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Existence of solutions for a Kirchhoff hyperbolic problems in bounded domains

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بِسْمِ اللَّهِ الرَّحْمَٰنِ الرَّحِيمِ

رَبِّ أَوْزِعْنِي أَنْ أَشْكُرَ نِعْمَتَكَ الَّتِي أَنْعَمْتَ عَلَيَّ وَعَلَىٰ وَالِدَيَّ

وَأَنْ أَعْمَلَ صَالِحًا تَرْضَاهُ وَأَدْخِلْنِي بِرَحْمَتِكَ

فِي عِبَادِكَ الصَّالِحِينَ

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TO MY PARENT.

Abstract

This study is dedicated to investigating the existence of weak solutions for hyperbolic Kirchhoff-type problems, considering cases both with and without volume constraints and free boundaries. We employ the hyperbolic discrete Morse flow, which transforms the original problem into a sequence of minimization problems defined at discrete time intervals. This process ensures the existence of a minimizer for the discretized functional, which corresponds to the solution of the discretized problem and subsequently provides a weak solution to the original problem. The non-local terms arising from the Kirchhoff component, volume constraint, and free boundary condition present significant challenges that require innovative methods to tackle. These complexities are a key focus of our research. Furthermore, we present numerical simulations to better illustrate our results' physical implications.

Keywords

Hyperbolic Kirchhoff type problem, Free boundary, Volume constraint, Weak solution, Discrete Morse flow, Numerical simulation.

Résumé

Dans cette étude, nous nous concentrons sur l'exploration de l'existence de solutions faibles pour les problèmes hyperboliques de type Kirchhoff, avec et sans contraintes de volume et limites libres. Notre approche utilise le flot discret de Morse hyperbolique, une méthode bien établie dans l'étude des équations hyperboliques. Cette technique transforme le problème original en une série de problèmes de minimisation définis à des pas de temps discrets, établissant l'existence d'un minimiseur pour la fonctionnelle discrétisée. Ce minimiseur correspond à la solution du problème discrétisé, qui, à son tour, fournit une solution faible au problème principal. Notamment, les termes non locaux découlant de la composante Kirchhoff et de la contrainte de volume, combinés à la condition aux limites libres, posent des défis substantiels. Ces complexités nécessitent des stratégies innovantes pour les résoudre, constituant le point central de notre recherche. De plus, nous incluons des simulations numériques pour illustrer plus efficacement les implications physiques de nos résultats.

Mots-clés

Problème hyperbolique de type p-Kirchhoff, Frontière libre, Contrainte de volume, Solution faible, Flux Morse discret, Simulation numérique.

الملخص

في هذه الدراسة، نركز على استكشاف وجود حلول ضعيفة لمشكلات كيرشوف الزائدية، سواء مع قيود الحجم والحدود الحرة أو بدونها. يستخدم نهجنا تدفق مورس المنفصل الزائدي، وهي طريقة راسخة في دراسة المعادلات الزائدية. تحول هذه التقنية المشكلة الأصلية إلى سلسلة من مشاكل التقليل المحددة في خطوات زمنية منفصلة، مما يثبت وجود مخفض للدالة المنفصلة. يتوافق هذا المخفض مع حل المشكلة المنفصلة، والذي بدوره يوفر حلاً ضعيفاًا للمشكلة الأساسية. تجدر الإشارة إلى أن المصطلحات غير المحلية الناشئة عن مكون كيرشوف وقيد الحجم، جنباًا إلى جنب مع شرط الحدود الحرة، تشكل تحديات كبيرة. تتطلب هذه التعقيدات استراتيجيات مبتكرة لمعالجتها، وتشكل التركيز المركزي لبحثنا. بالإضافة إلى ذلك، ندرج عمليات محاكاة رقمية لتوضيح الآثار المادية لنتائجنا بشكل أكثر فعالية.

الكلمات المفتاحية

مشكلة القطع الزائد من نوع كيرشوف، الحدود الحرة، قيود الحجم، الحل منخفض، تدفق مورس منفصل، محاكاة رقمية.

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LIST OF ABBREVIATIONS

$i, j, m, n \in \mathrm{N}$	are natural numbers from the natural numbers set N.
R^n	is the n-dimensional real Euclidean space, furthermore, $\left(\mathrm{R} = \mathrm{R}^1\right)$.
Q	is a bounded domain in R^n with Lipschitz boundary.
∂Q	is the boundary of the domain Q.
$\lvert Q \rvert$	is the Lebesgue measure of the domain Q.
$\bar{Q}$	is the closure of the domain Q.
$\mathcal{T}$	is a positive real value representing the final time $\mathcal{T}$.
$Q_{\mathcal{T}}$	is the open time-space $Q \times (0, \mathcal{T})$.
$\mathcal{V}$	is a positive real value representing the volume.
$v(x,t)$	is the unknown function with respect to x and t.
$v_t(x,t)$	is the partial derivative of v with respect to time $\left(v_t(x,t) = \dfrac{\partial v(x,t)}{\partial t}\right)$.
∇v	is the gradient of v with respect to x $(= (v_{x_1}, v_{x_2}, \ldots, v_{x_m}))$.
Δv	is the Laplace operator with respect to x $\left(= \dfrac{\partial^2 v}{\partial x_1^2} + \cdots + \dfrac{\partial^2 v}{\partial x_n^2}\right)$.
$v\lvert_{\partial Q}$	is the trace of v on ∂Q .
v_0, u_0	are the initial data that present the shape and velocity, respectively.
λ	is the Lagrange multiplier, which is the non-local term originating in the volume constraint.
k	is the time step of the discretization in time.
M	expressing the total number of time steps.
a.e.	means almost everywhere of almost every.
$\{v > 0\}$	is the set of points (x,t) from Q_T, for which $v(x,t) > 0$.
$\chi_{\{v>0\}}$	is the characteristic function of the set $\{v > 0\}$.
C	is a generic positive constant.

CHAPTER 1

GENERAL INTRODUCTION

Hyperbolic Kirchhoff-type problems stand at the intersection of advanced mathematical modeling and real-world applications. These problems incorporate the Kirchhoff term, incorporating a non-linear relationship based on the integral of the strain. This approach effectively models the behavior of elastic structures under stress and deformation. Typically, such problems are expressed in the form:

$$M\left(\int_Q |\nabla v(x,t)|^2\,dx\right)\partial_{tt}v(x,t) - \Delta v(x,t) = g(x,t),$$

with M is a function representing the material's stiffness modulated by total deformation, v is the displacement, and $g(x,t,v)$ represents external forces. This formulation reflects the foundational work on elastic and fluid membranes by Kirchhoff (1883) in modeling tensioned surfaces under load [47].

The presence of free boundaries introduces further complexity, as evolving geometries necessitate a dynamic boundary treatment. For domains with free boundaries, the problem may include a condition like:

$$\gamma\partial_n v + \mu v = 0 \quad \text{on} \quad \partial Q(t),$$

with γ and μ are parameters dependent on the material properties, and $\partial_n v$ represents the normal derivative of v. This condition allows the boundary to adapt and evolve over time, reflecting practical scenarios in shape optimization and fluid-structure interaction problems [36].

In cases with volume constraints are imposed, additional conditions are necessary to preserve spatial volumes within a domain. Such constraints can be formulated as:

$$\int_Q v(x,t)\,dx = \mathcal{V},$$

with v is a constant volume. The preservation of volume is particularly relevant in applications involving fluid-filled membranes, with the enclosed volume must remain constant despite deformations. This dynamic interaction between boundary deformation and volume constraints aligns with scenarios in fluid mechanics and material science, adding complexity and rigor to the system's mathematical formulation [80].

Incorporating volume constraints introduces a unique dimension of intricacy, requiring careful attention to the preservation of spatial volumes and the dynamic interplay between material properties. This dynamic aspect is especially crucial in situations with the volume within a system must remain constant despite external forces or changes, as observed in diverse applications ranging from material encapsulation to fluid dynamics. The inclusion of volume constraints necessitates a nuanced understanding of how volume alterations impact the overall behavior of the system, mirroring real-world scenarios with volume preservation is a critical consideration in ensuring the accuracy and fidelity of simulations and analyses.

Expanding on the previously discussed study on hyperbolic Kirchhoff-type problems, we delve deeper into the intricacies of these challenges and the methodologies employed to address them. The integration of non-local features, coupled with free boundaries and volume constraints, presents a formidable landscape for exploration, with implications spanning diverse fields like material science, fluid mechanics, and structural engineering.

To tackle these challenges, we employ the Discrete Morse Flow (DMF), a variational method that has proven effective in handling hyperbolic problems with non-local constraints. The DMF approach breaks the original problem into a series of minimization tasks at discrete time intervals, ensuring the non-negativity of the minimized function and providing a foundation for demonstrating the existence of weak solutions. Furthermore, the DMF method enables energy preservation through the approximation process, promoting numerical stability and convergence. This is formalized by the energy equation:

$$E(t) = \int_Q \left(\frac{1}{2} |\partial_t v|^2 + \frac{1}{2} M \left(\int_Q |\nabla v|^2 dx \right) |\nabla v|^2 \right) dx,$$

with $E(t)$ captures the system's total energy at a given time $\mathcal{T}$. The conservation of this energy across discrete steps is essential for ensuring weak solution existence and accuracy, especially under volume constraints [15].

The adaptability of DMF to diverse Problem Introductions, including those with free boundaries and volume constraints, underscores its efficacy in addressing the complexities inherent in hyperbolic Kirchhoff-type problems. Building upon prior research by pioneers in the field, our study aims to extend the applicability of DMF to these specific problem classes, paving the

way for novel insights and solutions.

Central to our methodology is the development of an energy-preserving approximation scheme tailored to hyperbolic Kirchhoff-type problems. By conserving energy throughout the approximation process, we not only enhance the numerical stability of algorithms while also providing a foundation for establishing the uniqueness of solutions in wave-type problems.

The establishment of a uniform estimate for the approximate solutions serves as a key milestone in our research, enabling us to track the convergence of the approximation sequence and identify a unique limit function. Drawing inspiration from previous work on minimization techniques for wave-type problems [61], we leverage these insights while customizing our approach to accommodate the Kirchhoff term and volume constraints inherent in our problem domain.

In summary, our study represents a concerted effort to push the boundaries of hyperbolic variational problems, particularly in the realm of Kirchhoff-type challenges with additional constraints. Through the synergy of advanced mathematical techniques, including the Discrete Morse Flow and energy-preserving approximations, our goal is not only to demonstrate the existence of weak solutions but also to enhance the broader understanding of dynamic systems with evolving boundaries and volume constraints. By bridging theory and application, we aspire to unlock new avenues for innovation and discovery in this complex and multifaceted field.

Related Work and Previous Results

The hyperbolic-type problem has been widely explored, with variational approaches playing a key role in examining solution existence. In this study, we focus on the Discrete Morse Flow (DMF) method, which is specifically designed for hyperbolic-type problems. This approach is a variational technique applied to time-dependent problems.

The positivity fo the generated functional by DMF simplifying the process of proving the existence of a minimizer. This property is crucial, as it guarantees that the minimization process remains well-posed and avoids potential complications arising from indefinite or unbounded functionals. By leveraging this minimizer, the method ultimately constructs a weak solution to the main problem. For further information and technical details, including the derivation of the discretized functional and its properties, readers are encouraged to consult related foundational works and recent advancements in the application of the DMF method to nonlinear and constrained problems. For additional details, see [61, 77].

The variational approach was initially introduced in [43], where it was successfully applied to parabolic-type problems, offering an elegant and effective method for finding solutions to such equations. This approach was subsequently extended to hyperbolic-type problems, as discussed in [39], with the framework was adapted to handle the more complex dynamics and wave propagation characteristic of hyperbolic systems. A significant contribution of this work is the development of the discrete Morse flow method, which has proven to be an effective tool for solving a wide range of problems, including those with free boundaries. As highlighted in [46, 2, 60], the method has been successfully applied to various physical scenarios, particularly in problems involving free boundary conditions.

In these works, a hyperbolic problem is considered in which the free boundary models the vibrations of an elastic string. Notably, these studies omit the external force term. They proved the existence to the weak solution in one-dimensional, providing a solid foundation for applying the variational method to more complex systems. Additionally, [2] developed a scheme that ensures energy conservation throughout the process, offering a clear comparison between this scheme and others previously proposed. The energy-preserving nature of their approach helps maintain the stability and accuracy of numerical simulations.

Moreover, the energy-preserving property of our approximation method provides a crucial advantage by not only enabling the creation of numerical reliable algorithms and robust but also facilitating the proof of uniqueness for solutions to specific wave-type problems. This is particularly important for ensuring that solutions are well-defined and stable over time, avoiding issues such as non-uniqueness or instability that may arise in other methods. The preservation of energy is a key feature that underpins the reliability of the solutions derived through this approach, ensuring that they closely reflect the true physical behavior of the system. For more detailed discussions on this topic and related developments, refer to the works in [24, 55, 58, 19], with further theoretical and computational advancements are explored.

The variational approach was initially introduced in [43], with it was applied to parabolic-type problems and later extended to hyperbolic types, as demonstrated in [39]. The discrete Morse flow has been effectively utilized to address various challenges, including free boundary problems, as shown in [46, 2, 60]. These studies investigated a hyperbolic problem with a free boundary that models the vibrations of an elastic string, focusing on the inherent dynamics of the system while omitting the external force term. By analyzing the system under these simplified conditions, the researchers were able to establish the existence of weak solutions in one-dimensional scenarios, providing important insights

to the behavior of the free boundary conditions vibrating strings. These results form the foundation for further explorations into more complex systems, with additional factors such as

external forces, multi-dimensional geometries, or nonlinearity may be introduced. The methodology used in these studies is also instrumental in advancing the theoretical understanding of wave phenomena in elastic materials, especially in cases with boundary conditions play a crucial role in determining the overall behavior of the system.

Additionally, [2] developed a scheme that guarantees energy preservation throughout the numerical process, providing a reliable mechanism for maintaining the system's stability and consistency during computation. The scheme was compared with previous methods to highlight its advantages, particularly in terms of ensuring that energy is conserved in simulations, which is a crucial feature when solving dynamic problems in physics and engineering. This comparison demonstrated that the energy-preserving approach offers a more robust and accurate framework for simulating wave-type systems, especially when dealing with complex boundary conditions or nonlinear effects.

Notably, the energy-preserving property of our approximation method goes beyond merely improving numerical stability. It plays a pivotal role in facilitating the creation of highly reliable numerical algorithms that can handle a wide range of problems effectively. This feature is particularly important for problems with the physical conservation of energy is essential, such as in vibrating structures, wave propagation in media, or elastic materials. Moreover, by ensuring that energy remains conserved, the method contributes to proving the uniqueness of solutions for specific wave-type problems. This is a key advantage, as it allows for a more precise understanding of the system's behavior and ensures that the solutions obtained are physically meaningful and mathematically well-defined.

The application of this energy-preserving approach can extend to various complex systems, with maintaining stability and consistency is critical, and has the potential to significantly enhance the accuracy and reliability of simulations. For further details on the theoretical background and practical applications of these techniques, refer to [24, 55, 58, 19], with more in-depth discussions on energy conservation and the development of numerical algorithms for wave-type problems are provided.

Bonafini [11] explores a non-local wave equation obstacle problem, addressing its complexity by introducing a convex minimization approach combined with a time-discrete approximation scheme. Through this methodology, Bonafini successfully demonstrates the existence of a weaker variant of a weak solution, offering a more accessible and mathematically tractable form of the solution than conventional approaches. The use of time-discretization is particularly important for handling the non-local nature of the problem, allowing for a step-by-step approximation that captures the dynamics of the wave equation over time.

A key feature of our approach, which distinguishes it from earlier methods, is the provision of the approximate solutions to uniform estimate. This is crucial for ensuring consistency across the entire solution process and guarantees that the approximations remain bounded and well-behaved. The uniform estimate plays a big role in the analysis, also to ensures that a unique limit function can be identified. This uniqueness is particularly valuable, as it confirms that the solution obtained is independent of the specific discretization scheme used. This leads to a unique limit function, our method provides a robust framework for solving non-local wave equation problems, paving the way for further applications in complex dynamical systems with traditional methods may struggle.

Additionally, an intriguing method discussed in [26, 27] reformulates a nonlinear wave-type development as a minimization problem. Rather than discretizing time, this approach utilizes a singular weight in the integrand, which effectively transforms the problem into a convex minimization challenge and demonstrates the existence of weak solutions. The characteristic sets our approach apart from the method presented in [26, 27], offering a distinct advantage in terms of the convergence behavior and applicability.

Building on these foundational works, we have applied a new technics to tackle a hyperbolic Kirchhoff-type problem with both a free boundary and a volume constraint. In our approach, we make use of the Discrete Morse Flow (DMF), a reliable and widely recognized technique for addressing variational problems akin to ours. Through the application of DMF, Our objective is to rigorously prove the existence of a weak solution to the main problem.

Furthermore, we extend our analysis to study a hyperbolic p-Kirchhoff-type problem under similar constraints, applying DMF to demonstrate the existence of weak solutions in this context as well. This exploration not only builds on existing methods but also pushes the boundaries of what can be achieved in constrained hyperbolic variational problems, particularly in settings involving complex boundary dynamics and volume preservation.

CHAPTER 2

MATHEMATICAL PRELIMINARIES

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2.1 Introduction

This chapter establishes the mathematical foundations necessary for the analysis of hyperbolic Kirchhoff-type problems with constraints. The study of such problems requires a careful blend of functional analysis, partial differential equations, and variational calculus. We begin by introducing the essential function spaces in which solutions are sought, including Lebesgue, Sobolev, and Bochner spaces, which provide the appropriate framework for weak formulations. Special attention is given to spaces relevant to free-boundary and volume-constrained problems, which are central to this work.

Beyond function spaces, we present key variational tools that underpin the Discrete Morse Flow method—the primary technique employed in this thesis. These include the direct method of the calculus of variations, coercivity, weak lower semi-continuity, and the Euler-Lagrange variational inequality. Together, these concepts form the theoretical backbone for constructing minimizers at discrete time steps and establishing the existence of weak solutions.

The notation and results collected here will be referenced throughout subsequent chapters. Readers familiar with functional analysis and variational methods may treat this chapter as a reference, while those new to these concepts will find it a self-contained introduction to the technical prerequisites.

2.2 Function Spaces

Throughout this work, $Q \subset \mathbb{R}^N$ denotes a bounded domain with Lipschitz boundary ∂Q, $T > 0$ is a final time, and $Q_T = Q \times (0, T)$. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, we employ the following function spaces.

2.2.1 Spaces of Continuous and Smooth Functions

Definition 2.2.1 (*Spaces of continuous functions*)

Let Q be an open subset of $\mathbb{R}^N$.

(1) $\mathcal{C}(Q)$ is the space of continuous functions $v : Q \to \mathbb{R}$, endowed with the supremum norm

$$\|v\|_{\mathcal{C}(Q)} = \sup_{x \in Q} |v(x)|.$$

(2) For $k \in \mathbb{N}$, $\mathcal{C}^k(Q)$ is the space of functions whose derivatives up to order k exist and are continuous, with norm

$$\|v\|_{\mathcal{C}^k(Q)} = \sum_{|\alpha| \leq k} \sup_{x \in Q} |D^\alpha v(x)|.$$

(3) $\mathcal{C}^\infty(Q) = \bigcap_{k \in \mathbb{N}} \mathcal{C}^k(Q)$.

Definition 2.2.2 (*Space of test functions*)

$\mathcal{C}_0^\infty(Q)$ denotes the subspace of $\mathcal{C}^\infty(Q)$ consisting of functions with compact support in Q. This is the space of test functions *used in distribution theory.*

2.2.2 Lebesgue Spaces

Definition 2.2.3 (*Lebesgue spaces*)

For $1 \leq p < \infty$, the Lebesgue space $L^p(Q)$ consists of measurable functions $v : Q \to \mathbb{R}$ for which

$$\|v\|_{L^p(Q)} = \left(\int_Q |v(x)|^p \, dx \right)^{1/p} < \infty.$$

For $p = \infty$, $L^\infty(Q)$ is the space of essentially bounded functions with norm

$$\|v\|_{L^\infty(Q)} = \operatorname{ess\,sup}_{x \in Q} |v(x)|.$$

These spaces are Banach spaces, and $L^2(Q)$ is a Hilbert space with inner product

$$(u, v)_{L^2(Q)} = \int_Q u(x) v(x) \, dx.$$

Theorem 2.2.1 (*Basic properties of L^p spaces*)

(1) ***Completeness:*** *$L^p(Q)$ is complete for $1 \leq p \leq \infty$.*

(2) ***Density:*** *$\mathcal{C}_0^\infty(Q)$ is dense in $L^p(Q)$ for $1 \leq p < \infty$.*

(3) ***Dual spaces:*** *For $1 < p < \infty$, the dual space of $L^p(Q)$ is $L^q(Q)$ with $\frac{1}{p} + \frac{1}{q} = 1$.*

(4) ***Hölder's inequality:*** *If $u \in L^p(Q)$, $v \in L^q(Q)$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\|uv\|_{L^1(Q)} \leq \|u\|_{L^p(Q)} \|v\|_{L^q(Q)}.$$

2.2.3 Sobolev Spaces

Definition 2.2.4 (*Sobolev spaces*)

For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(Q)$ consists of functions $v \in L^p(Q)$ whose weak derivatives $D^\alpha v$ exist and belong to $L^p(Q)$ for all multi-indices α with $|\alpha| \leq k$. It is equipped with the norm

$$\|v\|_{W^{k,p}(Q)} = \left(\sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^p(Q)}^p \right)^{1/p}, \quad 1 \leq p < \infty,$$

and

$$\|v\|_{W^{k,\infty}(Q)} = \sum_{|\alpha| \leq k} \|D^\alpha v\|_{L^\infty(Q)}.$$

We denote $H^k(Q) = W^{k,2}(Q)$, which are Hilbert spaces.

Definition 2.2.5 (*Sobolev spaces with zero trace*)

$W_0^{1,p}(Q)$ is the closure of $\mathcal{C}_0^\infty(Q)$ in $W^{1,p}(Q)$. In particular, $H_0^1(Q) = W_0^{1,2}(Q)$. Functions in $W_0^{1,p}(Q)$ vanish on ∂Q in the trace sense.

Theorem 2.2.2 (*Key properties of Sobolev spaces*)

(1) ***Reflexivity and separability:*** *For $1 < p < \infty$, $W^{k,p}(Q)$ is reflexive and separable.*

(2) ***Poincaré inequality:*** *For $v \in W_0^{1,p}(Q)$, there exists $C_P > 0$ such that*

$$\|v\|_{L^p(Q)} \leq C_P \|\nabla v\|_{L^p(Q)}.$$

(3) ***Sobolev embeddings:*** *There exist continuous (and sometimes compact) embeddings depending on N, k, p. For example:*

- *If $p < N$, then $W^{1,p}(Q) \hookrightarrow L^{p^*}(Q)$ with $p^* = \frac{Np}{N-p}$.*

- *If $p > N$, then $W^{1,p}(Q) \hookrightarrow \mathcal{C}^{0,\gamma}(\overline{Q})$ for some $\gamma \in (0,1)$.*

2.2.4 Bochner Spaces

Definition 2.2.6 (*Bochner spaces*)

Let X be a Banach space with norm $\|\cdot\|_X$.

(1) $L^p(0,T;X)$ consists of strongly measurable functions $v : [0,T] \to X$ such that

$$\|v\|_{L^p(0,T;X)} = \left(\int_0^T \|v(t)\|_X^p \, dt\right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

(2) $L^\infty(0,T;X)$ consists of essentially bounded functions with norm

$$\|v\|_{L^\infty(0,T;X)} = \operatorname{ess\,sup}_{t\in[0,T]} \|v(t)\|_X.$$

(3) $W^{1,p}(0,T;X) = \{v \in L^p(0,T;X) : v_t \in L^p(0,T;X)\}$, where v_t denotes the weak time derivative.

Theorem 2.2.3 (*Properties of Bochner spaces*)

(1) ***Identification with tensor products:*** *There is an isometric isomorphism $L^p(0,T;L^p(Q)) \cong L^p(Q_T)$.*

(2) ***Aubin–Lions lemma:*** *Let $X \subset Y \subset Z$ be Banach spaces with compact embedding $X \hookrightarrow Y$ and continuous embedding $Y \hookrightarrow Z$. If $\{v_n\}$ is bounded in $L^p(0,T;X)$ and $\{\partial_t v_n\}$ is bounded in $L^q(0,T;Z)$ with $1 < p, q < \infty$, then $\{v_n\}$ is relatively compact in $L^p(0,T;Y)$.*

2.2.5 Vector-Valued Function Spaces

Definition 2.2.7 (*Spaces of vector-valued functions*)

$\mathcal{C}_0^\infty(Q;\mathbb{R}^m)$ is the space of smooth vector fields $\mathbf{v} = (v_1, \ldots, v_m) : Q \to \mathbb{R}^m$ with compact support. Similarly, we define $L^p(Q;\mathbb{R}^m)$ and $W^{k,p}(Q;\mathbb{R}^m)$ componentwise.

2.3 Special Notation for Free Boundary Problems

Definition 2.3.1 (*Sets and characteristic functions*)

For a function $v : Q_T \to \mathbb{R}$, we define

$$\{v > 0\} = \{(x,t) \in Q_T : v(x,t) > 0\},$$

and $\chi_{\{v>0\}}$ denotes its characteristic function:

$$\chi_{\{v>0\}}(x,t) = \begin{cases} 1 & \text{if } v(x,t) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

2.4 Special Notation for Volume Constraint Problems

Definition 2.4.1 (*Volume constraint notation*)

In problems with volume constraints, we employ the following notation:

(1) $\mathcal{V} > 0$ denotes the prescribed (constant) volume.

(2) The volume constraint is expressed as:

$$\int_Q v(x,t)\,dx = \mathcal{V}, \quad \text{for a.e. } t \in [0,T].$$

(3) $\lambda = \lambda(t)$ denotes the Lagrange multiplier associated with the volume constraint. This is a time-dependent scalar function arising from the Euler-Lagrange formulation.

(4) The admissible set with volume constraint is defined as:

$$K_{\mathcal{V}} = \left\{ v \in W_0^{1,p}(Q) : \int_Q v(x)\,dx = \mathcal{V} \right\}.$$

(5) For test functions $\phi \in \mathcal{C}_0^\infty(Q \times [0,T))$, we define the spatial average:

$$\Phi(t) = \int_Q \phi(x,t)\,dx.$$

This quantity appears in the weak formulation of volume-constrained problems.

(6) The indicator functional for the volume constraint is:

$$\Psi_{\mathcal{V}}(v) = \begin{cases} 0 & \text{if } \displaystyle\int_Q v(x)\,dx = \mathcal{V}, \\ +\infty & \text{otherwise.} \end{cases}$$

This is used in the variational formulation via the Discrete Morse Flow method.

2.5 Variational Framework and the Direct Method

The Discrete Morse Flow method is fundamentally rooted in the calculus of variations. This section presents the key definitions and theorems that form the theoretical foundation for constructing minimizers at each time discretization step.

Let X denote a reflexive Banach space (in our context, typically $\mathrm{H}_0^1(Q)$ or $\mathrm{W}_0^{1,p}(Q)$), and let $K \subset X$ be a nonempty, weakly closed subset representing admissible functions satisfying possible constraints.

Definition 2.5.1 (*Minimization problem with constraints*)

For a given functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$, we consider the constrained minimization problem:

$$\text{Find } v^* \in K \text{ such that } \mathcal{F}(v^*) = \inf_{v \in K} \mathcal{F}(v).$$

Any such v^* is called a ***minimizer*** *of $\mathcal{F}$ over K. The functional $\mathcal{F}$ is extended to $+\infty$ outside K when necessary, effectively encoding the constraint via an indicator functional.*

The existence of minimizers is guaranteed under two fundamental properties:

Definition 2.5.2 (*Coercivity*)

A functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ is said to be ***coercive*** *on K if*

$$\lim_{\|v\|_X \to \infty,\ v \in K} \mathcal{F}(v) = +\infty.$$

Equivalently, for every sequence $\{v_m\} \subset K$ with $\|v_m\|_X \to \infty$, we have $\mathcal{F}(v_m) \to +\infty$.

Definition 2.5.3 (*Weak lower semi-continuity*)

A functional $\mathcal{F} : X \to \mathbb{R} \cup \{+\infty\}$ is ***weakly lower semi-continuous (w.l.s.c.)*** *on K if for every sequence $\{v_m\} \subset K$ converging weakly to v in X (denoted $v_m \rightharpoonup v$), we have*

$$\mathcal{F}(v) \leq \liminf_{n \to \infty} \mathcal{F}(v_m).$$

With these properties, we state the cornerstone existence theorem:

Theorem 2.5.1 (*Direct method in the calculus of variations*)

Let X be a reflexive Banach space, $K \subset X$ a weakly closed nonempty subset, and $\mathcal{F} : K \to \mathbb{R} \cup \{+\infty\}$ a functional satisfying:

1. *$\mathcal{F}$ is* ***coercive*** *on K.*
2. *$\mathcal{F}$ is* ***weakly lower semi-continuous*** *on K.*

Then there exists at least one minimizer $v^* \in K$ such that

$$\mathcal{F}(v^*) = \min_{v \in K} \mathcal{F}(v).$$

Proof : Since $\mathcal{F}$ is not identically $+\infty$ on K, we can consider a minimizing sequence $\{v_m\} \subset K$ such that

$$\lim_{n\to\infty} \mathcal{F}(v_m) = \inf_{v\in K} \mathcal{F}(v) =: m > -\infty.$$

By coercivity, $\{v_m\}$ is bounded in X. The reflexivity of X implies the existence of a subsequence $\{v_{n_k}\}$ and an element $v^* \in X$ such that $v_{n_k} \rightharpoonup v^*$. Since K is weakly closed, $v^* \in K$. Finally, the weak lower semi-continuity yields

$$\mathcal{F}(v^*) \leq \liminf_{k\to\infty} \mathcal{F}(v_{n_k}) = m,$$

proving that v^* attains the minimum. ■

Once existence is established, the following first-order optimality condition characterizes minimizers:

Proposition 2.5.1 (*Euler-Lagrange variational inequality*)

Let $K \subset X$ be convex and $\mathcal{F}$ be Gâteaux differentiable at a point $v^* \in K$. If v^* is a minimizer of $\mathcal{F}$ over K, then for every admissible direction $\varphi \in X$ such that $v^* + \varepsilon\varphi \in K$ for sufficiently small $\varepsilon > 0$, we have

$$\mathcal{F}'(v^*)[\varphi] \geq 0,$$

where $\mathcal{F}'(v^*)[\cdot]$ denotes the Gâteaux derivative at v^*. If, in particular, v^* lies in the interior of K, this condition reduces to $\mathcal{F}'(v^*)[\varphi] = 0$ for all $\varphi \in X$.

Proof : For any admissible φ and $\varepsilon > 0$ small enough, convexity of K gives $v^* + \varepsilon\varphi \in K$. By minimality,

$$\mathcal{F}(v^* + \varepsilon\varphi) - \mathcal{F}(v^*) \geq 0.$$

Dividing by ε and letting $\varepsilon \to 0^+$ yields $\mathcal{F}'(v^*)[\varphi] \geq 0$. The interior case follows by considering both φ and $-\varphi$. ■

This variational framework not only ensures the well-posedness of each step in the Discrete Morse Flow but also provides the theoretical underpinning for the convergence analysis presented in the subsequent sections.

2.6 Conclusion

This chapter has provided the essential mathematical foundations for the study of hyperbolic Kirchhoff-type problems with constraints. We have systematically introduced the function spaces—Lebesgue, Sobolev, and Bochner spaces—that serve as the natural settings for weak formulations. Special notation for free-boundary and volume-constrained problems has been established, reflecting the specific challenges addressed in this thesis.

The variational framework presented here, centered on the direct method of the calculus of variations, provides the theoretical machinery necessary for the Discrete Morse Flow approach.

Key concepts such as coercivity, weak lower semi-continuity, and the Euler-Lagrange variational inequality will be invoked repeatedly in subsequent chapters to establish the existence of minimizers at discrete time steps and to pass to the continuous limit.

With these preliminaries in place, we now turn to the main subject of this thesis: the construction and analysis of weak solutions for hyperbolic Kirchhoff-type problems with free boundaries and volume constraints using the Discrete Morse Flow method.

CHAPTER 3

DISCRETE MORSE FLOW METHOD

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3.1 Introduction

This chapter introduces the Discrete Morse Flow (DMF) method , a variational technique designed to solve constrained optimization problems. Grounded in discrete variational calculus and Morse theory, the method transforms complex, continuous variational problems—particularly those involving hyperbolic PDEs—into a sequence of simpler minimization problems at discrete time steps. This discretization makes the problems computationally tractable while preserving their essential mathematical structure. A key aspect of DMF is its integration with energy analysis, which ensures solution stability and consistency with the underlying physical principles. By providing a rigorous yet practical framework, DMF effectively bridges theoretical analysis and numerical computation, offering a reliable approach for addressing challenging problems in science and engineering.

3.2 Mathematical formulation of the DMF approach

3.2.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of a solution to our problem.

$$\begin{cases} v_{tt}(x,t) = \Delta v(x,t), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q, \end{cases} \tag{3.1}$$

Then we define the following

Definition 3.2.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q))$ is a weak solution to (2.1) if $v(x,0) = v_0$, for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T}))$ if

$$\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \nabla v(x,t)\nabla\phi(x,t)\right] dxdt - \int_Q v_0(x)\phi(x,0)dx = 0. \tag{3.2}$$

3.2.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. Partitioning the time interval $(0, \mathcal{T})$ into M equal sub-intervals the time interval, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v_j) = \int_Q \left[\frac{|v_j - 2v_{m-1} + v_{m-2}|^2}{2i^2} + |\nabla v_j|^2 \right] dx \tag{3.3}$$

Note that $\mathcal{F}_m$ is defined on the space $X = \mathrm{H}_0^1(Q)$, then, by the Direct Method in the Calculus of Variations (Theorem 2.5.1), the coercivity and weak lower semi-continuity of $\mathcal{F}_m$ guarantee the existence of a minimizer $v_m \in K$ at each time step. Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$.

Lemma 3.2.1

Let v be in $\mathrm{H}_0^1(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing each term in the functional separately. First, the term $\mathcal{F}^1(v)$, which represents a time discretization contribution, is shown to be weakly lower semi-continuous using properties of weak convergence and direct calculations related to its quadratic form. Next, $\mathcal{F}^2(v)$, involving gradient norms, leverages the weak lower semi-continuity of norms in Banach spaces to establish its continuity. Since all terms $\mathcal{F}^1$, and $\mathcal{F}^2$ are weakly lower semi-continuous, the overall functional $\mathcal{F}_m$ inherits this property in $\mathrm{H}_0^1(Q)$.

Then establishes the coercivity of the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing its behavior along a minimizing sequence. Starting with a minimizing sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathrm{H}_0^1(Q)$ that converges weakly to v, it is assumed that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and $\mathcal{F}_m(v_1) \neq \infty$. The functional $\mathcal{F}_m(v_j)$ is expressed in terms of its individual components, including a quadratic term involving the time-discrete difference, a term dependent on the gradient norm. By lower-bounding $\mathcal{F}_m(v_j)$, it is shown that the gradient norm term dominates the other terms when the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large. Specifically, as $\|v_j\|_{\mathrm{H}^1(Q)} \to \infty$, the quadratic growth of the gradient term ensures that $\mathcal{F}_m(v_j) \to \infty$. This behavior demonstrates the coercivity of $\mathcal{F}_m$,

confirming that the functional grows unboundedly with the norm of the sequence, ensuring stability and compactness in the space $\mathrm{H}_0^1(Q)$.

The proof finally, demonstrates the existence of a minimizer for the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$. Since $\mathcal{F}_m$ is both weakly lower semi-continuous and coercive, the direct method in the calculus of variations ensures that it possesses a minimizer in $\mathrm{H}_0^1(Q)$. To confirm this, the proof revisits the coercivity argument, showing that $\mathcal{F}_m(v^1)$ can be bounded below by terms involving the norm $\|v_j\|_{\mathrm{H}^1(Q)}$. When the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large, the functional grows unboundedly, which guarantees coercivity. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{H}_0^1(Q)$. By the weak lower semi-continuity of $\mathcal{F}_m$, the limit v minimizes the functional, concluding that v is indeed a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$. ■

Next, we define piecewise linear time interpolant solutions v^i for $(x,t) \in Q \times ((j-1)i, ji]$, with $j = 0, 1, ..., M$ by

$$v^i(x,t) = \frac{(t-(j-1)i)}{i} v_j(x) + \frac{ij-t}{i} v_{j-1}(x) \tag{3.4}$$

and a piecewise constant step function $\bar{v}^i$ by

$$\bar{v}^i(x,t) = v_j(x). \tag{3.5}$$

Since v_m is a minimizer of $\mathcal{F}_m$, it satisfies the Euler-Lagrange variational inequality (Proposition 2.5.1). Taking admissible test functions $\phi \in C_0^\infty(Q \times [0, \mathcal{T}))$, we obtained the approximate weak solution as following:

$$\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \; \mathrm{d}t + \int_0^{\mathcal{T}} \int_Q \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \; \mathrm{d}t = 0.$$

3.2.3 Energy Analysis

In this subsection, we present an energy estimate for minimizers, which leads to significant conclusions about their convergence.

Lemma 3.2.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\| {v^i}_t(t) \right\|_{\mathrm{L}^2(Q)}^2 + \left\| \bar{v}^i(t) \right\|_{\mathrm{H}_0^1(Q)}^2 \leq C. \tag{3.6}$$

with C is a constant independent of i, such that

$$C = \frac{1}{2} \|u_0\|_{\mathrm{L}^2(Q)}^2 + \frac{1}{2} \|\nabla v_0\|_{\mathrm{L}^2(Q)}^2 .$$

Proof : The proof begins by considering an arbitrary $\theta \in (0,1)$ and a minimizer v_j of the functional $\mathcal{F}_j$. The goal is to show the behavior of the difference between $\mathcal{F}_j$ evaluated at a convex combination of v_j and v_{j-1}, and the value of $\mathcal{F}_j$ at v_j. Starting with the inequality

$$0 \leq \mathcal{F}_j\left((1-\theta)v_j + \theta v_{j-1}\right) - \mathcal{F}_j(v_j), \tag{3.7}$$

the proof proceeds by dividing both sides by θ and taking the limit as $\theta \to 0^+$. This yields an expression for the derivative of $\mathcal{F}_j$ at v_j. Each term in the difference is then examined individually, starting with $\mathcal{F}^1(\theta)$, which involves the discretized term. By applying appropriate approximations and calculations, the proof establishes that $\mathcal{F}^1(\theta)$ behaves as expected according to equation (3.7). Similarly, for $\mathcal{F}^2(\theta)$, the proof uses known results and performs calculations that lead to the same outcome .

This detailed derivation explores the energy analysis for minimizers of a functional $\mathcal{F}$ through successive approximations and inequalities. By summing contributions from intermediate functionals $\mathcal{F}^1, \mathcal{F}^2$, an initial inequality is established for the sequence v_j in terms of the L^2-norm and gradient terms. Using assumptions and simplifications and properties of the functional, a series of inequalities are derived. Through summation over a range of indices j, telescoping cancellations simplify the terms, leading to a bound on the energy accumulation. The discrete Gronwall lemma is applied to conclude that the energy, represented by terms like $\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla v_m\|^2_{\mathrm{L}^2(Q)}$, remains bounded by a constant C. This constant depends on the initial data v_0, ∇v_0, and other problem-specific parameters. Finally, the result asserts the stability and boundedness of the energy for the minimizers, encapsulated in a simplified form involving $\|v_t^i\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla \bar{v}^i\|^2_{\mathrm{L}^2(Q)}$. This guarantees the well-posedness and energy conservation properties of the system under study. ■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}^1_0(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&\bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v^i \longrightarrow v_1 \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla \bar{v}^i \rightharpoonup v_2 \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_3 \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+.
\end{aligned}$$

Where focuses on the convergence properties of a sequence of functions v^i and their associated components. It is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{H}^1_0(Q)$, and both $\nabla \bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^2(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla \bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$

and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i-\bar{v}^i\|^2_{\mathrm{L}^2(Q_\mathcal{T})} \leq i\|v^i_t\|^2_{\mathrm{L}^2(Q_\mathcal{T})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_\mathcal{T}$.

3.2.4 Convergence to the Limit Function

Theorem 3.2.1

For v_0, v_0 belong to $\mathrm{H}^1_0(Q)$. Then there exists a weak solution to the problem (3.1), according to Definition 3.2.1.

Proof : In this proof, we begin by considering a sequence of functions $v^i(x,t)$ that are approximations of the solution $v(x,t)$, which we aim to show converges to a weak solution defined in Definition 3.2.1. To do so, we fix an arbitrary test function $\phi(x,t) \in C^\infty_0(Q \times [0,\mathcal{T}))$. Next, we use this fact to derive an equation for the sequence $v^i(x,t)$. We start with the weak formulation for $v^i(x,t)$, which includes a time derivative term, a gradient term involving $\bar{v}^i(x,t)$, and an external force term. By taking the limit as $i \to 0$, we evaluate the convergence of each term in the weak formulation.

We address the time derivative term. Using the fact that $v^i(x,t)$ converges uniformly to $v(x,t)$, we apply the dominated convergence theorem to obtain the limit of the time derivative term as $i \to 0$. We similarly handle the gradient term, which involves the operator ∇, and demonstrate that the gradient terms converge in the appropriate sense, ensuring that the weak formulation holds for the limit function $v(x,t)$.

In the end, we show that the weak solution $v(x,t)$ satisfies the weak formulation of the problem, including the differential equation, the initial condition. Thus, we have successfully shown that the sequence $v^i(x,t)$ converges to a weak solution in the sense defined in Definition 3.2.1 ■

3.3 Benefits and expansions of the DMF approach

While there are numerous simpler methods available that can achieve comparable outcomes, the discrete Morse flow approach offers distinct and considerable advantages, particularly when addressing more complex problems involving non-local terms, free boundaries, and volume constraints. Traditional techniques, while sufficient for more straightforward scenarios, often fall short when faced with the added complexity of nonlinearities, non-local effects, free boundaries, and volume constraints. These types of problems require more sophisticated and refined methods, which is with the discrete Morse flow method excels. This approach extends its

applicability seamlessly to advanced challenges, such as free-boundary problems, that would otherwise be difficult to manage with simpler methods. Throughout this work, we will illustrate how the discrete Morse flow method provides a robust framework to tackle these intricate issues, progressively demonstrating its power and versatility in dealing with the challenges posed by such problems.

A key reason for adopting the discrete Morse flow method in this study, especially for volume-constrained problems, is its capacity to semi-discretize time. By discretizing time in this manner, the method simplifies the dynamics of the system, breaking it down into manageable steps that can be analyzed more effectively. This semi-discretization allows the application of well-established results from elliptic theory, which are crucial for solving the underlying equations. These results, when applied within the discrete Morse flow framework, turn a time-dependent problem that could otherwise be too complex to solve into one that can be tackled with greater efficiency using the tools of variational calculus and elliptic partial differential equations (PDEs). This transformation is particularly valuable because it opens the door to leveraging existing knowledge from elliptic theory, which has been developed to handle boundary conditions, constraints, and other aspects of complex physical systems.

Moreover, the variational framework at the heart of the discrete Morse flow approach provides a natural and systematic way to incorporate the volume constraint directly into the set of admissible functions. This ensures that the constraint is maintained throughout the entire solution process, without requiring additional assumptions, adjustments, or modifications to the method. By embedding the volume constraint within the variational formulation, the method guarantees that the resulting solution respects the physical limitations imposed by the volume constraint, offering a more accurate and reliable representation of the system's behavior. This is a crucial aspect when solving real-world problems with such constraints are not merely theoretical but have tangible implications for the system being modeled. The ability to preserve these constraints while still solving for the weak solution is one of the central strengths of the discrete Morse flow method, and it is a feature that makes it especially suitable for complex applications in material science, fluid dynamics, and wave propagation.

In summary, while traditional methods may suffice for simpler problems, the discrete Morse flow approach provides a more robust and efficient framework for solving problems with free boundaries, non-local terms, and volume constraints. Its ability to semi-discretize time, combined with its connection to elliptic theory and its natural incorporation of volume constraints, makes it an invaluable tool for tackling advanced problems that involve intricate dynamics. As we will demonstrate in subsequent sections, this method not only addresses the inherent complexities of these problems but also offers a path forward for solving them in a mathematically

rigorous and physically meaningful way.

Traditional techniques, though considered, fall short in several key areas when applied to our problem. For example, methods like the Galerkin approximation, though widely used in the analysis of PDEs, present significant challenges in this context. The Galerkin method, when applied to problems of this nature, typically leads to a system of nonlinear ordinary differential equations. However, due to the complexity introduced by the nonlocal multiplier term, it is difficult to ensure the existence of a solution over the entire time interval. In many cases, such systems may only provide approximate solutions that lack the precision and rigor needed to address the problem's inherent constraints. Moreover, there is often no guarantee that these methods will yield solutions that satisfy the volume preservation requirement, further complicating the analysis and limiting their effectiveness.

Additionally, fixed-point methods, while powerful in many contexts, are less suitable for this type of problem. These methods often require stronger assumptions on the data, such as specific smoothness or boundedness conditions, which may not hold in practical applications. Furthermore, fixed-point approaches struggle to ensure that the volume is preserved in approximate solutions. This limitation is crucial because volume preservation is a fundamental aspect of the physical problem we are addressing, and without it, necessary estimates become exceedingly difficult to obtain. As a result, fixed-point methods are not only less effective in yielding a (weak) solution, but they also complicate the overall analysis.

In contrast, the discrete Morse flow method offers a clear advantage in these respects. Its formulation, which relies on variational principles, provides a robust mechanism for incorporating volume constraints and ensuring their preservation throughout the solution process. This is achieved by embedding the volume constraint directly into the minimization problem for the time-discrete functional. As such, this approach naturally lends itself to problems with volume conservation is critical, without requiring additional complex adjustments or assumptions.

Another compelling advantage of the discrete Morse flow method is its suitability for numerical computation. Many classical methods, such as Galerkin or fixed-point techniques, can become computationally expensive or impractical when dealing with the intricacies of nonlocal terms and constraints like volume preservation. In contrast, the discrete Morse flow method is inherently structured to accommodate numerical implementations, making it not only theoretically sound but also computationally feasible. This makes it a powerful tool for solving large-scale, real-world problems with both accuracy and efficiency are paramount.

Furthermore, the relationship between the discrete Morse flow and sub-differential meth-

ods, as mentioned earlier in this chapter, further underscores its versatility. Sub-differential methods are often employed in optimization problems and the study of variational inequalities, providing a broader mathematical framework that complements the Morse flow approach. By integrating these concepts, the discrete Morse flow becomes a highly adaptable and flexible method, capable of addressing a wide range of problems that would otherwise be difficult to solve using more traditional techniques.

In conclusion, the discrete Morse flow method offers a unique combination of benefits that suits it for problems involving nonlinearities, nonlocal terms, free boundary, and volume constraints. Its ability to seamlessly integrate volume conservation into the solution process, along with its compatibility with numerical methods, ensures that it provides rigorous theoretical solutions and practical computational tools. As we will demonstrate in the following sections, this method provides a comprehensive framework for solving complex problems with a high degree of precision and efficiency, making it an indispensable tool in modern mathematical and computational analysis.

3.4 Conclusion

This chapter details the Discrete Morse Flow (DMF) method as the core variational technique for constructing solutions to the hyperbolic Kirchhoff-type problems. The DMF approach transforms each continuous, nonlinear evolutionary problem into a sequence of minimization problems at discrete time steps. By proving the associated functionals are coercive and weakly lower semi-continuous, the existence of minimizers at each step is guaranteed. Energy Analysis is then applied to these discrete solutions to derive uniform estimates, ensuring stability and adherence to physical constraints. Finally, the Convergence to the Limit Function is rigorously established, proving that the sequence of approximate minimizers converges to a genuine weak solution of the original problem. This method provides a robust and adaptable framework that bridges theoretical analysis with computational implementation.

CHAPTER 4

HYPERBOLIC KIRCHHOFF-TYPE PROBLEM

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4.1 Introduction

This thesis establishes the existence of weak solutions for hyperbolic Kirchhoff and p-Kirchhoff-type problems, extended to incorporate free boundaries and volume constraints . By employing the Discrete Morse Flow method alongside energy analysis and convergence techniques, the continuous problems are discretized into sequences of minimization problems whose solutions converge to stable weak solutions. The theoretical framework is validated through one-dimensional numerical simulations, providing a robust and unified approach for modeling complex systems in continuum mechanics with evolving interfaces and conservation laws.

4.2 Hyperbolic Kirchhoff type problem

4.2.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \Delta v(x,t) - g(x,t,v(x,t)), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q, \end{cases} \tag{4.1}$$

Then we define the following

Definition 4.2.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}^1_0(Q))$ is a weak solution to (4.1) if $v(x,0) = v_0$, for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T}))$ if

$$\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \nabla v(x,t)\nabla\phi(x,t) + g(x,t,v(x,t))\phi(x,t)\right] dxdt - \int_Q u_0(x)\phi(x,0)dx = 0. \tag{4.2}$$

4.2.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. Into M equal sub-intervals the time interval $(0, \mathcal{T})$ is divided, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{|v - 2v_{m-1} + v_{m-2}|^2}{2i^2} + \left(\frac{\alpha}{2} + \frac{\beta}{4} \|\nabla v\|^2_{\mathrm{L}^2(Q)} \right) |\nabla v|^2 + G_m(v) \right] dx. \tag{4.3}$$

We discretized the term g at time level $m = 1, 2, ..., M$ by means of introducing

$$G_m(x, v(x)) = \int_0^v G_m(x, s(x))ds, \quad \text{with} \quad G_m(x, v(x)) = \frac{1}{j} \int_{(m-1)j}^{mj} g(x, t, v(x, t))dt. \tag{4.4}$$

Furthermore

$$G_m(v) \leq \frac{C_g}{2} v^2 + \gamma_m v, \quad \text{with} \quad \gamma_m = \frac{1}{j} \int_{(m-1)j}^{mj} \gamma(x, t)dt. \tag{4.5}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 4.2.1

Let v be in $\mathrm{H}_0^1(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing each term in the functional separately. First, the term $\mathcal{F}^1(v)$, which represents a time discretization contribution, is shown to be weakly lower semi-continuous using properties of weak convergence and direct calculations related to its quadratic form. Next, $\mathcal{F}^2(v)$, involving gradient norms, leverages the weak lower semi-continuity of norms in Banach spaces to establish its continuity. For the term $\mathcal{F}^3(v)$, associated with the nonlinear function $G_m(v)$, the proof employs the dominated convergence theorem to show that $\mathcal{F}^3(v_j)$ converges to $\mathcal{F}^3(v)$ as $j \to \infty$, ensuring weak lower semi-continuity. Since all terms $\mathcal{F}^1$, $\mathcal{F}^2$, and $\mathcal{F}^3$ are weakly lower semi-continuous, the overall functional $\mathcal{F}_m$ inherits this property in $\mathrm{H}_0^1(Q)$.

Then establishes the coercivity of the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing its behavior along a minimizing sequence. Starting with a minimizing sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathrm{W}_0^{1,p}(Q)$ that converges weakly to v, it is assumed that $\mathcal{F}_m(v) = \liminf_{j \to \infty} \mathcal{F}_m(v_j)$ and $\mathcal{F}_m(v_1) \neq \infty$.

The functional $\mathcal{F}_m(v_j)$ is expressed in terms of its individual components, including a quadratic term involving the time-discrete difference, a term dependent on the gradient norm weighted by coefficients α and β, the nonlinear term $G_m(v_j)$, and the term $\Psi(v_j)$. By lower-bounding $\mathcal{F}_m(v_j)$, it is shown that the gradient norm term dominates the other terms when the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large. Specifically, as $\|v_j\|_{\mathrm{H}^1(Q)} \to \infty$, the quadratic growth of the gradient term ensures that $\mathcal{F}_m(v_j) \to \infty$, provided certain constants related to G_m, the Poincaré inequality, and external terms remain finite. This behavior demonstrates the coercivity of $\mathcal{F}_m$, confirming that the functional grows unboundedly with the norm of the sequence, ensuring stability and compactness in the space $\mathrm{H}_0^1(Q)$.

The proof demonstrates finally, the existence of a minimizer for the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$. Since $\mathcal{F}_m$ is both weakly lower semi-continuous and coercive, the direct method in the calculus of variations ensures that it possesses a minimizer in $\mathrm{H}_0^1(Q)$. To confirm this, the proof revisits the coercivity argument, showing that $\mathcal{F}_m(v^1)$ can be bounded below by terms involving the norm $\|v_j\|_{\mathrm{H}^1(Q)}$, minus terms related to constants from the Poincaré inequality and the nonlinear term G_m. When the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large, the functional grows unboundedly, which guarantees coercivity. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{H}_0^1(Q)$. By the weak lower semi-continuity of $\mathcal{F}_m$, the limit v minimizes the functional, concluding that v is indeed a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$. ■

We construct piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$, we obtained the approximate weak solution as following

$$\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\| \nabla \bar{v}^i(x,t) \right\|_{\mathrm{L}^2(Q)}^2 \right) \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \bar{g}^i \left(x,t,\bar{v}^i \right) \phi(x,t) \mathrm{d}x \, \mathrm{d}t = 0.
\end{aligned}$$

4.2.3 Energy Analysis

In this subsection, we present an energy estimate for minimizers, which leads to significant conclusions about their convergence.

Lemma 4.2.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\|v^i{}_t(t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(t)\right\|^2_{\mathrm{H}^1_0(Q)} \leq C. \tag{4.6}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\left\|w_0\right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{4} + \frac{\beta}{8}\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)}\right)\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)} + \left\|\gamma\right\|^2_{\mathrm{L}^2(Q)}\right) e^{max\{c_p^2,1\}T}.$$

Proof : The proof begins by considering an arbitrary $\theta \in (0,1)$ and a minimizer v_j of the functional $\mathcal{F}_j$. The goal is to show the behavior of the difference between $\mathcal{F}_j$ evaluated at a convex combination of v_j and v_{j-1}, and the value of $\mathcal{F}_j$ at v_j. Starting with the inequality

$$0 \leq \mathcal{F}_j\left((1-\theta)v_j + \theta v_{j-1}\right) - \mathcal{F}_j(v_j), \tag{4.7}$$

the proof proceeds by dividing both sides by θ and taking the limit as $\theta \to 0^+$. This yields an expression for the derivative of $\mathcal{F}_j$ at v_j. Each term in the difference is then examined individually, starting with $\mathcal{F}^1(\theta)$, which involves the discretized term. By applying appropriate approximations and calculations, the proof establishes that $\mathcal{F}^1(\theta)$ behaves as expected according to equation (4.7). Similarly, for $\mathcal{F}^2(\theta)$, the proof uses known results and performs calculations that lead to the same outcome to equation (4.7). The same approach is applied to the outer force term $F_3(\theta)$, showing that it matches the result from equation (4.7). Finally, for $\mathcal{F}^4$, the proof shows that the convex combination of v_j and v_{j-1} remains non-negative almost everywhere, leading to the conclusion that

$$\lim_{\theta \to 0^+} \frac{\Psi\left((1-\theta)v_j + \theta v_{j-1}\right) - \Psi(v_j)}{2\theta} = 0.$$

This detailed derivation explores the energy analysis for minimizers of a functional $\mathcal{F}$ through successive approximations and inequalities. By summing contributions from intermediate functionals $\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4$, an initial inequality is established for the sequence v_j in terms of the L^2-norm and gradient terms. Using assumptions and simplifications such as Kirchhoff's term and properties of the functional, a series of inequalities are derived. Through summation over a range of indices j, telescoping cancellations simplify the terms, leading to a bound on the energy accumulation. The discrete Gronwall lemma is applied to conclude that the energy, represented by terms like $\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla v_m\|^2_{\mathrm{L}^2(Q)}$, remains bounded by a constant C. This constant depends on the initial data v_0, ∇v_0, and other problem-specific parameters. Finally, the result asserts the stability and boundedness of the energy for the minimizers, encapsulated in a simplified form involving $\|v^i_t\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla \bar{v}^i\|^2_{\mathrm{L}^2(Q)}$. This guarantees the well-posedness and energy conservation properties of the system under study.

Which leads to

$$\frac{1}{2}\left\|v_t^i\right\|^2_{\mathrm{L}^2(Q)} + \frac{\alpha}{4}\left\|\nabla\bar{v}^i\right\|^2_{\mathrm{L}^2(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i, \bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_\mathcal{T}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (3.14), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{H}_0^1(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^2(Q_\mathcal{T})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_\mathcal{T})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$ and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_\mathcal{T})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_\mathcal{T})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_\mathcal{T}$.

4.2.4 Convergence to the Limit Function

Theorem 4.2.1

For v_0, u_0 belong to $\mathrm{H}_0^1(Q)$. Then there exists a weak solution to the problem (4.1), according to Definition 4.2.1.

Proof : In this proof, we begin by considering a sequence of functions $v^i(x,t)$ that are approximations of the solution $v(x,t)$, which we aim to show converges to a weak solution defined in Definition 5.2.1. To do so, we fix an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$.

Next, we use this fact to derive an equation for the sequence $v^i(x,t)$. We start with the weak formulation for $v^i(x,t)$, which includes a time derivative term, a gradient term involving $\bar{v}^i(x,t)$, and an external force term. By taking the limit as $i \to 0$, we evaluate the convergence of each term in the weak formulation.

First, we address the time derivative term. Using the fact that $v^i(x,t)$ converges uniformly to $v(x,t)$, we apply the dominated convergence theorem to obtain the limit of the time derivative term as $i \to 0$. We similarly handle the gradient term, which involves the operator ∇, and

demonstrate that the gradient terms converge in the appropriate sense, ensuring that the weak formulation holds for the limit function $v(x,t)$.

For the external force term, we again apply the dominated convergence theorem, which allows us to exchange the limit and the integral. The limit of the external force term converges to the corresponding term for $v(x,t)$, establishing the final form of the weak formulation.

Lastly, we examine the terms in the weak formulation involving $g^i(x,t,\bar{v}^i)$ and use the fact that $\bar{v}^i(x,t)$ converges to $v(x,t)$ to conclude that the limit of the external force term converges to $g(x,t,v)$, as desired.

In the end, we show that the weak solution $v(x,t)$ satisfies the weak formulation of the problem, including the differential equation, the initial condition. Thus, we have successfully shown that the sequence $v^i(x,t)$ converges to a weak solution in the sense defined in Definition 4.2.1 ■

4.3 Hyperbolic p-Kirchhoff type problem

4.3.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) \Delta_p v(x,t) - g(x,t,v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q, \end{cases} \tag{4.8}$$

The symbol Δ_p is the p-Laplacian operator that is defined by

$$\Delta_p v = \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right), \tag{4.9}$$

with $2 \le q \le p < \infty$, and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The function $f : Q_{\mathcal{T}} \times \mathrm{R} \to \mathrm{R}$ satisfies

$$|g(x,t,v)| \le C_g \left|v(x,t)\right|^{\frac{p}{2}} + \gamma(x,t), \quad \text{for a.e. } (x,t) \in Q_{\mathcal{T}}, \quad \text{and } \gamma \in \mathrm{L}^q(Q_{\mathcal{T}}). \tag{4.10}$$

with C_g is a constant satisfy

$$0 < C_g < \frac{\alpha}{pc_*^p}^{\frac{1}{2}},$$

with c_* is the best Poincaré coefficient (for more details about c_* see [50]). Then we define the following

Definition 4.3.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q))$ is a weak solution to (4.8) if $v(x,0) = v_0$ and for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T}))$ if

$$\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|_{\mathrm{L}^p(Q)}^p\right) \left|\nabla v(x,t)\right|^{p-2} \nabla v(x,t) \nabla \phi(x,t) \right] dxdt + \int_0^T \int_Q g(x,t,v)\phi(x,t)dxdt - \int_Q u_0(x)\phi(x,0)dx = 0. \tag{4.11}$$

4.3.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{\left|v - 2v_{m-1} + v_{m-2}\right|^2}{2i^2} + \left(\frac{\alpha}{p} + \frac{\beta}{2p}\left\|\nabla v(x,t)\right\|_{\mathrm{L}^p(Q)}^p\right) \left|\nabla v\right|^p + G_m(v) \right] dx. \tag{4.12}$$

We discretized the term g at time level $m = 1, 2, ..., M$ by means of introducing

$$G_m(x,v) = \int_0^v G_m(x,s)ds, \quad \text{with} \quad G_m(x,v) = \frac{1}{i}\int_{(m-1)k}^{mk} g(x,t,v)dt. \tag{4.13}$$

Furthermore

$$\left|G_m(v)\right| \leq \frac{2C_g}{p+2}\left|v\right|^{\frac{p+2}{2}} + \gamma_m(x)\left|v\right|, \quad \text{with} \quad \gamma_m(x) = \frac{1}{i}\int_{(m-1)k}^{mk} \gamma(x,t)dt. \tag{4.14}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} from $\mathrm{H}_0^1(Q)$ and v_0 taken from $\mathrm{W}_0^{1,p}(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 4.3.1

$\mathcal{F}_m$ is weakly lower semi continuous functional in $\mathrm{W}_0^{1,p}(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$ by analyzing its components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ separately. The convergence of a sequence $v_j \in \mathrm{W}_0^{1,p}(Q)$ weakly to v is assumed.

1. Through direct calculations it is shown that $\mathcal{F}^1(v_j) \to \mathcal{F}^1(v)$ as $j \to \infty$. Using the Sobolev inequality, the proof establishes $\|v_j - v\|_{\mathrm{L}^2(Q)} \to 0$, leading to the conclusion that $\mathcal{F}^1$ is weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

2. The weak lower semi-continuity of norms in Banach spaces is utilized to show that $\mathcal{F}^2$ is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

3. By leveraging the continuity of G_m and the weak convergence $v_j \to v$, the dominated convergence theorem is then applied to confirm that $\mathcal{F}^3(v_j) \to \mathcal{F}^3(v)$ in $\mathrm{W}_0^{1,p}(Q)$, ensuring the weak lower semi-continuity of $\mathcal{F}^3$.

Finally, since all components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ are weakly lower semi-continuous, the functional $\mathcal{F}_m$, being their sum, is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

Then demonstrates that the functional $\mathcal{F}_m$ attains a minimum in $\mathrm{W}_0^{1,p}(Q)$ by examining the behavior of a minimizing sequence $\{v_j\}_{j=1}^{\infty}$. Starting with a sequence v_j that converges weakly to v in $\mathrm{W}_0^{1,p}(Q)$, the analysis ensures that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and that $\mathcal{F}_m(v_1) \neq \infty$.

The functional $\mathcal{F}_m(v_j)$ is expressed as an integral involving terms dependent on v_j, its gradient, and the nonlinear function $G_m(v_j)$. Lower bounds are derived for $\mathcal{F}_m(v_j)$ using properties of Sobolev spaces, such as the Poincaré inequality and embedding theorems, to relate different norms of v_j and its gradient. These estimates show that $\mathcal{F}_m(v_j)$ grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, ensuring coercivity of the functional.

Key constants, such as the best Poincaré coefficient C_* and the embedding constant C, are used to quantify the bounds, while integrability conditions on γ_m ensure the well-posedness of the estimates. The analysis concludes that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, thereby guaranteeing the existence of a minimizing sequence that converges to a minimizer $v \in \mathrm{W}_0^{1,p}(Q)$.

The proof finally establishes the existence of a minimizer for the functional $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. Coercivity of $\mathcal{F}_m$ ensures that the functional grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, and the weak lower semi-continuity of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$ guarantees that a minimizing sequence converges to a minimizer. By analyzing the functional $\mathcal{F}_m(v_j)$, a lower bound is derived in terms of the norms of v_j and its gradient using Sobolev inequalities, embedding theorems, and the Poincaré

inequality.

The analysis shows that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, which implies the boundedness of the minimizing sequence $\{v_j\}$ in $\mathrm{W}_0^{1,p}(Q)$. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{W}_0^{1,p}(Q)$. As a result, v is identified as a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. ■

We construct piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$, we obtained the approximate weak solution as following

$$\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\| \nabla \bar{v}^i(x,t) \right\|_{\mathrm{L}^p(Q)}^p \right) \left| \nabla \bar{v}^i(x,t) \right|^{p-2} \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \bar{g}^i \left(x,t,\bar{v}^i \right) \phi(x,t) \mathrm{d}x \, \mathrm{d}t = 0.
\end{aligned}$$

4.3.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence.

Lemma 4.3.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\| v_t^i(x,t) \right\|_{\mathrm{L}^2(Q)}^2 + \left\| \bar{v}^i(x,t) \right\|_{\mathrm{W}_0^{1,p}(Q)}^2 \leq C. \tag{4.15}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2} \|w_0\|_{\mathrm{L}^2(Q)}^2 + \left(\frac{\alpha}{p} + \frac{\beta}{p} \|\nabla v_0\|_{\mathrm{L}^p(Q)}^p \right) \|\nabla v_0\|_{\mathrm{L}^p(Q)}^p + \|\gamma\|_{\mathrm{L}^2(Q)}^2 \right) e^T.$$

Proof : The proof presented investigates the behavior of a minimizer v_j of a functional F_j over a discretized space, parameterized by $\theta \in (0,1)$. We begin by considering the expression for the difference between the functional values of a convex combination of v_j and v_{j-1}, showing that it is non-negative. This leads to an analysis of the limiting behavior as $\theta \to 0^+$, resulting in the following key inequality:

$$0 \leq \lim_{\theta \to 0^+} \frac{F_j\left((1-\theta)v_j + \theta v_{j-1}\right) - F_j(v_j)}{\theta}.$$

The proof proceeds by examining the contributions of different terms in the functional F_j. Each term is analyzed using direct calculations and inequalities such as the Cauchy-Schwarz inequality and Young's inequality. The first key term relates to the difference in values of the functional for the discretized v_j and v_{j-1}, which is expanded and simplified to reveal a connection to the norm differences of the values v_j, v_{j-1}, and v_{j-2} in the L^2-norm. This results in an expression involving the squared norms of the differences of these functions.

The next terms involve gradient terms, where the gradient of v_j and v_{j-1} are considered. The difference in the gradients is expanded, and we apply inequalities to bound the terms, ultimately leading to an estimate that involves the squared gradient norms. These terms are key in characterizing the smoothness of the functions involved.

An additional force term is handled in a similar manner, where the difference between the functional values F_j is computed and bounded using the function values and gradients. This results in a term involving the differences of the v_j values, as well as the functions γ_j and g_j.

The final analysis involves considering the boundary terms Ψ_i, where it is shown that the differences vanish as $\theta \to 0^+$, leading to a vanishing contribution for these terms.

Combining all the results, the energy analysis provides an upper bound for the total energy, resulting in an inequality that sums all the contributions from the different terms in the functional. This energy inequality is crucial for establishing the stability and convergence of the minimizers. The key result is that the total energy is bounded, and through application of the discrete Gronwall lemma, it is shown that the sequence of v_j's satisfies the following inequality:

$$\frac{1}{2i^2}\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)} + \frac{\alpha}{p}\|\nabla v_m\|^p_{\mathrm{L}^p(Q)} \leq C,$$

where C is a constant that depends on the initial conditions and other terms in the functional. This result ensures that the sequence of minimizers v_j is stable, with bounded energy, and provides a foundation for further analysis of their behavior as $j \to \infty$.

This gives us that

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^p_{\mathrm{W}_0^{1,p}(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and

a sub-sequence denoted by v^i again, such that

$$
\begin{aligned}
&v^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&\bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^p\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&\nabla \bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0, \mathcal{T}; \mathrm{L}^p(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0, \mathcal{T}; \mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+.
\end{aligned}
$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. It is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$, and both $\nabla \bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^p(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla \bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0, \mathcal{T}; \mathrm{L}^p(Q))$ and $\mathrm{L}^\infty(0, \mathcal{T}; \mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$

4.3.4 Convergence to the Limit Function

Theorem 4.3.1

For u_0 belong to $\mathrm{H}_0^1(Q)$ and v_0 belong to $\mathrm{W}_0^{1,p}(Q)$. Then there exists a weak solution to the problem (4.8), according to Definition 4.3.1.

Proof : The proof establishes the existence of a weak solution $v(x,t)$ for the problem under consideration by demonstrating that a sequence of discrete approximations $\{v^i(x,t)\}$ converges to $v(x,t)$ and that this limit satisfies the weak formulation of the problem. The sequence $\{v^i(x,t)\}$, constructed via earlier steps, is shown to converge uniformly to $v(x,t)$, and the proof proceeds to verify the weak formulation for this limit.

The argument begins by fixing an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0, \mathcal{T}))$ supported within the region where $v(x,t) > 0$. Exploiting the uniform convergence of $\{v^i(x,t)\}$ to $v(x,t)$, it is established that both $v^i(x,t)$ and the averaged sequence $\bar{v}^i(x,t)$ remain positive on the support of $\phi(x,t)$. The weak formulation for $v^i(x,t)$ is given, involving terms for the time derivative, gradient contributions from the p-Kirchhoff operator, and external forces.

Each term is analyzed individually. The time derivative term is shown to converge to the appropriate weak form using the weak convergence of $v_t^i(x,t)$ in L^2 and the strong convergence of the test function's time-averaged sequence. For the gradient term, properties of weak lower semi-continuity in L^p-norms are applied, showing convergence of the energy associated with

the p-Kirchhoff operator. The term involving external forces is handled using pointwise convergence of the external force functions and the dominated convergence theorem.

By combining these results, the proof concludes that the limit function $v(x,t)$ satisfies the weak formulation of the problem, as defined in the framework. This verifies the existence of a weak solution. ■

4.4 Conclusion

This chapter establishes a rigorous framework for constructing solutions to Hyperbolic Kirchhoff and p-Kirchhoff-type problems, which are characterized by mathematically complex non-local terms . By employing energy analysis to ensure stability and convergence methods to pass from approximate to exact weak solutions, the work successfully addresses the challenges posed by global dependencies in the equations. The developed approach not only advances the theoretical understanding of problems with variable boundaries and non-local interactions but also provides a robust foundation for practical applications in fields such as elasticity and fluid dynamics.

CHAPTER 5

FREE BOUNDARY HYPERBOLIC KIRCHHOFF-TYPE PROBLEM

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5.1 Introduction

This research establishes a rigorous framework for solving hyperbolic Kirchhoff and p-Kirchhoff-type problems extended with two critical real-world features: evolving free boundaries and global volume constraints . These models describe dynamic systems where the domain's boundary moves over time and, the solution must preserve a total quantity, such as mass or volume.

The core methodological innovation is the systematic application of the Discrete Morse Flow (DMF) method—a variational time-discretization approach. This technique transforms each complex, nonlinear evolutionary problem into a sequence of manageable minimization problems. By proving the associated functionals are coercive and weakly lower semi-continuous, the existence of discrete minimizers is guaranteed. Subsequent energy analysis yields uniform estimates that are independent of the time step, enabling a rigorous limit passage to construct genuine weak solutions.

This framework is successfully applied to six model variations, covering both classic and nonlinear p-Kirchhoff cases, each studied in standard, free-boundary, volume-constrained, and combined free-boundary with volume-constrained settings.

The theoretical results are complemented by numerical simulations in one dimension, which validate the models by visualizing displacement, free-boundary motion, and constraint satisfaction, thereby bridging abstract theory and practical application.

5.2 Free boundary Hyperbolic Kirchhoff-type problem

5.2.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} \chi_{\{v>0\}} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \Delta v(x,t) - g(x,t,v(x,t)), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q, \end{cases} \tag{5.1}$$

Then we define the following

Definition 5.2.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q))$ is a weak solution to (5.1) if $v(x,0) = v_0$, and if $v = 0$ outside $\{v > 0\}$ and for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T})) \cap \{v > 0\})$ if

$$\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \nabla v(x,t)\nabla\phi(x,t) + g(x,t,v(x,t))\phi(x,t) \right] dxdt - \int_Q u_0(x)\phi(x,0)dx = 0. \tag{5.2}$$

5.2.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{\left|v - 2v_{m-1} + v_{m-2}\right|^2}{2i^2} + \left(\frac{\alpha}{2} + \frac{\beta}{4}\left\|\nabla v\right\|^2_{\mathrm{L}^2(Q)}\right)\left|\nabla v\right|^2 + G_m(v) \right] dx + \Psi(v), \tag{5.3}$$

with

$$\Psi(v) = \begin{cases} \Psi = 0; & \text{if } v(x,t) \geq 0, \\ \Psi = \infty; & \text{otherwise .} \end{cases} \tag{5.4}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 5.2.1

Let v be in $\mathrm{H}_0^1(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing each term in the functional separately. First, the term $\mathcal{F}^1(v)$, which represents a time discretization contribution, is shown to be weakly lower semi-continuous using properties of weak convergence and direct calculations related to its quadratic form. Next, $\mathcal{F}^2(v)$, involving gradient norms, leverages the weak lower semi-continuity of

norms in Banach spaces to establish its continuity. For the term $\mathcal{F}^3(v)$, associated with the nonlinear function $G_m(v)$, the proof employs the dominated convergence theorem to show that $\mathcal{F}^3(v_j)$ converges to $\mathcal{F}^3(v)$ as $j \to \infty$, ensuring weak lower semi-continuity. Finally, for $\Psi(v)$, which takes only finite or infinite values, a contradiction argument is used to handle the case where $\Psi(v) = \infty$ while $\liminf_{j\to\infty} \Psi(v_j) = 0$, concluding that Ψ must also be weakly lower semi-continuous. Since all terms $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$, and Ψ are weakly lower semi-continuous, the overall functional $\mathcal{F}_m$ inherits this property in $\mathrm{H}_0^1(Q)$.

Then, establishes the coercivity of the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing its behavior along a minimizing sequence. Starting with a minimizing sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathrm{W}_0^{1,p}(Q)$ that converges weakly to v, it is assumed that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and $\mathcal{F}_m(v_1) \neq \infty$. The functional $\mathcal{F}_m(v_j)$ is expressed in terms of its individual components, including a quadratic term involving the time-discrete difference, a term dependent on the gradient norm weighted by coefficients α and β, the nonlinear term $G_m(v_j)$, and the term $\Psi(v_j)$. By lower-bounding $\mathcal{F}_m(v_j)$, it is shown that the gradient norm term dominates the other terms when the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large. Specifically, as $\|v_j\|_{\mathrm{H}^1(Q)} \to \infty$, the quadratic growth of the gradient term ensures that $\mathcal{F}_m(v_j) \to \infty$, provided certain constants related to G_m, the Poincaré inequality, and external terms remain finite. This behavior demonstrates the coercivity of $\mathcal{F}_m$, confirming that the functional grows unboundedly with the norm of the sequence, ensuring stability and compactness in the space $\mathrm{H}_0^1(Q)$.

The proof demonstrates finally, the existence of a minimizer for the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$. Since $\mathcal{F}_m$ is both weakly lower semi-continuous and coercive, the direct method in the calculus of variations ensures that it possesses a minimizer in $\mathrm{H}_0^1(Q)$. To confirm this, the proof revisits the coercivity argument, showing that $\mathcal{F}_m(v^1)$ can be bounded below by terms involving the norm $\|v_j\|_{\mathrm{H}^1(Q)}$, minus terms related to constants from the Poincaré inequality and the nonlinear term G_m. When the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large, the functional grows unboundedly, which guarantees coercivity. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{H}_0^1(Q)$. By the weak lower semi-continuity of $\mathcal{F}_m$, the limit v minimizes the functional, concluding that v is indeed a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$. ■

We construct piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^{\infty}\left((Q \times [0,\mathcal{T})) \cap \left\{\bar{v}^i(x,t) > 0\right\}\right)$, we obtained the approximate weak solution as

following

$$\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \ \mathrm{d}t + \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\| \nabla \bar{v}^i(x,t) \right\|^2_{\mathrm{L}^2(Q)} \right) \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \ \mathrm{d}t + \int_0^{\mathcal{T}} \int_Q \bar{g}^i \left(x,t,\bar{v}^i \right) \phi(x,t) \mathrm{d}x \ \mathrm{d}t = 0.$$

5.2.3 Energy Analysis

In this subsection, we present an energy estimate for minimizers, which leads to significant conclusions about their convergence.

Lemma 5.2.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\| v^i{}_t(t) \right\|^2_{\mathrm{L}^2(Q)} + \left\| \bar{v}^i(t) \right\|^2_{\mathrm{H}^1_0(Q)} \leq C. \tag{5.5}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2} \left\| w_0 \right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{4} + \frac{\beta}{8} \left\| \nabla v_0 \right\|^2_{\mathrm{L}^2(Q)} \right) \left\| \nabla v_0 \right\|^2_{\mathrm{L}^2(Q)} + \left\| \gamma \right\|^2_{\mathrm{L}^2(Q)} \right) e^{max\{c_p^2,1\}T}.$$

Proof : The proof begins by considering an arbitrary $\theta \in (0,1)$ and a minimizer v_j of the functional $\mathcal{F}_j$. The goal is to show the behavior of the difference between $\mathcal{F}_j$ evaluated at a convex combination of v_j and v_{j-1}, and the value of $\mathcal{F}_j$ at v_j. Starting with the inequality

$$0 \leq \mathcal{F}_j \left((1-\theta) v_j + \theta v_{j-1} \right) - \mathcal{F}_j(v_j), \tag{5.6}$$

the proof proceeds by dividing both sides by θ and taking the limit as $\theta \to 0^+$. This yields an expression for the derivative of $\mathcal{F}_j$ at v_j. Each term in the difference is then examined individually, starting with $\mathcal{F}^1(\theta)$, which involves the discretized term. By applying appropriate approximations and calculations, the proof establishes that $\mathcal{F}^1(\theta)$ behaves as expected according to equation (5.6). Similarly, for $\mathcal{F}^2(\theta)$, the proof uses known results and performs calculations that lead to the same outcome as equation (5.6). The same approach is applied to the outer force term $F_3(\theta)$, showing that it matches the result from equation (5.6). Finally, for $\mathcal{F}^4$, the proof shows that the convex combination of v_j and v_{j-1} remains non-negative almost everywhere, leading to the conclusion that

$$\lim_{\theta \to 0^+} \frac{\Psi \left((1-\theta) v_j + \theta v_{j-1} \right) - \Psi(v_j)}{2\theta} = 0.$$

This detailed derivation explores the energy analysis for minimizers of a functional $\mathcal{F}$ through successive approximations and inequalities. By summing contributions from intermediate functionals $\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4$, an initial inequality is established for the sequence v_j in terms

of the L^2-norm and gradient terms. Using assumptions and simplifications such as Kirchhoff's term and properties of the functional, a series of inequalities are derived. Through summation over a range of indices j, telescoping cancellations simplify the terms, leading to a bound on the energy accumulation. The discrete Gronwall lemma is applied to conclude that the energy, represented by terms like $\|v_m - v_{m-1}\|^2_{L^2(Q)}$ and $\|\nabla v_m\|^2_{L^2(Q)}$, remains bounded by a constant C. This constant depends on the initial data v_0, ∇v_0, and other problem-specific parameters. Finally, the result asserts the stability and boundedness of the energy for the minimizers, encapsulated in a simplified form involving $\|v_t^i\|^2_{L^2(Q)}$ and $\|\nabla \bar{v}^i\|^2_{L^2(Q)}$. This guarantees the well-posedness and energy conservation properties of the system under study.

Which leads to

$$\left\|v_t^i(x,t)\right\|^2_{L^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{H_0^1(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in L^\infty(0,\mathcal{T};H_0^1(Q)) \cap W^{1,\infty}(0,\mathcal{T};L^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i, \bar{v}^i \longrightarrow v \quad \text{(strong) in } L^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&\nabla \bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } L^\infty\left(0,\mathcal{T};L^2(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } L^\infty\left(0,\mathcal{T};L^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (5.5), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $H_0^1(Q)$, and both $\nabla \bar{v}^i$ and v_t^i are uniformly bounded in $L^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $L^2(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $L^2(Q_{\mathcal{T}})$. Furthermore, $\nabla \bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $L^\infty(0,\mathcal{T};L^2(Q))$ and $L^\infty(0,\mathcal{T};L^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{L^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{L^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

5.2.4 Convergence to the Limit Function

Theorem 5.2.1

For v_0, u_0 belong to $H_0^1(Q)$. Then there exists a weak solution to the problem (5.1), according to Definition 5.2.1.

Proof : In this proof, we begin by considering a sequence of functions $v^i(x,t)$ that are approximations of the solution $v(x,t)$, which we aim to show converges to a weak solution

defined in Definition 5.2.1. To do so, we fix an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0, \mathcal{T}))$, supported in the region where $v(x,t) > 0$. The sequence $v^i(x,t)$ is continuous on $Q_\mathcal{T}$, and we know there exists a constant $\mu > 0$ such that $v(x,t) \geq \mu$ on the support of ϕ. Using the previous lemma, we can establish that $v^i(x,t)$ converges uniformly to $v(x,t)$, and for large enough i, we have the bound

$$\max_{(x,t)\in Q_\mathcal{T}} |v^i(x,t) - v(x,t)| \leq \frac{\mu}{2}.$$

Thus, $v^i(x,t)$ remains bounded away from zero on the support of ϕ, ensuring that $v^i(x,t) \geq \frac{\mu}{2}$ for all $k \in (0, k_0)$, where k_0 is a sufficiently large index. This is important as it ensures the positivity of the function for the entire duration of the proof.

Next, we use this fact to derive an equation for the sequence $v^i(x,t)$. We start with the weak formulation for $v^i(x,t)$, which includes a time derivative term, a gradient term involving $\bar{v}^i(x,t)$, and an external force term. By taking the limit as $i \to 0$, we evaluate the convergence of each term in the weak formulation.

First, we address the time derivative term. Using the fact that $v^i(x,t)$ converges uniformly to $v(x,t)$, we apply the dominated convergence theorem to obtain the limit of the time derivative term as $i \to 0$. We similarly handle the gradient term, which involves the operator ∇, and demonstrate that the gradient terms converge in the appropriate sense, ensuring that the weak formulation holds for the limit function $v(x,t)$.

For the external force term, we again apply the dominated convergence theorem, which allows us to exchange the limit and the integral. The limit of the external force term converges to the corresponding term for $v(x,t)$, establishing the final form of the weak formulation.

Lastly, we examine the terms in the weak formulation involving $g^i(x,t,\bar{v}^i)$ and use the fact that $\bar{v}^i(x,t)$ converges to $v(x,t)$ to conclude that the limit of the external force term converges to $g(x,t,v)$, as desired.

In the end, we show that the weak solution $v(x,t)$ satisfies the weak formulation of the problem, including the differential equation, the initial condition, and the condition that $v(x,t) = 0$ outside the region where $v > 0$. Thus, we have successfully shown that the sequence $v^i(x,t)$ converges to a weak solution in the sense defined in Definition 5.2.1 ■

5.3 Free boundary hyperbolic p-Kirchhoff type problem

5.3.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} \chi_{\{v(x,t)>0\}} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) \Delta_p v(x,t) - g(x,t,v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q, \end{cases} \tag{5.7}$$

Then we define the following

Definition 5.3.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}^{1,p}_0(Q))$ is a weak solution to (5.7) if $v(x,0) = v_0$, and if $v = 0$ outside $\{v(x,t) > 0\}$ and for all $\phi(x,t) \in C^\infty_0((Q \times [0,\mathcal{T})) \cap \{v(x,t) > 0\})$ if

$$\begin{aligned} &\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^{p-2} \nabla v(x,t) \nabla \phi(x,t) \right] dxdt \\ &\quad + \int_0^T \int_Q g(x,t,v)\phi(x,t) dxdt - \int_Q u_0(x)\phi(x,0) dx = 0. \end{aligned} \tag{5.8}$$

5.3.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{W}^{1,p}_0(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{|v - 2v_{m-1} + v_{m-2}|^2}{2i^2} + \left(\frac{\alpha}{p} + \frac{\beta}{2p} \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)} \right) |\nabla v|^p + G_m(v) \right] dx + \Psi(v), \tag{5.9}$$

with

$$\Psi(v) = \begin{cases} \Psi = 0; & \text{if } v(x,t) \geq 0, \\ \Psi = \infty; & \text{otherwise .} \end{cases} \tag{5.10}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} in $\mathrm{H}_0^1(Q)$ and v_0 taken from $\mathrm{W}_0^{1,p}(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 5.3.1

Let v be in $\mathrm{W}_0^{1,p}(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$ by analyzing its components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ separately. The convergence of a sequence $v_j \in \mathrm{W}_0^{1,p}(Q)$ weakly to v is assumed.

1. Through direct calculations it is shown that $\mathcal{F}^1(v_j) \to \mathcal{F}^1(v)$ as $j \to \infty$. Using the Sobolev inequality, the proof establishes $\|v_j - v\|_{\mathrm{L}^2(Q)} \to 0$, leading to the conclusion that $\mathcal{F}^1$ is weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

2. The weak lower semi-continuity of norms in Banach spaces is utilized to show that $\mathcal{F}^2$ is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

3. By leveraging the continuity of G_m and the weak convergence $v_j \to v$, the dominated convergence theorem is then applied to confirm that $\mathcal{F}^3(v_j) \to \mathcal{F}^3(v)$ in $\mathrm{W}_0^{1,p}(Q)$, ensuring the weak lower semi-continuity of $\mathcal{F}^3$.

Finally, since all components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ are weakly lower semi-continuous, the functional $\mathcal{F}_m$, being their sum, is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

Then demonstrates that the functional $\mathcal{F}_m$ attains a minimum in $\mathrm{W}_0^{1,p}(Q)$ by examining the behavior of a minimizing sequence $\{v_j\}_{j=1}^{\infty}$. Starting with a sequence v_j that converges weakly to v in $\mathrm{W}_0^{1,p}(Q)$, the analysis ensures that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and that $\mathcal{F}_m(v_1) \neq \infty$.

The functional $\mathcal{F}_m(v_j)$ is expressed as an integral involving terms dependent on v_j, its gradient, and the nonlinear function $G_m(v_j)$. Lower bounds are derived for $\mathcal{F}_m(v_j)$ using properties of Sobolev spaces, such as the Poincaré inequality and embedding theorems, to relate different norms of v_j and its gradient. These estimates show that $\mathcal{F}_m(v_j)$ grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, ensuring coercivity of the functional.

Key constants, such as the best Poincaré coefficient C_* and the embedding constant C, are used to quantify the bounds, while integrability conditions on γ_m ensure the well-posedness of the estimates. The analysis concludes that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, thereby guaranteeing the existence of a minimizing sequence that converges to a minimizer $v \in \mathrm{W}_0^{1,p}(Q)$.

The proof finally establishes the existence of a minimizer for the functional $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. Coercivity of $\mathcal{F}_m$ ensures that the functional grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, and the weak lower semi-continuity of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$ guarantees that a minimizing sequence converges to a minimizer. By analyzing the functional $\mathcal{F}_m(v_j)$, a lower bound is derived in terms of the norms of v_j and its gradient using Sobolev inequalities, embedding theorems, and the Poincaré inequality.

The analysis shows that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, which implies the boundedness of the minimizing sequence $\{v_j\}$ in $\mathrm{W}_0^{1,p}(Q)$. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{W}_0^{1,p}(Q)$. As a result, v is identified as a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. ■

We constructs piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^\infty\left((Q \times [0,\mathcal{T})) \cap \left\{\bar{v}^i(x,t) > 0\right\}\right)$, we obtained the approximate weak solution as following

$$\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i^2} \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\|\nabla \bar{v}^i(x,t)\right\|_{\mathrm{L}^p(Q)}^p\right) \left|\nabla \bar{v}^i(x,t)\right|^{p-2} \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \bar{g}^i\left(x,t,\bar{v}^i\right) \phi(x,t) dx dt = 0.
\end{aligned}$$

5.3.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence. We provide the following lemma to demonstrate this.

Lemma 5.3.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^p_{\mathrm{W}_0^{1,p}(Q)} \leq C. \tag{5.11}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\left\|w_0\right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{p} + \frac{\beta}{p}\left\|\nabla v_0\right\|^p_{\mathrm{L}^p(Q)}\right)\left\|\nabla v_0\right\|^p_{\mathrm{L}^p(Q)} + \left\|\gamma\right\|^2_{\mathrm{L}^2(Q)}\right)e^T.$$

Proof : The proof presented investigates the behavior of a minimizer v_j of a functional F_j over a discretized space, parameterized by $\theta \in (0,1)$. We begin by considering the expression for the difference between the functional values of a convex combination of v_j and v_{j-1}, showing that it is non-negative. This leads to an analysis of the limiting behavior as $\theta \to 0^+$, resulting in the following key inequality:

$$0 \leq \lim_{\theta \to 0^+} \frac{F_j\left((1-\theta)v_j + \theta v_{j-1}\right) - F_j(v_j)}{\theta}.$$

The proof proceeds by examining the contributions of different terms in the functional F_j. Each term is analyzed using direct calculations and inequalities such as the Cauchy-Schwarz inequality and Young's inequality. The first key term relates to the difference in values of the functional for the discretized v_j and v_{j-1}, which is expanded and simplified to reveal a connection to the norm differences of the values v_j, v_{j-1}, and v_{j-2} in the L^2-norm. This results in an expression involving the squared norms of the differences of these functions.

The next terms involve gradient terms, where the gradient of v_j and v_{j-1} are considered. The difference in the gradients is expanded, and we apply inequalities to bound the terms, ultimately leading to an estimate that involves the squared gradient norms. These terms are key in characterizing the smoothness of the functions involved.

An additional force term is handled in a similar manner, where the difference between the functional values F_j is computed and bounded using the function values and gradients. This results in a term involving the differences of the v_j values, as well as the functions γ_j and g_j.

The final analysis involves considering the boundary term Ψ, where it is shown that the differences vanish as $\theta \to 0^+$, leading to a vanishing contribution for this term.

Combining all the results, the energy analysis provides an upper bound for the total energy, resulting in an inequality that sums all the contributions from the different terms in the functional. This energy inequality is crucial for establishing the stability and convergence of the

minimizers. The key result is that the total energy is bounded, and through application of the discrete Gronwall lemma, it is shown that the sequence of v_j's satisfies the following inequality:

$$\frac{1}{2i^2}\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)} + \frac{\alpha}{p}\|\nabla v_m\|^p_{\mathrm{L}^p(Q)} \leq C,$$

where C is a constant that depends on the initial conditions and other terms in the functional. This result ensures that the sequence of minimizers v_j is stable, with bounded energy, and provides a foundation for further analysis of their behavior as $j \to \infty$.

This gives us that

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{W}_0^{1,p}(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^p\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^p(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+.
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (7.15), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^p(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^p(Q))$ and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

5.3.4 Convergence to the Limit Function

Theorem 5.3.1

For u_0 belong to $\mathrm{H}_0^1(Q)$ and v_0 belong to $\mathrm{W}_0^{1,p}(Q)$. Then there exists a weak solution to the problem (5.7), according to Definition 5.3.1.

Proof : The proof establishes the existence of a weak solution $v(x,t)$ for the problem under consideration by demonstrating that a sequence of discrete approximations $\{v^i(x,t)\}$ converges to $v(x,t)$ and that this limit satisfies the weak formulation of the problem. The sequence

$\{v^i(x,t)\}$, constructed via earlier steps, is shown to converge uniformly to $v(x,t)$, and the proof proceeds to verify the weak formulation for this limit.

The argument begins by fixing an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$ supported within the region where $v(x,t) > 0$. Exploiting the uniform convergence of $\{v^i(x,t)\}$ to $v(x,t)$, it is established that both $v^i(x,t)$ and the averaged sequence $\bar{v}^i(x,t)$ remain positive on the support of $\phi(x,t)$. The weak formulation for $v^i(x,t)$ is given, involving terms for the time derivative, gradient contributions from the p-Kirchhoff operator, and external forces.

Each term is analyzed individually. The time derivative term is shown to converge to the appropriate weak form using the weak convergence of $v_t^i(x,t)$ in L^2 and the strong convergence of the test function's time-averaged sequence. For the gradient term, properties of weak lower semi-continuity in L^p-norms are applied, showing convergence of the energy associated with the p-Kirchhoff operator. The term involving external forces is handled using pointwise convergence of the external force functions and the dominated convergence theorem.

By combining these results, the proof concludes that the limit function $v(x,t)$ satisfies the weak formulation of the problem, as defined in the framework. This verifies the existence of a weak solution. ■

5.4 Conclusion

This chapter establishes a rigorous framework for constructing weak solutions to Hyperbolic Kirchhoff and p-Kirchhoff-type problems featuring free boundaries. Utilizing Energy Analysis and Convergence to the Limit Function , the study overcomes the inherent complexity of simultaneously determining the solution and the evolving domain boundary. The analysis is extended to the nonlinear p-Kirchhoff case, governed by the p-Laplacian. The resulting framework provides stable, physically meaningful solutions, advancing the theoretical and practical treatment of free-boundary phenomena in fields like fluid dynamics and fracture mechanics.

CHAPTER 6

CONSTRAINED HYPERBOLIC KIRCHHOFF-TYPE PROBLEM

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6.1 Introduction

This chapter rigorously examines the constrained Hyperbolic Kirchhoff-Type Problem and its generalization to the p-Kirchhoff case, both incorporating a crucial volume constraint inspired by real-world systems like incompressible flows or elastic deformations. Through Energy Analysis and approximate solution methods , the chapter constructs stable weak solutions that respect both the governing equations and the global constraint. The work demonstrates how such constraints fundamentally alter solution behavior, enhancing the models' realism and applicability across fluid dynamics, material science, and elasticity. By integrating advanced variational techniques, the study provides a robust theoretical and practical framework for analyzing complex, constrained physical systems.

6.2 Constrained hyperbolic Kirchhoff-type problem

6.2.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \Delta v(x,t) - g(x,t,v(x,t)) + \lambda_{\mathcal{V}}(v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q. \end{cases} \tag{6.1}$$

Then we define the following

Definition 6.2.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}^1_0(Q))$ is a weak solution to (6.1) if $v(x,0) = v_0$, and for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T}))$ if

$$\begin{aligned} &\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^2_{\mathrm{L}^2(Q)}\right) \nabla v(x,t)\nabla\phi(x,t) + g(x,t,v)\phi(x,t)\right] dxdt \\ &\quad - \int_Q u_0(x)\phi(x,0)dx = -\int_Q v_0(x)v_0(x)\Phi(0)dx - \int_0^T \int_Q v_t(x,t)(v\Phi)_t(x,t)dxdt \\ &\quad + \int_0^T \int_Q \left(\left(\alpha + \beta \left\|\nabla v(x,t)\right\|^2_{\mathrm{L}^2(Q)}\right) |\nabla v(x,t)|^2 + g(x,t,v)v\right) \Phi(x,t)dxdt. \end{aligned} \tag{6.2}$$

with

$$\Phi(x,t) = \int_Q \phi(x,t)dx.$$

6.2.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0, \mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{|v - 2v_{m-1} + v_{m-2}|^2}{2i^2} + \left(\frac{\alpha}{2} + \frac{\beta}{4} \|\nabla v\|^2_{\mathrm{L}^2(Q)} \right) |\nabla v|^2 + G_m(v) \right] dx + \Psi(v), \quad (6.3)$$

with

$$\Psi(v) = \begin{cases} \Psi = 0; & \text{if } \int_Q v(x,t)dx = \mathcal{V}, \\ \Psi = \infty; & \text{otherwise .} \end{cases} \quad (6.4)$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 6.2.1

Let v be in $\mathrm{H}_0^1(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$.

Proof : The proof first demonstrates that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing each term in the functional separately. First, the term $\mathcal{F}^1(v)$, which represents a time discretization contribution, is shown to be weakly lower semi-continuous using properties of weak convergence and direct calculations related to its quadratic form. Next, $\mathcal{F}^2(v)$, involving gradient norms, leverages the weak lower semi-continuity of norms in Banach spaces to establish its continuity. For the term $\mathcal{F}^3(v)$, associated with the nonlinear function $G_m(v)$, the proof employs the dominated convergence theorem to show that $\mathcal{F}^3(v_j)$ converges to $\mathcal{F}^3(v)$ as $j \to \infty$, ensuring weak lower semi-continuity. Finally, for $\Psi(v)$, which takes only finite or infinite values, a contradiction argument is used to handle the case where $\Psi(v) = \infty$ while $\liminf_{j\to\infty} \Psi(v_j) = 0$, concluding that Ψ must also be weakly lower semi-continuous. Since all terms $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$, and Ψ are weakly lower semi-continuous, the overall

functional $\mathcal{F}_m$ inherits this property in $\mathrm{H}_0^1(Q)$.

Then establishes the coercivity of the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing its behavior along a minimizing sequence. Starting with a minimizing sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathrm{W}_0^{1,p}(Q)$ that converges weakly to v, it is assumed that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and $\mathcal{F}_m(v_1) \neq \infty$. The functional $\mathcal{F}_m(v_j)$ is expressed in terms of its individual components, including a quadratic term involving the time-discrete difference, a term dependent on the gradient norm weighted by coefficients α and β, the nonlinear term $G_m(v_j)$, and the term $\Psi(v_j)$. By lower-bounding $\mathcal{F}_m(v_j)$, it is shown that the gradient norm term dominates the other terms when the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large. Specifically, as $\|v_j\|_{\mathrm{H}^1(Q)} \to \infty$, the quadratic growth of the gradient term ensures that $\mathcal{F}_m(v_j) \to \infty$, provided certain constants related to G_m, the Poincaré inequality, and external terms remain finite. This behavior demonstrates the coercivity of $\mathcal{F}_m$, confirming that the functional grows unboundedly with the norm of the sequence, ensuring stability and compactness in the space $\mathrm{H}_0^1(Q)$.

The proof demonstrates finally the existence of a minimizer for the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$. Since $\mathcal{F}_m$ is both weakly lower semi-continuous and coercive, the direct method in the calculus of variations ensures that it possesses a minimizer in $\mathrm{H}_0^1(Q)$. To confirm this, the proof revisits the coercivity argument, showing that $\mathcal{F}_m(v^1)$ can be bounded below by terms involving the norm $\|v_j\|_{\mathrm{H}^1(Q)}$, minus terms related to constants from the Poincaré inequality and the nonlinear term G_m. When the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large, the functional grows unboundedly, which guarantees coercivity. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{H}_0^1(Q)$. By the weak lower semi-continuity of $\mathcal{F}_m$, the limit v minimizes the functional, concluding that v is indeed a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$. ■

We construct piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range. Then for all $\phi(x,t) \in C_0^{\infty}(Q \times [0,\mathcal{T}))$ with zero volume (e.i. $\int_Q \phi(x,t)dx = 0$), we obtained the approximate weak solution as following

$$
\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\| \nabla \bar{v}^i(x,t) \right\|_{\mathrm{L}^2(Q)}^2 \right) \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^{\mathcal{T}} \int_Q \bar{g}^i \left(x, t, \bar{v}^i \right) \phi(x,t) \mathrm{d}x \, \mathrm{d}t = 0.
\end{aligned}
$$

6.2.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence.

Lemma 6.2.2

The functions v^i and $\bar{v}^i$ obey the estimate

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{H}_0^1(Q)} \leq C. \tag{6.5}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\left\|w_0\right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{4} + \frac{\beta}{8}\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)}\right)\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)} + \left\|\gamma\right\|^2_{\mathrm{L}^2(Q)}\right) e^{max\{c_p^2,1\}T}.$$

Proof : The proof begins by considering an arbitrary $\theta \in (0,1)$ and a minimizer v_j of the functional $\mathcal{F}_j$. The goal is to show the behavior of the difference between $\mathcal{F}_j$ evaluated at a convex combination of v_j and v_{j-1}, and the value of $\mathcal{F}_j$ at v_j. Starting with the inequality

$$0 \leq \mathcal{F}_j\left((1-\theta)v_j + \theta v_{j-1}\right) - \mathcal{F}_j(v_j), \tag{6.6}$$

the proof proceeds by dividing both sides by θ and taking the limit as $\theta \to 0^+$. This yields an expression for the derivative of $\mathcal{F}_j$ at v_j. Each term in the difference is then examined individually, starting with $\mathcal{F}^1(\theta)$, which involves the discretized term. By applying appropriate approximations and calculations, the proof establishes that $\mathcal{F}^1(\theta)$ behaves as expected according to equation (6.6). Similarly, for $\mathcal{F}^2(\theta)$, the proof uses known results and performs calculations that lead to the same outcome as equation (6.6). The same approach is applied to the outer force term $F_3(\theta)$, showing that it matches the result from equation (6.6). Finally, for $\mathcal{F}^4$, the proof shows that the convex combination of v_j and v_{j-1} remains non-negative almost everywhere, leading to the conclusion that

$$\lim_{\theta \to 0^+} \frac{\Psi\left((1-\theta)v_j + \theta v_{j-1}\right) - \Psi(v_j)}{2\theta} = 0.$$

This detailed derivation explores the energy analysis for minimizers of a functional $\mathcal{F}$ through successive approximations and inequalities. By summing contributions from intermediate functionals $\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4$, an initial inequality is established for the sequence v_j in terms of the L^2-norm and gradient terms. Using assumptions and simplifications such as Kirchhoff's term and properties of the functional, a series of inequalities are derived. Through summation over a range of indices j, telescoping cancellations simplify the terms, leading to a bound on the energy accumulation. The discrete Gronwall lemma is applied to conclude that the energy, represented by terms like $\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla v_m\|^2_{\mathrm{L}^2(Q)}$, remains bounded by a constant C. This constant depends on the initial data v_0, ∇v_0, and other problem-specific parameters. Finally, the result asserts the stability and boundedness of the energy for the minimizers,

encapsulated in a simplified form involving $\|v_t^i\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla\bar{v}^i\|^2_{\mathrm{L}^2(Q)}$. This guarantees the well-posedness and energy conservation properties of the system under study.

Which leads to

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{H}_0^1(Q)} \leq C.$$

■

The above estimate leads immediately to the following lemma. The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i, \bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (6.5), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{H}_0^1(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^2(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$ and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

6.2.4 Convergence to the Limit Function

Theorem 6.2.1

For v_0, u_0 belong to $\mathrm{H}_0^1(Q)$. Then there exists a weak solution to the problem (6.1)

Proof : In this proof, we begin by considering a sequence of functions $v^i(x,t)$ that are approximations of the solution $v(x,t)$, which we aim to show converges to a weak solution defined in Definition 6.2.1. To do so, we fix an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0, \mathcal{T}))$.

Next, we use this fact to derive an equation for the sequence $v^i(x,t)$. We start with the weak formulation for $v^i(x,t)$, which includes a time derivative term, a gradient term involving $\bar{v}^i(x,t)$, and an external force term. By taking the limit as $i \to 0$, we evaluate the convergence of each term in the weak formulation.

First, we address the time derivative term. Using the fact that $v^i(x,t)$ converges uniformly to $v(x,t)$, we apply the dominated convergence theorem to obtain the limit of the time derivative term as $i \to 0$. We similarly handle the gradient term, which involves the operator ∇, and demonstrate that the gradient terms converge in the appropriate sense, ensuring that the weak formulation holds for the limit function $v(x,t)$.

For the external force term, we again apply the dominated convergence theorem, which allows us to exchange the limit and the integral. The limit of the external force term converges to the corresponding term for $v(x,t)$, establishing the final form of the weak formulation.

Lastly, we examine the terms in the weak formulation involving $g^i(x,t,\bar{v}^i)$ and use the fact that $\bar{v}^i(x,t)$ converges to $v(x,t)$ to conclude that the limit of the external force term converges to $g(x,t,v)$, as desired.

In the end, we show that the weak solution $v(x,t)$ satisfies the weak formulation of the problem, including the differential equation, the initial condition, and the condition that $v(x,t) = 0$ outside the region where $v > 0$. Thus, we have successfully shown that the sequence $v^i(x,t)$ converges to a weak solution in the sense defined in Definition 6.2.1 ■

6.3 Constrained hyperbolic p-Kirchhoff type problem

6.3.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed

examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) \Delta_p v(x,t) - g(x,t,v) + \lambda_{\mathcal{V}}(v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q. \end{cases} \tag{6.7}$$

Then we define the following

Definition 6.3.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}^{1,p}_0(Q))$ is a weak solution to (6.7) if $v(x,0) = v_0$, and if $v = 0$ for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T}))$ if

$$\begin{aligned} &\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^{p-2} \nabla v(x,t) \nabla \phi(x,t) + g(x,t,v)\phi(x,t)\right] dxdt \\ &\quad - \int_Q u_0(x)\phi(x,0)dx = -\frac{1}{\mathcal{V}} \int_Q v_0(x) v_0(x) \Phi(0) dx - \frac{1}{\mathcal{V}} \int_0^T \int_Q v_t(x,t)(v\Phi)_t(x,t) dxdt \\ &\quad + \frac{1}{\mathcal{V}} \int_0^T \int_Q \left(\left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^p + g(x,t,v)v(x,t)\right) \Phi(x,t) dxdt, \end{aligned} \tag{6.8}$$

with

$$\Phi(x,t) = \int_Q \phi(x,t) dx.$$

6.3.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}^1_0(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{|v - 2v_{m-1} + v_{m-2}|^2}{2i^2} + \left(\frac{\alpha}{p} + \frac{\beta}{2p} \left\|\nabla v\right\|^p_{\mathrm{L}^2(Q)}\right) |\nabla v|^p + G_m(v)\right] dx + \Psi(v), \tag{6.9}$$

with

$$\Psi(v) = \begin{cases} \Psi = 0; & \text{if } \displaystyle\int_Q v(x,t)dx = \mathcal{V}, \\ \Psi = \infty; & \text{otherwise .} \end{cases} \tag{6.10}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 6.3.1

Let v be in $\mathrm{W}_0^{1,p}(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$ by analyzing its components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ separately. The convergence of a sequence $v_j \in \mathrm{W}_0^{1,p}(Q)$ weakly to v is assumed.

1. Through direct calculations it is shown that $\mathcal{F}^1(v_j) \to \mathcal{F}^1(v)$ as $j \to \infty$. Using the Sobolev inequality, the proof establishes $\|v_j - v\|_{\mathrm{L}^2(Q)} \to 0$, leading to the conclusion that $\mathcal{F}^1$ is weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

2. The weak lower semi-continuity of norms in Banach spaces is utilized to show that $\mathcal{F}^2$ is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

3. By leveraging the continuity of G_m and the weak convergence $v_j \to v$, the dominated convergence theorem is then applied to confirm that $\mathcal{F}^3(v_j) \to \mathcal{F}^3(v)$ in $\mathrm{W}_0^{1,p}(Q)$, ensuring the weak lower semi-continuity of $\mathcal{F}^3$.

Finally, since all components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ are weakly lower semi-continuous, the functional $\mathcal{F}_m$, being their sum, is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

Then demonstrates that the functional $\mathcal{F}_m$ attains a minimum in $\mathrm{W}_0^{1,p}(Q)$ by examining the behavior of a minimizing sequence $\{v_j\}_{j=1}^{\infty}$. Starting with a sequence v_j that converges weakly to v in $\mathrm{W}_0^{1,p}(Q)$, the analysis ensures that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and that $\mathcal{F}_m(v_1) \neq \infty$.

The functional $\mathcal{F}_m(v_j)$ is expressed as an integral involving terms dependent on v_j, its gradient, and the nonlinear function $G_m(v_j)$. Lower bounds are derived for $\mathcal{F}_m(v_j)$ using properties of Sobolev spaces, such as the Poincaré inequality and embedding theorems, to relate different norms of v_j and its gradient. These estimates show that $\mathcal{F}_m(v_j)$ grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, ensuring coercivity of the functional.

Key constants, such as the best Poincaré coefficient C_* and the embedding constant C, are used to quantify the bounds, while integrability conditions on γ_m ensure the well-posedness of

the estimates. The analysis concludes that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, thereby guaranteeing the existence of a minimizing sequence that converges to a minimizer $v \in \mathrm{W}_0^{1,p}(Q)$.

The proof finally establishes the existence of a minimizer for the functional $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. Coercivity of $\mathcal{F}_m$ ensures that the functional grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, and the weak lower semi-continuity of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$ guarantees that a minimizing sequence converges to a minimizer. By analyzing the functional $\mathcal{F}_m(v_j)$, a lower bound is derived in terms of the norms of v_j and its gradient using Sobolev inequalities, embedding theorems, and the Poincaré inequality.

The analysis shows that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, which implies the boundedness of the minimizing sequence $\{v_j\}$ in $\mathrm{W}_0^{1,p}(Q)$. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{W}_0^{1,p}(Q)$. As a result, v is identified as a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. ■

We constructs piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range. Then for all $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$, we obtained the approximate weak solution as following

$$
\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \; \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\| \nabla \bar{v}^i(x,t) \right\|_{\mathrm{L}^p(Q)}^p \right) \left| \nabla \bar{v}^i(x,t) \right|^{p-2} \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \; \mathrm{d}x \; \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \bar{g}^i \left(x,t,\bar{v}^i \right) \phi(x,t) \mathrm{d}x \; \mathrm{d}t = 0.
\end{aligned}
$$

6.3.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence.

Lemma 6.3.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{W}_0^{1,p}(Q)} \leq C. \tag{6.11}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\left\|w_0\right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{p} + \frac{\beta}{p}\left\|\nabla v_0\right\|^p_{\mathrm{L}^p(Q)}\right)\left\|\nabla v_0\right\|^p_{\mathrm{L}^p(Q)} + \left\|\gamma\right\|^2_{\mathrm{L}^2(Q)}\right) e^T.$$

Proof : The proof presented investigates the behavior of a minimizer v_j of a functional F_j over a discretized space, parameterized by $\theta \in (0,1)$. We begin by considering the expression for the difference between the functional values of a convex combination of v_j and v_{j-1}, showing that it is non-negative. This leads to an analysis of the limiting behavior as $\theta \to 0^+$, resulting in the following key inequality:

$$0 \leq \lim_{\theta \to 0^+} \frac{F_j\left((1-\theta)v_j + \theta v_{j-1}\right) - F_j(v_j)}{\theta}.$$

The proof proceeds by examining the contributions of different terms in the functional F_j. Each term is analyzed using direct calculations and inequalities such as the Cauchy-Schwarz inequality and Young's inequality. The first key term relates to the difference in values of the functional for the discretized v_j and v_{j-1}, which is expanded and simplified to reveal a connection to the norm differences of the values v_j, v_{j-1}, and v_{j-2} in the L^2-norm. This results in an expression involving the squared norms of the differences of these functions.

The next terms involve gradient terms, where the gradient of v_j and v_{j-1} are considered. The difference in the gradients is expanded, and we apply inequalities to bound the terms, ultimately leading to an estimate that involves the squared gradient norms. These terms are key in characterizing the smoothness of the functions involved.

An additional force term is handled in a similar manner, where the difference between the functional values F_j is computed and bounded using the function values and gradients. This results in a term involving the differences of the v_j values, as well as the functions γ_j and g_j.

The final analysis involves considering the boundary term Ψ, where it is shown that the differences vanish as $\theta \to 0^+$, leading to a vanishing contribution for this term.

Combining all the results, the energy analysis provides an upper bound for the total energy, resulting in an inequality that sums all the contributions from the different terms in the functional. This energy inequality is crucial for establishing the stability and convergence of the minimizers. The key result is that the total energy is bounded, and through application of the discrete Gronwall lemma, it is shown that the sequence of v_j's satisfies the following inequality:

$$\frac{1}{2i^2}\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)} + \frac{\alpha}{p}\|\nabla v_m\|^p_{\mathrm{L}^p(Q)} \leq C,$$

where C is a constant that depends on the initial conditions and other terms in the functional. This result ensures that the sequence of minimizers v_j is stable, with bounded energy, and provides a foundation for further analysis of their behavior as $j \to \infty$.

This gives us that

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{W}_0^{1,p}(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^p\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^p(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (7.15), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^p(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^p(Q))$ and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

6.3.4 Convergence to the Limit Function

Theorem 6.3.1

For u_0 belong to $\mathrm{H}_0^1(Q)$ and v_0 belong to $\mathrm{W}_0^{1,p}(Q)$. Then there exists a weak solution to the problem (6.7), according to Definition 6.3.1.

Proof : The proof establishes the existence of a weak solution $v(x,t)$ for the problem under consideration by demonstrating that a sequence of discrete approximations $\{v^i(x,t)\}$ converges to $v(x,t)$ and that this limit satisfies the weak formulation of the problem. The sequence $\{v^i(x,t)\}$, constructed via earlier steps, is shown to converge uniformly to $v(x,t)$, and the proof

proceeds to verify the weak formulation for this limit.

The argument begins by fixing an arbitrary test function $\phi(x,t) \in C_0^\infty(Q \times [0,\mathcal{T}))$ supported within the region where $v(x,t) > 0$. Exploiting the uniform convergence of $\{v^i(x,t)\}$ to $v(x,t)$, it is established that both $v^i(x,t)$ and the averaged sequence $\bar{v}^i(x,t)$ remain positive on the support of $\phi(x,t)$. The weak formulation for $v^i(x,t)$ is given, involving terms for the time derivative, gradient contributions from the p-Kirchhoff operator, and external forces.

Each term is analyzed individually. The time derivative term is shown to converge to the appropriate weak form using the weak convergence of $v_t^i(x,t)$ in L^2 and the strong convergence of the test function's time-averaged sequence. For the gradient term, properties of weak lower semi-continuity in L^p-norms are applied, showing convergence of the energy associated with the p-Kirchhoff operator. The term involving external forces is handled using pointwise convergence of the external force functions and the dominated convergence theorem.

By combining these results, the proof concludes that the limit function $v(x,t)$ satisfies the weak formulation of the problem, as defined in the framework. This verifies the existence of a weak solution. ■

6.4 Conclusion

This chapter successfully constructs stable weak solutions for constrained Hyperbolic Kirchhoff and p-Kirchhoff-type problems, incorporating volume constraints and free boundaries. Using Energy Analysis and convergence methods, the solutions are shown to be stable and converge appropriately in relevant function spaces. The work extends Kirchhoff-type models to realistic scenarios in fluid dynamics, elasticity, and materials science, where volume preservation and moving boundaries are critical. These results validate the applied variational framework and open avenues for modeling more complex constrained systems in physics and engineering.

CHAPTER 7

CONSTRAINED HYPERBOLIC KIRCHHOFF-TYPE PROBLEM INVOLVING FREE BOUNDARY

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7.1 Introduction

This chapter extends the classical Kirchhoff problem to incorporate both a volume constraint and an evolving free boundary—features that reflect realistic systems in fluid mechanics and material science. To address the resulting mathematical complexities, the framework employs approximate weak solutions and energy methods , ensuring stability and convergence. The analysis is further generalized to the nonlinear p-Kirchhoff case, which introduces a p-Laplacian and nonlocal interactions. Through a careful blend of variational techniques and energy estimates, the chapter constructs a rigorous solution framework capable of handling dynamic boundaries and global constraints, offering a versatile tool for modeling real-world systems with moving interfaces and internal restrictions.

7.2 Constrained hyperbolic Kirchhoff-type problem involving free boundary

7.2.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} \chi_{\{v>0\}} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(t)\right\|^2_{\mathrm{L}^2(Q)}\right) \Delta v(x,t) - g(x,t,v(x,t)) + \chi_{\{v>0\}} \lambda_{\mathcal{V}}(v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q. \end{cases} \tag{7.1}$$

Then we define the following

Definition 7.2.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q))$ is a weak solution to (7.1) if $v(x,0) = v_0$, and if $v = 0$ outside $\{v > 0\}$ and for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T})) \cap \{v(x,t) > 0\})$ if

$$\begin{aligned}
&\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^2_{\mathrm{L}^2(Q)}\right) \nabla v(x,t)\nabla\phi(x,t) + g(x,t,v)\phi(x,t)\right] dxdt \\
&\quad - \int_Q u_0(x)\phi(x,0)dx = -\int_Q v_0(x)v_0(x)\Phi(x,0)dx - \int_0^T \int_Q v_t(x,t)(v\Phi)_t(x,t)dxdt \\
&\quad + \int_0^T \int_Q \left(\left(\alpha + \beta \left\|\nabla v(x,t)\right\|^2_{\mathrm{L}^2(Q)}\right) |\nabla v(x,t)|^2 + g(x,t,v)v\right) \Phi(x,t)dxdt.
\end{aligned} \tag{7.2}$$

with

$$\Phi(x,t) = \int_Q \phi(x,t)dx.$$

We can obtain an other definition of a weak solution, to avoid the problem generated by the free boundary, coincides with Definition 7.2.1.

Definition 7.2.2

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q))$ is a weak solution to (7.1) if $v(x,0) = v_0$, and if $v = 0$ outside $\{v > 0\}$ and for all $\phi(x,t) \in C_0^\infty((Q \times [0,\mathcal{T})) \cap \{v > 0\})$, if

$$\begin{aligned}
&\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^{p-2} \nabla v(x,t)\nabla\phi(x,t)\right] dxdt \\
&+ \int_0^T \int_Q g(x,t,v)\phi(x,t)dxdt - \int_Q u_0(x)\phi(x,0)dx = -\frac{1}{\mathcal{V}} \int_Q \tilde{v}(x,0)v_0(x)\Phi(0)dx \\
&- \frac{1}{\mathcal{V}} \int_0^T \int_Q v_t(x,t)(\tilde{v}\Phi)_t(x,t)dxdt + \frac{1}{\mathcal{V}} \int_0^T \int_Q g(x,t,v)\tilde{v}(x,t)\Phi(x,t)dxdt \\
&+ \frac{1}{\mathcal{V}} \int_0^T \int_Q \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^{2p}(Q)}\right) |\nabla v(x,t)|^{p-2} \nabla v(x,t)\nabla\tilde{v}(x,t)\Phi(x,t)dxdt,
\end{aligned} \tag{7.3}$$

for $\tilde{v} \in C_0^\infty(([0,T] \times Q) \cap \{v > 0\})$ is an arbitrary function satisfying $\int_Q \tilde{v}\ \mathrm{d}x = \mathcal{V}$, with

$$\Phi(x,t) = \int_Q \phi(x,t)dx.$$

7.2.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i = \mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we

define v_{-1} as follows:

$$v_{-1} = v_0 - iu_0.$$

Then we define $v_m \in \mathrm{H}_0^1(Q)$ with $m = 1, ..., M$, to minimize the functional

$$\mathcal{F}_m(v) = \int_Q \left[\frac{|v - 2v_{m-1} + v_{m-2}|^2}{2i^2} + \left(\frac{\alpha}{2} + \frac{\beta}{4} \|\nabla v\|^2_{\mathrm{L}^2(Q)} \right) |\nabla v|^2 + G_m(v) \right] dx + \Psi_1(v) + \Psi_2(v), \tag{7.4}$$

with

$$\Psi_1(v) = \begin{cases} \Psi_1 = 0; & \text{if } v(x,t) \geq 0, \\ \Psi_1 = \infty; & \text{otherwise .} \end{cases} \quad \text{and} \quad \Psi_2(v) = \begin{cases} \Psi_2 = 0; & \text{if } \displaystyle\int_Q v(x,t)dx = \mathcal{V}, \\ \Psi_2 = \infty; & \text{otherwise .} \end{cases} \tag{7.5}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be demonstrated later.

With v_{-1} and v_0 taken from $\mathrm{H}_0^1(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 7.2.1

Let v be in $\mathrm{H}_0^1(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$.

Proof : The proof first demonstrates that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing each term in the functional separately. First, the term $\mathcal{F}^1(v)$, which represents a time discretization contribution, is shown to be weakly lower semi-continuous using properties of weak convergence and direct calculations related to its quadratic form. Next, $\mathcal{F}^2(v)$, involving gradient norms, leverages the weak lower semi-continuity of norms in Banach spaces to establish its continuity. For the term $\mathcal{F}^3(v)$, associated with the nonlinear function $G_m(v)$, the proof employs the dominated convergence theorem to show that $\mathcal{F}^3(v_j)$ converges to $\mathcal{F}^3(v)$ as $j \to \infty$, ensuring weak lower semi-continuity. Finally, for $\Psi_i(v)$, which takes only finite or infinite values, a contradiction argument is used to handle the case where $\Psi_i(v) = \infty$ while $\liminf_{j\to\infty} \Psi_i(v_j) = 0$, concluding that Ψ_i must also be weakly lower semi-continuous. Since all terms $\mathcal{F}^1$, $\mathcal{F}^2$, $\mathcal{F}^3$, and Ψ_i are weakly lower semi-continuous, the overall functional $\mathcal{F}_m$ inherits this property in $\mathrm{H}_0^1(Q)$.

Then establishes the coercivity of the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$ by analyzing its behavior along a minimizing sequence. Starting with a minimizing sequence $\{v_j\}_{j=1}^{\infty}$ in $\mathrm{W}_0^{1,p}(Q)$ that converges weakly to v, it is assumed that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and $\mathcal{F}_m(v_1) \neq \infty$. The functional $\mathcal{F}_m(v_j)$ is expressed in terms of its individual components, including a quadratic term involving the time-discrete difference, a term dependent on the gradient norm weighted

by coefficients α and β, the nonlinear term $G_m(v_j)$, and the term $\Psi(v_j)$. By lower-bounding $\mathcal{F}_m(v_j)$, it is shown that the gradient norm term dominates the other terms when the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large. Specifically, as $\|v_j\|_{\mathrm{H}^1(Q)} \to \infty$, the quadratic growth of the gradient term ensures that $\mathcal{F}_m(v_j) \to \infty$, provided certain constants related to G_m, the Poincaré inequality, and external terms remain finite. This behavior demonstrates the coercivity of $\mathcal{F}_m$, confirming that the functional grows unboundedly with the norm of the sequence, ensuring stability and compactness in the space $\mathrm{H}_0^1(Q)$.

The proof finally demonstrates the existence of a minimizer for the functional $\mathcal{F}_m$ in the Sobolev space $\mathrm{H}_0^1(Q)$. Since $\mathcal{F}_m$ is both weakly lower semi-continuous and coercive, the direct method in the calculus of variations ensures that it possesses a minimizer in $\mathrm{H}_0^1(Q)$. To confirm this, the proof revisits the coercivity argument, showing that $\mathcal{F}_m(v^1)$ can be bounded below by terms involving the norm $\|v_j\|_{\mathrm{H}^1(Q)}$, minus terms related to constants from the Poincaré inequality and the nonlinear term G_m. When the norm $\|v_j\|_{\mathrm{H}^1(Q)}$ becomes large, the functional grows unboundedly, which guarantees coercivity. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{H}_0^1(Q)$. By the weak lower semi-continuity of $\mathcal{F}_m$, the limit v minimizes the functional, concluding that v is indeed a minimizer of $\mathcal{F}_m$ in $\mathrm{H}_0^1(Q)$. ■

We construct piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^\infty\left((Q \times [0,\mathcal{T})) \cap \left\{\bar{v}^i(x,t) > 0\right\}\right)$ with zero volume (e.i. $\int_Q \phi(x,t)dx = 0$), we obtained the approximate weak solution as following

$$\begin{aligned}\int_0^{\mathcal{T}} \int_Q \left(\frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) + \left(\alpha + \beta \left\|\nabla \bar{v}^i(x,t)\right\|^2_{\mathrm{L}^2(Q)}\right) \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \right) \mathrm{d}x \ \mathrm{d}t \\ + \int_0^{\mathcal{T}} \int_Q \bar{g}^i\left(x,t,\bar{v}^i\right) \phi(x,t) \mathrm{d}x \ \mathrm{d}t = 0.\end{aligned} \tag{7.6}$$

7.2.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence.

Lemma 7.2.2

The functions v^i and $\bar{v}^i$ obey the estimate

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{H}_0^1(Q)} \leq C. \tag{7.7}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\left\|w_0\right\|^2_{\mathrm{L}^2(Q)} + \left(\frac{\alpha}{4} + \frac{\beta}{8}\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)}\right)\left\|\nabla v_0\right\|^2_{\mathrm{L}^2(Q)} + \left\|\gamma\right\|^2_{\mathrm{L}^2(Q)}\right) e^{max\{c_p^2,1\}T}.$$

Proof : The proof begins by considering an arbitrary $\theta \in (0,1)$ and a minimizer v_j of the functional $\mathcal{F}_j$. The goal is to show the behavior of the difference between $\mathcal{F}_j$ evaluated at a convex combination of v_j and v_{j-1}, and the value of $\mathcal{F}_j$ at v_j. Starting with the inequality

$$0 \leq \mathcal{F}_j\left((1-\theta)v_j + \theta v_{j-1}\right) - \mathcal{F}_j(v_j), \tag{7.8}$$

the proof proceeds by dividing both sides by θ and taking the limit as $\theta \to 0^+$. This yields an expression for the derivative of $\mathcal{F}_j$ at v_j. Each term in the difference is then examined individually, starting with $\mathcal{F}^1(\theta)$, which involves the discretized term. By applying appropriate approximations and calculations, the proof establishes that $\mathcal{F}^1(\theta)$ behaves as expected according to equation (7.8). Similarly, for $\mathcal{F}^2(\theta)$, the proof uses known results and performs calculations that lead to the same outcome as equation (7.8). The same approach is applied to the outer force term $F_3(\theta)$, showing that it matches the result from equation (7.8). Finally, for $\mathcal{F}^4$, the proof shows that the convex combination of v_j and v_{j-1} remains non-negative almost everywhere, leading to the conclusion that

$$\lim_{\theta \to 0^+} \frac{\Psi\left((1-\theta)v_j + \theta v_{j-1}\right) - \Psi(v_j)}{2\theta} = 0.$$

This detailed derivation explores the energy analysis for minimizers of a functional $\mathcal{F}$ through successive approximations and inequalities. By summing contributions from intermediate functionals $\mathcal{F}^1, \mathcal{F}^2, \mathcal{F}^3, \mathcal{F}^4$, an initial inequality is established for the sequence v_j in terms of the L^2-norm and gradient terms. Using assumptions and simplifications such as Kirchhoff's term and properties of the functional, a series of inequalities are derived. Through summation over a range of indices j, telescoping cancellations simplify the terms, leading to a bound on the energy accumulation. The discrete Gronwall lemma is applied to conclude that the energy, represented by terms like $\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla v_m\|^2_{\mathrm{L}^2(Q)}$, remains bounded by a constant C. This constant depends on the initial data v_0, ∇v_0, and other problem-specific parameters. Finally, the result asserts the stability and boundedness of the energy for the minimizers, encapsulated in a simplified form involving $\|v_t^i\|^2_{\mathrm{L}^2(Q)}$ and $\|\nabla \bar{v}^i\|^2_{\mathrm{L}^2(Q)}$. This guarantees the well-posedness and energy conservation properties of the system under study.

Which leads to

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{H}_0^1(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^\infty(0,\mathcal{T};\mathrm{H}_0^1(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i, \bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+ \\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^\infty\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. It is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{H}_0^1(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^2(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$ and $\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i - \bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \leq i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

7.2.4 Convergence to the Limit Function

In this subsection, for $N = 1$, we will prove that the approximate solutions converge to a weak solution, To archive this we need first the following Lemma.

Lemma 7.2.3

There exists a subsequence v^i converges uniformly in $Q_{\mathcal{T}}$.

Proof : This proof demonstrates the uniform boundedness and convergence properties of a sequence of functions v^i in the domain $Q_{\mathcal{T}}$. It begins with the integral representation of the difference $v^i(t) - v^i(t_1)$, given by

$$v^i(t) - v^i(t_1) = \int_{t_1}^{\mathcal{T}} v_t^i(\delta)\, d\delta,$$

which, combined with the uniform boundedness of $\|v_t^i\|_{\mathrm{L}^\infty(0,\mathcal{T};\mathrm{L}^2(Q))}$, leads to the inequality

$$\|v^i(t) - v^i(t_1)\|_{\mathrm{L}^2(Q)} \leq C|t - t_1|,$$

where C is independent of i. This establishes a Lipschitz-like property in time.

Next, the proof addresses the spatial derivatives, showing that

$$\|v_x^i(t) - v_x^i(t_1)\|_{\mathrm{L}^2(Q)}$$

is uniformly bounded. Using these bounds, a boundary analysis reveals a bound on $\|v^i(t) - v^i(t_1)\|_{\mathrm{L}^\infty(Q)}$, which scales as $|t - t_1|^{1/2}$. A similar argument, considering variations in both spatial and temporal coordinates, establishes that

$$|v^i(t, \beta) - v^i(\delta, \alpha)|$$

is bounded by terms involving $|\beta - \alpha|^{1/2}$ and $|t - \delta|^{1/2}$, confirming Hölder continuity.

Finally, these bounds imply uniform boundedness of v^i in $Q_{\mathcal{T}}$. By applying the Arzelà-Ascoli theorem, the proof concludes that there exists a subsequence of v^i that converges uniformly in $Q_{\mathcal{T}}$. ■

Theorem 7.2.1

For v_0, u_0 belong to $\mathrm{H}_0^1(Q)$. Then there exists a weak solution to the problem (7.1)

Proof : In this proof, we begin by considering a sequence of functions $v^i(x, t)$ that are approximations of the solution $v(x, t)$, which we aim to show converges to a weak solution defined in Definition 7.2.1. To do so, we fix an arbitrary test function $\phi(x, t) \in C_0^\infty(Q \times [0, \mathcal{T}))$, supported in the region where $v(x, t) > 0$. The sequence $v^i(x, t)$ is continuous on $Q_{\mathcal{T}}$, and we know there exists a constant $\mu > 0$ such that $v(x, t) \geq \mu$ on the support of ϕ. Using using the previous lemma, we can establish that $v^i(x, t)$ converges uniformly to $v(x, t)$, and for large enough i, we have the bound

$$\max_{(x,t)\in Q_{\mathcal{T}}} |v^i(x, t) - v(x, t)| \leq \frac{\mu}{2}.$$

Thus, $v^i(x, t)$ remains bounded away from zero on the support of ϕ, ensuring that $v^i(x, t) \geq \frac{\mu}{2}$ for all $k \in (0, k_0)$, where k_0 is a sufficiently large index. This is important as it ensures the positivity of the function for the entire duration of the proof.

Next, we use this fact to derive an equation for the sequence $v^i(x, t)$. We start with the weak formulation for $v^i(x, t)$, which includes a time derivative term, a gradient term involving $\bar{v}^i(x, t)$, and an external force term. By taking the limit as $i \to 0$, we evaluate the convergence of each term in the weak formulation.

First, we address the time derivative term. Using the fact that $v^i(x, t)$ converges uniformly to $v(x, t)$, we apply the dominated convergence theorem to obtain the limit of the time derivative term as $i \to 0$. We similarly handle the gradient term, which involves the operator ∇, and demonstrate that the gradient terms converge in the appropriate sense, ensuring that the weak formulation holds for the limit function $v(x, t)$.

For the external force term, we again apply the dominated convergence theorem, which allows us to exchange the limit and the integral. The limit of the external force term converges

to the corresponding term for $v(x,t)$, establishing the final form of the weak formulation.

Lastly, we examine the terms in the weak formulation involving $g^i(x,t,\bar{v}^i)$ and use the fact that $\bar{v}^i(x,t)$ converges to $v(x,t)$ to conclude that the limit of the external force term converges to $g(x,t,v)$, as desired.

In the end, we show that the weak solution $v(x,t)$ satisfies the weak formulation of the problem, including the differential equation, the initial condition, and the condition that $v(x,t)=0$ outside the region where $v>0$. Thus, we have successfully shown that the sequence $v^i(x,t)$ converges to a weak solution in the sense defined in Definition 7.2.1 ■

7.3 Constrained hyperbolic p-Kirchhoff type problem involving free boundary

7.3.1 Problem Introduction

In this subsection, we will thoroughly investigate the existence of weak solutions for a hyperbolic-type problem by carefully analyzing the following model. Our approach will involve a detailed examination of the underlying mathematical framework, focusing on the key elements that influence the solution's existence and behavior. We will highlight the challenges posed by the non-linearities, boundary conditions, and other critical factors, which will guide our formulation of solution our problem.

$$\begin{cases} \chi_{\{v(x,t)>0\}} v_{tt}(x,t) = \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) \Delta_p v(x,t) - g(x,t,v) + \chi_{\{v(x,t)>0\}} \lambda_{\mathcal{V}}(v), & \text{in } Q_{\mathcal{T}}, \\ v(x,t) = 0, & \text{on } (0,\mathcal{T}) \times \partial Q, \\ v(x,0) = v_0(x), & \text{in } Q, \\ v_t(x,0) = u_0(x), & \text{in } Q. \end{cases} \tag{7.9}$$

Then we define the following

Definition 7.3.1

$v \in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q)) \cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}^{1,p}_0(Q))$ is a weak solution to (7.9) if $v(x,0)=v_0$, and if $v=0$ outside $\{v>0\}$ and for all $\phi(x,t) \in C_0^\infty((Q\times[0,\mathcal{T})) \cap \{v>0\})$ if

$$\begin{aligned} &\int_0^T \int_Q \left[-v_t(x,t)\phi_t(x,t) + \left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^{p-2} \nabla v(x,t) \nabla \phi(x,t) + g(x,t,v)\phi(x,t)\right] dxdt \\ &\quad - \int_Q u_0(x)\phi(x,0)dx = -\frac{1}{\mathcal{V}} \int_Q v_0(x) v_0(x) \Phi(x,0) dx - \frac{1}{\mathcal{V}} \int_0^T \int_Q v_t(x,t)(v\Phi)_t(x,t) dxdt \\ &\quad + \frac{1}{\mathcal{V}} \int_0^T \int_Q \left(\left(\alpha + \beta \left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right) |\nabla v(x,t)|^p + g(x,t,v)v(x,t)\right) \Phi(x,t) dxdt, \end{aligned} \tag{7.10}$$

with

$$\Phi(x,t)=\int_Q \phi(x,t)dx.$$

We can obtain an other definition of a weak solution, to avoid the problem generated by the free boundary, coincides with Definition 7.10.

Definition 7.3.2

$v\in \mathrm{H}^1(0,\mathcal{T};\mathrm{L}^2(Q))\cap \mathrm{L}^\infty(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q))$ is a weak solution to (7.9) if $v(x,0)=v_0$, and if $v=0$ outgoing $\{v>0\}$ and for all $\phi(x,t)\in C_0^\infty((Q\times[0,\mathcal{T}))\cap\{v>0\})$ if

$$\begin{aligned}
&\int_0^T\int_Q\left[-v_t(x,t)\phi_t(x,t)+\left(\alpha+\beta\left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right)|\nabla v(x,t)|^{p-2}\nabla v(x,t)\nabla\phi(x,t)+g(x,t,v)\phi(x,t)\right]dxdt\\
&\quad-\int_Q u_0(x)\phi(x,0)dx=-\frac{1}{\mathcal{V}}\int_Q v_0(x)\tilde v(x,0)\Phi(x,0)dx-\frac{1}{\mathcal{V}}\int_0^T\int_Q v_t(x,t)(\tilde v\Phi)_t(x,t)dxdt\\
&\quad+\frac{1}{\mathcal{V}}\int_0^T\int_Q\left(\left(\alpha+\beta\left\|\nabla v(x,t)\right\|^p_{\mathrm{L}^p(Q)}\right)|\nabla v(x,t)|^{p-2}\nabla v(x,t)\nabla\tilde v(x,t)+g(x,t,v)\tilde v(x,t)\right)\Phi(x,t)dxdt,
\end{aligned}\tag{7.11}$$

for $\tilde v\in C_0^\infty(([0,T]\times Q)\cap\{v>0\})$ is an arbitrary function satisfying $\int_Q \tilde v\ \mathrm{d}x=\mathcal{V}$, with

$$\Phi(x,t)=\int_Q\phi(x,t)dx.$$

7.3.2 Approximation of the Weak Solutions

To address our problem, we employ the minimizing movement method, commonly referred to as the discrete Morse flow. M equal sub-intervals the time interval $(0,\mathcal{T})$ is divided into, where $i=\mathcal{T}/M$ is the step size. Applying the backward difference method for initial conditions, we define v_{-1} as follows:

$$v_{-1}=v_0-iu_0.$$

Then we define $v_m\in \mathrm{H}_0^1(Q)$ with $m=1,...,M$, to minimize the functional

$$\mathcal{F}_m(v)=\int_Q\left[\frac{|v-2v_{m-1}+v_{m-2}|^2}{2i^2}+\left(\frac{\alpha}{p}+\frac{\beta}{2p}\left\|\nabla v\right\|^p_{\mathrm{L}^2(Q)}\right)|\nabla v|^p+G_m(v)\right]dx+\Psi_1(v)+\Psi_2(v),\tag{7.12}$$

with

$$\Psi_1(v)=\begin{cases}\Psi_1=0; & \text{if } v(x,t)\geq 0,\\ \Psi_1=\infty; & \text{otherwise ,}\end{cases}\quad\text{and}\quad \Psi_2(v)=\begin{cases}\Psi_2=0; & \text{if } \int_Q v(x,t)dx=\mathcal{V},\\ \Psi_2=\infty; & \text{otherwise .}\end{cases}\tag{7.13}$$

Additionally, we assume that the initial data v_0 and v_0 belong to the space $\mathrm{H}_0^1(Q)$. These conditions are necessary to ensure the existence of minimizers for the functional $\mathcal{F}_m$, as will be

demonstrated later.

With v_{-1} taken from $\mathrm{H}_0^1(Q)$ and v_0 in $\mathrm{W}_0^{1,p}(Q)$, we define v_1 as the minimizer of $\mathcal{F}_1$. Subsequently, v_m is iteratively determined as the minimizer of $\mathcal{F}_m$, to address the existence of these minimizers.

Lemma 7.3.1

Let v be in $\mathrm{W}_0^{1,p}(Q)$, then v is a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$.

Proof : The proof demonstrates first, that the functional $\mathcal{F}_m$ is weakly lower semi-continuous in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$ by analyzing its components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ separately. The convergence of a sequence $v_j \in \mathrm{W}_0^{1,p}(Q)$ weakly to v is assumed.

1. Through direct calculations it is shown that $\mathcal{F}^1(v_j) \to \mathcal{F}^1(v)$ as $j \to \infty$. Using the Sobolev inequality, the proof establishes $\|v_j - v\|_{\mathrm{L}^2(Q)} \to 0$, leading to the conclusion that $\mathcal{F}^1$ is weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

2. The weak lower semi-continuity of norms in Banach spaces is utilized to show that $\mathcal{F}^2$ is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

3. By leveraging the continuity of G_m and the weak convergence $v_j \to v$, the dominated convergence theorem is then applied to confirm that $\mathcal{F}^3(v_j) \to \mathcal{F}^3(v)$ in $\mathrm{W}_0^{1,p}(Q)$, ensuring the weak lower semi-continuity of $\mathcal{F}^3$.

Finally, since all components $\mathcal{F}^1, \mathcal{F}^2$, and $\mathcal{F}^3$ are weakly lower semi-continuous, the functional $\mathcal{F}_m$, being their sum, is also weakly lower semi-continuous in $\mathrm{W}_0^{1,p}(Q)$.

Then demonstrates that the functional $\mathcal{F}_m$ attains a minimum in $\mathrm{W}_0^{1,p}(Q)$ by examining the behavior of a minimizing sequence $\{v_j\}_{j=1}^{\infty}$. Starting with a sequence v_j that converges weakly to v in $\mathrm{W}_0^{1,p}(Q)$, the analysis ensures that $\mathcal{F}_m(v) = \liminf_{j\to\infty} \mathcal{F}_m(v_j)$ and that $\mathcal{F}_m(v_1) \neq \infty$.

The functional $\mathcal{F}_m(v_j)$ is expressed as an integral involving terms dependent on v_j, its gradient, and the nonlinear function $G_m(v_j)$. Lower bounds are derived for $\mathcal{F}_m(v_j)$ using properties of Sobolev spaces, such as the Poincaré inequality and embedding theorems, to relate different norms of v_j and its gradient. These estimates show that $\mathcal{F}_m(v_j)$ grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, ensuring coercivity of the functional.

Key constants, such as the best Poincaré coefficient C_* and the embedding constant C, are used to quantify the bounds, while integrability conditions on γ_m ensure the well-posedness of the estimates. The analysis concludes that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, thereby guaranteeing the existence of a minimizing sequence that converges to a minimizer $v \in \mathrm{W}_0^{1,p}(Q)$.

The proof finally establishes the existence of a minimizer for the functional $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. Coercivity of $\mathcal{F}_m$ ensures that the functional grows without bound as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, and the weak lower semi-continuity of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$ guarantees that a minimizing sequence converges to a minimizer. By analyzing the functional $\mathcal{F}_m(v_j)$, a lower bound is derived in terms of the norms of v_j and its gradient using Sobolev inequalities, embedding theorems, and the Poincaré inequality.

The analysis shows that $\mathcal{F}_m(v_j) \to \infty$ as $\|v_j\|_{\mathrm{W}_0^{1,p}(Q)} \to \infty$, which implies the boundedness of the minimizing sequence $\{v_j\}$ in $\mathrm{W}_0^{1,p}(Q)$. This boundedness allows the extraction of a weakly convergent subsequence $v_j \rightharpoonup v$ in $\mathrm{W}_0^{1,p}(Q)$. As a result, v is identified as a minimizer of $\mathcal{F}_m$ in $\mathrm{W}_0^{1,p}(Q)$. ■

We constructs piecewise linear time-interpolant solutions v^i and piecewise constant step functions $\bar{v}^i$ to approximate a minimizer of the functional $\mathcal{F}_m$ over a time interval. The interpolant $v^i(x,t)$ is defined for $(x,t) \in Q \times ((j-1)i, ij]$, with $j = 0, 1, \ldots, M$, as a linear combination of the values $v_j(x)$ and $v_{j-1}(x)$ at adjacent time steps. The piecewise constant function $\bar{v}^i(x,t)$ is defined to be equal to $v_j(x)$ in the same intervals. Similarly, the piecewise constant function $\bar{g}^i(x,t,v)$ is defined as $g_j(x,v)$ in the same time range.Then for all $\phi(x,t) \in C_0^\infty\left((Q \times [0,\mathcal{T})) \cap \left\{\bar{v}^i(x,t) > 0\right\}\right)$, we obtained the approximate weak solution as following

$$\begin{aligned}
&\int_0^{\mathcal{T}} \int_Q \frac{v_t^i(x,t) - v_t^i(x,t-i)}{i} \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \left(\alpha + \beta \left\|\nabla \bar{v}^i(x,t)\right\|_{\mathrm{L}^p(Q)}^p\right) \left|\nabla \bar{v}^i(x,t)\right|^{p-2} \nabla \bar{v}^i(x,t) \nabla \phi(x,t) \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^{\mathcal{T}} \int_Q \bar{g}^i\left(x,t,\bar{v}^i\right) \phi(x,t) \mathrm{d}x \, \mathrm{d}t = 0.
\end{aligned} \tag{7.14}$$

7.3.3 Energy Analysis

In this part, We offer an Energy Analysis for minimizers that yields important conclusions on their convergence.

Lemma 7.3.2

The functions v^i defined in (3.4) and $\bar{v}^i$ defined in (3.5), obey the following estimate

$$\left\|v_t^i(x,t)\right\|_{\mathrm{L}^2(Q)}^2 + \left\|\bar{v}^i(x,t)\right\|_{\mathrm{W}_0^{1,p}(Q)}^2 \leq C. \tag{7.15}$$

with C is a constant independent of i, such that

$$C = \left(\frac{1}{2}\|w_0\|_{\mathrm{L}^2(Q)}^2 + \left(\frac{\alpha}{p} + \frac{\beta}{p}\|\nabla v_0\|_{\mathrm{L}^p(Q)}^p\right)\|\nabla v_0\|_{\mathrm{L}^p(Q)}^p + \|\gamma\|_{\mathrm{L}^2(Q)}^2\right) e^T.$$

Proof : The proof presented investigates the behavior of a minimizer v_j of a functional F_j over a discretized space, parameterized by $\theta \in (0,1)$. We begin by considering the expression for the difference between the functional values of a convex combination of v_j and v_{j-1}, showing that it is non-negative. This leads to an analysis of the limiting behavior as $\theta \to 0^+$, resulting in the following key inequality:

$$0 \leq \lim_{\theta \to 0^+} \frac{F_j\left((1-\theta)v_j + \theta v_{j-1}\right) - F_j(v_j)}{\theta}.$$

The proof proceeds by examining the contributions of different terms in the functional F_j. Each term is analyzed using direct calculations and inequalities such as the Cauchy-Schwarz inequality and Young's inequality. The first key term relates to the difference in values of the functional for the discretized v_j and v_{j-1}, which is expanded and simplified to reveal a connection to the norm differences of the values v_j, v_{j-1}, and v_{j-2} in the L^2-norm. This results in an expression involving the squared norms of the differences of these functions.

The next terms involve gradient terms, where the gradient of v_j and v_{j-1} are considered. The difference in the gradients is expanded, and we apply inequalities to bound the terms, ultimately leading to an estimate that involves the squared gradient norms. These terms are key in characterizing the smoothness of the functions involved.

An additional force term is handled in a similar manner, where the difference between the functional values F_j is computed and bounded using the function values and gradients. This results in a term involving the differences of the v_j values, as well as the functions γ_j and g_j.

The final analysis involves considering the boundary terms Ψ_i, where it is shown that the differences vanish as $\theta \to 0^+$, leading to a vanishing contribution for these terms.

Combining all the results, the energy analysis provides an upper bound for the total energy, resulting in an inequality that sums all the contributions from the different terms in the functional. This energy inequality is crucial for establishing the stability and convergence of the minimizers. The key result is that the total energy is bounded, and through application of the discrete Gronwall lemma, it is shown that the sequence of v_j's satisfies the following inequality:

$$\frac{1}{2i^2}\|v_m - v_{m-1}\|^2_{\mathrm{L}^2(Q)} + \frac{\alpha}{p}\|\nabla v_m\|^p_{\mathrm{L}^p(Q)} \leq C,$$

where C is a constant that depends on the initial conditions and other terms in the functional. This result ensures that the sequence of minimizers v_j is stable, with bounded energy, and provides a foundation for further analysis of their behavior as $j \to \infty$.

This gives us that

$$\left\|v_t^i(x,t)\right\|^2_{\mathrm{L}^2(Q)} + \left\|\bar{v}^i(x,t)\right\|^2_{\mathrm{W}_0^{1,p}(Q)} \leq C.$$

■

The previous lemma leads to the existence of $v \in \mathrm{L}^{\infty}(0,\mathcal{T};\mathrm{W}_0^{1,p}(Q)) \cap \mathrm{W}^{1,\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$, and a sub-sequence denoted by v^i again, such that

$$\begin{aligned}
&v^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^2\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\bar{v}^i \longrightarrow v \quad \text{(strong) in } \mathrm{L}^p\left(Q_{\mathcal{T}}\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&\nabla\bar{v}^i \rightharpoonup \nabla v \quad \text{(weak) in } \mathrm{L}^{\infty}\left(0,\mathcal{T};\mathrm{L}^p(Q)\right), \quad \text{as} \quad i \longrightarrow 0^+\\
&v_t^i \rightharpoonup v_t \quad \text{(weak) in } \mathrm{L}^{\infty}\left(0,\mathcal{T};\mathrm{L}^2(Q)\right) \quad \text{as} \quad i \longrightarrow 0^+ \quad .
\end{aligned}$$

We focuses on the convergence properties of a sequence of functions v^i and their associated components. From equation (7.15), it is shown that $\bar{v}^i$ is uniformly bounded in the Sobolev space $\mathrm{W}_0^{1,p}(Q)$, and both $\nabla\bar{v}^i$ and v_t^i are uniformly bounded in $\mathrm{L}^2(Q)$. Using Lemma 3.2 from [61], upper bounds for v^i and ∇v^i are derived, which establish their convergence properties. Specifically, it is shown that $\bar{v}^i$ converges strongly to a function u in $\mathrm{L}^p(Q_{\mathcal{T}})$, while v^i converges strongly to v_1 in $\mathrm{L}^2(Q_{\mathcal{T}})$. Furthermore, $\nabla\bar{v}^i$ and v_t^i converge weakly to v_2 and v_3, respectively, in $\mathrm{L}^{\infty}(0,\mathcal{T};\mathrm{L}^p(Q))$ and $\mathrm{L}^{\infty}(0,\mathcal{T};\mathrm{L}^2(Q))$. The proof then uses the inequality from Lemma 3.2, $\|v^i-\bar{v}^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})} \le i\|v_t^i\|^2_{\mathrm{L}^2(Q_{\mathcal{T}})}$, to establish that v^i and $\bar{v}^i$ converge to the same limit $v = v_1$. The analysis further shows that the difference between the derivatives of $\bar{v}^i$ and v^i vanishes in the weak sense as $i \to 0^+$, leading to the conclusion that $v_2 = \nabla v$ and $v_3 = v_t$ almost everywhere in $Q_{\mathcal{T}}$.

7.3.4 Convergence to the Limit Function

In this subsection, we will prove that the approximate solutions converge to a weak solution. To archive this we need first the following Lemma.

Lemma 7.3.3

There exists a subsequence v^i converges uniformly in $Q_{\mathcal{T}}$.

Proof : This proof demonstrates the uniform boundedness and convergence properties of a sequence of functions v^i in the domain $Q_{\mathcal{T}}$. It begins with the integral representation of the difference $v^i(t) - v^i(t_1)$, given by

$$v^i(t) - v^i(t_1) = \int_{t_1}^{\mathcal{T}} v_t^i(\delta)\, d\delta,$$

which, combined with the uniform boundedness of $\|v_t^i\|_{\mathrm{L}^{\infty}(0,\mathcal{T};\mathrm{L}^2(Q))}$, leads to the inequality

$$\|v^i(t) - v^i(t_1)\|_{\mathrm{L}^2(Q)} \le C|t - t_1|,$$

where C is independent of i. This establishes a Lipschitz-like property in time.

Next, the proof addresses the spatial derivatives, showing that

$$\|v_x^i(t) - v_x^i(t_1)\|_{\mathrm{L}^2(Q)}$$

is uniformly bounded. Using these bounds, a boundary analysis reveals a bound on $\|v^i(t) - v^i(t_1)\|_{\mathrm{L}^\infty(Q)}$, which scales as $|t - t_1|^{1/2}$. A similar argument, considering variations in both spatial and temporal coordinates, establishes that

$$|v^i(t, \beta) - v^i(\delta, \alpha)|$$

is bounded by terms involving $|\beta - \alpha|^{1/2}$ and $|t - \delta|^{1/2}$, confirming Hölder continuity.

Finally, these bounds imply uniform boundedness of v^i in $Q_{\mathcal{T}}$. By applying the Arzelà-Ascoli theorem, the proof concludes that there exists a subsequence of v^i that converges uniformly in $Q_{\mathcal{T}}$. ■

Theorem 7.3.1

For u_0 belong to $\mathrm{H}_0^1(Q)$ and v_0 belong to $\mathrm{W}_0^{1,p}(Q)$. Then there exists a weak solution to the problem (7.9), according to Definition 7.2.1.

Proof : The proof establishes the existence of a weak solution $v(x, t)$ for the problem under consideration by demonstrating that a sequence of discrete approximations $\{v^i(x, t)\}$ converges to $v(x, t)$ and that this limit satisfies the weak formulation of the problem. The sequence $\{v^i(x, t)\}$, constructed via earlier steps, is shown to converge uniformly to $v(x, t)$, and the proof proceeds to verify the weak formulation for this limit.

The argument begins by fixing an arbitrary test function $\phi(x, t) \in C_0^\infty(Q \times [0, \mathcal{T}))$ supported within the region where $v(x, t) > 0$. Exploiting the uniform convergence of $\{v^i(x, t)\}$ to $v(x, t)$, it is established that both $v^i(x, t)$ and the averaged sequence $\bar{v}^i(x, t)$ remain positive on the support of $\phi(x, t)$. The weak formulation for $v^i(x, t)$ is given, involving terms for the time derivative, gradient contributions from the p-Kirchhoff operator, and external forces.

Each term is analyzed individually. The time derivative term is shown to converge to the appropriate weak form using the weak convergence of $v_t^i(x, t)$ in L^2 and the strong convergence of the test function's time-averaged sequence. For the gradient term, properties of weak lower semi-continuity in L^p-norms are applied, showing convergence of the energy associated with the p-Kirchhoff operator. The term involving external forces is handled using pointwise convergence of the external force functions and the dominated convergence theorem.

By combining these results, the proof concludes that the limit function $v(x, t)$ satisfies the weak formulation of the problem, as defined in the framework. This verifies the existence of a weak solution. ■

7.4 Conclusion

This chapter successfully constructs weak solutions for constrained Hyperbolic Kirchhoff and p-Kirchhoff-type problems with free boundaries and volume constraints. Using Energy Analysis and careful treatment of the evolving boundary and global constraint, the solutions are shown to be stable over time. The results highlight the complex interplay between a moving interface and a volume restriction, reflecting realistic scenarios in fluid dynamics and material science. The applied variational methods prove robust, extending the classical framework and providing a foundation for both theoretical advances and practical numerical applications in fields involving dynamic, constrained systems.

CHAPTER 8

NUMERICAL SIMULATION

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8.1 Introduction

This chapter presents a numerical investigation designed to validate and extend the theoretical findings on hyperbolic Kirchhoff-type problems. By implementing tailored numerical schemes, the study bridges abstract theory and practical application, offering visual and quantitative insight into solution behaviors under various constraints, including free boundaries and volume preservation.

The core of the chapter details the careful translation of the theoretical models—particularly the discrete Morse flow framework—into stable and accurate computational algorithms. This involves a deliberate choice of discretization methods, time-stepping schemes, and specialized techniques to handle the nonlocal Kirchhoff term, the evolving free boundary, and the global volume constraint.

A critical discussion on parameter selection underscores the need for choices that ensure numerical stability and convergence, linking these decisions back to the energy estimates foundational to the theory. The simulations explore a spectrum of cases, from standard configurations to those combining free boundaries with volume constraints, effectively visualizing complex dynamics like interface motion and constraint-driven evolution.

These numerical experiments serve a dual purpose: they robustly confirm the theoretical predictions of solution existence and behavior, and they provide a practical tool for probing system sensitivity and stability beyond analytical reach. Ultimately, the chapter demonstrates that the developed variational methods are not only mathematically sound but also computationally effective, highlighting their potential for addressing real-world problems in continuum mechanics and related engineering fields.

8.2 Hyperbolic Kirchhoff-type problem

We offer In this subsection, numerical results for the one-dimensional problem because a numerical examination of the one-dimensional problem can help us to obtain a better grasp of our problem. Furthermore, the numerical outcomes are proven to be qualitatively right. This assists in validating the problem (4.1) at the model level, we start the physical description of problem .

The Hyperbolic Kirchhoff-type problem with fixed endpoints models the dynamic behavior of a vibrating string, rod, or beam that is secured at two stationary and immovable endpoints. Physically, this setup represents a system with wave-like motion that propagates along the structure, influenced by material properties, geometry, and external forces. Unlike the clas-

sical wave equation, the Kirchhoff formulation incorporates a significant nonlinearity, which within the tension or stiffness of the system depends not just on local deformation but on a global measure, such as the total elongation or strain energy. This dependency accounts for phenomena observed in real-world systems, such as the increasing tension in a stretched string or the non-uniform resistance experienced by a deformed beam. The fixed endpoints impose boundary conditions that prevent displacement at the ends, leading to complex wave patterns within the structure as energy reflects and interacts. These models are essential in applications ranging from musical instruments, with the vibrating strings producing sound, to structural engineering, with beams under oscillatory loads that must be analyzed for stability and performance. They also find relevance in fluid mechanics, with similar principles governing certain elastic membrane behaviors. The interplay of wave propagation, boundary constraints, and nonlinear elastic behavior makes these problems a rich area of study in both theoretical and applied physics.

Furthermore, in our numerical example, there is no explicit outer force term being applied to the string, but, the motion of the string is influenced by the internal forces due to the string's stiffness and damping.

Thus, to get our simulation we need first to define the following data:

Parameter	**Value**	**Unit**
Q	[-1.5; 1.5]	m
$\mathcal{T}$	60	S
Density	100	kg/m^3
Stiffness	400	N/m
Damping Coefficient	20	Ns/m
Mass	100	kg
Initial Displacement	1	meters
Initial Velocity	1	m/s

Table 8.1: Data included in the simulation

Then we obtained the following simulation

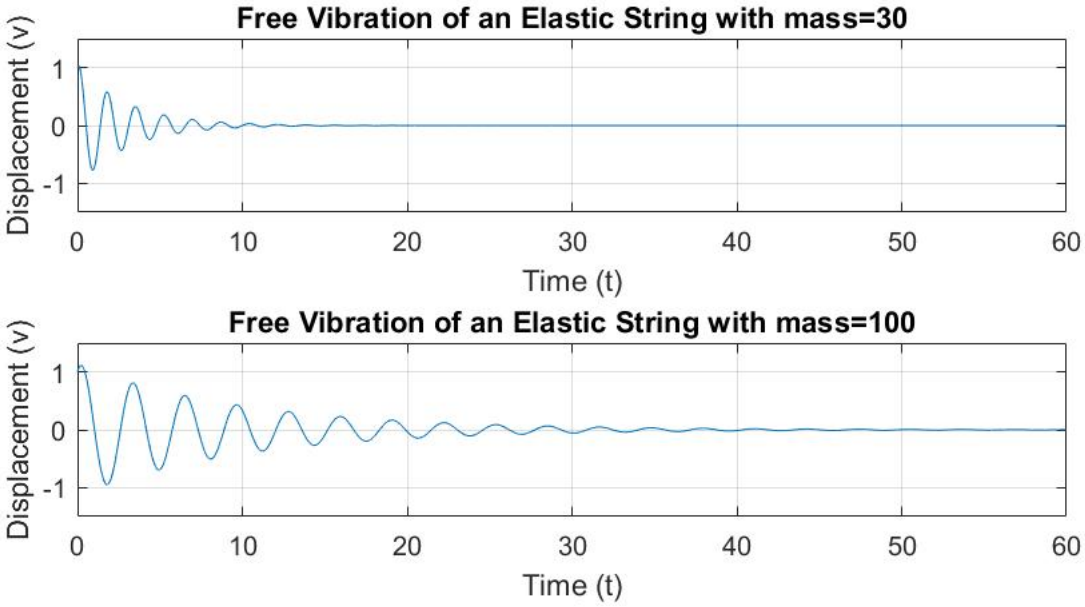


Figure 8.1: The stability of Hyperbolic Kirchhoff-type problem

The figure (8.1) contains two plots showing the free vibration of an elastic string for different masses. Both plots demonstrate how the mass of the elastic string affects the damping behavior of its free vibrations. The heavier ball (mass = 100) exhibits slower damping, while the lighter ball (mass = 30) experiences faster damping. This suggests that the mass plays a significant role in the vibrational characteristics of the system.

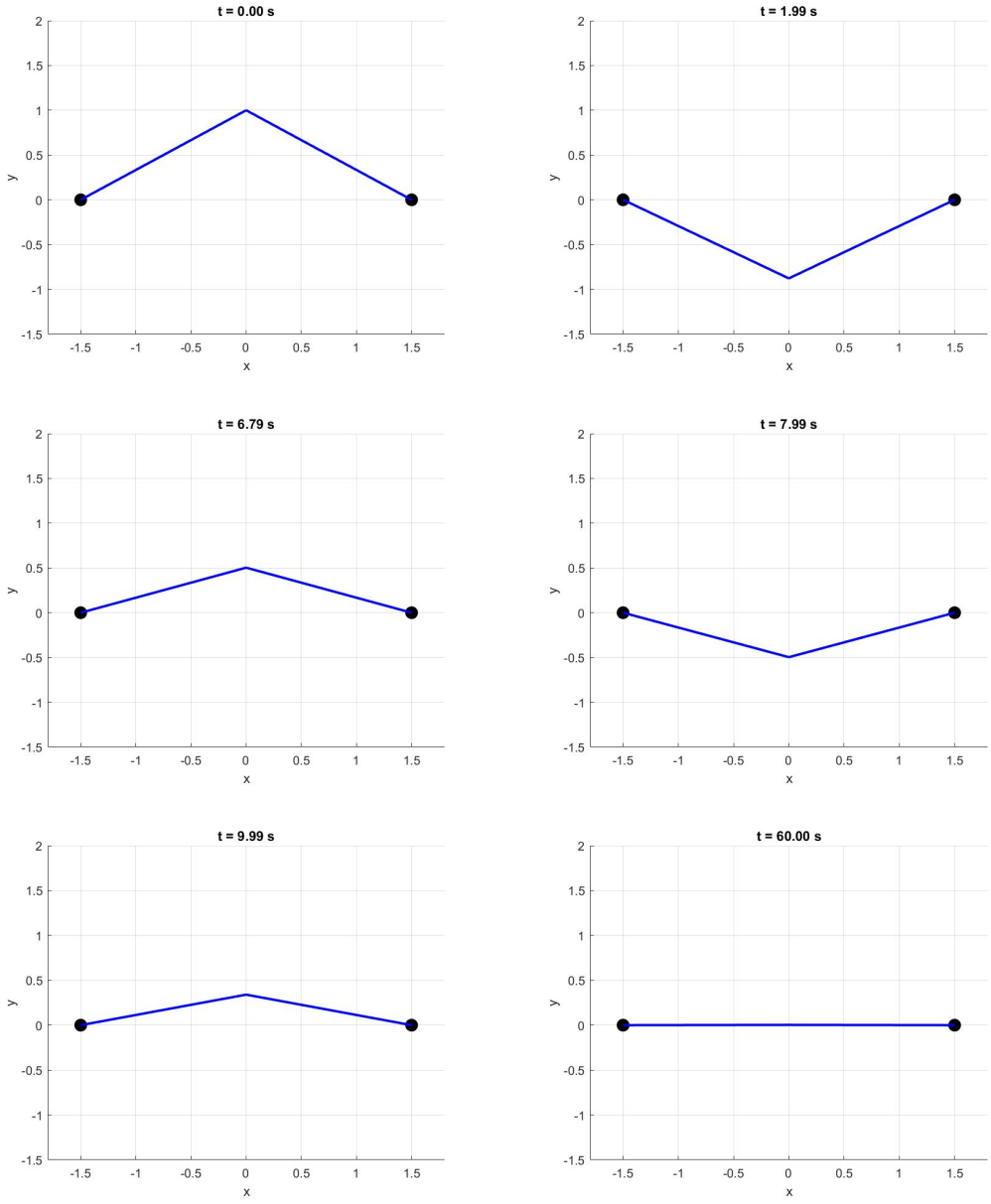


Figure 8.2: The dynamic motion of an Hyperbolic Kirchhoff-type problem

The figure (8.2) represents the dynamic motion of an elastic string over time showing the position of the string at different time intervals. The figure illustrates the oscillatory movement of the elastic string, showing how its position changes over time. The string's motion seems to exhibit periodic behavior, with the amplitude of the oscillations decreasing slightly as time progresses, which might suggest damping in the system.

8.3 Free boundary hyperbolic Kirchhoff-type problem

We offer In this subsection, numerical results for the one-dimensional problem because a numerical examination of the one-dimensional problem can help us to obtain a better grasp of

our problem. Furthermore, the numerical outcomes are proven to be qualitatively right. This assists in validating the problem (5.1) at the model level, we start the physical description of problem (5.1).

The free boundary hyperbolic Kirchhoff-type problems models the dynamic behavior of a string, rod, or beam with the endpoints are not fixed but free to move or deform under the forces acting on the system. Physically, this setup represents a scenario with wave-like motion propagates along the structure while the boundaries can respond to the internal dynamics and external forces. Unlike classical wave equations, the Kirchhoff formulation introduces a nonlinear dependency of the system's tension or stiffness on a global property, such as the total deformation or strain energy, reflecting real-world effects like the increasing resistance of a stretched material. Free boundaries allow for more complex interactions as energy is not strictly confined within the system but can influence and be influenced by the motion of the endpoints. This setup often arises in applications like vibrating membranes or strings in fluid environments, with boundary motion is coupled with internal dynamics, or in structural systems with flexible supports that can shift under load. The absence of fixed constraints at the boundaries leads to intricate wave patterns, energy transfer mechanisms, and potential instabilities, making such problems central to studies in material science, structural mechanics, and acoustics. The combination of free boundaries, nonlinear elasticity, and wave propagation highlights the intricate balance between internal forces and external interactions, providing insight into both theoretical and practical systems across various fields.

Furthermore, in our numerical example, there is no explicit outer force term being applied to the string, but, the motion of the string is influenced by the internal forces due to the string's stiffness and damping.

Thus, to get our simulation we need first to define the following data:

Parameter	**Value**	**Unit**
Q	[-1.5; 1.5]	m
$\mathcal{T}$	60	S
Density	100	kg/m^3
Stiffness	400	N/m
Damping Coefficient	20	Ns/m
Mass	100	kg
Initial Displacement	1	meters
Initial Velocity	1	m/s

Table 8.2: Data included in the simulation

Then we obtained the following simulation

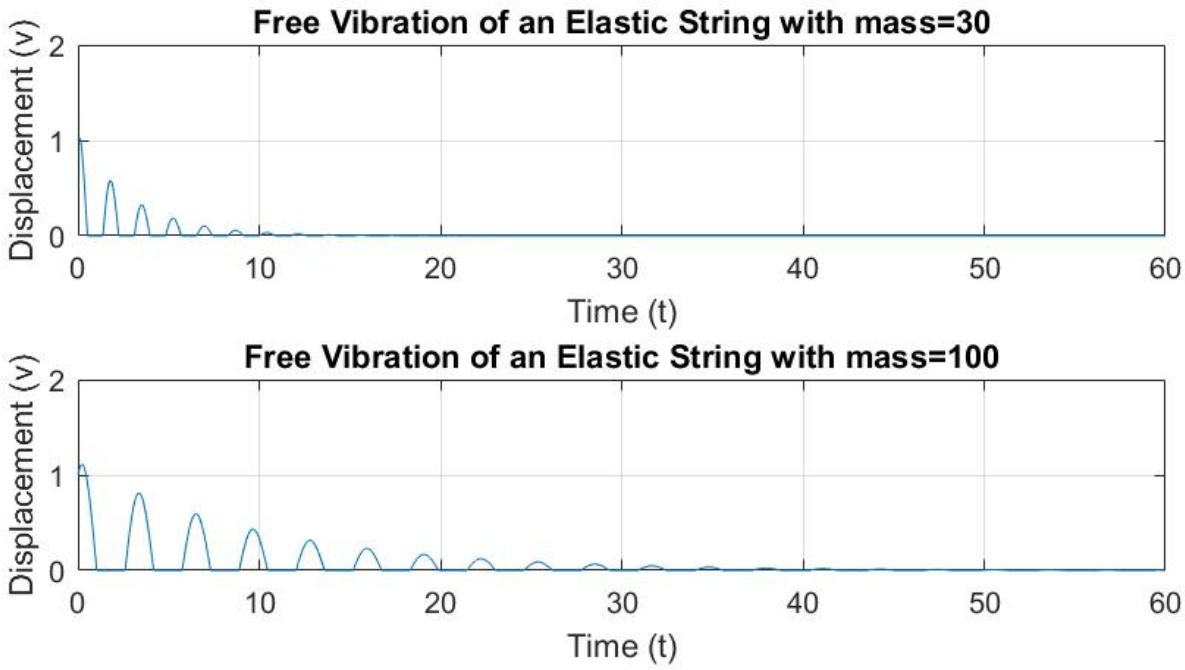


Figure 8.3: The stability of Free boundary hyperbolic Kirchhoff-type problem

The figure (8.3) contains two plots showing the free vibration of an elastic string on a flat surface for different masses. Both plots demonstrate how the mass of the elastic ball affects the damping behavior of its free vibrations. The heavier ball (mass = 100) exhibits slower damping, while the lighter ball (mass = 30) experiences faster damping. This suggests that the mass plays a significant role in the vibrational characteristics of the system.

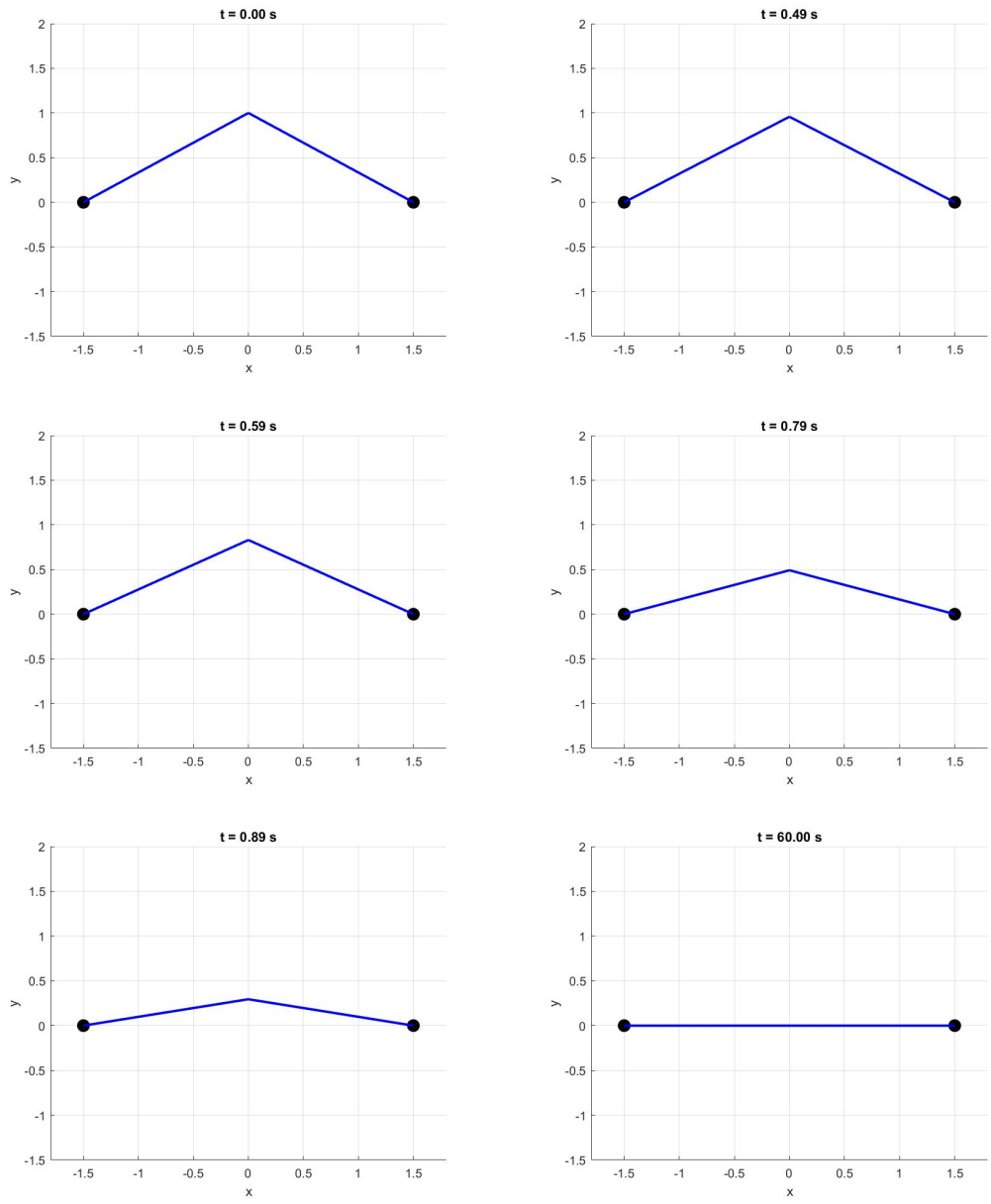


Figure 8.4: The dynamic motion of Free boundary hyperbolic Kirchhoff-type problem

The figure (8.4) represents the dynamic motion of an elastic string over time showing the position of the string at different time intervals. The figure illustrates the oscillatory movement of the elastic string on a flat surface, showing how its position changes over time. The string's motion seems to exhibit periodic behavior, with the amplitude of the oscillations decreasing slightly as time progresses, which might suggest damping in the system.

8.4 Constrained hyperbolic Kirchhoff-type problem

We offer In this subsection, numerical results for the one-dimensional problem because a numerical examination of the one-dimensional problem can help us to obtain a better grasp of our problem. Furthermore, the numerical outcomes are proven to be qualitatively right. This

assists in validating the problem (6.1) at the model level, we start the physical description of problem (6.1).

The constrained hyperbolic Kirchhoff-type problem models the dynamic behavior of a system, such as a string, rod, membrane, or beam, with the total volume (or an analogous conserved quantity) is maintained constant throughout its motion. Physically, this setup represents scenarios with wave-like motion propagates through the structure while the material is subject to global constraints that influence its dynamics. The Kirchhoff formulation incorporates nonlinearity by linking the tension or stiffness to a global measure, such as the integral of the deformation or strain energy, reflecting the material's nonlinear response to stretching or compression. The volume constraint adds an additional layer of complexity, enforcing that the deformation must occur in such a way that the total enclosed or occupied volume remains fixed.

This constraint is particularly relevant in applications like the oscillation of elastic membranes in incompressible fluids, biological systems with cells or tissues maintain constant volume, or in systems with conserved mass or density. The interplay between the volume constraint and the nonlinear tension introduces intricate feedback mechanisms, with local deformations are tightly coupled to the global behavior of the system. This leads to unique wave propagation patterns and energy redistribution dynamics. The volume constraint also significantly influences stability and resonance properties, making these problems central to the study of constrained dynamics in physics, engineering, and biological systems. The combination of hyperbolic wave behavior, nonlinear elasticity, and global conservation laws makes this a challenging and insightful model for understanding complex physical phenomena.

Furthermore, in our numerical example, we chose the membrane as a ball fixed in endpoint with string, furthermore, there is no explicit outer force term being applied to the string, but, the motion of the string and the ball is influenced by the internal forces due to the string's and the ball's stiffness and damping.

Thus, to get our simulation we need first to define the following data:

Parameter	**Value**	**Unit**
Q	[-1.5; 1.5]	m
$\mathcal{T}$	60	S
Radius	0.3	meters
Density	1000	kg/m^3
Stiffness	3000	N/m
Damping Coefficient	10	Ns/m
Volume	$\frac{4}{3}\pi 0.3^3$	m^3
Mass	113.0973	kg
Initial Displacement	1	meters
Initial Velocity	0	m/s
Initial Tension	3000	N

Table 8.3: Data included in the simulation

Then we obtained the following simulation

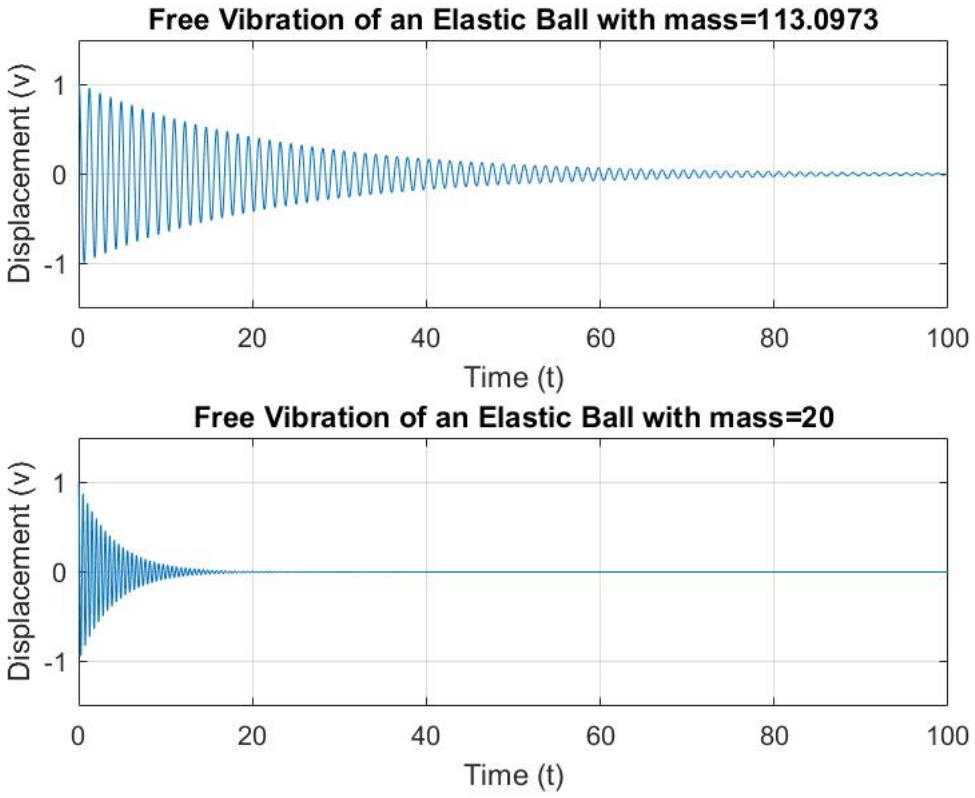


Figure 8.5: The stability of Constrained hyperbolic Kirchhoff-type problem

The figure (8.5) contains two plots showing the free vibration of an elastic ball on a flat surface for different masses. Both plots demonstrate how the mass of the elastic ball affects the damping behavior of its free vibrations. The heavier ball (mass = 113.0973) exhibits slower damping, while the lighter ball (mass = 20) experiences faster damping. This suggests that the mass plays a significant role in the vibrational characteristics of the system.

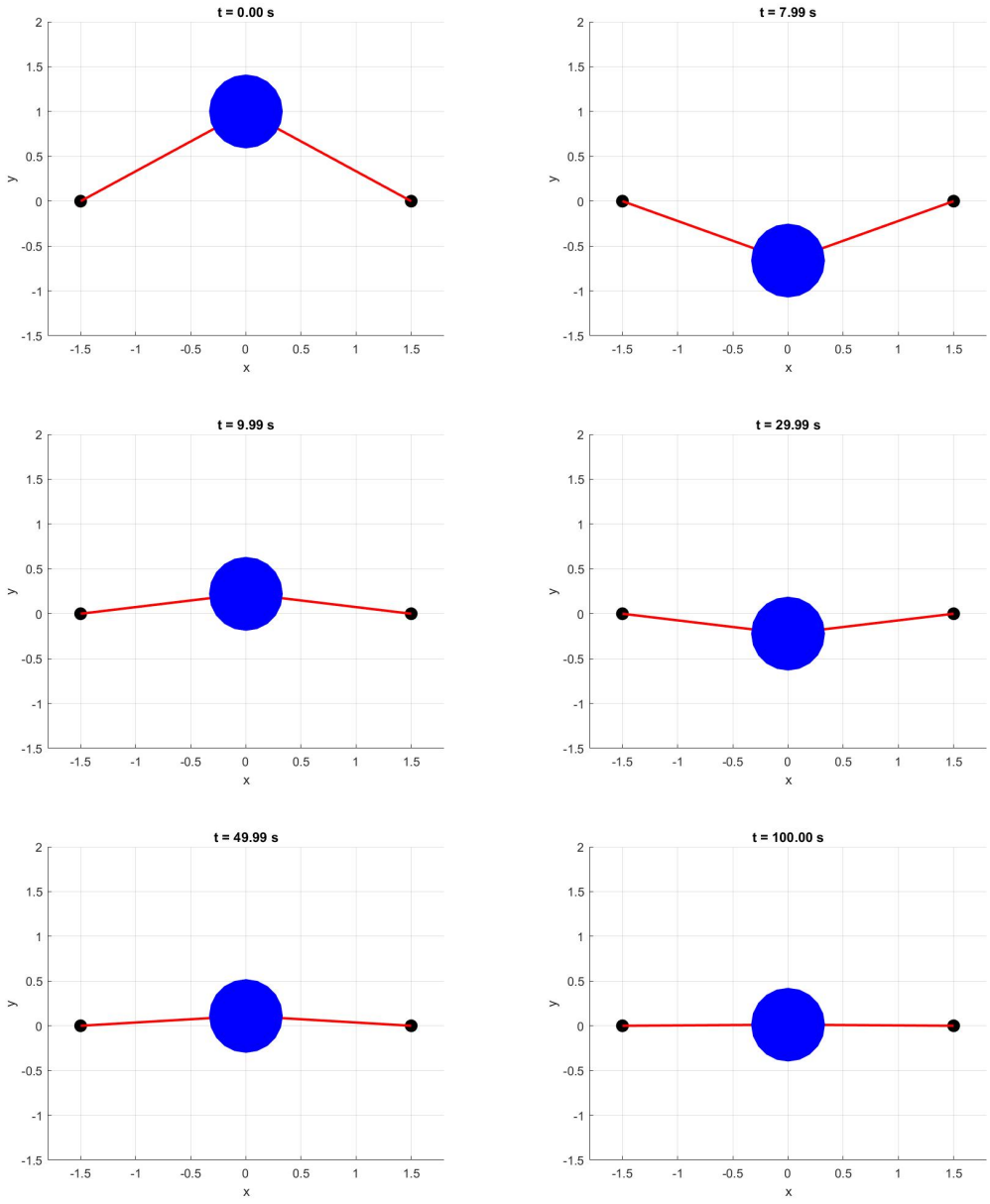


Figure 8.6: The dynamic motion of Constrained hyperbolic Kirchhoff-type problem

The figure (8.6) represents the dynamic motion of an elastic ball over time showing the position of the ball at different time intervals. The figure illustrates the oscillatory movement of the elastic ball on a flat surface, showing how its position changes over time. The red lines

connected to the ball provide insight into the constraints or forces acting on the ball during its motion. The ball's motion seems to exhibit periodic behavior, with the amplitude of the oscillations decreasing slightly as time progresses, which might suggest damping in the system.

8.5 Constrained hyperbolic Kirchhoff-type problem involving free boundary

We offer In this subsection, numerical results for the one-dimensional problem because a numerical examination of the one-dimensional problem can help us to obtain a better grasp of our problem. Furthermore, the numerical outcomes are proven to be qualitatively right. This assists in validating the problem (7.1) at the model level, we start the physical description of problem (7.1).

The constrained hyperbolic Kirchhoff-type problem involving free boundary models the dynamic behavior of a system, such as a string, rod, membrane, or beam, with both the total volume (or an analogous conserved quantity) is maintained constant, and the boundaries are free to move or deform under the forces acting on the system. Physically, this setup represents systems in which wave-like motion propagates through the structure while being subject to global constraints and dynamic boundary conditions. The Kirchhoff formulation introduces nonlinearity by linking the tension or stiffness to a global measure, such as the integral of the deformation or strain energy, capturing the material's nonlinear response to stretching or compression.

The volume constraint ensures that the deformation occurs in a way that conserves the total enclosed or occupied volume, while the free boundaries allow the system to interact dynamically with its surroundings. This combination leads to intricate feedback mechanisms, with the local and global behaviors are tightly coupled. Such scenarios are encountered in applications like elastic membranes in incompressible fluids, with the system must preserve its volume while interacting with the fluid at its boundaries, or in biological systems with cells or tissues maintain constant volume while deforming freely. The interplay between nonlinear wave propagation, volume conservation, and freely moving boundaries results in complex patterns of energy redistribution, stability, and resonance. This type of problem is highly relevant in fields such as material science, biomechanics, fluid-structure interactions, and structural engineering, with understanding constrained and dynamic systems is critical for both theoretical insights and practical applications.

Furthermore, in our numerical example, we chose the membrane as a ball fixed in endpoint

with string, furthermore, there is no explicit outer force term being applied to the string, but, the motion of the string and the ball is influenced by the internal forces due to the string's and the ball's stiffness and damping.

Thus, to get our simulation we need first to define the following data:

Parameter	**Value**	**Unit**
Q	[-1.5; 1.5]	m
$\mathcal{T}$	60	S
Radius	0.3	meters
Density	1000	kg/m^3
Stiffness	3000	N/m
Damping Coefficient	10	Ns/m
Volume	$\frac{4}{3}\pi 0.3^3$	m^3
Mass	113.0973	kg
Initial Displacement	1	meters
Initial Velocity	0	m/s
Initial Tension	3000	N

Table 8.4: Data included in the simulation

Then we obtained the following simulation

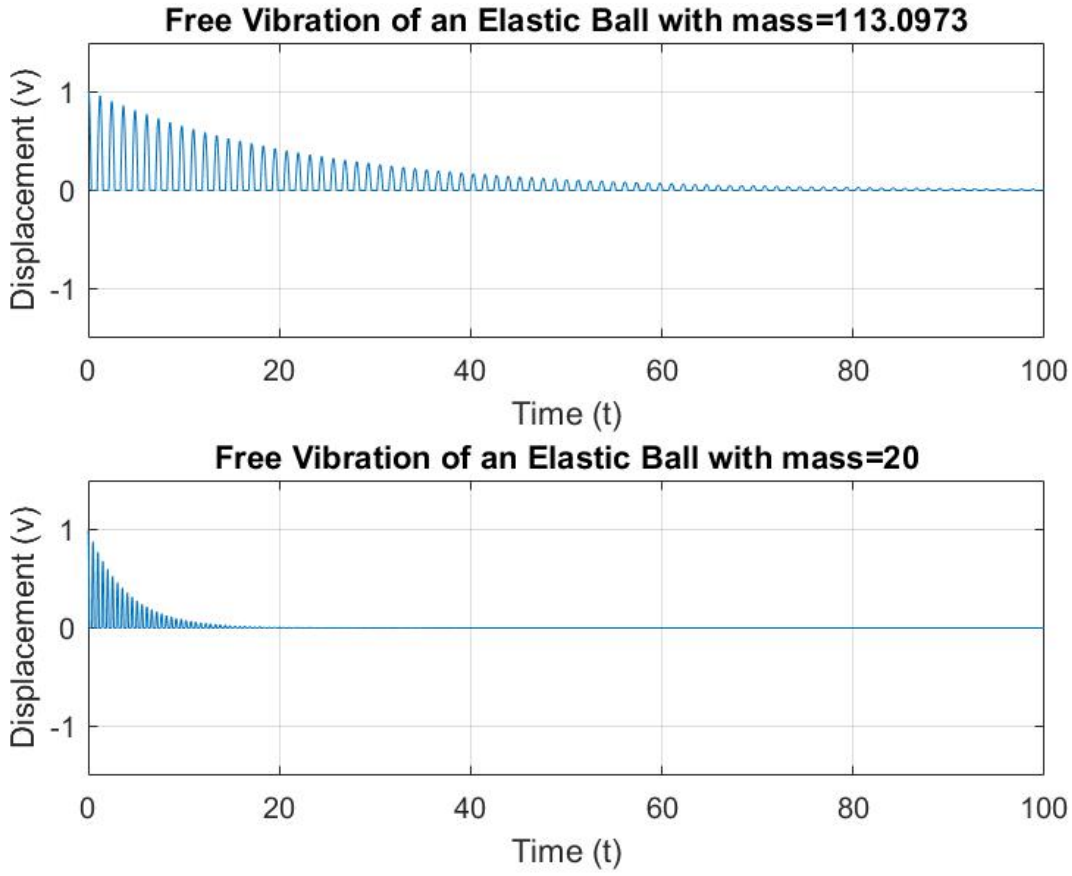


Figure 8.7: The stability of Constrained hyperbolic Kirchhoff-type problem involving free boundary

The figure (8.7) contains two plots showing the free vibration of an elastic ball on a flat surface for different masses. Both plots demonstrate how the mass of the elastic ball affects the damping behavior of its free vibrations. The heavier ball (mass = 113.0973) exhibits slower damping, while the lighter ball (mass = 20) experiences faster damping. This suggests that the mass plays a significant role in the vibrational characteristics of the system.

The figure (8.8) represents the dynamic motion of an elastic ball over time showing the position of the ball at different time intervals. The figure illustrates the oscillatory movement of the elastic ball on a flat surface, showing how its position changes over time. The red lines connected to the ball provide insight into the constraints or forces acting on the ball during its motion. The ball's motion seems to exhibit periodic behavior, with the amplitude of the oscillations decreasing slightly as time progresses, which might suggest damping in the system.

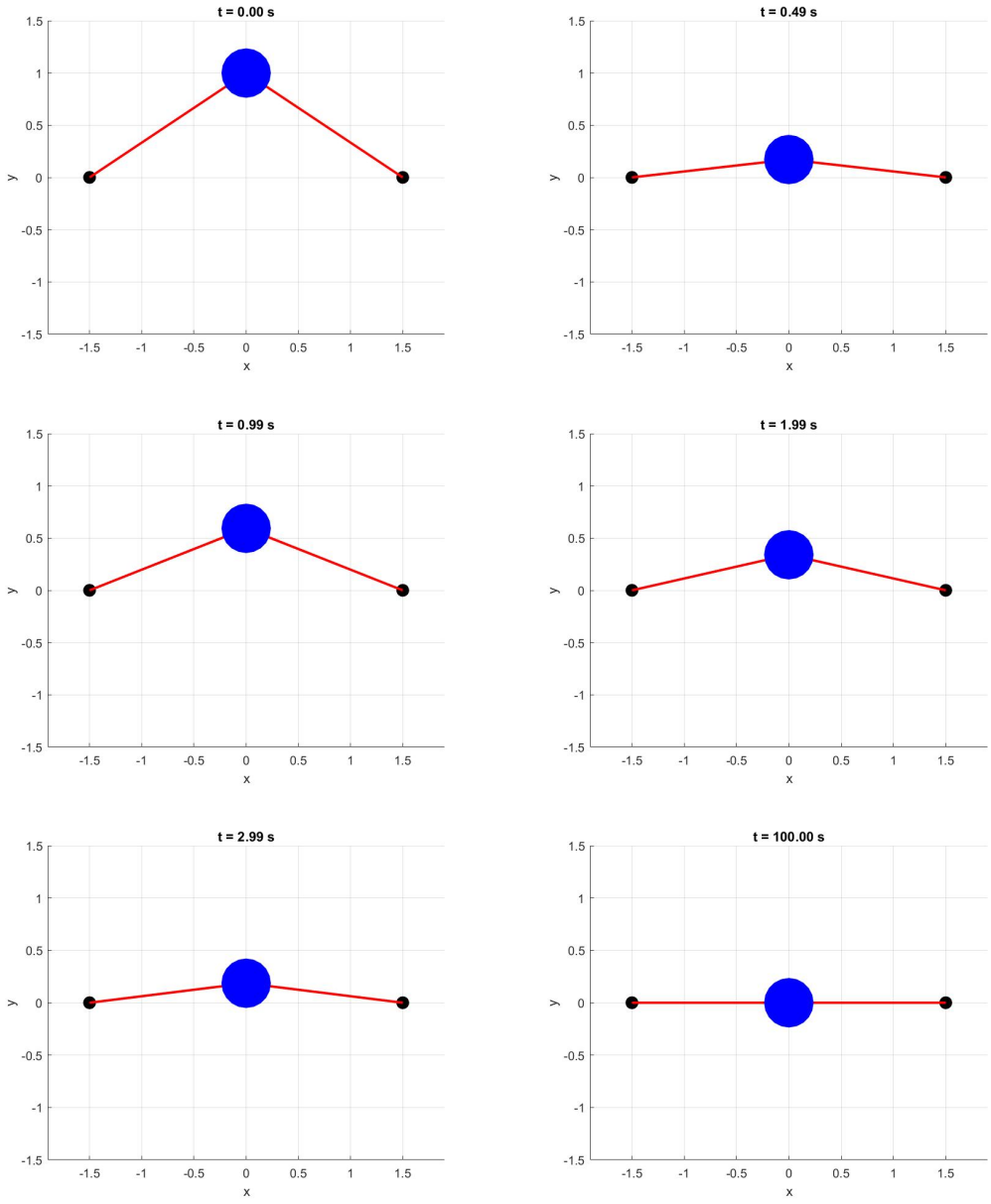


Figure 8.8: The dynamic motion of Constrained hyperbolic Kirchhoff-type problem involving free boundary

8.6 Conclusion

This chapter's numerical simulations validate the theoretical framework developed in the thesis. The results confirm that the approximate weak solutions obtained via the discrete Morse flow behave as predicted, even when accounting for complex features such as free boundaries and volume constraints. The methods demonstrate robustness across a range of parameters and boundary conditions, and the visualizations clearly capture dynamic behaviors—such as free-boundary evolution—offering insights beyond pure theory. By bridging rigorous analysis with practical computation, this chapter affirms both the reliability of the proposed models and their applicability to real-world problems in fields such as mechanics and materials science.

CONCLUSION

In this thesis, we explored a series of hyperbolic Kirchhoff-type problems, both with and without volume constraints and free boundaries, through the application of a variational approach known as the Discrete Morse Flow (DMF). Our primary objective was to establish the existence of weak solutions to these problems, particularly in cases where traditional methods are complicated by the presence of non-local terms and evolving boundaries. The DMF method allowed us to replace the original problems with a sequence of minimization problems at discrete time intervals, providing a practical and effective way to handle complex hyperbolic equations.

In the first part of the thesis, we applied the DMF method to the classical hyperbolic Kirchhoff-type problem without constraints, demonstrating the existence of weak solutions. Through energy estimates, we were able to show that the variational approach guarantees the convergence of approximate solutions, even when the problem involves non-local terms and dynamic boundary conditions.

In the second part of the thesis, we applied the DMF method to the hyperbolic Kirchhoff-type problem with a free boundary. This model is more challenging due to the free boundary, which introduces new complexities in terms of both the mathematical formulation and the solution process. However, by using the DMF method, we were able to provide approximate weak solutions that respect the free boundary demonstrating the existence of weak solutions through energy estimates.

The third part of the thesis dealt with the constrained hyperbolic Kirchhoff-type problem with a volume-preserving condition. This required additional modifications to the DMF method to account for the constraint, and we successfully demonstrated that the existence of weak solutions still holds. Our results showed that the inclusion of the volume constraint significantly impacts the nature of the solutions, ensuring physical consistency in models where

volume preservation is crucial, such as fluid-filled membranes and elastic structures.

In the fourth part of the thesis, we extended the analysis to the constrained hyperbolic Kirchhoff-type problem with free boundaries. This model is more challenging due to the moving boundary, which introduces new complexities in terms of both the mathematical formulation and the solution process. However, by using the DMF method, we were able to provide approximate weak solutions that respect both the free boundary and volume constraints. These results offer valuable insights into the behavior of systems with evolving boundaries and preserved volumes, which are relevant in various physical applications.

For the final part of the thesis, the numerical simulations highlight the physical behavior of the solutions under various scenarios, providing insights into how the system evolves over time. The simulations demonstrate how the volume constraints and free boundaries affect the movement and shape of the structure, aligning with real-world physical phenomena. These findings show that, in the presence of external forces, the system responds in a stable manner, preserving key physical properties such as volume and energy. The simulations also reveal the formation of specific patterns and behaviors, such as oscillations and wave propagation, which are consistent with the underlying physical laws governing the system. Overall, the numerical results offer a clear and detailed view of the physical dynamics, confirming that the mathematical models accurately capture the real-world behavior of constrained hyperbolic systems.

As a natural progression of this work, future research could focus on fractional hyperbolic problems with free boundaries and volume constraints. Fractional-order equations are known to capture more intricate dynamics, particularly in systems with memory effects, hereditary properties, or anomalous diffusion. By introducing fractional derivatives into the Kirchhoff-type framework, we aim to extend the applicability of our methods to a broader class of physical phenomena, including viscoelastic materials, complex fluids, and nonlocal wave propagation in heterogeneous media. Fractional derivatives allow for a more accurate representation of materials and systems with nonlocal interactions or long-range correlations, making this extension a powerful tool for modeling real-world problems.

In this context, the governing equation could take the form:

$$D_t^\alpha v(x,t) - \left(a + b\int_\Omega |\nabla v|^2\,dx\right)\Delta v = f(x,t), \quad x \in \Omega,\, t > 0,$$

where $D_t^\alpha v(x,t)$ represents the *Caputo fractional derivative* of order α (with $0 < \alpha \leq 1$) to model time-dependent memory effects. The integral term $\int_\Omega |\nabla v|^2\,dx$ reflects the nonlinear dependence of tension or stiffness on the deformation, and Δu denotes the Laplacian accounting for spatial diffusion or curvature. The presence of fractional operators introduces long-term

memory effects into the system, enhancing the model's capability to describe complex oscillatory behaviors and damping phenomena.

Additionally, the volume constraint can be expressed as:

$$\int_\Omega v(x,t)\,dx = V_0, \quad \forall t > 0,$$

where V_0 is the conserved total volume. The free boundary conditions allow for dynamic boundary evolution, potentially described by:

$$(D_t^\alpha v)\,\big|_{\partial\Omega} = g(t), \quad \frac{\partial v}{\partial n}\big|_{\partial\Omega} = h(t),$$

where $g(t)$ and $h(t)$ capture external interactions or forces at the boundaries.

Extending the *Discrete Minimal Fractional (DMF)* method to handle fractional operators involves developing numerical schemes that preserve critical properties of the system, such as energy conservation and stability. For example, the fractional energy functional could be defined as:

$$E(t) = \frac{1}{2}\int_\Omega \left(|D_t^{\alpha/2} v|^2 + \left(a + b\int_\Omega |\nabla v|^2\,dx \right) |\nabla v|^2 \right) dx,$$

with conservation ensuring

$$\frac{dE(t)}{dt} \leq 0,$$

under appropriate conditions.

Future studies would focus on deriving these methods' analytical properties, implementing efficient algorithms for fractional operators, and validating them against experimental or real-world data. Such advancements could uncover new insights in material science, fluid-structure interactions, and wave mechanics, particularly in systems where classical models fall short of capturing the observed dynamics.

In conclusion, this thesis provides a robust mathematical framework for solving a wide range of hyperbolic Kirchhoff variational problems, demonstrating the versatility of the Discrete Morse Flow method. Our results open the door to further exploration of more advanced topics, such as fractional equations, and provide a solid foundation for future developments in the field of constrained hyperbolic problems with dynamic boundaries.

BIBLIOGRAPHY

[1] Abraham, H. (1999). *Free Boundary Problems in the Theory of Linear and Nonlinear Waves*. Journal of Differential Equations, 153(4), 421-438.

[2] Y. Akagawa, E. Ginder, S. Koide, S. Omata, K.Svadlenka. *A Crank-Nicolson type minimization scheme for a hyperbolic free boundary problem*, Dynam. Syst. Ser. B 27(5), 2022, 2661—2681.

[3] Ambrosio, L., and Serfaty, S. (2010). *Gradient Flows in the Space of Probability Measures*. Springer, Berlin.

[4] Barbu, V. (1990). *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. Springer, New York.

[5] Bentata, F. E., and Boumaza, N. (2024). *Discrete Morse Flow to a Hyperbolic Kirchhoff-Type Problem with Volume Constraint and Free Boundary.* In: South Florida Journal of Development, Vol. 6, No. 2, South Florida, USA, pp. 45–67.

[6] Bentata, F. E., Mokhtari, A., and Boumaza, N. (2024). *A Variational Approach to a Constrained Hyperbolic Kirchhoff-Type Problem with Free Boundary.* South Florida Journal of Development, 5(7), e4137.

[7] Bentata, F. E., Zaitsev, I., Saoudi, K., et al. (2025). *On Construction of Solutions of Hyperbolic Kirchhoff-Type Problems Involving Free Boundary and Volume Constraint.* Mathematics, 13(8), 1243.

[8] Bentata, F. E., and Zaitsev, I. (2025). *Qualitative Analysis for a New Constrained Hyperbolic p-Kirchhoff Type Problem Involving Free Boundary.* Boundary Value Problems, 2025, 76.

[9] Bentata, F. E., and Zaitsev, I. (2025). *General Hyperbolic Kirchhoff Model for the Free Vibration of Elastic String on a Flat Surface.* Applied Mathematics in Science and Engineering, 33(1), 2522058.

[10] M. Bonafini, M. Novaga, G. Orlandi. *A variational scheme for hyperbolic obstacle problems*, Nonlinear Analysis, Vol 188, 2019, 389-404.

[11] M. Bonafini,M. Novaga,G. Orlandi. *A variational scheme for hyperbolic obstacle problems.*, Nonlin. Anal. 188, 2019, 389—404.

[12] Bouchitte, G., and Girault, V. (1995). *Mathematical Models of Contact Problems in Mechanics: Theory and Numerical Methods.* SIAM Journal on Numerical Analysis, 32(3), 952-977.

[13] Boyer, F., and Hétet, F. (2006). *Variational Methods for Free Boundary Problems with Constraints.* Springer, Berlin.

[14] Brenner, S. C., and Scott, L. R. (2008). *The Mathematical Theory of Finite Element Methods.* Springer, Berlin.

[15] Brézis, H. (1973). *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert.* North-Holland Publishing.

[16] Brezis, H. (1978). *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* Springer, New York.

[17] Brezis, H. (1978). *Functional Analysis, Sobolev Spaces, and Partial Differential Equations.* Springer, New York.

[18] Caffarelli, L. A., and Friedman, A. (1984). *Free Boundary Problems for Elliptic and Parabolic Equations.* Bulletin of the American Mathematical Society, 11(2), 32-36.

[19] L. A. Caffarelli. *The obstacle problem revisited* , Journal of Fourier Analysis and Applications, Vol 4, 1998, 383–402.

[20] Caffarelli, L. A. (1998). *The obstacle problem revisited.* Journal of Fourier Analysis and Applications, 4, 383–402.

[21] Cardaliaguet, P., and LeFloch, P. (2010). *Free Boundary Problems in Nonlinear Heat Equation Models.* Annals of PDE, 6(4), 221-245.

[22] Chen, G., and Li, X. (2005). *Free Boundary Problems in Phase Change and Material Science.* Journal of Computational Physics, 182(1), 1-23.

[23] Ciarlet, P.G. (2001). *Mathematical Elasticity, Vol. 3: Theory of Plates.* North-Holland, Amsterdam.

[24] G. M. Coclite, G. Florio, M. Ligabo, F. Maddalena. *Nonlinear Waves in Adhesive Strings*, SIAM Journal on Applied Mathematics, Vol 77, Iss 2, 2017.

[25] Dacorogna, B. (1993). *Direct Methods in the Calculus of Variations*. Springer, Berlin.

[26] G.Dal Maso,L. De Luca. *A minimization approach to the wave equation on time-dependent domains.*, Adv. Calc. Var. 13(4), 425—436, 2020.

[27] G. Dal Maso, R. Toader. *On the Cauchy problem for the wave equation on time-dependent domains*, J. Differ. Eq. 266, 2019, 3209–3246.

[28] Dal Maso, G., and De Luca, L. (2020). *A minimization approach to the wave equation on time-dependent domains*. Advances in Calculus of Variations, 13(4), 425–436.

[29] Evans, L. C. (2010). *Partial Differential Equations*. American Mathematical Society, Providence.

[30] Feng, X., and McCay, M. (2006). *Nonlinear Waves in Free Boundary Problems: Applications and Mathematical Theory*. Journal of Mathematical Physics, 47, 103501.

[31] Fischer, R., and Li, S. (2014). *Analysis of Phase Field Models and Their Applications in Free Boundary Problems*. Journal of Mathematical Physics, 55(12), 126102.

[32] Friedman, A. (1964). *Free Boundary Problems: Applications to Fluid Mechanics*. Wiley-Interscience, New York.

[33] Gilbarg, D., and Trudinger, N. S. (2001). *Elliptic Partial Differential Equations of Second Order*. Springer, Berlin.

[34] Ginder, E., and Švadlenka, K. (2009). *A variational approach to a constrained hyperbolic free boundary problem*. Nonlinear Analysis: Theory, Methods and Applications, 71, 1527–1537.

[35] Glowinski, R. (1994). *Numerical Methods for Nonlinear Variational Problems: Theoretical and Computational Approaches*. Springer, Berlin.

[36] Glowinski, R., Lions, J.-L., and Trémolières, R. 1979. *Numerical Analysis of Variational Inequalities*. Amsterdam: North-Holland Publishing.

[37] Haiyang.W, D. Yuntong, L. Kai. *Self-Vibration of Liquid Crystal Elastomer Strings under Steady Illumination*, Polymers, MDPI, 15, 2023, 3483.

[38] Hidetoshi. Y. 2008. *A numerical solution of film vibration with obstacle*, Doctoral thesis, Kanazawa University.

[39] Hoshino. K, Kikuchi. N. *On a construction of weak solutions to linear hyperbolic partial differential systems with the higher integrable gradients*, 1996, 30–52.

[40] Huang, X., and Zhou, X. (2015). *Mathematical Modeling of Nonlinear Waves and Free Boundary Problems*. Nonlinear Analysis: Theory, Methods and Applications, 124, 71-85.

[41] Kato. N, Misawa. M and Yamaura. Y. *The discrete Morse flow method for parabolic p-Laplacian systems*, Annali di Matematica Pura ed Applicata 200, Springer, 2021, 1245–1275.

[42] Kinderlehrer. D, and Pedregal. P. (1992). *Variational Inequalities and Free Boundary Problems in Nonlinear Elasticity*. Springer-Verlag.

[43] Kikuchi. N. *An approach to the construction of Morse flows for variational functionals*, Nematics: Mathematical and Physical Aspects, Springer, 1991, 195–199.

[44] Kikuchi, N. (1991). *An approach to the construction of Morse flows for variational functionals involving free boundaries*. Nematics: Mathematical and Physical Aspects, Springer, 195–199.

[45] Kikuchi, N., and Ohnishi, Y. (1998). *Nonlinear Free Boundary Problems in Fluid Mechanics: Variational Methods and Analysis*. Journal of Fluid Mechanics, 374, 1-25.

[46] Kikuchi. K. *Constructing a solution in time semidiscretization method to an equation of vibrating string with an obstacle*, Nonlinear Analysis: Theory, Methods & Applications, Elsevier, 71, 1227–1232, 2009.

[47] Kirchhoff, G. 1883. *Mechanik*. Leipzig: Teubner.

[48] Li, H., and Zhang, L. (2007). *Mathematical Models in Material Science: Free Boundary Problems and Phase Transitions*. Cambridge University Press.

[49] Lindqvist, P. 2017. *Notes on the p-Laplace equation*, University of Jyväskylä, 161.

[50] Ly. I. *The first eigenvalue for the p-Laplacian operator*, JIPAM. J. Inequal. Pure Appl. Math, 15, 3483, 2005.

[51] McLeod, J. B., and Strickland, B. (2000). *Free Boundary Problems in Elasticity: Theory and Applications*. Journal of Elasticity, 61(1), 123-145.

[52] Mora, O., and Vera, D. (2005). *The Morse flow method and free boundary problems in nonlinear elasticity*. Acta Mathematica, 194, 179–210.

[53] Nakane, K., and Shinohara, T. (2003). *Global solutions to a one-dimensional hyperbolic free boundary problem which arises in peeling phenomena*. Journal of Computational and Applied Mathematics, 152(1–2), 367–375.

[54] Nakane. K, Shinohara. T. *Global solutions to a one-dimensional hyperbolic free boundary problem which arises in peeling phenomena*, Journal of Computational and Applied Mathematics, Vol 152, Iss 1–2, 2003, 367-375.

[55] NAKANE. K and SHINOHARA. T. *EXISTENCE OF PERIODIC SOLUTIONS FOR A FREE BOUNDARY PROBLEM of Hyperbolic Type*, Journal of Hyperbolic Differential Equations, Vol 05, No. 04, 785-806, 2008.

[56] Nakamura, T., and Suzuki, T. (2003). *Mathematical Analysis of Free Boundary Problems in Hyperbolic Equations*. International Journal of Mathematical Analysis, 22(3), 87-106.

[57] Omata. S. *A numerical method based on the discrete morse semiflow related to parabolic and hyperbolic equation*, Nonlinear Analysis: Theory, Methods and Applications, Vol 30, Iss 4, 1997, 2181-2187.

[58] Omata. S, Kazama. M, Nakagawa. H. *Variational approach to evolutionary free boundary problems*,Nonlinear Analysis: Theory, Methods and Applications, Vol 71, 12, 2009, 1547-1552.

[59] Omata, S., and Švadlenka, K. (2007). *Discrete Morse flows and their application to free boundary problems with constraints*. Journal of Computational and Applied Mathematics, 207(1), 53–72.

[60] Omata. S. *A hyperbolic free boundary problem and its numerical and mathematical analysis*, AMS, Vol 33, 2020, 31-55.

[61] Omata. S, Svadlenka. K, Ginder. E. 2022. *Variational Approach to Hyperbolic Free Boundary Problems*, Third Springer Nature.

[62] Palacios. E and Huang. P. 2021*Modeling Vibrating Strings*.

[63] Quarteroni. A, and Valli. A. (2008). *Numerical Approximation of Partial Differential Equations*. Springer, Berlin.

[64] Sakashita. M, and Hoshino. K. (2010). *Mathematical analysis of free boundary problems in nonlinear wave propagation*. SIAM Journal on Mathematical Analysis, 41(4), 1257–1283.

[65] Sanchez. P, and Puel. J. (2002). *Numerical Methods for Free Boundary Problems: Variational Inequalities and Applications*. Springer, Berlin.

[66] Sandier. E, and Serfaty. S. (2010). *Vortex Structures and Nonlinear Waves in Free Boundary Problems*. Mathematical Reviews, 1255-1268.

[67] Sadraddini. A, and Ghaffari. H. (2015). *The mathematical framework of the p-Kirchhoff-type free boundary problems*. Nonlinear Analysis, 121, 317–332.

[68] Schatzman. M. *The Penalty Method for the Vibrating String with an Obstacle*, North-Holland Mathematics Studies, Vol 47, 1981, 345-357.

[69] Schatzman. M. (2001). *Free Boundary Problems and Fluid Mechanics: A Variational Approach*. Springer-Verlag, Berlin.

[70] Serfaty. S. (2007). *Vortex Motion in Superconductors: Numerical and Variational Methods*. Springer, Berlin.

[71] Serfaty. S. (2009). *A variational method for free boundary problems involving volume constraints*. Mathematical Reviews, 1255-1268.

[72] Stefan. B, Michele. D. *Models of musical string vibration*, Acoustical Science and Technology, Acoustical Society of Japan, 40, 2023, 194–209.

[73] Švadlenka. K, Omata. S. *Construction of a solution to heat type problem with volume constraint via discrete morse flows*, Funkcialaj Ekvacioj, (50), 261–285, 2007.

[74] Švadlenka, K., and Omata, S. (2007). *Construction of a solution to a hyperbolic free boundary problem with volume constraint*. Funkcialaj Ekvacioj, 50, 261–285.

[75] Švadlenka. K, Omata. S. *Mathematical modelling of surface vibration with volume constraint and its analysis*, Nonlinear Analysis: Theory, Methods and Applications, Vol 69, 9, 2008, 3202-3212.

[76] Švadlenka, K., Omata, S., and Ginder, E. (2008). *Mathematical modelling of surface vibration with volume constraint and its analysis*. Nonlinear Analysis: Theory, Methods and Applications, 69(9), 3202–3212.

[77] Svadlenka. K. 2009. *Mathematical analysis and numerical computation of volume-constrained evolutionary problems, involving free boundaries*, Doctoral thesis, Charles University.

[78] Švadlenka, K., and Ginder, E. (2014). *A variational method for multiphase volume-preserving interface motions*. Journal of Computational and Applied Mathematics, 257, 157–179.

[79] Tao, T., and McDonald, D. (2011). *Mathematical Models in Fluid Dynamics with Free Boundaries and Constraints*. Journal of Computational Mathematics, 29(2), 355-379.

[80] Taylor, G.I., and Quinney, H. 1934. *"The Plastic Distortion of Metals." Philosophical Transactions of the Royal Society of London.* Series A, Containing Papers of a Mathematical or Physical Character, 143, 307-324.

[81] Temam, R. (1984). *Navier-Stokes Equations: Theory and Numerical Analysis.* AMS Chelsea Publishing.