ty$$

also belongs to C.

Formally:

$$C \subseteq V \text{ is convex } \iff \forall x, y \in C,\ \forall t \in [0,1],\ (1-t)x + ty \in C.$$

Definition 0.11 (Totally bounded) *Page 89, [6]*

A subset $A \subset X$ is **totally bounded** if for every $\varepsilon > 0$, there exists a finite set of points $\{x_1, x_2, \ldots, x_n\} \subset X$ such that

$$A \subset \bigcup_{i=1}^{n} B(x_i, \varepsilon)$$

where $B(x_i, \varepsilon)$ is the open ball centered at x_i with radius ε.

Definition 0.12 (Compact Global Attractor) *Pages 23-24, [7]*

Let (X, d) be a metric space and $\{S(t)\}_{t\geq 0}$ a semigroup of continuous operators on X. A set $\mathcal{A} \subset X$ is called a **compact global attractor** if:

1. **Invariance:** $S(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$.
2. **Compactness:** $\mathcal{A}$ is compact in X.
3. **Attractiveness:** For every bounded subset $B \subset X$,

$$\lim_{t\to\infty} \operatorname{dist}(S(t)B, \mathcal{A}) = 0.$$

Definition 0.13 (L^p Spaces) *Pages 125-130, [26]*

Let $(X, \mathcal{A}, \mu)$ be a measure space. For $1 \leq p < \infty$, the space $L^p(X)$ is defined as the set of (equivalence classes of) measurable functions $f : X \to \mathbb{R}$ such that

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{1/p} < \infty.$$

For $p = \infty$, the space $L^\infty(X)$ consists of essentially bounded measurable functions with norm

$$\|f\|_\infty = \inf\{M \geq 0 : |f(x)| \leq M \text{ for almost every } x \in X\}.$$

Definition 0.14 (Norms on L^p Spaces) *Pages 130-132, [26]*

For $1 \leq p < \infty$, the norm on the space $L^p(X)$ is defined by

$$\|f\|_p = \left(\int_X |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}.$$

For $p = \infty$, the norm is given by the essential supremum norm

$$\|f\|_\infty = \operatorname{ess\,sup}_{x \in X} |f(x)|,$$

which means the smallest number $M \geq 0$ such that $|f(x)| \leq M$ for almost every $x \in X$.

Definition 0.15 (Point Dissipative Dynamical System) *Page 321, [38]*

A dynamical system $(X, \{T(t)\}_{t \geq 0})$ on a metric space X is **point dissipative** if there exists a bounded set $B \subset X$ such that for every $x \in X$, there exists $t_x \geq 0$ with:

$$T(t)x \in B \quad \text{for all } t \geq t_x.$$

This implies that all trajectories eventually enter and remain in the bounded set B.

Definition 0.16 (Positively Invariant Set) *Page 2, [31]*

Let $S \subseteq \mathbb{R}^n$ be a set, and consider the dynamical system:

$$\dot{x}(t) = f(x(t))$$

The set S is called **positively invariant** if every solution $x(t)$ with initial condition $x(0) \in S$ satisfies:

$$x(t) \in S \quad \forall t \geq 0$$

That is, trajectories starting in S remain in S for all future time.

Definition 0.17 (Minimizing Sequence) *Section 1.2 [8]* A sequence (x_n) in a metric space X is called a **minimizing sequence** for a functional $J : X \to \mathbb{R} \cup \{+\infty\}$ if

$$\lim_{n \to \infty} J(x_n) = \inf_{x \in X} J(x).$$

In other words, the values of $J(x_n)$ approach the infimum of J on X as n grows.

Definition 0.18 (Admissible Set) *Page 5, [33]*

An **admissible set** is a collection of functions (or controls, states, etc.) that satisfy the constraints of a variational or optimization problem. These constraints may include boundary conditions, integrability, smoothness, or inequality constraints.

For example, the admissible set of controls is given by:

$$\mathcal{U}_{ad} = \left\{ u \in L^2(0,T) \mid u_{min} \leq u(t) \leq u_{max}, \text{ a.e. on } (0,T) \right\}$$

Definition 0.19 (C_0-Semigroup) *Page 328, [2]*

Let X be a Banach space. A family $\{T(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ is called a **C_0-semigroup** if:

1. $T(0) = I$, the identity operator on X,
2. $T(t+s) = T(t)T(s)$ for all $t, s \geq 0$, $For every x \in X$, $\lim_{t\downarrow 0} \|T(t)x - x\| = 0$.

3. **Definition 0.20 (Dense Set)** *Page 32, [28]*

Let X be a topological space and $A \subseteq X$. The set A is said to be **dense in** X if for every point $x \in X$, and every neighborhood U of x, the intersection $A \cap U \neq \emptyset$. Equivalently, $\overline{A} = X$, where $\overline{A}$ denotes the closure of A.

Definition 0.21 (Sobolev Space) *Page 57, [1]*

Let $\Omega \subseteq \mathbb{R}^n$ be an open subset. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the **Sobolev space** $W^{k,p}(\Omega)$ is defined as the set of functions $u \in L^p(\Omega)$ whose weak derivatives of order up to k also belong to $L^p(\Omega)$. That is:

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \text{ for all } |\alpha| \leq k \}$$

where $D^\alpha u$ denotes the weak derivative of u corresponding to the multi-index α.

Definition 0.22 (Alpha and Omega Limit Sets) *Pages 126-127, [17]*

Let ϕ_t be a flow on a metric space X and $x \in X$. The **alpha-limit set** $\alpha(x)$ and the **omega-limit set** $\omega(x)$ of the point x are defined as follows:

$$\alpha(x) = \{ y \in X \mid \exists\, t_n \to -\infty \text{ with } \phi_{t_n}(x) \to y \}$$

$$\omega(x) = \{y \in X \mid \exists\, t_n \to +\infty \text{ with } \phi_{t_n}(x) \to y\}$$

Definition 0.23 (Lower semicontinuity) *Pages 33-35, [25]*

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *lower semicontinuous* at $x_0 \in X$ if for every $\alpha < f(x_0)$, there exists a neighborhood U of x_0 such that

$$f(x) > \alpha \quad \forall x \in U.$$

Equivalently,

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

Definition 0.24 (Lyapunov function) *Page 93, [21]*

Consider the system:

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \quad f(0) = 0.$$

A continuously differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called a *Lyapunov function* near the origin if:

1. $V(0) = 0$,
2. $V(x) > 0$ for all $x \neq 0$ (positive definite),
3. $\dot{V}(x) = \nabla V(x) \cdot f(x) \leq 0$ for all x near the origin.

If $\dot{V}(x) < 0$ for all $x \neq 0$, the origin is said to be *asymptotically stable*.

Definition 0.25 (Arzelà–Ascoli Theorem) *Page 158, [28]*

Let K be a compact metric space, and let $\mathcal{F} \subset C(K)$ be a family of real-valued continuous functions on K. If $\mathcal{F}$ is equicontinuous and pointwise relatively compact (i.e., for each $x \in K$, the set $\{f(x) : f \in \mathcal{F}\}$ has compact closure in $\mathbb{R}$), then $\mathcal{F}$ is relatively compact in $C(K)$ with the uniform norm.

Definition 0.26 (LaSalle's Invariance Principle Theorem) *Page 125, [21]*

Let $D \subset \mathbb{R}^n$ be a compact positively invariant set for the autonomous system $\dot{x} = f(x)$, where f is continuous. Suppose there exists a continuously differentiable function $V : D \to \mathbb{R}$ such that:

1. $\dot{V}(x) \leq 0$ for all $x \in D$.

Let $E = \{x \in D \mid \dot{V}(x) = 0\}$, and let M be the largest invariant set contained in E. Then every solution starting in D approaches M as $t \to \infty$.

Definition 0.27 (Gâteaux derivative) *Page 22, [11]*

Let X and Y be normed vector spaces and $F : X \to Y$ be a function. The *Gâteaux derivative* of F at $x \in X$ in the direction $h \in X$ is defined as

$$D_G F(x)(h) = \lim_{t \to 0} \frac{F(x + th) - F(x)}{t},$$

provided this limit exists. If it exists for all $h \in X$, then F is said to be Gâteaux differentiable at x.

Definition 0.28 (Uniform Persistence) *(Page 36, [16])*

Let $X \subset \mathbb{R}^n$, and let $\phi : [0, \infty) \times X \to X$ be a semiflow. The system is said to be uniformly persistent on a positively invariant set $M \subset X$ if there exists $\eta > 0$ such that

$$\liminf_{t \to \infty} d(\phi(t, x), \partial \widehat{M}) \geq \eta, \quad \forall x \in M,$$

where $\widehat{M} \subset X$ is a compact subset of M and $\partial \widehat{M}$ denotes its boundary.

CHAPTER 1

Stability analysis of an age-structured epidemic model

In this chapter, we formulate an age-structured SEIRI model to describe the transmission dynamics of an infectious disease. Based on the information and discussion provided in the introduction, we propose a model in which the population is divided into four compartments: susceptible, latent, infectious, and recovered individuals. In this framework, susceptible individuals may have different levels of susceptibility, the latency period varies among latent individuals, and recovered individuals may become reinfected. Precisely, let $s(t, a)$ represent the density of susceptible individuals at time t and susceptibility age a, $e(t, \theta)$ denote the density of latent individuals (those infected but not yet infectious) at time t with latency age θ, $I(t)$ be the number of infectious individuals at time t, and $R(t)$ be the number of recovered individuals at time t. We assume that all recruitment into the population occurs exclusively in the susceptible class at a constant flux Λ. New infections occur at the rate $I(t)\int_0^\infty \beta(a)s(t, a)\,da$, where $\beta(a)$ is the transmission rate that depends on the susceptibility age. Latent individuals progress to the infectious class at rate $\psi(\theta)$. Recovered individuals may be reinfected at rate γ, and infectious individuals recover at rate σ. A natural death rate μ is assumed to apply to all individuals. Under these assumptions, the disease transmission dynamics can be described by the following

mathematical model:

$$\begin{cases} \dfrac{\partial s(t,a)}{\partial t} + \dfrac{\partial s(t,a)}{\partial a} = -\mu s(t,a) - \beta(a) I(t) s(t,a), \\ \dfrac{\partial e(t,\theta)}{\partial t} + \dfrac{\partial e(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta)) e(t,\theta), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta) e(t,\theta)\, d\theta + \gamma R(t) - (\mu + \sigma) I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu + \gamma) R(t), \end{cases} \tag{1.1}$$

with the boundary conditions

$$s(t,0) = \Lambda, \quad e(t,0) = I(t) \int_0^\infty \beta(a) s(t,a)\, da, \tag{1.2}$$

and initial conditions

$$s(0,a) = s_0(a), \quad e(0,\theta) = e_0(\theta), \quad I(0) = I_0, \quad R(0) = R_0. \tag{1.3}$$

Assumption A: For system (1.1)-(1.3), we assume that:

(i) $\Lambda, \mu, \sigma, \gamma > 0$, $\psi \in L^\infty_+(0,\infty)$ and $\beta \in L^\infty_+(0,\infty) \cap L^1_+(0,\infty)$;

(ii) β is a Lipschitz continuous function on $\mathbb{R}_+$;

(iii) Let $a \in [0, a^*]$, $\theta \in [0, \theta^*]$, where a and θ represent the maximum age of susceptibility and latency, which may be infinite. If $a^*, \theta^* = +\infty$, we assume that $s(t,a) = 0$, $e(t,\theta) = 0$ for sufficiently large values of a and θ.

1.1 Existence and uniqueness

In this part, we study the existence and uniqueness of solutions in order to ensure that the system (1.1)-(1.3) is well-defined and satisfies the required mathematical consistency by using the integrated semi-group approach [3]. For this purpose, we will first reformulate system (1.1)-(1.3) as a non-densely defined Cauchy problem by proceeding as follows:

We consider the space

$$Y = \mathbb{R} \times L^1(0, +\infty).$$

Then, we define the linear operators $A_j : D(A_j) \subset Y \to Y$ as follows:

$$A_1 \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} -\phi_1(0) \\ -\phi_1'(a) - \mu\phi_1(a) \end{pmatrix}, \quad A_2 \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} -\phi_2(0) \\ -\phi_2'(\theta) - \omega(\theta)\phi_2(\theta) \end{pmatrix},$$

where $\omega(\theta) = \mu + \psi(\theta)$, with $D(A_j) = \{0\} \times W^{1,1}(\mathbb{R}_+, \mathbb{R})$, $j = 1, 2$.

Hence, it us important to observe that A_j are non-densely defined operators (since $D(A_j)$ are not dense in Y). Moreover, for any $\lambda \in \rho(A_1)$, where $\rho(A_1)$ denotes the resolvent set of A_1, and $\begin{pmatrix} 0 \\ \phi_1 \end{pmatrix} \in D(A_1)$. If

$$(\lambda I - A_1)^{-1} \begin{pmatrix} c_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_1 \end{pmatrix},$$

then, a straightforward calculation yields

$$\phi_1(a) = c_1 e^{-(\lambda+\mu)a} + \int_0^a e^{-(\lambda+\mu)(a-s)} y_1(s)\, ds, \quad \text{for all } a \in \mathbb{R}_+. \tag{1.4}$$

Similarly, for any $\lambda \in \rho(A_2)$, where $\rho(A_2)$ denotes the resolvent set of A_2, and $\begin{pmatrix} 0 \\ \phi_2 \end{pmatrix} \in D(A_2)$. So, if

$$(\lambda I - A_2)^{-1} \begin{pmatrix} c_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi_2 \end{pmatrix},$$

then, we obtain

$$\phi_2(\theta) = c_2 e^{-\int_0^\theta (\lambda+\omega_2(s))\, ds} + \int_0^\theta e^{-\int_s^\theta (\lambda+\omega(\sigma))\, d\sigma} y_2(s)\, ds, \quad \text{for all } \theta \in \mathbb{R}_+. \tag{1.5}$$

Accordingly, it is clear to notice that system (1.1)-(1.3) is equivalent to the following system:

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} 0 \\ s(t,\cdot) \end{pmatrix} = A_1 \begin{pmatrix} 0 \\ s(t,\cdot) \end{pmatrix} + \begin{pmatrix} \Lambda - \beta(a)I(t)s(t,a) \\ 0 \end{pmatrix}, \\ \dfrac{d}{dt}\begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix} = A_2 \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix} + \begin{pmatrix} I(t)\displaystyle\int_0^\infty \beta(a)s(t,a)da \\ 0 \end{pmatrix}, \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi\theta)e(t,\theta)d\theta + \gamma R(t) - (\mu+\sigma)I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu+\gamma)R(t). \end{cases} \tag{1.6}$$

Now, by setting

$$\hat{s}(t) = \begin{pmatrix} 0 \\ s(t,\cdot) \end{pmatrix}, \quad \hat{e}(t) = \begin{pmatrix} 0 \\ e(t,\cdot) \end{pmatrix}.$$

Thus, we can rewrite system (1.6) as non-densely defined Cauchy problem

$$\begin{cases} \dfrac{d\hat{s}(t)}{dt} = A_1\hat{s}(t) + F_1\left(\hat{s}(t), \hat{e}(t), I(t), R(t)\right), \\ \dfrac{d\hat{e}(t)}{dt} = A_2\hat{e}(t) + F_2\left(\hat{s}(t), \hat{e}(t), I(t), R(t)\right), \\ \dfrac{dI(t)}{dt} = -(\mu+\sigma)I(t) + F_3\left(\hat{s}(t), \hat{e}(t), I(t), R(t)\right), \\ \dfrac{dR(t)}{dt} = -(\mu+\gamma)R(t) + F_4\left(\hat{s}(t), \hat{e}(t), I(t), R(t)\right), \end{cases} \tag{1.7}$$

where

$$F_1\left(\hat{s}(t),\hat{e}(t),I(t),R(t)\right)=\begin{pmatrix}\Lambda-\beta(a)I(t)s(t,a)\\0\end{pmatrix},$$

$$F_2\left(\hat{s}(t),\hat{e}(t),\hat{i}(t),R(t)\right)=\begin{pmatrix}I(t)\int_0^\infty\beta(a)s(t,a)da\\0\end{pmatrix},$$

$$F_3\left(\hat{s}(t),\hat{e}(t),\hat{i}(t),R(t)\right)=\int_0^\infty\psi\theta)e(t,\theta)d\theta+\gamma R(t),$$

$$F_4\left(\hat{s}(t),\hat{e}(t),\hat{i}(t),R(t)\right)=\sigma I(t).$$

Next, define the spaces

$$\mathbb{Y}=\mathrm{Y}\times\mathrm{Y}\times\mathbb{R}\times\mathbb{R},\quad\text{and}\quad\mathbb{Y}_+=\mathrm{Y}_+\times\mathrm{Y}_+\times\mathbb{R}_+\times\mathbb{R}_+.$$

Define the linear operator $A:D(A)\subset\mathbb{Y}\to\mathbb{Y}$ by

$$A\begin{pmatrix}\hat{s}(t)\\\hat{e}(t)\\I(t)\\R(t)\end{pmatrix}=\begin{pmatrix}A_1\hat{s}(t)\\A_2\hat{e}(t)\\-(\mu+\sigma)I(t)\\-(\mu+\gamma)R(t)\end{pmatrix},\tag{1.8}$$

with $D=D(A_1)\times D(A_2)\times\mathbb{R}\times\mathbb{R}$. Thus, we have

$$\overline{D(A)}=\{0\}\times L^1(0,\infty)\times\{0\}\times L^1(0,\infty)\times\mathbb{R}\times\mathbb{R},$$

and hence, $D(A)$ is not dense in $\mathbb{Y}$. Moreover, we define the nonlinear function $F:\overline{D(A)}\to\mathbb{Y}$ by

$$F\begin{pmatrix}\hat{s}(t)\\\hat{e}(t)\\I(t)\\R(t)\end{pmatrix}=\begin{pmatrix}F_1\left(\hat{s}(t),\hat{e}(t),I(t),R(t)\right)\\F_2\left(\hat{s}(t),\hat{e}(t),I(t),R(t)\right)\\F_3\left(\hat{s}(t),\hat{e}(t),I(t),R(t)\right)\\F_4\left(\hat{s}(t),\hat{e}(t),I(t),R(t)\right)\end{pmatrix}.\tag{1.9}$$

Set $\mathbb{Y}_0:=\overline{D(A)}$ and its positive cone $\mathbb{Y}_{0+}:=\overline{D(A)}\cap\mathbb{Y}_+$. Then, we can reformulate system

(1.1)-(1.3) as the following abstract Cauchy problem:

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t) + F(u(t)), \quad t \geq 0, \\ u(0) = u_0 \in \mathbb{Y}_{0+}, \end{cases} \tag{1.10}$$

where $u(t) = (\hat{s}(t), \hat{e}(t), I(t), R(t))$ and $u_0 = (\hat{s}_0, \hat{e}_0, I_0, R_0)$. From the definition of the linear operator A given by (1.8) together with the expressions (1.4) and (1.5), it follows that:

H1. A is a Hill-Yosida operator with $(-\mu, \infty) \subset \rho(A)$, i.e., there exists $M \geq 0$ such that

$$\|(\lambda I - A)^{-n}\| \leq \frac{M}{(\mathrm{Re}(\lambda) + \mu)^n}, \quad \text{for } \mathrm{Re}(\lambda) > -\mu \text{ and } n \geq 1.$$

Moreover, from (1.9) we can check that

H2. F is locally Lipschitz operator, i.e., for all $C > 0$, there exists $L(C) > 0$ such that

$$\|F(\xi) - F(\tilde{\xi})\|_X \leq L\|\xi - \tilde{\xi}\|,$$

for all $\xi, \tilde{\xi} \in \mathbb{Y} \cap \bar{B}(0, C)$, where $\bar{B}(0, C)$ is the ball of $\mathbb{Y}$, centered in 0 and with radius $C > 0$. Suppose that the initial condition u_0 satisfies the following compatibility condition:

$$s_0(0) = \Lambda, \quad \text{and} \quad e_0(0) = I_0 \int_0^\infty \beta(a) s_0(a)\, da. \tag{1.11}$$

Then, we can derive the following result:

Proposition 1.1 *Under H1 and H2, for each $u_0 \in \mathbb{Y}_0$, problem (1.10) has a unique continuous solution $u \in C([0, t_{\max}), \mathbb{Y})$, for some $t_{\max} > 0$.*

Again, with the help of the expressions (1.4) and (1.5), it can be deduced that

H3. A is resolvent positive, i.e.,

$$(\lambda I - A)^{-1}\mathbb{Y}_+ \subset \mathbb{Y}_+, \quad \text{for all } \lambda \in \rho(A).$$

Furthermore, the formula (1.9) of the non-linearity F implies that

H4. For every $r > 0$, there exists $\alpha \geq 0$ such that

$$F(u) + \alpha u \in \mathbb{Y}_+, \quad \text{for all } u \in B(0, r) \cap \mathbb{Y}_{+0},$$

where $B(0,r)$ denotes the ball of $\mathbb{Y}$, centered in $0 \in \mathbb{Y}$ and with radius r. Then, we can obtain the non-negativity of solution of problem (1.10) as follows:

Proposition 1.2 *Under H3 and H4, for each $u_0 \in \mathbb{Y}_{+0}$, the solution u of problem (1.10) is satisfying*

$$u(t) \in \mathbb{Y}_{+0}, \quad \textit{for all } t \in [0, t_{\max}).$$

Define the space

$$\mathcal{Y} := L^1_+(0,\infty) \times L^1_+(0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+,$$

equipped with the norm

$$\|(x_1,x_2,x_3,x_4)\|_X = \int_0^\infty |x_1(a)|da + \int_0^\infty |x_2(\theta)|d\theta + |x_3| + |x_4|,$$

where $L^1_+(0,\infty)$ is the set space of integrable functions from $(0,\infty)$ into $\mathbb{R}_+ = [0,+\infty)$. Note that the initial condition of the system (1.1)-(1.3) can be written as follows

$$x_0 = (s_0(\cdot), e_0(\cdot), I_0, R_0) \in \mathcal{Y}.$$

Then, for each $x_0 = (s_0(\cdot), e_0(\cdot), I_0, R_0) \in \mathcal{Y}$ satisfying the compatibility condition (1.11), system (1.1)-(1.3) has a unique continuous solution $(s(t,\cdot), e(t,\cdot), I(t), R(t)) \in C([0,t_{\max}), \mathcal{Y})$. Next, we pass to show that this solution exists globally, i.e., $(s(t,\cdot), e(t,\cdot), I(t), R(t)) \in C(\mathbb{R}_+, \mathcal{Y})$.

Proposition 1.3 *Let $x_0 \in \mathcal{Y}$. Then, the solution $(s(t,\cdot), e(t,\cdot), I(t), R(t))$ of system (1.1)-(1.3) exists globally.*

Proof To extend the domain of existence from $t \in [0,t_{\max})$ to $t \in \mathbb{R}_+$, it suffices to show that the solution does not blow-up in finite time. Let $x_0 \in \mathcal{Y}_t$ and $s(t,\cdot), e(t,\cdot), I(t), R(t)) \in C([0,t_{\max}), \mathcal{Y})$ be the solution of system (1.1)-(1.3). Suppose by contradiction that $t_{\max} < \infty$.
From system (1.1)-(1.2), we can have

$$\frac{d}{dt}\left(\int_0^\infty s(t,a) + \int_0^\infty e(t,\theta) + I(t) + R(t)\right) = \Lambda - \int_0^\infty \beta(a)I(t)s(t,a)$$

$$-\mu\int_0^\infty s(t,a) - \mu\int_0^\infty e(t,\theta) - \int_0^\infty \psi(\theta)e(t,\theta) + I(t)\int_0^\infty \beta(a)s(t,a)da$$

$$+\int_0^\infty \psi(\theta)e(t,\theta)d\theta + \gamma R(t) - \mu I(t) - \delta I(t) + \delta I(t) - \mu R(t) - \gamma R(t).$$

Then,

$$\frac{d}{dt}\left(\underbrace{\int_0^\infty s(t,a)+\int_0^\infty e(t,\theta)+I(t)+R(t)}_{x(t)}\right)=\Lambda-\mu\underbrace{\int_0^\infty s(t,a)da+\int_0^\infty e(t,a)d\theta+I(t)+R(t))}_{x(t)}$$

Hence, we obtain

$$\frac{d}{dt}x(t)=\Lambda-\mu x(t)$$

Via Gronwall, we have

$$x(t)=\left(x(0)-\frac{\Lambda}{\mu}\right)e^{-\mu t}+\frac{\Lambda}{\mu}.$$

By passing to the limit as t goes to $t_{\max}$, we get

$$\lim_{t\to t_{\max}}\|x(t)\|_{\mathcal{Y}}<\infty. \tag{1.12}$$

In conclusion,

$$\lim_{t\to t_{\max}}\left(\|s(t,\cdot)\|_1+\|e(t,\cdot)\|_1+I(t)+R(t)\right)<\infty,$$

which gives a contradiction, and hence, we have claimed that $t_{\max}=+\infty$. This completes the proof of Proposition 1.3.

So we can define the semi-flow $U:\mathbb{R}_+\times\mathcal{Y}\to\mathcal{Y}$ generated by the system (1.1)-(1.3) as follows

$$U(t,x_0)=(s(t,\cdot),e(t,\cdot),I(t),R(t)),\quad t\geq 0,x_0\in\mathcal{Y}. \tag{1.13}$$

Hence

$$\|U(t,x_0)\|_{\mathcal{Y}}=\|(s(t,\cdot),e(t,\cdot),I(t),R(t))\|_{\mathcal{Y}}=\int_0^\infty s(t,a)da+\int_0^\infty e(t,\theta)d\theta+I(t)+R(t).$$

Moreover, we define

$$\Sigma:=\left\{(s,e,I,R)\in\mathcal{Y}:\int_0^\infty s(t,a)da+\int_0^\infty e(t,\theta)d\theta+I(t)+R(t)\leq\frac{\Lambda}{\mu}\right\}.$$

Then, for the semi-flow $U(t,x_0)$ defined in (1.13), we obtain

Proposition 1.4 *Consider the system (1.1)-(1.3). Then*

(i) The set Σ is positively invariant under the semi-flow $U(t,x_0)_{t\leq 0}$, i.e., $U(t,x_0)\in\Sigma$ for all

$x_0 \in \Sigma$, $t \geq 0$;

(ii) The semi-flow $U(t, x_0)$ is point dissipative, and the set Σ attracts every point in $\mathcal{Y}$.

Proof From the system (1.1)-(1.3), we get

$$\begin{aligned}
\frac{d}{dt}\|U(t,x_0)\|_{\mathcal{Y}} &= \int_0^\infty \frac{\partial s(t,a)}{\partial t}da + \int_0^\infty \frac{\partial e(t,\theta)}{\partial t}d\theta + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} \\
&= \underbrace{-\int_0^\infty \frac{\partial s(t,a)}{\partial a}da}_{s(t,0)} - \mu\int_0^\infty s(t,a)da - \underbrace{I(t)\int_0^\infty \beta(a)s(t,a)da}_{e(t,0)} \\
&\quad \underbrace{-\int_0^\infty \frac{\partial e(t,\theta)}{\partial \theta}d\theta}_{e(t,0)} - \mu\int_0^\infty e(t,\theta)d\theta - \int_0^\infty \psi(\theta)e(t,\theta)d\theta + \int_0^\infty \psi(\theta)e(t,\theta)d\theta \\
&\quad - \mu I(t) - \mu R(t). \\
&\leq \Lambda - \mu\|U(t,x_0)\|_{\mathcal{Y}}
\end{aligned}$$

Via Gronwall, we get

$$\|U(t,x_0)\|_{\mathcal{Y}} \leq \frac{\Lambda}{\mu} - e^{-\mu t}\left(\frac{\Lambda}{\mu} - \|x_0\|_{\mathcal{Y}}\right), \tag{1.14}$$

which implies that $U(t,x_0) \in \Sigma$, $t \geq 0$, for any solution of system (1.1)-(1.3) satisfying $x_0 \in \Sigma$. Hence, $\{U(t,x_0)\}_{t\geq 0}$ is positively invariant for the set Σ. Furthermore, passing to the limit as $t \to \infty$ in (1.14), we get $\limsup_{t\to\infty}\|U(t,x_0)\|_{\mathcal{Y}} \leq \frac{\Lambda}{\mu}$, for any $x_0 \in \Sigma$. As a result, the semi-flow $\{U(t,x_0)\}_{t\geq 0}$ is point dissipative, and the set Σ attracts all points in $\mathcal{Y}$. This completes the proof of Proposition 1.4.

From 1.4, we obtain the following

Proposition 1.5 *Let $D \geq \frac{\Lambda}{\mu}$ be a constant. If $x_0 \in \mathcal{Y}$ and $\|x_0\|_{\mathcal{Y}} \leq D$, then*

(i) $0 \leq \int_0^\infty s(t,a)da, \int_0^\infty e(t,\theta)d\theta, I(t), R(t) \leq D$, for all $t \geq 0$;

(ii) there exists $\varepsilon_1 > 0$ such that

$$\liminf_{t\to\infty}\int_0^\infty s(t,a)da \geq \varepsilon_1, \quad \text{with } \varepsilon_1 := \frac{\Lambda}{\mu + D\|\beta\|_\infty}.$$

1.2 Existence and local stability of equilibria

In this section, we address the existence and local asymptotic stability of the equilibria of system (1.1)–(1.3). We begin by introducing the threshold parameter known as the basic reproduction

number, which quantifies the expected number of secondary infections caused by a single infected individual. The basic reproduction number, denoted by $\mathcal{R}_0$, is a key epidemiological indicator that determines whether an infectious disease can infect and persist in a fully susceptible population. Specifically, $\mathcal{R}_0$ represents the average number of secondary cases generated by one infectious individual during their entire infectious period. For example The basic reproduction number R_0 is estimated to be approximately 2.5 for COVID-19 (and up to 8–12 for Omicron), and around 1.0 for tuberculosis (typically ranging from 0.5 to 1.5). The calculation of $\mathcal{R}_0$ is typically based on the next-generation matrix method, which separates new infection terms from other transitions in the model, allowing us to derive an explicit expression for this threshold value. Following the argument presented in [9], the basic reproduction number of system (1.1)–(1.3) is given by the following expression

$$\mathcal{R}_0 = \frac{\Lambda}{\mu+\sigma}\xi_1\xi_2 + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)}, \tag{1.15}$$

where

$$\xi_1 = \int_0^\infty \beta(a)\pi_1(a)\,da, \quad \xi_2 = \int_0^\infty \psi(\theta)\pi_2(\theta)\,d\theta,$$

and

$$\pi_1(a) = e^{-\mu a}, \quad \pi_2(\theta) = e^{-\int_0^\theta(\mu+\psi(\tau)}.$$

Next, we turn to the main objective of this section. We begin with the following result

Lemma 1.1 *System (1.1)-(1.3) always admits a unique disease-free equilibrium, given by*

$$E^0 = (s^0(a), 0, 0, 0),$$

such that

$$s^0(a) = \Lambda\pi_1(a), \quad \textit{for all } a \in \mathbb{R}^+. \tag{1.16}$$

Proof In the absence of disease transmission, there are no infected individuals, i.e., $e(t,\theta) = I(t) = R(t) = 0$. Setting the time derivatives in system (1.1)-(1.3) equal to zero, the disease-free equilibrium must satisfy the following first-order ordinary differential equation

$$\begin{cases} \frac{ds^0(a)}{da} = -\mu s^0(a), \\ s^0(0) = \Lambda, \end{cases}$$

which yields the solution

$$s^0(a) = \Lambda\pi_1(a), \quad \forall a \in \mathbb{R}_+.$$

Therefore, System (1.1)-(1.3) always has a unique disease-free equilibrium given by E^0= ($s^0(a)$, $0, 0, 0$), which proves Lemma 1.1. Now, we will apply the linearization method as described in Webb [37, Section 4.5] to establish the following

Theorem 1.1 *Let $\mathcal{R}_0$ be given by (1.15). Then, E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof To examine the local stability of E^0, we begin by linearizing the system (1.1)–(1.3) around E^0. So, we introduce small perturbation variables as follows

$$\bar{s}(t,a) = s(t,a) - s^0(a), \quad \bar{e}(t,\theta) = e(t,\theta), \quad I(t) = I(t), \quad \text{and} \quad \bar{R}(t) = R(t).$$

The linearized system is then expressed as

$$\begin{cases} \dfrac{\partial \bar{s}(t,a)}{\partial t} + \dfrac{\partial \bar{s}(t,a)}{\partial a} = -\mu\bar{s}(t,a) - \beta(a)s^0(a)\bar{I}(t), \\ \dfrac{\partial \bar{e}(t,\theta)}{\partial t} + \dfrac{\partial \bar{e}(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta))\bar{e}(t,\theta), \\ \dfrac{d\bar{I}(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta)\bar{e}(t,\theta)d\theta + \gamma\bar{R}(t) - (\mu+\sigma)\bar{I}(t), \\ \dfrac{d\bar{R}(t)}{dt} = \sigma\bar{I}(t) - (\mu+\gamma)\bar{R}(t), \\ \bar{s}(t,0) = 0, \\ \bar{e}(t,0) = \bar{I}(t)\displaystyle\int_0^\infty \beta(a)s^0(a)da. \end{cases} \tag{1.17}$$

We now set

$$\bar{s}(t,a) = \phi_1(a)e^{\lambda t}, \quad \bar{e}(t,\theta) = \phi_2(\theta)e^{\lambda t}, \quad I(t) = \phi_3 e^{\lambda t}, \quad \text{and} \quad \bar{R}(t) = \phi_4 e^{\lambda t},$$

for all $\lambda \in \mathbb{R}$, where $(\phi_1(a), \phi_2(\theta), \phi_3, \phi_4)$ will be determined later. Consequently, the linearized

system (1.17) becomes

$$\begin{cases} \dfrac{d\phi_1(a)}{da} = -(\lambda+\mu)\phi_1(a) - \beta(a)s^0(a)\phi_3, \\ \dfrac{d\phi_2(\theta)}{d\theta} = -(\lambda+\mu+\psi(\theta))\phi_2(\theta), \\ (\lambda+\mu+\sigma)\phi_3 = \displaystyle\int_0^\infty \psi(\theta)\phi_2(\theta)d\theta + \gamma\phi_4, \\ (\lambda+\mu+\gamma)\phi_4 = \sigma\phi_3, \\ \phi_1(0) = 0, \\ \phi_2(0) = \phi_3 \displaystyle\int_0^\infty \beta(a)s^0(a)da. \end{cases} \tag{1.18}$$

It is important to observe that the first equation in system (1.18) is decoupled from the rest. as a result, our analysis can be restricted to the subsystem involving only to the disease-transmission classes

$$\begin{cases} \dfrac{d\phi_2(\theta)}{d\theta} = -(\lambda+\mu+\psi(\theta))\phi_2(\theta), \\ \phi_2(0) = \phi_3 \displaystyle\int_0^\infty \beta(a)s^0(a)da, \\ (\lambda+\mu+\sigma)\phi_3 = \displaystyle\int_0^\infty \psi(\theta)\phi_2(\theta)d\theta + \gamma\phi_4, \\ (\lambda+\mu+\gamma)\phi_4 = \sigma\phi_3. \end{cases} \tag{1.19}$$

Solving the ordinary differential equation in system (1.19), we find that

$$\phi_2(\theta) = \phi_3\pi_2(\theta)e^{-\lambda\theta}\int_0^\infty \beta(a)s^0(a)da. \tag{1.20}$$

At this point, we need to show $\lambda+\mu+\sigma \neq 0$ and $\lambda+\mu+\gamma \neq 0$. Assume, For the sake of contradiction that $\lambda+\mu+\sigma = 0$. Substituting (1.20) into the third equation of system (1.19), we immediately obtain $\phi_3 = 0$. Then, Using the last equation of (1.19), it follows that $\lambda+\mu+\gamma = 0$, which contradicts our assumption. Similarly, conclude that $\lambda+\mu+\gamma \neq 0$. Hence, from the last equation of system (1.19), we deduce that

$$\phi_4 = \frac{\sigma}{\lambda+\mu+\gamma}\phi_3. \tag{1.21}$$

Substituting the expressions for $\phi_2(\theta)$ and ϕ_4 into the third equation of (1.19), we obtain

$$\phi_3 = \frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma}\int_0^\infty \beta(a)s^0(a)da + \frac{\sigma\gamma}{(\lambda+\mu+\gamma)(\lambda+\mu+\sigma)}\phi_3,$$

where

$$\hat{\xi}_2(\lambda) = \int_0^\infty \psi(\theta)\pi_2(\theta)e^{-\lambda\theta}d\theta,$$

denotes the Laplace transform of the function $\psi(\theta)\pi_2(\theta)$. As a result, the characteristic equation of (1.17) at E^0 can be written as

$$\frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma}\int_0^\infty \beta(a)s^0(a)da + \frac{\sigma\gamma}{(\lambda+\mu+\gamma)(\lambda+\mu+\sigma)} = 1. \tag{1.22}$$

Let us denote the left-hand side of equation (1.22) by $\Gamma(\lambda)$, it is clear that $\Gamma(\lambda)$ is a continuously differentiable function and satisfies the following propertirs

$$\lim_{\lambda\to-\infty}\Gamma(\lambda) = +\infty, \quad \lim_{\lambda\to+\infty}\Gamma(\lambda) = 0, \quad \text{and} \quad \Gamma'(\lambda) < 0.$$

Hence, Γ is a monotonically decreasing function with respect to $\lambda \in \mathbb{R}$. Given that $\Gamma(0) = \mathcal{R}_0 > 0$, the Intermediate Value Theorem guarantess the existence of a positive real root of the equation $\Gamma(\lambda) = 1$ this implies that E^0 is unstable when $\mathcal{R}_0 > 1$. On the other hand, we can prove that all roots of equation (1.22) have negative real parts when $\mathcal{R}_0 < 1$. Suppose, to the contrary, that equation (1.22) admits an eigenvalue λ such that $\mathrm{Re}(\lambda) \geq 0$. More precisely, consider the case where (1.22) has a real root $\lambda = \lambda_0 \geq 0$ with no imaginary part. Then, we obtain

$$\begin{aligned}
1 &= \frac{\hat{\xi}_2(\lambda_0)}{\lambda_0+\mu+\sigma}\int_0^\infty \beta(a)\, s^0(a)\, da + \frac{\sigma\gamma}{(\lambda_0+\mu+\gamma)(\lambda_0+\mu+\sigma)} \\
&\leq \frac{\xi_2}{\mu+\sigma}\int_0^\infty \beta(a)\, s^0(a)\, da + \frac{\sigma\gamma}{(\mu+\gamma)(\mu+\sigma)} \\
&= \mathcal{R}_0.
\end{aligned}$$

This leads to a contradiction with the assumption that $\mathcal{R}_0 < 1$. Therefore, any eigenvalue λ of equation (1.22) must have a negative real part when $\mathcal{R}_0 < 1$, which implies that E^0 is locally asymptotically stable. In conclusion, we have proven that E^0 is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$. This completes the proof of Theorem 1.1

Next, we proceed by proving the existence of the endemic equilibrium and analyzing its local

asymptotic stability. We begin by introducing the following

Lemma 1.2 *Suppose that $\mathcal{R}_0 > 1$. In this case system (1.1)-(1.3) admits a unique endemic equilibrium, such that*

$$E^* = (s^*(a), e^*(\theta), I^*, R^*),$$

satisfying the following

$$\begin{aligned} s^*(a) &= s^0(a)\, e^{-I^* \int_0^a \beta(\tau)\, d\tau}, \\ e^*(\theta) &= \pi_2(\theta)\, I^* \int_0^\infty \beta(a)\, s^0(a)\, e^{-I^* \int_0^a \beta(\tau)\, d\tau}\, da, \\ R^* &= \frac{\sigma}{\mu+\gamma} I^*, \end{aligned}$$

where I^* is given as the solution of equation (1.12).

Remark 1.1 *Observe that if we set $I^* = 0$, the system naturally the disease-free equilibrium $E^0 = (s^0(a), 0, 0, 0)$.*

Proof The equilibrium equations are obtained by setting all time derivatives in system to zero. Let $E^* = (s^*(a), e^*(\theta), I^*, R^*)$ represent the endemic equilibrium of system (1.1)-(1.3). It must then satisfy the following system that is independent of time

$$\begin{cases} \dfrac{ds^*(a)}{da} = -\mu s^*(a) - I^* \beta(a) s^*(a), \\ \dfrac{de^*(\theta)}{d\theta} = -(\mu + \psi(\theta)) e^*(\theta), \\ 0 = \displaystyle\int_0^\infty \psi(\theta) e^*(\theta) d\theta + \gamma R^* - (\mu+\sigma) I^*, \\ 0 = \sigma I^* - (\mu+\gamma) R^*, \\ s^*(0) = \Lambda, \\ e^*(0) = I^* \displaystyle\int_0^\infty \beta(a) s^*(a) da. \end{cases} \tag{1.23}$$

By integrating the first and second ODE of system (1.23), we obtain

$$s^*(a) = s^0(a) e^{-I^* \int_0^a \beta(\tau) d\tau}, \quad \text{for all } a \in \mathbb{R}_+,$$

and

$$e^*(\theta) = \pi_2(\theta) I^* \int_0^\infty \beta(a) s^0(a) e^{-I^* \int_0^a \beta(\tau) d\tau} da, \quad \text{for all } \theta \in \mathbb{R}_+,$$

A straightforward computation of the fourth equation in system (1.23) yields

$$R^* = \frac{\sigma}{\mu+\gamma} I^*.$$

By substituting the expressions for $e^*(\theta)$ and R^* into the third equation of system (1.23), We obtain

$$\frac{\xi_2}{\mu+\sigma} + I^* \int_0^\infty \beta(a) s^0(a) e^{-I^* \int_0^a \beta(\tau)d\tau} da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)} I^* = I^*.$$

Since $I^* > 0$, we may divide both sides of the above equation by I^*, which yields

$$\frac{\xi_2}{\mu+\sigma} \int_0^\infty \beta(a) s^0(a) e^{-I^* \int_0^a \beta(r)dr} da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)} = 1. \tag{1.24}$$

Denoting the left-hand side of (1.24) by $P(I^*)$, Clearly, P is monotonically decreasing function tending 0 as $I^* \to \infty$. and we have $P(0) = \mathcal{R}_0 > 1$. Hence, the equation $P(I^*) = 1$ admits a unique positive solution $I^* > 0$. As a resut, when $\mathcal{R}_0 > 1$, system (1.23) possesses a unique positive endemic equilibrium $E^* = (s^*(a), e^*(\theta), I^*, R^*)$. This completes the proof of lemma 1.2.

We now proceed to examine the local asymptotic stability of E^*. Aiming to understand the behavior and spread the disease once it becomes endemic in the population.

Theorem 1.2 *The endemic equilibrium E^* of system (1.1)-(1.3) is locally asymptotically stable provided that $\mathcal{R}_0 > 1$.*

Proof To analyze the local stability of E^*, we linearize system (1.1)-(1.3) around E^*. So, we introduce the perturbation variables

$$\bar{s}(t,a) = s(t,a) - s^*(a), \quad \bar{e}(t,\theta) = e(t,\theta) - e^*(\theta), \quad \bar{I}(t) = I(t) - I^*, \quad \text{and} \quad \bar{R}(t) = R(t) - R^*.$$

These perturbations are assumed to satisfy the following linearized system

$$\begin{cases} \dfrac{\partial \bar{s}(t,a)}{\partial t} + \dfrac{\partial \bar{s}(t,a)}{\partial a} = -\mu \bar{s}(t,a) - I^* \beta(a) \bar{s}(t,a) - \beta(a) s^*(a) \bar{I}(t), \\ \dfrac{\partial \bar{e}(t,\theta)}{\partial t} + \dfrac{\partial \bar{e}(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta)) \bar{e}(t,\theta), \\ \dfrac{d\bar{I}(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta) \bar{e}(t,\theta) d\theta + \gamma \bar{R}(t) - (\mu + \sigma) \bar{I}(t), \\ \dfrac{d\bar{R}(t)}{dt} = \sigma \bar{I}(t) - (\mu + \gamma) \bar{R}(t), \\ \bar{s}(t,0) = 0, \\ \bar{e}(t,0) = I^* \displaystyle\int_0^\infty \beta(a) \bar{s}(t,a) da + \bar{I}(t) \int_0^\infty \beta(a) s^*(a) da. \end{cases} \tag{1.25}$$

Next, we propose an exponential form for the solution of system (1.25) as follows:

$\bar{s}(t,a) = \phi_1(a) e^{\lambda t}$, $\bar{e}(t,\theta) = \phi_2(\theta) e^{\lambda t}$, $\bar{I}(t) = \phi_3 e^{\lambda t}$, and $\bar{R}(t) = \phi_4 e^{\lambda t}$,

where $\phi_1(a)$, $\phi_2(\theta)$, ϕ_3, ϕ_4 is to be determined later. Thus, system (1.25) is rewritten as

$$\begin{cases} \dfrac{d\phi_1(a)}{da} = -(\lambda + \mu + I^* \beta(a)) \phi_1(a) - \beta(a) s^*(a) \phi_3, \\ \dfrac{d\phi_2(\theta)}{d\theta} = -(\lambda + \mu + \psi(\theta)) \phi_2(\theta), \\ (\lambda + \mu + \sigma) \phi_3 = \displaystyle\int_0^\infty \psi(\theta) \phi_2(\theta) d\theta + \gamma \phi_4, \\ (\lambda + \mu + \gamma) \phi_4 = \sigma \phi_3. \\ \phi_1(0) = 0, \\ \phi_2(0) = I^* \displaystyle\int_0^\infty \beta(a) \phi_1(a) da + \phi_3 \int_0^\infty \beta(a) s^*(a) da. \end{cases} \tag{1.26}$$

Solving the first and second differential equations of system (1.26), we find

$$\phi_1(a) = -\phi_3 \int_0^a \beta(\tau) s^*(\tau) e^{-I^* \int_\tau^a \beta(\ell) d\ell} e^{-(\lambda+\mu)(a-\tau)} d\tau, \tag{1.27}$$

and

$$\phi_2(\theta) = e^{-\lambda\theta} \pi_2(\theta) \left[-I^* \int_0^\infty \beta(a) \left(\int_0^a \beta(\tau) s^*(\tau) e^{-I^* \int_\tau^a \beta(\ell) d\ell} e^{(\lambda+\mu)(a-\tau)} d\tau \right) da \right. \tag{1.28}$$

$$\left. + \int_0^\infty \beta(a) s^*(a) da \right] \phi_3.$$

As in the earlier analysis , we can confirm that $\lambda + \mu + \sigma \neq 0$ and $\lambda + \mu + \gamma \neq 0$. Consequently,

the third and fourth equations of system (1.26) yield the following expressions:

$$\phi_3 = \frac{1}{\lambda+\mu+\sigma}\int_0^\infty \psi(\theta)\phi_2(\theta)d\theta + \frac{\sigma\gamma}{(\lambda+\mu+\sigma)(\lambda+\mu+\gamma)}\phi_3. \tag{1.29}$$

Substituting (1.28) into the equation (1.29), we find

$$\phi_3 = \frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma}\Big[-I^*\int_0^\infty \beta(a)\int_0^a \beta(\tau)s^*(\tau)e^{-I^*\int_\tau^a \beta(\ell)d\ell}e^{(\lambda+\mu)(a-\tau)}d\tau da$$

$$+\int_0^\infty \beta(a)s^*(a)da\Big]\phi_3 + \frac{\sigma\gamma}{(\lambda+\mu+\sigma)(\lambda+\mu+\gamma)}\phi_3.$$

Dividing the both sides of the previous equation by $\phi_3 \neq 0$, we obtain the following expression

$$1 = \frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma}\Big[-I^*\int_0^\infty \beta(a)\int_0^a \beta(\tau)s^*(\tau)e^{-I^*\int_\tau^a \beta(\ell)d\ell}e^{-(\lambda+\mu)(a-\tau)}d\tau da$$

$$+\int_0^\infty \beta(a)s^*(a)da\Big] + \frac{\sigma\gamma}{(\lambda+\mu+\sigma)(\lambda+\mu+\gamma)}.$$

This represents the characteristic equation of the linearized system (1.25) evaluated atE^*. Accordingly, it can be expressed as

$$1 + \frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma} - I^*\int_0^\infty \beta(a)\int_0^a \beta(\tau)s^*(\tau)e^{-I^*\int_\tau^a \beta(\ell)d\ell}e^{-(\lambda+\mu)(a-\tau)}d\tau da$$

$$= \frac{\hat{\xi}_2(\lambda)}{\lambda+\mu+\sigma}\int_0^\infty \beta(a)s^*(a)da + \frac{\sigma\gamma}{(\lambda+\mu+\sigma)(\lambda+\mu+\gamma)}.$$

Now, let us assume for contradiction that the equation (1.2) admits a complex root λ with $\mathrm{Re}\,(\lambda) \geq 0$. In particular, Suppose it has a real root $\lambda = \lambda_0 \geq 0$ with no imaginary part. Therefore, on one hand, we have

$$\left|\frac{\hat{\xi}_2(\lambda_0)}{\lambda_0+\mu+\sigma}\int_0^\infty \beta(a)s^*(a)da + \frac{\sigma\gamma}{(\lambda_0+\mu+\sigma)(\lambda_0+\mu+\gamma)}\right|$$

$$\leq \left|\frac{\hat{\xi}_2}{\mu+\sigma}\int_0^\infty \beta(a)s^*(a)da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)}\right| = 1.$$

Using equation (1.24), we obtain the expression above. we also have

$$\left|1+\frac{\hat{\xi}_2(\lambda_0)}{\lambda_0+\mu+\sigma}-I^*\int_0^\infty \beta(a)\int_0^a \beta(\tau)s^*(\tau)e^{-I^*\int_\tau^a \beta(\ell)d\ell}e^{-(\lambda_0+\mu)(a-\tau)}d\tau da\right|>1.$$

Observe that the left-hand side of the equation (1.2) is strictly greater than one, while the right-hand side is strictly less than one, this contradiction implies that all roots of equation (1.2) must have negative real parts. Consequently, E^* is locally asymptotically stable. This concludes the proof of Theorem 1.2.

1.3 Asymptotic smoothness

To demonstrate the existence of a compact global attractor, it is necessary to prove that the semi-flow U, generated by the system (1.1)-(1.3) is asymptotically smooth. As stated in [18], the semi-flow U is said to be asymptotically smooth if every forward invariant bounded closed set is attracted by a nonempty compact set.

Analogous to that of [37], integrating the equations of $s(t,a)$ and $e(t,\theta)$ on the system (1.1)-(1.3) along the characteristic lines $t-a=\text{const}$, $t-\theta=\text{const}$, respectively, we get

$$s(t,a)=\begin{cases}\Lambda\pi_1(a)e^{-\int_0^a \beta(\tau)I(\tau+t-a)\,d\tau}, & t>a,\\ s_0(a-t)\,e^{-\int_{a-t}^a \beta(\tau)I(\tau+t-a)d\tau}\dfrac{\pi_1(a)}{\pi_1(a-t)}, & t\le a,\end{cases}\tag{1.30}$$

and

$$e(t,\theta)=\begin{cases}M(t-\theta)I(t-\theta)\pi_2(\theta), & t>\theta.\\ e_0(\theta-t)\dfrac{\pi_2(\theta)}{\pi_2(\theta-t)}, & t\le\theta,\end{cases}\tag{1.31}$$

where

$$M(t)=\int_0^\infty \beta(a)s(t,a)\,da.\tag{1.32}$$

Before going on, we present the following proposition

Proposition 1.6 *The function $M(t)$ Lipschitz-continuous on $\mathbb{R}_+$.*

Additionally, We present the following two Lemmas from [30], which provide the essential tools to show the asymptotic smoothness of the semi-flow U generated by system (1.1)-(1.3).

Lemma 1.3 *(Theorem 3.2 [30]). The semi-flow $U : \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ is asymptotically smooth if there are maps $U_1, U_2 : \mathbb{R}_+ \times \mathcal{Y} \to \mathcal{Y}$ such that $U(t, x_0) = U_1(t, x_0) + U_2(t, x_0)$, and the following statements hold for any bounded closed set $\mathscr{A} \subset \mathcal{Y}$, which is forward invariant under U:*

(i) $\lim_{t\to\infty} \operatorname{diam} U_1(t, \mathscr{A}) = 0$;

(ii) There exists $t_{\mathscr{A}} > 0$ such that $U_2(t, \mathscr{A})$ has compact closure for each $t \geq t_{\mathscr{A}}$.

Since $\mathcal{Y}$ is an infinite dimensional space, $L^1_+(0, +\infty)$ is a subset of $\mathcal{Y}$. In infinite dimensional space, boundedness alone does not imply precompactness. Therefore, to establish the precompactness of $\mathcal{Y}$, we need to apply the following lemma:

Lemma 1.4 *(Theorem B.2 [30]). A set $\mathscr{A} \in L^1_+(0, +\infty)$ has a compact closure if the following statements hold*

(i) $\sup_{f\in\mathscr{A}} \int_0^\infty |f(\ell)|d\ell < \infty$;

(ii) $\lim_{r\to\infty} \int_r^\infty |f(\ell)|d\ell \to 0$ *uniformly in* $f \in \mathscr{A}$;

(iii) $\lim_{h\to 0^+} \int_0^\infty |f(\ell + h) - f(\ell)|d\ell \to 0$ *uniformly in* $f \in \mathscr{A}$;

(iv) $\lim_{h\to 0^+} \int_0^\infty |f(\ell)|d\ell \to 0$ *uniformly in* $f \in \mathscr{A}$.

After the above preparation, we can deal with this result:

Theorem 1.3 *The semi-flow U generated by system (1.1)-(1.3) is asymptotically smooth.*

Proof Firstly, we decompose $U(t, x_0)$ into two parts U_1 and U_2 defined, respectively, as follows: $U_1(t, x_0) = (\varphi_s(t, \cdot), \varphi_e(t, \cdot), 0, 0, 0), and \quad U_2(t, x_0) = (\hat{s}(t, \cdot), \hat{e}(t, \cdot), I(t), R(t))$, where

$$\varphi_s(t, a) = \begin{cases} 0, & t > a, \\ s_0(a - t)e^{-\int_{a-t}^{a} \beta(\tau)I(\tau+t-a)d\tau} \frac{\pi_1(a)}{\pi_1(a-t)}, & t \leq a. \end{cases} \tag{1.33}$$

$$\varphi_e(t, \theta) = \begin{cases} 0, & t > \theta, \\ e_0(\theta - t)\frac{\pi_2(\theta)}{\pi_2(\theta-t)}, & t \leq \theta. \end{cases} \tag{1.34}$$

$$\hat{s}(t,a) = \begin{cases} \Lambda\pi_1(a)e^{-\int_0^a \beta(\tau)(\tau+t-a)d\tau}, & t > a, \\ 0, & t \le a. \end{cases} \tag{1.35}$$

and

$$\hat{e}(t,\theta) = \begin{cases} I(t-\theta)M(t-\theta)\pi_2(\theta), & t > \theta, \\ 0, & t \le \theta. \end{cases} \tag{1.36}$$

For $x_0 \in \mathscr{A}$, where $\mathscr{A} \subset \mathcal{Y}$ is a bounded set, such that $r > \frac{\Lambda}{\mu}$ is a bound for $\mathscr{A}$, we obtain

$$\begin{aligned} \|U_1(t,x_0)\|_{\mathcal{Y}} &= \|\varphi_s(t,\cdot)\|_1 + \|\varphi_e(t,\cdot)\|_1 \\ &= \int_t^\infty s_0(a-t)e^{-\int_{a-t}^a \beta(\tau)I(\tau+t-a)d\tau}\frac{\pi_1(a)}{\pi_1(a-t)}da + \int_0^\infty e_0(\theta-t)\frac{\pi_2(\theta)}{\pi_2(\theta-t)}d\theta \\ &= \int_0^\infty s_0(a)e^{-\int_a^{a+t} \beta(\tau)I(\tau+t-a)d\tau}\frac{\pi_1(t+a)}{\pi_1(a)}da + \int_0^\infty e_0(\theta)\frac{\pi_2(t+\theta)}{\pi_2(\theta)}d\theta. \end{aligned}$$

Hence, we have

$$\|U_1(t,x_0)\|_{\mathcal{Y}} \le e^{-\mu t}(\|s_0\|_{L^1} + \|e_0\|_{L^1}) \le e^{-\mu t}\|x_0\|_{\mathcal{Y}} \le re^{-\mu t}.$$

Therefore,

$$\lim_{t\to\infty} U_1(t,\mathscr{A}) = 0,$$

and the condition (i) in Lemma 1.3 holds. Next, we move to show that $U_2(t,\mathscr{A})$ has compact closure for each $t \ge 0$. According to (i) of Proposition 1.5, we have $I(t)$ and $R(t)$ remain in the compact set $[0,D]$ where $D \ge \frac{\Lambda}{\mu}$ is a bound for $\mathscr{A}$. Thus, it remains only to verify that $\hat{s}(t,a)$ and $\hat{e}(t,\theta)$ remain in precompact subset of $L^1_+(0,+\infty)$ which is independent of x_0. To this end, it needs to verify that (i)-(iv) in Lemma 1.3. From (i) of Proposition 1.5, we could easily conclude that (i), (ii) and (iv) in Lemma 1.3 are hold. Now, we pass to claim the statement (iii) in Lemma 1.3. So, for sufficiently small $h \in (0,t)$, we have

$$
\begin{aligned}
&\int_0^{\infty} |\hat{e}(t,\theta+h)-\hat{e}(t,\theta)|\,d\theta \\
&= \int_0^{t-h} |\hat{e}(t,\theta+h)-\hat{e}(t,\theta)|\,d\theta + \int_{t-h}^{t} |\hat{e}(t,\theta)|\,d\theta \\
&= \int_0^{t-h} \Big| I(t-\theta-h)\,M(t-\theta-h)\,\pi_2(\theta+h) - I(t-\theta)\,M(t-\theta)\,\pi_2(\theta)\Big|\,d\theta \\
&\quad + \int_{t-h}^{t} I(t-\theta)\,M(t-\theta)\,\pi_2(\theta)\dot{d}\theta.
\end{aligned}
$$

Thus, it yields

$$
\begin{aligned}
&\int_0^{\infty} |\hat{e}(t,\theta+h)-\hat{e}(t,\theta)|\,d\theta \\
&\le \underbrace{\int_0^{t-h} |I(t-\theta-h)\,M(t-\theta-h)\,\pi_2(\theta+h) - \pi_2(\theta)|\;d\theta}_{\Xi_1} \\
&\quad + \underbrace{\int_0^{t-h} |I(t-\theta-h)\,M(t-\theta-h) - I(t-\theta)\,M(t-\theta)|\;\pi_2(\theta)\,d\theta}_{\Xi_2} \\
&\quad + \underbrace{\int_{t-h}^{t} I(t-\theta)\,M(t-\theta)\,\pi_2(\theta)\,d\theta}_{\Xi_3}.
\end{aligned}
$$

At this stage, from (i) of Proposition 1.5, it yields

$$
\begin{aligned}
\Xi_1 &\le D^2\|\beta\|_{\infty} \int_0^{t-h} |\pi_2(\theta+h)-\pi_2(\theta)|\,d\theta \\
&\le D^2\|\beta\|_{\infty} \left[\int_0^{t-h} \pi_2(\theta)\,d\theta - \int_0^{t-h} \pi_2(\theta+h)\,d\theta\right] \\
&= D^2\|\beta\|_{\infty} \left[\int_0^{h} \pi_2(\theta)\,d\theta - \int_{t-h}^{t} \pi_2(\theta)\,d\theta\right] \\
&\le D^2\|\beta\|_{\infty} h. \qquad (1.37)
\end{aligned}
$$

Furthermore, from the equation of $I(t)$ in system (1.1), it follows

$$
\left|\frac{dI(t)}{dt}\right| \le (\|\psi\|_{\infty} + \mu + \gamma + \sigma)D := L_I.
$$

Which implies that $\left|\frac{dI(t)}{dt}\right|$ has bound L_I and $I(t)$ is Lipschitz continuous on $(0,+\infty)$ with

Lipschitz coefficient L_I. Therefore, we have

$$\begin{aligned}\Xi_2 &\leq \int_0^{t-h} |I(t-\theta-h)\, M(t-\theta-h) - M(t-\theta)|\, \pi_2(\theta)\, d\theta \\ &\quad + \int_0^{t-h} M(t-\theta)\, |I(t-\theta-h) - I(t-\theta)|\, \pi_2(\theta)\, d\theta \\ &\leq (DL_M + D\|\beta\|_\infty L_I)\, h, \end{aligned} \tag{1.38}$$

where L_I and L_M are the Lipschitz constants of $I(t)$ and $M(t)$, respectively. Similarly, we could obtain

$$\Xi_3 = \int_{t-h}^{t} I(t-\theta)M(t-\theta)\pi_2(\theta)d\theta \leq 2D^2\|\beta\|_\infty h. \tag{1.39}$$

In conclusion, by collecting (1.37)-(1.39), we could write

$$\|\hat{e}(t,\cdot+h) - \hat{e}(t,\cdot)\|_1 \leq \hat{C}h, \quad \hat{C} = (2D^2\|\beta\|\infty + DL_M + D\|\beta\|_\infty L_I)h.$$

Therefore, $\lim_{h\to 0} \|\hat{e}(t,\cdot+h) - \hat{e}(t,\cdot)\|_1 = 0$ uniformly for any x_0 in $\mathscr{A}$, and hence $\hat{e}(t,\theta)$ remains in a precompact subset $\mathscr{A}_{\hat{e}}$ of $L_1^+(0,\infty)$. Similarly, we can claim that $\hat{s}(t,a)$ remains in precompact subset $\mathscr{A}_{\hat{s}}$ of $L_1^+(0,\infty)$. Thus, $U_1(t,x_0) \subseteq \mathscr{A}_{\hat{s}} \times \mathscr{A}_{\hat{e}} \times [0,D] \times [0,D]$, which is compact in $\mathcal{Y}$. It follows that $U_2(t,\mathscr{A})$ has compact closure, which implies that (iii) in Lemma 1.4 holds. Consequently, the semi-flow generated by system (1.1)-(1.3) is asymptotically smooth. This completes the proof of Theorem 1.3

According to [30], we can introduce the following useful result of the existence of a global attractor for semi-flow $\{U(t,x_0)\}_{t\geq 0}$:

Theorem 1.4 *The semi-flow $\{U(t,x_0)\}_{t\geq 0}$ has a global attractor $\mathscr{A}$ contained in $\mathcal{Y}$, which attracts the bounded sets of $\mathcal{Y}$.*

1.4 Uniform persistence

In this part, we shall demonstrate the uniform persistence of system (1.1)-(1.3) if $\mathcal{R}_0 > 1$, which indicates that there will always exist infectious individuals if the infectivity is initially exist. Let us first define

$$\hat{\mathcal{M}} := \left\{(s(t,\cdot), e(t,\cdot), I(t), R(t)) \in \mathcal{Y} : \int_0^\infty e(t,\theta)d\theta + I(t) + R(t) > 0\right\}$$

and

$$\partial\hat{\mathcal{M}} := \mathcal{Y} \setminus \hat{\mathcal{M}}.$$

Theorem 1.5 *For the semi-flow $\{U(t,x_0)\}_{t\geq 0}$ generated by system 1-3, the sets $\hat{\mathcal{M}}$ and $\partial\hat{\mathcal{M}}$ are positively invariant. Furthermore, for every $\zeta \in \partial\hat{\mathcal{M}}$, we have $U(t,x_0)\zeta \to E^0$, as t goes to infinity.*

Proof Let $(s_0(\cdot), e_0(\cdot), I_0, R_0) \in \hat{\mathcal{M}}$. Define

$$H(t) = I(t) + R(t) + \int_0^\infty e(t,\theta)d\theta.$$

Therefore, we find

$$\begin{aligned}\frac{dH(t)}{dt} &= e(t,0) - \mu\int_0^\infty e(t,\theta)\,d\theta - \mu I(t) - \mu R(t)\\ &\geq -\mu H(t)\end{aligned}$$

Moreover, we have $H(0) > 0$. Then, it yields

$$H(t) \geq H(0)e^{-\mu t} > 0.$$

which means that $U(t,x_0)\hat{\mathcal{M}} \subset \hat{\mathcal{M}}$, and hence, $\hat{\mathcal{M}}$ is positively invariant. For any $(s_0(\cdot), e_0(\cdot), I_0, R_0) \in \partial\hat{\mathcal{M}}$, we have $I(0) = I_0 = 0$, $R(0) = R_0 = 0$, and $\int_0^\infty e_0(\theta)d\theta = 0$. Thus, $\partial\hat{\mathcal{M}}$ is positively invariant. Next, let $(s_0(\cdot), e_0(\cdot), I_0, R_0) \in \hat{\mathcal{M}}$. Then, we have

$$\begin{cases}\dfrac{\partial e(t,\theta)}{\partial t} + \dfrac{\partial e(t,\theta)}{\partial \theta} = -(\mu+\psi(\theta))\,e(t,\theta),\\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta)\,e(t,\theta)\,d\theta + \gamma R(t) - (\mu+\sigma)\,I(t),\\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu+\gamma)\,R(t),\\ e(t,0) = I(t)\displaystyle\int_0^\infty \beta(a)\,s(t,a)\,da,\\ e(0,\theta)=e_0(\theta), \quad I(0) = 0, \quad R(0) = 0.\end{cases} \tag{1.40}$$

From (i) of Proposition 1.5, it follows

$$\int_0^\infty \beta(a)s(t,a)da \leq D\|\beta\|_\infty, \quad \text{as } t \to \infty.$$

Consequently, by comparison principle, it results

$$e(t,\theta) \le \tilde{e}(t,\theta), \quad I(t) \le \tilde{I}(t), \quad \text{and} \quad R(t) \le \tilde{R}(t), \tag{1.41}$$

where $(\tilde{e}(t,\theta), \tilde{I}(t), \tilde{R}(t))$ is the solution of the following system

$$\begin{cases} \dfrac{\partial \tilde{e}(t,\theta)}{\partial t} + \dfrac{\partial \tilde{e}(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta))\tilde{e}(t,\theta), \\ \dfrac{d\tilde{I}(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta)\tilde{e}(t,\theta)d\theta + \sigma\tilde{I}(t), \\ \dfrac{d\tilde{R}(t)}{dt} = \sigma\tilde{I}(t) - (\mu+\gamma)\tilde{R}(t), \\ \tilde{e}(t,0) = D\|\beta\|_\infty \tilde{I}(t), \\ \tilde{e}(0,\theta) = e_0(\theta), \quad I(0) = 0, \quad R(0) = 0. \end{cases} \tag{1.42}$$

By solving the first equation of system (1.42) along the characteristic line $t - \theta =$ constant, we obtain

$$\tilde{e}(t,\theta) = \begin{cases} D\|\beta\|_\infty \tilde{I}(t-\theta)\pi_2(\theta), & t > \theta \ge 0, \\ e_0(\theta - t)\dfrac{\pi_2(\theta)}{\pi_2(\theta - t)}, & \theta \ge t \ge 0. \end{cases} \tag{1.43}$$

Furthermore, solving the third equation of (**??**) while taking into account that $R(0) = 0$, we find

$$\tilde{R}(t) = \sigma \int_0^t e^{-(\mu+\gamma)\tau}\tilde{I}(t-\tau)d\tau. \tag{1.44}$$

Substituting (1.43) and (1.44)) into the second equation of system (1.42), we arrive at

$$\begin{cases} \dfrac{d\tilde{I}(t)}{dt} = D\|\beta\|_\infty \displaystyle\int_0^t \psi(\theta)\pi_2(\theta)\tilde{I}(t-\theta)d\theta + \sigma\gamma\int_0^t e^{-(\mu+\gamma)\tau}\tilde{I}(t-\tau)d\tau + \mathcal{F}(t) - (\mu+\sigma)\tilde{I}(t), \\ I(0) = 0, \end{cases} \tag{1.45}$$

where $\mathcal{F}$ is defined by

$$\mathcal{F}(t) = \int_t^\infty \psi(\theta)e_0(\theta - t)\frac{\pi_2(\theta)}{\pi_2(\theta - t)}d\theta.$$

It is clear that $\mathcal{F}(t) = 0$ as $t \to \infty$. Therefore, the equation (1.33) has a unique solution $\tilde{I}(t) = 0$. Consequently, from (1.29), it yields that $\tilde{e}(t,\theta) = 0$, for $0 \le \theta < t$. Moreover, for $0 \le t \le \theta$, we have

$$\left\| \tilde{e}(t,\theta) \,\right|_1 = \left\| \, e_0(\theta - t)\frac{\pi_2(\theta)}{\pi_2(\theta - t)} \right\|_1 \le e^{-\mu t}\|e_0\|_1. \tag{1.46}$$

This means that $\tilde{e}(t,\theta) \to 0$ as $t \to \infty$. Next, from the equation of $\tilde{R}(t)$ in system (1.45), we have $\tilde{R}(t) = 0$. In conclusion, from (1.41) it follows that $I(t) = 0, R(t) = 0$ and $e(t,\theta) = 0$ as $t \to \infty$. Moreover, we can conclude from the first equation of system (1.1)-(1.3) that $\lim\limits_{t\to\infty} s(t,a) = s^0(a)$. This completes the proof Thorem 1.5.

Next, we will show the uniform persistence of $\{U(t,x_0)\}_{t\geq 0}$ generated by system (1.1)-(1.3).

Theorem 1.6 *Suppose $\mathcal{R}_0 > 1$. The semi-flow $\{U(t,x_0)\}_{t\geq 0}$ is uniformly persistent with respect to $(\hat{\mathfrak{M}}, \partial\hat{\mathfrak{M}})$, i.e., there exists a constant $\varepsilon > 0$, such that*

$$\liminf_{t\to\infty} d(U(t,x_0)\zeta, \partial\hat{\mathfrak{M}}) \geq \varepsilon,$$

for any $\zeta \in \hat{\mathfrak{M}}$.

Proof As already shown, E^0 is globally asymptotically stable in $\partial\hat{\mathfrak{M}}$, then according to Theorem 4.2 in Hale and Waltman [18], we need to study the behavior of solutions starting in $\hat{\mathfrak{M}}$ in some neighborhood of E^0. Therefore, we should demonstrate that

$$W^s(E^0) \cap \hat{\mathfrak{M}} = \emptyset,$$

where

$$W^s(E^0) = \left\{\zeta \in \mathcal{Y} : \lim_{t\to\infty} U(t,x_0)\zeta = E^0\right\}.$$

To this end, we will proceed by contradiction. So, assuming that there exists $\tilde{\zeta} \in W^s(E^0) \cap \hat{\mathfrak{M}}$. Hence, there exists $t_1 > 0$ so that $I(t_1) + R(t_1) + \int_0^\infty e(t_1,\theta)d\theta > 0$ since $\tilde{\zeta} \in \hat{\mathfrak{M}}$. Therefore, it yields

$$I(t) + R(t) + \int_0^\infty e(t,\theta)d\theta > 0, \quad t \geq t_1 > 0.$$

On the other hand, due to $\mathcal{R}_0 > 1$, we can pick an $\varepsilon_0 > 0$ small enough so that

$$\frac{\xi_2}{\mu+\sigma}\int_0^\infty \beta(a)(s^0(a) - \varepsilon_0)da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)} > 1. \tag{1.47}$$

Since $\tilde{\zeta} \in W^s(E^0)$, we have $\lim\limits_{t\to\infty}(s(t,a), e(t,\theta), 0, 0) = (s^0(a), 0, 0, 0, 0)$. Then, for the above chosen ε_0, there exists $t_2 > 0$ such that

$$s(t,a) > s^0(a) - \varepsilon_0, \quad \text{for all } t \geq t_2. \tag{1.48}$$

Now, setting

$$J(t) = I(t) + \frac{\gamma}{\mu+\gamma} R(t) + \int_0^\infty \Phi(\theta) e(t,\theta) d\theta, \tag{1.49}$$

where $\Phi(\theta) : \mathbb{R}_+ \to \mathbb{R}_+$ is given by

$$\Phi(\theta) = \int_\theta^\infty \psi(r) e^{-\int_\theta^\tau (\mu+\psi(\ell))d\ell} d\tau. \tag{1.50}$$

It is clear that $\Phi(0) = \xi_2$ and

$$\frac{d\Phi(\theta)}{d\theta} = (\mu + \psi(\theta))\Phi(\theta) - \psi(\theta).$$

Next, differentiating $J(t)$ with respect to time t gives

$$\begin{aligned}\frac{dJ(t)}{dt} &= -\int_0^\infty \Phi(\theta) \left[\partial -\partial\theta e(t,\theta) + (\mu+\psi(\theta))e(t,\theta)\right] d\theta + \frac{dI(t)}{dt} + \frac{\gamma}{\mu+\gamma}\frac{dR(t)}{dt}. \\ &= \Phi(0)e(t,0) - \int_0^\infty \psi(\theta) e(t,\theta) d\theta + \frac{dI(t)}{dt} + \frac{\gamma}{\mu+\gamma}\frac{dR(t)}{dt}.\end{aligned}$$

Recalling that $e(t,0) = I(t)\int_0^\infty \beta(a)s^0(a,t)da$, and by applying (1.48), we obtain

$$\begin{aligned}\frac{dJ(t)}{dt} &\geq \xi_2 I(t) \int_0^\infty \beta(a)(s^0(a) - e_0)\, da + \frac{\sigma\gamma}{\mu+\gamma} I(t) - (\mu+\sigma)I(t) \\ &= (\mu+\sigma)\left[\frac{\xi_2}{\mu+\sigma}\int_0^\infty \beta(a)(s^0(a) - e_0)\, da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)} - 1\right] I(t) \\ &\geq (\mu+\sigma)\left[\frac{\xi_2}{\mu+\sigma}\int_0^\infty \beta(a)(s^0(a) - e_0)\, da + \frac{\sigma\gamma}{(\mu+\sigma)(\mu+\gamma)} - 1\right] I(t) \\ &\geq 0.\end{aligned}$$

Obviously, $J(t)$ is a non-decreasing function for $t \geq t_2$. Therefore, $J(t) \geq J(t_0) > 0$ for all $t \geq t_0$ with $t_0 = \max\{t_1, t_2\}$. This prevents $(e(t,\theta), I(t), R(t))$ converging to $(0,0,0)$ as $t \to \infty$, which contradicts the assumption of $\tilde{\zeta} \in W^s(E^0)$. Hence, $W^s(E^0) \cap \hat{\mathcal{M}} = \emptyset$. This proves Theorem 1.6.

Before investigating the global stability results in the next section, it is necessary to define the total trajectory which refers to the solution of (1.1)-(1.3) extended to $\mathbb{R}$. A total trajectory of U is defined as a function φ :R$\to$ $\mathcal{Y}$, $\varphi(t) = (s(t,\cdot), e(t,\cdot), I(t), R(t))$ such that $U(r, \varphi(t)) = \varphi(t+r)$, for all $t \in \mathbb{R}$ and $r \in \mathbb{R}_+$. Setting $s_r(t,a) = s(t+r,a)$, $e_r(t,\theta) = e(t+r,\theta)$, $I_r(t) = I(t+r)$, and $R_r(t) = R(t+r)$, we obtain

$$
\begin{cases}
s_r(t,a) = \begin{cases} \Lambda\pi_1(a)e^{-\int_0^a \beta(\tau)(I(\tau+t+r-a))d\tau}, & t > a, \\ s(r,a-t)e^{-\int_{a-t}^a \beta(\tau)(I(\tau+t+r-a))d\tau}\dfrac{\pi_1(a)}{\pi_1(a-t)}, & t \le a, \end{cases} \\
e_r(t,\theta) = \begin{cases} e(t+r-\theta,0)\pi_2(\theta), & t > \theta, \\ e(r,\theta-t)\dfrac{\pi_2(\theta)}{\pi_2(\theta-t)}, & t \le \theta, \end{cases} \\
\dfrac{dI_r(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta)e_r(t,\theta)d\theta + \gamma R_r(t) - (\mu+\sigma)I_r(t), \\
\dfrac{dR_r(t)}{dt} = (\mu+\gamma)R_r(t).
\end{cases}
$$

For $\tau \ge r \in \mathbb{R}_+$, with $t = \tau - r$, we arrive at

$$
\begin{cases}
s_r(\tau,a) = \begin{cases} \Lambda\pi_1(a)e^{-\int_0^a \beta(\ell)(I(\ell+\tau-a))d\ell}, & \tau - r > a, \\ s(r,a-(\tau-r))e^{-\int_{a-(\tau-r)}^a \beta(\ell)(I(\ell+\tau-a))d\ell}\dfrac{\pi_1(a)}{\pi_1(a-(\tau-r))}, & \tau - r \le a, \end{cases} \\
e_r(\tau,\theta) = \begin{cases} e(\tau-\theta,0)\pi_2(\theta), & \tau - r > \theta, \\ e(r,\theta-(\tau-r))\dfrac{\pi_2(\theta)}{\pi_2(\theta-(\tau-r))}, & \tau - r \le \theta, \end{cases} \\
\dfrac{dI_r(t)}{dt} = \displaystyle\int_0^{\tau-r} \psi(\theta)\pi_2(\theta)e(\tau-\theta,0)d\theta + \int_{\tau-r}^\infty \psi(\theta)e(\tau,\theta-(\tau-r))\frac{\pi_2(\theta)}{\pi_2(\theta-(\tau-r))}d\theta \\
\qquad + \gamma R_r(t) - (\mu+\sigma)I_r(t), \\
\dfrac{dR_r(t)}{dt} = \sigma I_r(t) - (\mu+\gamma)R_r(t).
\end{cases}
\tag{1.51}
$$

Letting $r \to -\infty$ in system (1.51) and substituting τ by t, we get for all $t \in \mathbb{R}$, $a, \theta \in \mathbb{R}_+$,

$$\begin{cases} s(t,a) = \Lambda \pi_1(a) e^{-\int_0^a \beta(\tau) I(t+\tau-a) d\tau}, \\ e(t,\theta) = e(t-\theta,0)\pi_2(\theta), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta)\pi_2(\theta) e(t-\theta,0) d\theta + \gamma R(t) - (\mu+\sigma) I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu+\gamma) R(t). \end{cases} \tag{1.52}$$

1.5 Global stability

In this section, we present the core findings of this study. Specifically , we aim to establish the global asymptotic stability of each equilibrium of the epidemic model described by system (1.1)-(1.3). Before we proceed, we define $G : (0, +\infty) \to \mathbb{R}_+$ as follows

$$G(x) = x - 1 - \ln x. \tag{1.53}$$

Clearly, the derivative is $G'(x) = 1 - \frac{1}{x}$ which implies that G is decreasing on $(0, 1]$, and increasing on $[1, +\infty)$. It has also only one extremum, which is a global minimum at $x = 1$. Additionally, we have $1 - x + \ln x \leq 0$ for $x > 0$, and the equality holds if and only if $x = 1$. This function is typically used to construct the Lyapunov functional in Volterra-Lotka systems [12]. Now, we start by establishing the global asymptotic stability of E^0 of system (1.1)-(1.3).

Theorem 1.7 *The disease-free equilibrium E^0 is globally asymptotically stable, if $\mathcal{R}_0 < 1$.*

Proof Let $\varphi(t) = (s(t,\cdot), e(t,\cdot), I(t), R(t))$ be a total trajectory of system (1.1)-(1.3) in $\mathcal{A}$. Construct the Lyapunov functional as follows:

$$V(t) = \xi_2 Y_1(t) + Y_2(t) + Y_3(t) + Y_4(t), \tag{1.54}$$

such that

$$\begin{aligned} Y_1(t) &= \int_0^\infty s^0(a)\, G\left(\frac{s(t,a)}{s^0(a)}\right) da, \\ Y_2(t) &= \int_0^\infty \Phi(\theta)\, e(t,\theta) d\theta, \\ Y_3(t) &= I(t), \\ Y_4(t) &= \frac{\gamma}{\mu+\gamma} R(t). \end{aligned}$$

Moreover, in view of Proposition 1.5, $s(t,a)$ is bounded and bounded away from 0. Consequently, $V(t)$ is bounded and hence well-defined on $\varphi(t)$. Moreover, by (1.39), $V(t)$ is nonnegative definite with global minimum 0 at E^0. Now, differentiating $Y_1(t)$ along the trajectory of system (1.1)-(1.3) with respect to t yields

$$\begin{aligned} \frac{dY_1(t)}{dt} &= \int_0^\infty s^0(a) \frac{\partial}{\partial t} G\left(\frac{s(t,a)}{s^0(a)}\right) da. \\ &= -\int_0^\infty \left(1 - \frac{s^0(a)}{s(t,a)}\right) \left[\frac{\partial}{\partial a} s(t,a) + \mu s(t,a) + \beta(a) I(t) s(t,a)\right] da. \end{aligned} \tag{1.55}$$

It is clear that

$$\left(1 - \frac{s^0(a)}{s(t,a)}\right) \left[\frac{\partial}{\partial a} s(t,a) + \mu s(t,a)\right] = s^0(a) \frac{\partial}{\partial a} G\left(\frac{s(t,a)}{s^0(a)}\right).$$

Therefore, the expression (1.55) can be simplified to

$$\frac{dY_1(t)}{dt} = -\int_0^\infty s^0(a) \frac{\partial}{\partial a} G\left(\frac{s(t,a)}{s^0(a)}\right) da - I(t) \int_0^\infty \left(1 - \frac{s^0(a)}{s(t,a)}\right) \beta(a) s(t,a) da.$$

Integrating by parts the first term on the right-hand side of equation (1.5), we obtain

$$\begin{aligned} \frac{dY_1(t)}{dt} = \underbrace{s^0(0) G\left(\frac{s(t,0)}{s^0(0)}\right)}_{=0} - \mu \int_0^\infty s^0(a) G\left(\frac{s(t,a)}{s^0(a)}\right) da \\ \underbrace{- I(t) \int_0^\infty \beta(a) s(t,a) da}_{e(t,0)} + I(t) \int_0^\infty \beta(a) s^0(a) da. \end{aligned}$$

Observe that $G\left(\frac{s(t,0)}{s^0(0)}\right) = G(1) = 0$. Thus, we have

$$\frac{dY_1(t)}{dt} = -e(t,0) + I(t) \int_0^\infty \beta(a) s^0(a) da - \mu \int_0^\infty s^0(a) G\left(\frac{s(t,a)}{s^0(a)}\right) da. \tag{1.56}$$

Similarly, we differentiate $Y_2(t)$ along the solutions of system (1.1)-(1.3) to obtain

$$\begin{aligned}\frac{dY_2(t)}{dt} &= -\int_0^\infty \Phi(\theta)\left[\frac{\partial}{\partial\theta}e(t,\theta)+(\mu+\psi(\theta))e(t,\theta)\right]d\theta\\ &= -\int_0^\infty \Phi(\theta)\frac{\partial}{\partial\theta}e(t,\theta)d\theta-\int_0^\infty(\mu+\psi(\theta))\Phi(\theta)e(t,\theta)d\theta\\ &= \xi_2 e(t,0)-\int_0^\infty \psi(\theta)e(t,\theta)d\theta.\end{aligned} \tag{1.57}$$

Moreover, the time derivatives of $Y_3(t)$ and $Y_4(t)$ are given, respectively, by

$$\frac{dY_3(t)}{dt} = \int_0^\infty \psi(\theta)e(t,\theta)d\theta-(\mu+\sigma)I(t)+\gamma R(t), \tag{1.58}$$

and

$$\frac{dY_4(t)}{dt} = \frac{\sigma\gamma}{\mu+\gamma}I(t)-\gamma R(t). \tag{1.59}$$

Substituting (1.56)-(1.59) into (1.54) and after doing some rearrangement, we obtain $\frac{dY(t)}{dt}$ along the trajectories of system (1.1)-(1.3) in the following form

$$\frac{dY(t)}{dt} = (\mu+\sigma)\left[(\mathcal{R}_0-1)I(t)-\frac{\mu}{\mu+\sigma}\int_0^\infty s^0(a)G\left(\frac{s(t,a)}{s^0(a)}\right)da\right].$$

Since $\mathcal{R}_0<1$, $\frac{dY(t)}{dt}\leq 0$ holds, i.e., $Y(t)$ is non-decreasing over time. Due to the boundedness of $V(t)$ on $\varphi(.)$, the α-limit set of $\varphi(t)$ must be contained in $\mathcal{M}_0$, the largest compact invariant subset in $\left\{\frac{dY(t)}{dt}=0\right\}$. Let's claim that $\mathcal{M}_0=E^0$. Recalling that $\frac{dY(t)}{dt}=0$ holds when $s(t,a)=s^0(a)$ and $I(t)=0$. Then, from (1.52) it follows that $e(t,\theta)=0$ and $R(t)=0$, for all $t\in\mathbb{R}$. Therefore, our claim holds. This implies that the α-limit consists only of E^0. Hence, $\varphi\equiv E^0$. So, $\mathcal{A}=\{E^0\}$. In conclusion, following the LaSalle's invariance principle (see, e.g., ([15], [36]), we can conclude that E^0 is globally asymptotically stable if $\mathcal{R}_0<1$. Which completes proof of Theorem 1.7

Next, we jump to show the global asymptotic stability of the endemic equilibrium E^* of system (1.1)-(1.3).

Theorem 1.8 *The endemic equilibrium E^* is globally asymptotically stable, if $\mathcal{R}_0>1$.*

Proof Let $\varphi(t)=(s(t,\cdot),e(t,\cdot),I(t),R(t))$ be the total U-trajectory in $\mathcal{A}$, for all $t\in\mathbb{R}$. Define the Lyapunov function by

$$W(t) = W_1(t)+W_2(t)+W_3(t)+W_4(t). \tag{1.60}$$

Where

$$W_1(t) = \int_0^\infty \frac{s^*(a)}{e^*(0)} G\left(\frac{s(t,a)}{s^*(a)}\right) da,$$
$$W_2(t) = \int_0^\infty \Psi(\theta) \frac{e(t,\theta)}{e^*(\theta)} d\theta,$$
$$W_3(t) = \frac{I}{\xi_2 e^*(0)} G\left(\frac{I(t)}{I^*}\right),$$
$$W_4(t) = \frac{\gamma R}{(\mu+\gamma)\xi_2 e^*(0)} G\left(\frac{R(t)}{R^*}\right),$$

with

$$\Psi(\theta) = \frac{\int_0^\theta \psi(\tau) e^*(\tau) d\tau}{\int_0^\theta \psi(\theta) e^*(\theta) d\theta}.$$

It is clear that $\Psi(0) = 1$, and $\Psi'(\theta) = \frac{\psi(\theta)e^*(\theta)}{\xi_2 e^*(0)}$. According to Proposition 1.5, $s(t,a)$, $e(t,\theta)$, $I(t)$ and $R(t)$ are bounded, and they are also bounded away from 0. This means that $W(t)$ is bounded on $\varphi(t)$, and hence $W(t)$ is well-defined. Thus, $W(t) \geq 0$ with global minimum 0 at E^*. Differentiating W_1 along the trajectory of system (1.1)-(1.3) with respect to time t yields

$$\begin{aligned}\frac{dW_1(t)}{dt} &= \int_0^\infty s^*(a) \frac{\partial}{\partial t} G\left(\frac{s(t,a)}{s^*(a)}\right) da \\ &= -\frac{1}{e^*(0)} \int_0^\infty \left(1 - \frac{s^*(a)}{s(t,a)}\right) \left[\frac{\partial}{\partial a} s(t,a) + \mu s(t,a)\right] da \\ &\quad - \frac{I(t)}{e^*(0)} \int_0^\infty \left(1 - \frac{s^*(a)}{s(t,a)}\right) \beta(a) s(t,a) da.\end{aligned} \tag{1.61}$$

Taking into account that

$$\left(1 - \frac{s^*(a)}{s(t,a)}\right) \left[\frac{\partial}{\partial a} s(t,a) + \mu s(t,a)\right] = s^*(a) \frac{\partial}{\partial a} G\left(\frac{s(t,a)}{s^*(a)}\right) - \left(1 - \frac{s^*(a)}{s(t,a)}\right) I^* \beta(a) s(t,a),$$

Then, (1.61) can be expressed as follows

$$\begin{aligned}\frac{dW_1(t)}{dt} &= -\frac{1}{e(0)} \int_0^\infty s^*(a) \frac{\partial}{\partial a} G\left(\frac{s(t,a)}{s^*(a)}\right) da + \frac{I^*}{e^*(0)} \int_0^\infty \left(1 - \frac{s^*(a)}{s(t,a)}\right) \beta(a) s(t,a) da \\ &\quad - \frac{I(t)}{e^*(0)} \int_0^\infty \left(1 - \frac{s^*(a)}{s(t,a)}\right) \beta(a) s(t,a) da.\end{aligned}$$

Recalling that $e^*(0) = I^* \int_0^\infty \beta(a)s^*(a)da$. Therefore, a straightforward calculation gives

$$\begin{aligned}\frac{dW_1(t)}{dt} = &\underbrace{\frac{s^*(0)}{e^*(0)}G\left(\frac{s(t,0)}{s^*(0)}\right)}_{=0} - \int_0^\infty (\mu+\beta(a)I^*)\frac{s^*(a)}{e^*(a)}G\left(\frac{s(t,a)}{s^*(a)}\right)da \\ &+ \frac{\int_0^\infty \beta(a)s(t,a)da}{\int_0^\infty \beta(a)s^*(a)da} - \frac{e(t,0)}{e^*(0)} + \frac{I(t)}{I^*} - 1.\end{aligned} \tag{1.62}$$

Similarly, we can get the derivative of $W_2(t)$ along the solutions of (1.1)-(1.3) as

$$\begin{aligned}\frac{dW_2(t)}{dt} &= \int_0^\infty \Psi(\theta)\frac{\partial}{\partial t}\left(\frac{e(t,\theta)}{e^*(0)}\right)d\theta \\ &= -\int_0^\infty \Psi(\theta)\frac{1}{e^*(0)}\left(1-\frac{e^*(\theta)}{e(t,\theta)}\right)\left[\frac{\partial}{\partial\theta}e(t,\theta) + (\mu+\psi(\theta))e(t,\theta)\right]d\theta.\end{aligned} \tag{1.63}$$

Considering

$$e^*(\theta)\frac{\partial}{\partial\theta}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right) = \left(1-\frac{e^*(\theta)}{e(t,\theta)}\right)\left[\frac{\partial}{\partial\theta}e(t,\theta) + (\mu+\psi(\theta))e(t,\theta)\right].$$

Then, (1.63) is converted to

$$\frac{dW_2(t)}{dt} = -\int_0^\infty \Psi(\theta)\frac{\partial}{\partial\theta}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right)d\theta.$$

By integration by parts, we have

$$\begin{aligned}\frac{dW_2(t)}{dt} &= \Psi(0)G\left(\frac{e(t,0)}{e^*(0)}\right) - \frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e(t,\theta]}{e^*(\theta)}G\left(\frac{e(t,\theta)}{e(\theta)}\right)d\theta \\ &= \frac{e(t,0)}{e^*(0)} - 1 - \ln\frac{I(t)}{I^*} - \ln\frac{\int_0^\infty \beta(a)s(t,a)da}{\int_0^\infty \beta(a)s^*(a)da} \\ &\quad - \frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right)d\theta.\end{aligned} \tag{1.64}$$

Combining (1.62) and (1.64), it results

$$\begin{aligned}\frac{dW_1(t)}{dt} + \frac{dW_2(t)}{dt} = &G\left(\frac{I(t)}{I^*}\right) + G\left(\frac{\int_0^\infty \beta(a)s(t,a)da}{\int_0^\infty \beta(a)s^*(a)da}\right) \\ &- \int_0^\infty (\mu+\beta(a)I^*)\frac{s^*(a)}{e^*(0)}G\left(\frac{s(t,a)}{s^*(a)}\right)da \\ &- \frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right)d\theta.\end{aligned} \tag{1.65}$$

By Jensen's inequality, we have

$$G\left(\frac{\int_0^\infty \beta(a)s(t,a)da}{\int_0^\infty \beta(a)s^*(a)da}\right) \leq \frac{1}{\int_0^\infty \beta(a)s^*(a)da}\int_0^\infty \beta(a)s^*(a)G\left(\frac{s(t,a)}{s^*(a)}\right)da.$$

Hence, it follows

$$\begin{aligned}\frac{dW_1(t)}{dt}+\frac{dW_2(t)}{dt} &= G\left(\frac{I(t)}{I^*}\right)-\mu\int_0^\infty \frac{s^*(a)}{e^*(0)}G\left(\frac{s(t,a)}{s^*(a)}\right)da \\ &\quad -\frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right)d\theta.\end{aligned} \tag{1.66}$$

Now, we turn to differentiate $W_3(t)$ along the trajectory of (1.1)-(1.3) with respect to t as

$$\begin{aligned}\frac{dW_3(t)}{dt} &= \frac{1}{\xi_2 e^*(0)}\left(1-\frac{I^*}{I(t)}\right)\left[\int_0^\infty \psi(\theta)e(t,\theta)d\theta+\gamma R(t)-(\sigma+\mu)I(t)\right] \\ &= \frac{1}{\xi_2 e^*(0)}\left[\int_0^\infty \psi(\theta)e(t,\theta)d\theta+\gamma R(t)-(\sigma+\mu)I(t)\right. \\ &\quad \left.-I^*\int_0^\infty \psi(\theta)\frac{e(t,\theta)}{I(t)}d\theta-\gamma I^*\frac{R(t)}{I(t)}+(\sigma+\mu)I^*\right].\end{aligned}$$

Applying the equation

$$(\sigma+\mu)I^* = \int_0^\infty \psi(\theta)e^*(\theta)d\theta+\gamma R^*.$$

Therefore, we can get

$$\begin{aligned}\frac{dW_3(t)}{dt} &= \frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}\left[1+\frac{e(t,\theta)}{e^*(\theta)}-\frac{I^*e(t,\theta)}{I(t)e^*(\theta)}-\frac{I(t)}{I^*}\right]d\theta \\ &\quad +\frac{\gamma R^*}{\xi_2 e^*(0)}\left[1+\frac{R(t)}{R^*}-\frac{I^*R(t)}{R^*I(t)}-\frac{I(t)}{I^*}\right].\end{aligned}$$

After straightforward calculation, we can get

$$\begin{aligned}\frac{dW_3(t)}{dt} &= \frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{e(t,\theta)}{e^*(\theta)}\right)d\theta-G\left(\frac{I(t)}{I^*}\right) \\ &\quad -\frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{I^*e(t,\theta)}{I(t)e^*(\theta)}\right)d\theta \\ &\quad +\frac{\gamma R}{\xi_2 e^*(0)}\left[G\left(\frac{R(t)}{R^*}\right)-G\left(\frac{I^*R(t)}{R^*I(t)}\right)-G\left(\frac{I(t)}{I^*}\right)\right].\end{aligned} \tag{1.67}$$

Finally, the derivative of $W_4(t)$ could be expressed as follows

$$\begin{aligned}\frac{dW_4(t)}{dt} &= \frac{\gamma}{(\mu+\gamma)\xi_2 e^*(0)}\left(1-\frac{R^*}{R(t)}\right)[\sigma I(t)-(\mu+\gamma)R(t)] \\ &= \frac{\gamma}{(\mu+\gamma)\xi_2 e^*(0)}\left[\sigma I(t)-(\mu+\gamma)R(t)-\sigma R^*\frac{I(t)}{R(t)}+(\mu+\gamma)R^*\right].\end{aligned}$$

Applying the equation $\sigma I^* = (\mu+\gamma)R^*$, it yields

$$\begin{aligned}\frac{dW_4(t)}{dt} &= -\frac{\gamma R}{\xi_2 e^*(0)}\left[1-\frac{R^* I(t)}{I^* R(t)}-\frac{R(t)}{R^*}+\frac{I(t)}{I^*}\right] \\ &= -\frac{\gamma R}{\xi_2 e^*(0)}\left[G\left(\frac{I(t)}{I}\right)-G\left(\frac{R(t)}{R}\right)-G\left(\frac{R^* I(t)}{I^* R(t)}\right)\right].\end{aligned} \tag{1.68}$$

In summary, collecting (1.66), (1.67) and (1.68) gives

$$\begin{aligned}\frac{dW(t)}{dt} &= -\mu\int_0^\infty \frac{s^*(a)}{e^*(0)}G\left(\frac{s(t,a)}{s^*(a)}\right)da-\frac{1}{\xi_2}\int_0^\infty \psi(\theta)\frac{e^*(\theta)}{e^*(0)}G\left(\frac{I^* e(t,\theta)}{I(t)e^*(\theta)}\right)d\theta \\ &\quad -\frac{\gamma R^*}{\xi_2 e^*(0)}\left[G\left(\frac{I^* R(t)}{I(t)R^*}\right)+G\left(\frac{R^* I(t)}{I^* R(t)}\right)\right].\end{aligned}$$

Therefore, $\frac{dW(t)}{dt} \le 0$; in addition $\frac{dW}{dt} = 0$ holds if and only if $s(t,a) = s^*(a)$, $e(t,\theta) = e^*(\theta)$, $I(t) = I^*$, and $R(t) = R^*$. Due to the boundedness of $W(t)$ on $\varphi(t)$, the α-limit set of $\varphi(t)$ must be contained in $\mathcal{M}_1$, the largest compact invariant subset in $\left\{\frac{dW}{dt} = 0\right\}$. Therefore, $\mathcal{M}_1 = \{E^*\}$, and hence, the α-limit consists only of E^*. Furthermore, we have $W(t)$ is strictly decreasing along trajectory $\varphi(t)$ in A and it is distinct from E^*. Since $W(t)$ has the minimum at $\varphi(t) = E^*$, we must have $\varphi(t) \equiv E^*$. This means for any total trajectory $\varphi(t)$ in $\mathcal{A}$, which consists entirely of total trajectories, we have $\mathcal{A} = \{E^*\}$. In conclusion, by LaSalle's invariance principle, E^* is globally asymptotically stable if $R_0 > 1$. This completes the proof of Theorem 1.8.

CHAPTER 2

Optimal control

In this chapter, we study the optimal control problem associated with the proposed epidemic model (1.1)-(1.3). We introduce a control variable $u(t)$, representing the treatment rate of infected individuals, and formulate an objective functional that aims to minimize the number of susceptible, latent, and infected individuals, while accounting for the cost of treatment.

We begin by proving the existence of an optimal control using appropriate mathematical analysis tools. Then, we use the Gâteaux derivative of the objective functional to derive the necessary conditions for optimality and obtain an explicit expression for the optimal control $u^*(t)$ based on the associated adjoint system.

Next, we move to the numerical part, where we solve the model numerically with control using a suitable numerical algorithm. This allows us to evaluate the effectiveness of the optimal strategy and compare it to the uncontrolled case.

This numerical study highlights the role of treatment in reducing the number of infections and increasing the number of recoveries, thereby contributing to improved epidemic control strategies.

2.1 Optimal Control Strategies

Optimal control theory [29] is a mathematical framework derived from the calculus of variations, designed to determine the best possible intervention strategy for a dynamical system to meet specific objectives. In the context of epidemic modeling, this theory helps public health authorities to identify optimal ways to reduce disease transmission and minimize the effect of

infection.

In our system model described by the equations (1.1)-(1.3), we introduce a time-dependent control variable $u(t)$ representing the treatment applied to infected individuals. The control belongs to the admissible set:

$$U_{ad} = \left\{ u \in L^2(0,T) \,\middle|\, 0 \leq u(t) \leq \ell \leq 1,\ \forall t \in [0,T] \right\},$$

where ℓ denotes the maximum feasible level of intervention due to resource limitations.

The control term $u(t)I(t)$ represents the rate at which infected individuals are removed from the infected class due to the treatment intervention. This term is subtracted from the equation governing the infected class and added to the recovered class, assuming successful intervention leads to recovery.

The objective of the control problem is to reduce the total number of susceptible, latent, and infected individuals over time, while minimizing the intervention cost. This can be expressed through the objective functional as follows

$$J(u(t)) = \int_0^T \left(S(t) + E(t) + I(t) + \frac{B_1}{2} u^2(t) \right) dt, \tag{2.1}$$

where $S(t) = \int_0^\infty s(t,a)da$, $E(t) = \int_0^\infty e(t,\theta)d\theta$, and $B_1 > 0$ is a weighting constant reflecting the cost or social impact of the control effort.

The optimal control problem aims to find the function $u^*(t)$ that minimizes $J(u)$ subject to the age-structured SEIRI epidemic model, incorporating the control in the infected equation and hence, the optimal control problem can be expressed as follows:

$$\begin{cases} \dfrac{\partial s(t,a)}{\partial t} + \dfrac{\partial s(t,a)}{\partial a} = -\mu s(t,a) - \beta(a) I(t) s(t,a), \\ \dfrac{\partial e(t,\theta)}{\partial t} + \dfrac{\partial e(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta)) e(t,\theta), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta) e(t,\theta)\, d\theta + \gamma R(t) - (\mu + \sigma) I(t) - u(t) I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu + \gamma) R(t) + u(t) I(t), \end{cases} \tag{2.2}$$

with boundary conditions

$$s(t,0) = \Lambda, \quad e(t,0) = I(t) \int_0^\infty \beta(a) s(t,a)\, da, \tag{2.3}$$

and initial conditions

$$s(0,a) = s_0(a), \quad e(0,\theta) = e_0(\theta), \quad I(0) = I_0, \quad \text{and} \quad R(0) = R_0. \tag{2.4}$$

This formulation ensures that the number of infected individuals is actively reduced while maximizing recovery through optimal use of available interventions.

Theorem 2.1 *There exists an optimal control variable $u^* \in U_{ad}$ such that $J(u^*) = \inf_{u \in U_{ad}} J(u)$, subject to the control system (2.2)-(2.4).*

Proof Now, let $(u_n)_{n\in\mathbb{N}} \subset U_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in U_{ad}} J(u).$$

Since the admissible set U_{ad} is closed, convex and bounded set in $L^2(0,T)$, and the functional J is coercive, the sequence (u_n) is bounded in $L^2(0,T)$. By the Banach-Alaoglu Theorem [14], there exists a subsequence and a function $u^* \in U_{ad}$ such that

$$u_n \rightharpoonup u^* \quad \text{weakly in } L^2(0,T).$$

The corresponding state variables $(x_n) = (S_n(t), E_n(t), I_n(t), R_n(t))$ associated with (u_n), are uniformly bounded and equicontinuous with respect to time t. By the Arzelà-Ascoli Theorem [13], one can extract a subsequence such that the states converge uniformly to a limit $x^* = (S^*, E^*, I^*, R^*)$ associated with the control u^*.

Using standard lower semi-continuity arguments, the cost functional satisfies

$$\liminf_{n\to\infty} J(u_n) \geq J(u^*).$$

By definition of (u_n) as a minimizing sequence, we also have

$$\lim_{n\to\infty} J(u_n) = \inf_{u\in U_{ad}} J(u) \leq J(u^*).$$

Combining both inequalities yields

$$J(u^*) = \inf_{u\in U_{ad}} J(u),$$

which proves that u^* is an optimal control. This completes the proof of Theorem 2.1.

To determine the necessary conditions for optimality, we apply the Gâteaux derivative approach. Starting with a given a control $u(t)$, we consider another control defined as: $u^\epsilon(t) = u(t) + \epsilon v(t)$, where $v(t)$ is a variation function and $0 < \epsilon < 1$ is a small parameter. The variation function can take any sign, since small perturbations in $u(t)$ (whether positive or negative) still ensure that $u^\epsilon(t) \in U_{ad}$, i.e, the perturbed control remains admissible within the perdefind bounds.

Let $S = S(u), E = E(u), I = I(u), R = R(u)$, and $S^\epsilon = S(u^\epsilon), E^\epsilon = E(u^\epsilon), I^\epsilon = I(u^\epsilon), R^\epsilon = R(u^\epsilon)$. Then, the state equations corresponding to the control u^ϵ are given by

$$\begin{cases} \dfrac{\partial s^\epsilon(t,a)}{\partial t} + \dfrac{\partial s^\epsilon(t,a)}{\partial a} = -\mu s^\epsilon(t,a) - \beta(a) I^\epsilon(t) s^\epsilon(t,a), \\ \dfrac{\partial e^\epsilon(t,\theta)}{\partial t} + \dfrac{\partial e^\epsilon(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta)) e^\epsilon(t,\theta), \\ \dfrac{dI^\epsilon(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta) e^\epsilon(t,\theta)\, d\theta + \gamma R^\epsilon(t) - (\mu+\sigma) I(t)^\epsilon - u^\epsilon(t) I^\epsilon(t), \\ \dfrac{dR^\epsilon(t)}{dt} = \sigma I^\epsilon(t) - (\mu+\gamma) R^\epsilon(t) + u^\epsilon(t) I^\epsilon(t), \end{cases} \tag{2.5}$$

with boundary conditions

$$s^\epsilon(t,0) = \Lambda, \quad e^\epsilon(t,0) = I^\epsilon(t) \int_0^\infty \beta(a) s^\epsilon(t,a)\, da, \tag{2.6}$$

and initial conditions

$$s^\epsilon(0,a) = s_0(a), \quad e^\epsilon(0,\theta) = e_0(\theta), \quad I^\epsilon(0) = I_0, \quad \text{and} \quad R^\epsilon(0) = R_0, \tag{2.7}$$

where $S^\epsilon(t) = \displaystyle\int_0^\infty s^\epsilon(t,a) da$ and $E^\epsilon(t) = \displaystyle\int_0^\infty e^\epsilon(t,\theta) d\theta$.Now, we find the difference quotient $\frac{S^\epsilon - S}{\epsilon}$ and similarly for E, I, and R. By subtracting system (2.2) from (2.5), we find

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{s^\epsilon(t,a) - s(t,a)}{\epsilon}\right) + \frac{\partial}{\partial a}\left(\frac{s^\epsilon(t,a) - s(t,a)}{\epsilon}\right) &= -\mu\left(\frac{s^\epsilon(t,a) - s(t,a)}{\epsilon}\right) \\ &\quad - \beta(a)\left(\frac{I^\epsilon(t) - I(t)}{\epsilon}\right) s^\epsilon(t,a) \\ &\quad - \beta(a)\left(\frac{s^\epsilon(t,a) - s(t,a)}{\epsilon}\right) I(t). \end{aligned}$$

Moreover, we have

$$\frac{\partial}{\partial t}\left(\frac{e^\epsilon(t,\theta)-e(t,\theta)}{\epsilon}\right)+\frac{\partial}{\partial \theta}\left(\frac{e^\epsilon(t,\theta)-e(t,\theta)}{\epsilon}\right)=-(\mu+\psi(\theta))\left(\frac{e^\epsilon(t,\theta)-e(t,\theta)}{\epsilon}\right).$$

Also,

$$\begin{aligned}\frac{d}{dt}\left(\frac{I^\epsilon(t)-I(t)}{\epsilon}\right)=&\int_0^\infty \psi(\theta)\left(\frac{e^\epsilon(t,\theta)-e(t,\theta)}{\epsilon}\right)d\theta-(\mu+\sigma+u(t))\left(\frac{I^\epsilon(t)-I(t)}{\epsilon}\right)\\&+\gamma\left(\frac{R^\epsilon(t)-R(t)}{\epsilon}\right)+v(t)I^\epsilon(t).\end{aligned}$$

Further, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{R^\epsilon(t)-R(t)}{\epsilon}\right)=&(\sigma+u(t))\left(\frac{I^\epsilon(t)-I(t)}{\epsilon}\right)da-(\mu+\gamma)\left(\frac{R^\epsilon(t)-R(t)}{\epsilon}\right)\\&+v(t)I^\epsilon(t),\end{aligned}$$

with

$$\frac{s^\epsilon(t,0)-s(t,0)}{\epsilon}=0,$$

$$\frac{e^\epsilon(t,0)-e(t,0)}{\epsilon}=I^\epsilon(t)\int_0^\infty\beta(a)\left(\frac{s^\epsilon(t,a)-s(t,a)}{\epsilon}\right)da+\left(\frac{I^\epsilon(t)-I(t)}{\epsilon}\right)\int_0^\infty\beta(a)s(t,a)da,$$

and

$$\frac{s^\epsilon(0,a)-s(0,a)}{\epsilon}=0,\quad \frac{e^\epsilon(0,\theta)-e(0,\theta)}{\epsilon}=0,\quad \frac{I^\epsilon(0)-I(0)}{\epsilon}=0,\quad \frac{R^\epsilon(0)-R(0)}{\epsilon}=0.$$

Letting $\epsilon\to 0$, then $\frac{s^\epsilon(t,a)-s(t,a)}{\epsilon}\to\bar{s}$, and same for the others. Thus, we obtain

$$\begin{cases}\dfrac{\partial\bar{s}(t,a)}{\partial t}+\dfrac{\partial\bar{s}(t,a)}{\partial a}=-\mu\bar{s}(t,a)-\beta(a)\bar{I}(t)s(t,a)-\beta(a)I(t)\bar{s}(t,a),\\ \dfrac{\partial\bar{e}(t,\theta)}{\partial t}+\dfrac{\partial\bar{e}(t,\theta)}{\partial\theta}=-(\mu+\psi(\theta))\bar{e}(t,\theta),\\ \dfrac{d\bar{I}(t)}{dt}=\displaystyle\int_0^\infty\psi(\theta)\bar{e}(t,\theta)\,d\theta+\gamma\bar{R}(t)-(\mu+\sigma+u(t))\bar{I}(t)-v(t)I(t),\\ \dfrac{d\bar{R}(t)}{dt}=(\sigma+u(t))\bar{I}(t)-(\mu+\gamma)\bar{R}(t)+v(t)I(t),\end{cases}\tag{2.8}$$

with boundary conditions

$$\bar{s}(t,0)=0, \quad \bar{e}(t,0)=I(t)\int_0^\infty \beta(a)\bar{s}(t,a)+\bar{I}(t)\int_0^\infty \beta(a)s(t,a)\,da, \tag{2.9}$$

and initial conditions

$$\bar{s}(0,a)=0, \quad \bar{e}(0,\theta)=0, \quad \bar{I}(0)=0, \quad \text{and} \quad \bar{R}(0)=0. \tag{2.10}$$

In order to find the adjoint system, we rewrite the fisrt equation in (2.8) as

$$\begin{aligned} 0 &= \left\langle\!\!\left\langle \frac{\partial}{\partial t}\bar{s}(t,a)+\frac{\partial}{\partial a}\bar{s}(t,a)+\mu\bar{s}(t,a)+I(t)\beta(a)\bar{s}(t,a)+\bar{I}(t)\beta(a)s(t,a), \lambda_1(t,a)\right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \bar{s}(t,a), -\frac{\partial}{\partial t}\lambda_1(t,a)-\frac{\partial}{\partial a}\lambda_1(t,a)+\beta(a)I(t)\lambda_1(t,a)+\mu\lambda_1(t,a)\right\rangle\!\!\right\rangle \\ &\quad +\int_0^T\int_0^\infty \beta(a)s(t,a)\bar{I}(t)\lambda_1(t,a)\,da\,dt, \end{aligned} \tag{2.11}$$

with boundary conditions

$$\bar{s}(0,a)=0, \quad \bar{s}(t,\infty)=0, \quad \text{and} \quad \lambda_1(T,a)=0,$$

where $\langle\!\langle f,g\rangle\!\rangle=\int_0^T\int_0^\infty fg\,da\,dt$. Similarly, the second equation in (2.8) becomes

$$\begin{aligned} 0 &= \left\langle\!\!\left\langle \frac{\partial}{\partial t}\bar{e}(t,\theta)+\frac{\partial}{\partial \theta}\bar{e}(t,\theta)+(\mu+\psi(\theta))\bar{e}(t,\theta),\ \lambda_2(t,\theta)\right\rangle\!\!\right\rangle \\ &= \left\langle\!\!\left\langle \bar{e}(t,\theta),\ -\frac{\partial}{\partial t}\lambda_2(t,\theta)-\frac{\partial}{\partial \theta}\lambda_2(t,\theta)+(\mu+\psi(\theta))\lambda_2(t,\theta)\right\rangle\!\!\right\rangle \\ &\quad -\int_0^T\int_0^\infty \left[\bar{s}(t,a)I(t)+s(t,a)\bar{I}(t)\right]\beta(a)\lambda_2(t,0)\,da\,dt, \end{aligned} \tag{2.12}$$

with conditions

$$\bar{e}(0,\theta)=0, \quad \bar{e}(t,\infty)=0, \quad \text{and} \quad \lambda_2(T,\theta)=0.$$

Similarly, the third and fourth equations in (2.8) can be written in the form of

$$
\begin{aligned}
0 &= \left\langle \frac{d\bar{I}}{dt} - \int_0^\infty \psi(\theta)\bar{e}(t,\theta)\,d\theta + \gamma\bar{R}(t) - (\mu+\sigma+u(t))\bar{I}(t) + v(t)I(t), \lambda_3(t) \right\rangle \\
&= \left\langle \bar{I}(t), -\frac{d}{dt}\lambda_3(t) + (\mu+\sigma+u(t))\lambda_3(t) \right\rangle \\
&\quad - \int_0^T \int_0^\infty \psi(\theta)\bar{e}(t,\theta)\lambda_3(t)\,d\theta\,dt - \int_0^T \gamma\lambda_3(t)\bar{R}(t) + \int_0^T \lambda_3(t)v(t)I(t)\,dt, \qquad (2.13)
\end{aligned}
$$

with

$$\bar{I}(0) = 0, \quad \text{and} \quad \lambda_3(T) = 0.$$

where $\langle f, g\rangle = \int_0^T fg\,dt$. By the same way in last equation of (2.8), we can find

$$
\begin{aligned}
0 &= \left\langle \frac{d\bar{R}}{dt} - (\sigma+u(t))\bar{I}(t) + (\mu+\gamma)\bar{R}(t) - v(t)I(t), \lambda_4(t) \right\rangle \\
&= \left\langle \bar{R}(t), -\frac{d}{dt}\lambda_4(t) + (\mu+\gamma)\lambda_4(t) \right\rangle \\
&\quad - \int_0^T \int_0^\infty (\sigma+u(t))\bar{I}(t)\lambda_4(t)\,da\,dt - \int_0^T \lambda_4(t)v(t)I(t)\,dt, \qquad (2.14)
\end{aligned}
$$

with

$$\bar{R}(0) = 0, \quad \text{and} \quad \lambda_4(T) = 0.$$

Respectively, thus, we derive the Gâteaux derivative [20] of $J(u)$ as follows

$$0 \le J'(u) = \int_0^T \left[\bar{S}(t) + \bar{E}(t) + \bar{I}(t) + B_1 v(t)u(t)\right] dt. \qquad (2.15)$$

Next, we define the Lagrangian L function as follows:

$$
\begin{aligned}
\mathcal{L}(\bar{S},\bar{E},\bar{I},\bar{R}) = &\int_0^T \left[\bar{S}(t)+\bar{E}(t)+\bar{I}(t)+B_1 v(t)u(t)\right] dt \\
&- \int_0^T \int_0^\infty \bar{s}(t,a)\left[-\frac{\partial}{\partial t}\lambda_1(t,a) - \frac{\partial}{\partial a}\lambda_1(t,a) + \beta(a)I(t)\lambda_1(t,a) + \mu\lambda_1(t,a)\right] da\, dt \\
&- \int_0^T \int_0^\infty \beta(a)s(t,a)\bar{I}(t)\lambda_1(t,a)\, da\, dt \\
&- \int_0^T \int_0^\infty \bar{e}(t,\theta)\left[-\frac{\partial}{\partial t}\lambda_2(t,\theta) - \frac{\partial}{\partial \theta}\lambda_2(t,\theta) + (\mu+\psi(\theta))\lambda_2(t,\theta)\right] d\theta\, dt \\
&+ \int_0^T \int_0^\infty \left[\bar{s}(t,a)I(t) + s(t,a)\bar{I}(t)\right]\beta(a)\lambda_2(t,0)\, da\, dt \\
&- \int_0^T \bar{I}(t)\left[-\frac{d}{dt}\lambda_3(t) + (\mu+\sigma+u(t))\lambda_3(t)\right] dt \\
&+ \int_0^T \int_0^\infty \psi(\theta)\bar{e}(t,\theta)\lambda_3(t)\, d\theta\, dt \\
&+ \int_0^T \gamma\lambda_3(t)\bar{R}(t)\, dt - \int_0^T \lambda_3(t)v(t)I(t)\, dt \\
&- \int_0^T \bar{R}(t)\left[-\frac{d}{dt}\lambda_4(t) + (\mu+\gamma)\lambda_4(t)\right] dt \\
&+ \int_0^T \int_0^\infty (\sigma+u(t))\bar{I}(t)\lambda_4(t)\, da\, dt + \int_0^T \lambda_4(t)v(t)I(t)\ dt. \qquad (2.16)
\end{aligned}
$$

By solving

$$
\frac{\partial \mathcal{L}}{\partial \bar{S}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \bar{E}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \bar{I}} = 0, \quad \frac{\partial \mathcal{L}}{\partial \bar{R}} = 0,
$$

We obtain the following adjoint system:

$$
\begin{cases}
\dfrac{\partial}{\partial t}\lambda_1(t,a) + \dfrac{\partial}{\partial a}\lambda_1(t,a) = \beta(a)I(t)\lambda_1(t,a) + \mu\lambda_1(t,a) - \beta(a)I(t)\lambda_2(t,0) - 1, \\
\dfrac{\partial}{\partial t}\lambda_2(t,\theta) + \dfrac{\partial}{\partial \theta}\lambda_2(t,\theta) = (\mu+\psi(\theta))\lambda_2(t,\theta) - \psi(\theta)\lambda_3(t) - 1, \\
\qquad \dfrac{d}{dt}\lambda_3(t) = (\mu+\sigma+u(t))\lambda_3(t) - (\sigma+u(t))\lambda_4(t) + \lambda_1(t)\displaystyle\int_0^\infty \beta(a)s(t,a)\, da \\
\qquad\qquad\qquad - \lambda_2(t,0)\displaystyle\int_0^\infty \beta(a)s(t,a)\, da - 1, \\
\qquad \dfrac{d}{dt}\lambda_4(t) = (\mu+\gamma)\lambda_4(t) - \gamma\lambda_3(t),
\end{cases} \qquad (2.17)
$$

with transversality conditions

$$
\lambda_1(T,a) = 0, \quad \lambda_2(T,\theta) = 0, \quad \lambda_3(T) = 0, \quad \text{and} \quad \lambda_4(T) = 0. \qquad (2.18)
$$

Theorem 2.2 *If u^* in U_{ad} is an optimal control that minimizes (2.1), and $(S^*(t), E^*(t), I^*(t), R^*(t))$ and $(\lambda_1(t,a), \lambda_2(t,\theta), \lambda_3(t), \lambda_4(t))$ are the corresponding state and adjoint variables, respectively, then*

$$u^*(t) = \min\left\{\ell, \max\left\{0, \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_1}\right\}\right\}. \tag{2.19}$$

Proof Using system (2.17), we can rewrite relation (2.15) as follows:

$$\begin{aligned}
0 \le J'(u) &= \int_0^T \left[\bar{S}(t) + \bar{E}(t) + \bar{I}(t) + B_1 v(t)u(t)\right] dt \\
&= \int_0^T \bar{S}(t)\left[-\frac{\partial}{\partial t}\lambda_1(t,a) - \frac{\partial}{\partial a}\lambda_1(t,a) + \beta(a)I(t)\lambda_1(t,a) + \mu\lambda_1(t,a) - \beta(a)I(t)\lambda_2(t,0)\right] dt \\
&\quad + \int_0^T \bar{E}(t)\left[-\frac{\partial}{\partial t}\lambda_2(t,\theta) + \frac{\partial}{\partial \theta}\lambda_2(t,\theta)(\mu + \psi(\theta))\lambda_2(t,\theta) - \psi(\theta)\lambda_3(t)\right] dt \\
&\quad + \int_0^T \bar{I}(t)\left[-\frac{d}{dt}\lambda_3(t) + (\mu + \sigma + u(t))\lambda_3(t) - (\sigma + u(t))\lambda_4(t) + \lambda_1(t)\int_0^\infty \beta(a)s(t,a)\,da\right. \\
&\quad \left. - \lambda_2(t,0)\int_0^\infty \beta(a)s(t,a)da\right] dt + \int_0^T \bar{R}(t)\left[-\frac{d}{dt}\lambda_4(t) + (\mu + \gamma)\lambda_4(t) - \gamma\lambda_3(t)\right] dt \\
&\quad + \int_0^T B_1 v(t)u(t)\,dt.
\end{aligned} \tag{2.20}$$

Now, keeping in mind equations (2.11), (2.12), (2.13), and (2.14), we can write equation (2.20) as:

$$0 \le \int_0^T v(t)\left[I(t)(\lambda_4(t) - \lambda_3(t)) + B_1 u(t)\right] dt, \quad \text{for all } u(t) \in U_{ad}.$$

Therefore, on this set, in the case $v(t) \neq 0$, the rest of the integrand must be equal to zero, thus

$$u(t) = \frac{(\lambda_3(t) - \lambda_4(t))I(t)}{B_1}.$$

Hence, by taking the upper and lower bound into account, we obtain

$$u^*(t) = \min\left\{\ell, \max\left\{0, \frac{(\lambda_3(t) - \lambda_4(t))I^*(t)}{B_1}\right\}\right\}. \tag{2.21}$$

This completes the proof of Theorem 2.2.

Clearly, the optimal control presented in formula (2.21) lies within the bounds $0 \le u \le \ell$. Thus,

we obtained the optimal system:

$$\begin{cases} \dfrac{\partial s(t,a)}{\partial t} + \dfrac{\partial s(t,a)}{\partial a} = -\mu s(t,a) - \beta(a) I(t) s(t,a), \\ \dfrac{\partial e(t,\theta)}{\partial t} + \dfrac{\partial e(t,\theta)}{\partial \theta} = -(\mu + \psi(\theta)) e(t,\theta), \\ \dfrac{dI(t)}{dt} = \displaystyle\int_0^\infty \psi(\theta) e(t,\theta)\, d\theta + \gamma R(t) - (\mu + \sigma) I(t) - u^*(t) I(t), \\ \dfrac{dR(t)}{dt} = \sigma I(t) - (\mu + \gamma) R(t) + u^*(t) I(t), \end{cases} \tag{2.22}$$

with boundary conditions

$$s(t,0) = \Lambda, \quad e(t,0) = I(t) \int_0^\infty \beta(a) s(t,a)\, da, \tag{2.23}$$

and initial conditions

$$s(0,a) = s_0(a), \quad e(0,\theta) = e_0(\theta), \quad I(0) = I_0, \quad \text{and} \quad R(0) = R_0, \tag{2.24}$$

where $u^*(t)$ is defind in relation (2.21).

2.2 numerical simulations

To validate the effectiveness of the analytically derived optimal system (2.22), obtained from the original model (1.1)-(1.3) through the incorporation of a control variable $u(t)$, a numerical simulation was performed. This simulation involved solving both the state and adjoint equations using a combination of forward and backward finite difference methods [24]. The initial population distributions for the susceptible, exposed, infected, and recovered compartments were adopted from the study by Thater et al. [32]. For the selection and justification of parameter values, please refer to Table 1.

The age-dependent functions $\sigma(a)$ and $\xi(a)$ were assumed to take the form of a constant multiple of the exponential function e^{-a}. This choice is motivated not only by the monotonic decreasing nature of e^{-a}, but also by its desirable mathematical properties such as positivity, boundedness, and its natural decay to zero as age increases.

To ensure sufficient resolution in the simulation, small step sizes were used for both time and age, specifically $\Delta t = \Delta a = 0.05$. The maximum age considered for infection was set to 40 years, and the total duration for the application of the control strategy was set to 40 days

Parameters	Values
ν	0.0136
Λ	15.844
μ	0.1
l	1
B_1	1
$\sigma(a)$	$0.005e^{-a}$
$\xi(a)$	$0.002e^{-a}$

Table 2.1: Parameters values used in numerical simulation

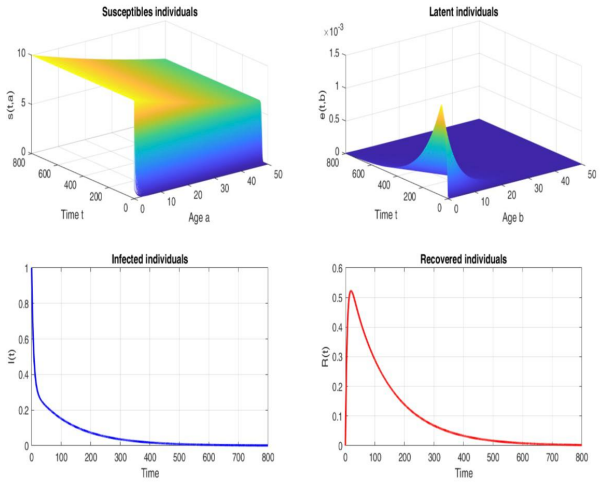


Fig.1. Asymptotic behavior of the solution of the system (1.1), with $\beta = 0.002$. In this case, we obtain $R_0 < 1$. Leading to the extinction of the epidemic.

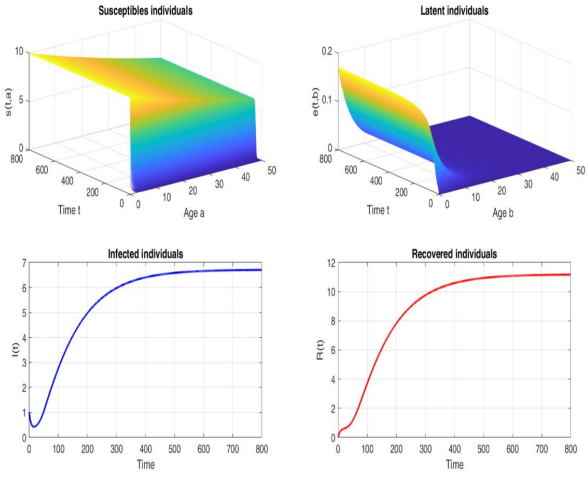


Fig.2. Asymptotic behavior of the solution of the system (1.1), with $\beta = 0.005$. In this case, we obtain $R_0 > 1$. Leading to the disease spreading.

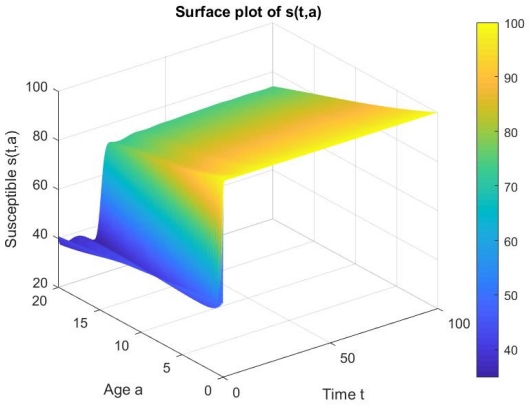


Figure 2.1: Surface plot of s(t,a)

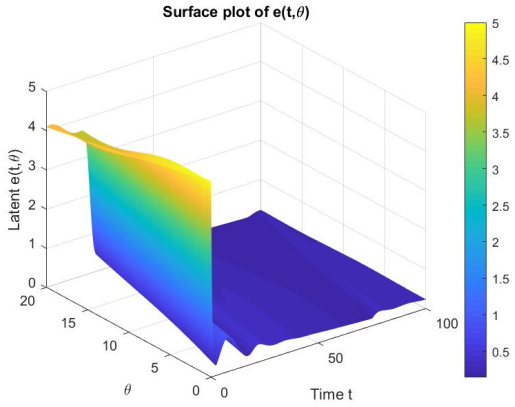


Figure 2.2: Surface plot of e(t,a)

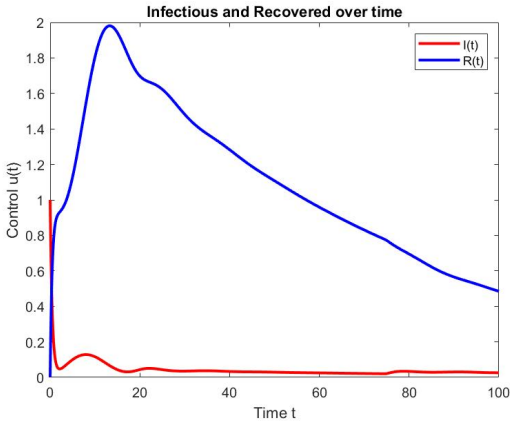


Figure 2.3: Infectious and Recovered over time

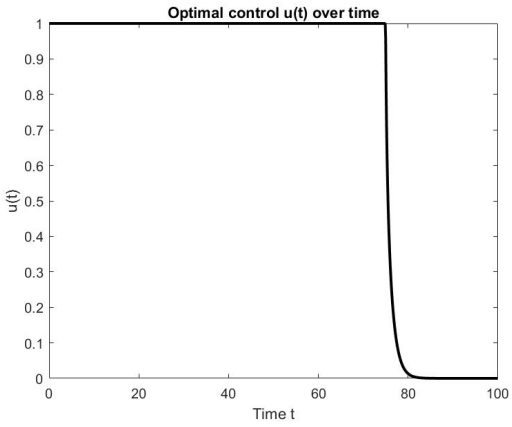


Figure 2.4: Optimal control $u(t)$ over time

Fig.3. The plots illustrate the dynamics of the epidemic model under the control strategy. 1. Susceptible population s(t,a); 2. Exposed population e(t,a); 3. Infected population I(t) and recovered population R(t); 4. All compartments under the effect of the optimal control strategy.

Conclusion

This study aims to analyze the dynamics of infectious disease spread using a mathematical model that takes into account the age structure of the population and the incubation period. By examining the stability of the system, we were able to understand the behavior of equilibrium points in both the presence and absence of the disease. Additionally, we applied optimal control theory to identify effective strategies for reducing disease transmission, using numerical simulations to evaluate the impact of health interventions such as targeted vaccination. The results showed that these strategies could significantly contribute to improving public health and reducing the overall impact of the disease on the community.

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