People's Democratic Republic
Ministry of Higher Education and
Scientific Research
Larbi Tebessi University- Tebessa
Faculty of the Exactes Sciences and Sciences of
Nature and life
Departement: Mathematics

Final dissertation
For the graduation of MASTER
Domaine: Mathematics and Computer Science
Field: Mathematics
Option: Partial Differential Equations and Applications
Theme

Control methods for inverse problems

Presented by:
ALI LINDA

Before the jury :

Dr, BERRAH KHALED	MCA	Larbi Tebessi University	President
Dr, HAFDALLAH Abdelhak	MCA	Larbi Tebessi University	Supervisor
Dr, AMIROUCHE LAADJEL	MAA	Larbi Tebessi University	Examiner

Date of Dissertation : 03/06/2025


بسم الله الرحمن الرحيم

الاهداء

" و آخر دعواهم أن الحمد لله رب العالمين "

الحمد لله عند البدء و عند الختام

الحمد لله الذي وفقني لتثمين هذه الخطوة في مسيراتي الدراسية بمذكرتي هذه ثمرة الجهد والنجاح بفضله تعالى

انتهت الرحلة. لم تكن الرحلة قصيرة لقد كانت طويلة مليئة بالإخفاقات والنجاحات ولم تكن سهلة ولم يكن الحلم قريبا ولكني فخورة بكفاحي لتحقيق احلامي وفي اللحظة أكثر فخرا اهدي نجاحي هذا

لنفسي الطموحة " يا من صبرتي رغم العثرات يا من وصلتي رغم التعب يا من آمنتي بأن الخطوة الأولى مهما كانت صغيرة تقود دائما الى التقدم يا من بكيتي بصمت ثم ابتسمتي للناس ها أنت تلمسين ثمار صبرك وايمانك بذاتك فخورة بك يا انا ".

الى امي

الى من غرست في قلبي حب الطموح وسقت روحي بالدعاء والصبر

الى من كانت تفرح لفرحي و تدمع لدمعتي الى حبيبتي و ملاكي في الحياة يا من كنتي النور في عتمتي .....امي الغالية حفظها الله .

الى ابي

الى من رباني وكافح من أجلي الى المصباح الذي انار دربي ولمن احمل اسمه بكل افتخار طاب بك العمر يا سيد الرجال و طبت لي عمرا الى من

كلله الله بالهيبة والوقار أرجو من الله ان يمد في عمرك لترى ثمارا قد حان قطفها واهديك هذا الإنجاز فخرا لك ..والدي العزيز .

الى اخواتي

الغاليات كبرت بين ضحكاتكن ونضجت على حبكن انتن اماني الذي لا يتغير انتن الفوضى الجميلة في ايامي و الجدال الذي احبه و الدعم الذي ما يصف نجاحي ما يسوى كثيرا لو ما انتم اخواتي نجاحي اليوم لا يكتمل الا بكن شكرا لانكن كنتن الأمان و السند والضحكة التي لا تنتهي .

الى اخواني

الى من جمعنا معهم بيت واحد و كانوا خير سند الى ضلع الثابت الذي لا يميل الى من امنوا بي وجودكم الذي منحني القوة رغم كل الهوشات انتم الطمأنينة في وسط العاصفة والصبر في لحظات التحدي شكرا لكم من الأعماق على وجود لا يعوض وعلى حب لا يشترى .

الى عائلتي الثانية

الى من سيشاركني الحياة يوما كنت دائما الدعم الذي لا يغيب والذي استشعره قبل ان يقال والروح التي تمنحني دفعة أمل

الى من أراه سندا ورفيق درب الى من يؤمن بي شكرا لك على الثقة ووجودك بجانبي حفظك الله اهديك هذا البحث حروف كتبت بالإصرار وصفحات من نور الامل .

الى امي الثانية

الى من لم تنجبني ولكنها ولدت في قلبي حبا لا يموت واحتوتني كانني قطعة منها شكرا لك على وجودك وعلى دعمك وتشجيعك ادع الله ان يحفظك ويبارك في عمرك اهديك هذا العمل المتواضع .

الى براعم المستقبل والكتاكيت الصغار انتم زهرات الحياة وامل الغد الجميل والى أصهارنا الكرام انتم لستم فقط اهلا بالمصاهرة بل اهل

بالمحبة والاحترام بكم اكتملت العائلة اهديكم هذا العمل دمتم لنا اخوة في القلب وسندا في الحياة .

Contents

Abstract ii
Acknowledgments iii
Notations abbreviations iv
List of scientific activities vi
Introduction vi

1 Inverse problems and control problems **1**
1.1 Inverse problems 1
1.1.1 Inverse Problems Examples 3
1.1.2 Well-posed and ill-posed problems 6
1.1.3 Ill-posed Problems Examples 6
1.2 Control Problems 7
1.2.1 Control Problem Examples 9

2 Observability inverse problem **11**
2.1 Observability problem 11
2.2 Exact observability for the plate equation 14
2.2.1 Exact observability for the plate equation 15

3 Inverse problem of an SIS epidemic, reaction-diffusion model by optimal control method **18**
3.1 Statement of the problem 19
3.2 Revaluation of **SIS** inverse problem into an optimal control problem 20
3.3 Existence and Necessary Conditions for a Minimizer's Optimality 21

Appendices 34
Bibliography 35

Abstract

In this memory, we have made a detailed study of the inverse problem and in particular the observation problem and analyzed an epidemiological model of the type SIS using optimal control methods with the aim of identifying unknown coefficients such as transition and recovery coefficients .

The inverse problem is transformed into an optimal control problem in which a cost function is defined that minimizes the difference between the model results and the observed data at the final time and is often represented by differential or integral equations, where the results of the theory such as existence and necessary conditions for optimality are proved .

Key-words : inverse problem; control problem; observability; optimal control .

ملخص

لقد قمنا في هذه المذكرة، بدراسة مفصلة للمشكلة العكسية و بشكل خاص مشكلة الملاحظة وتحليل نموذج وبائي من نوع SIS باستخدام طرق التحكم الأمثل بهدف تحديد المعاملات المجهولة مثل معاملات الانتقال والشفاء .
حيث يتم تحويل المشكلة العكسية إلى مشكلة التحكم الأمثل و يتم فيها تعريف دالة تكلفة تقلل الفرق بين النتائج النموذج والبيانات الملاحظة في الوقت النهائي وغالباً ما يتم تمثيلها بمعادلات تفاضلية أو تكاملية، حيث يتم إثبات نتائج النظرية مثل الوجود والشروط الضرورية لتحقيق الأمثلية.

الكلمات الرئيسية : مشكلة العكسية؛ مشكلة التحكم؛ إمكانية الملاحظة؛ التحكم الأمثل .

Acknowledgments

Praise be to Allah until the praise reaches its end, through whose grace all blessings are bestowed, and peace and blessings be upon the most honorable First of all, I thank God for making me live to see this message through this message. In this place, I can only say that in the sanctuary of science, one stands certain that every achievement has beacons and every milestone step has someone who holds his hand at the critical moment and gives him light when the darkness intensifies, and among these noble people is a scientific figure and an unparalleled human value, **Dr. Abdelhak Hafdadallah**, who has my full thanks and appreciation, and who did not spare me his knowledge and experience, and was the best guide and support for me during the preparation period of this research and his continuous encouragement that had a great impact in achieving this memorial.

I would also like to express my gratitude and thanks to **Dr. Khaled Berrah** for chairing the discussion committee .

I would also like to thank **Dr. Laadjel Amirouche** for agreeing to be part of the discussion Committee as an examiner and to take a look at my work .

I would like to thank, **Prof. Elhadj Zeraoulia** for his valuable advice and guidance, he is one of the pillars of the university for his .

I also extend my sincere thanks and giving to my parents who made sur to provide the appropriate conditions for this work .

And I would also like to thank every hand that accompanied us in this work, whether from near or far .

Notations & abbreviations

$\mathbb{R}^n$	A space composed of all vectors with n real numbers .
$\mathbb{R}_+$	Set of positive real numbers .
$\|.\|_H$	A norm in Hilbert space H .
$(.,.)_H$	A scalar product in Hilbert space H .
$\operatorname{div} f = \nabla.f = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$	Divergence .
C^2	The class of functions with continuous first and second derivative .
$\frac{\partial y}{\partial t} = \nabla y.t$	The conormal derivative .
$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$	The laplacien operator .
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$	The gradient operator .
$M_n(\mathbb{R})$	The space of square matrices of order in$\mathbb{R}$.
A	The adjoint operator of A .
$A = A^*$	Self-adjoint operator.
$\mathcal{L}(X, Y)$	The space of linear bounded operators from X to$Y$.
$D(A)$	The domain of definition of the operator A .
$L^2(\Omega)$	The space of measurable functions of summable squares in Ω .
$L^2(0, T; X)$	The bounded linear operator space .
$\ker A$	The kernel of the operator A .
iff	If and only if .
SIS	Susceptible-Infectious-Susceptible .
IPP	Inverse problem Parameter .

List of Figures

Figures N°	Title	Page
1.1	Direct Problem and Inverse Problem .	2
1.2	Direct Problem and Inverse Problem .	3
1.3	Forwards and Backwards Heat Equation .	4
1.4	Inverse Problem Modeling .	4
1.5	Inverse Stefan Problem For Ice Heating .	5
1.6	A Control Problem .	7
2.1	Observed Inverse Problem .	15
3.1	Diagram of The **SIS** Model .	18
3.2	Inverse Problem for The Model .	20

Introduction

Inverse problems arise when the objective is to ascertain specific parameter values that govern a phenomenon, and this determination is made by observing the effects produced by that phenomenon. The realm of inverse problems is broad and encompasses numerous scientific disciplines, including but not limited to medical imaging techniques, radar technology, quantum mechanics, and various methods of image processing . Furthermore, the study of inverse problems extends to issues encountered in remote sensing applications, where the goal is to estimate the properties of a surface or object based on measurements taken at a distance .

A significant challenge associated with inverse problems is their inherent difficulty in finding solutions . This difficulty often stems from the fact that these problems are frequently not posed correctly from a mathematical or computational standpoint . This incorrect formulation can lead to instability and inaccuracies in the solutions obtained .

In the context of inverse problems, we will delve into the definitions of well-posed and ill-posed problems . These definitions will be based on the criteria established by Hadamard, which are centered on three fundamental aspects of a solution : its existence, its uniqueness, and its stability . A problem is considered well-posed if a solution exists, that solution is unique, and the solution is stable with respect to small changes in the input data . Conversely, if any of these three conditions are not met, the problem is classified as ill-posed, presenting additional complexities in finding reliable solutions .

Mathematical control theory stands as a subject of immense importance and profound influence, impacting numerous fields and shaping various scientific endeavors . Its significance stems from its capacity to provide a framework for understanding and manipulating the behavior of dynamic systems, making it an indispensable tool across a spectrum of disciplines . This widespread applicability renders it a valuable asset not only in economics, where it informs models of market behavior and policy interventions, but also in many other scientific disciplines that grapple with the control and optimization of complex processes .

At its core, the general control problem addresses the fundamental ability to exert influence over the state of a system, guiding its trajectory through a series of actions and inputs . More specifically, this field of study delves into the question of whether it is indeed possible, given the inherent constraints and underlying dynamics that govern a particular system, to navigate from a defined starting point, commonly referred to as the initial state, to a meticulously predetermined and highly desirable goal, which we formally define as a specific objective . The pursuit of control therefore centers on the feasibility of steering a system towards a target state . Furthermore, the concept of control inherently introduces the crucial element of time, thereby adding another

layer of complexity to the problem. This temporal dimension directly concerns whether the aforementioned transition from the initial state to the desired objective can be successfully achieved within a predetermined time frame, a period which is typically characterized by a fixed initial time and a defined duration . Therefore, mathematical control theory essentially investigates the possibilities and limitations of guiding a given system from any arbitrary initial configuration to a very specific and pre-defined target state, all while adhering to a specified and constrained period of time . This investigation seeks to provide a rigorous understanding of how systems can be effectively manipulated to achieve desired outcomes within practical temporal constraints .

Relation : how to applying control methods to solve inverse problem

every inverse problem can be transformed into on optimal control

This memoir is consists into three chapters :

In the first chapter, we will first provide a reminder of some of the concepts and definitions that are necessary for the inverse problem, as well as examples . Second, we will cover some concepts and definitions that are necessary for the control problem, as well as examples .

In Chapter two, we will discuss the concepts and definitions of observability, as well as some of the theories we will use, and we will take an application to the observability of a plate equation with a potential .

The last chapter will be devoted to how to apply optimal control methods to solving inverse problem where we will restructure the model SIS as an inverse problem to determine the transition and recovery coefficients .

Chapter 1

Inverse problems and control problems

The aim of this chapter is to carefully define some basic concepts related to the inverse problem furthermore, we will carefully introduce a series of necessary definitions, and proceed to illustrate the concepts through a selection of examples, we will then shift our focus towards defining the control problem and illustrating it through examples, which contributes to the completeness of this memory .

1.1 Inverse problems

Inverse Problem is a vibrant and expanding branch of mathematics that has found numerous applications . In this section, we will define some basic notions of the inverse problem .

Definition 1.1 *(**Direct problem**) : Direct Problem relates to the equation :*

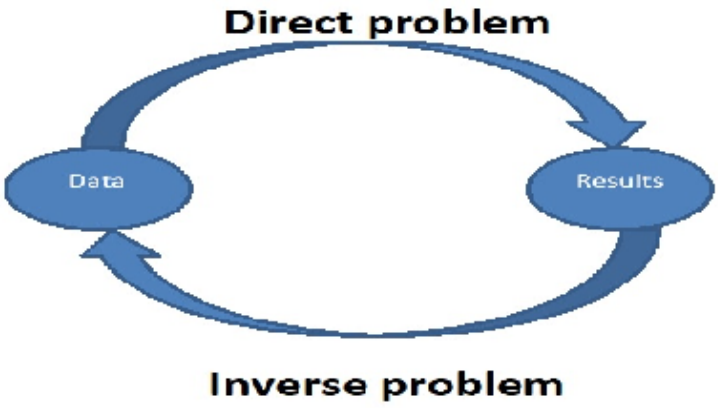
$$y = A\,(x)$$

Such that $A : E \longrightarrow F$ an application and E , F be two linear spaces
where :
y : is the unknown term.
x : is given data.

Definition 1.2 *(**Linear inverse problem**) : Let E , F be two linear spaces and $A : E \longrightarrow F$ is a operator linear, so the linear inverse problems can be represent such that an equation of the form :*

$$y = A(x)$$

where :
y : is the measured quantities or given data .
x : is the unknown term or unknown parameters or inputs .



$FIGURE$ 1.1 – Direct Problem and Inverse Problem .

Definition 1.3 ***(Nonlinear inverse problem)*** *: For nonlinear inverse problems, the link between observations and model parameters is more intricate . So that* $G : E \longrightarrow F$, E , F *is two linear spaces this relationship can then be expressed as :*

$$y = G(x)$$

where :
G : is operator non-linear .
General Definition :
In another words inverse problem it is the process of making inferences about the causes from knowing of the results .
An inverse problem in mathematics can be stated in the following manner :

$$y = A\,(x)$$

where :
y : is the data that got measure .
x : is unknown parameters .
A : is operator linear or non-linear .
Inverse problems are ill-posed problems in general .

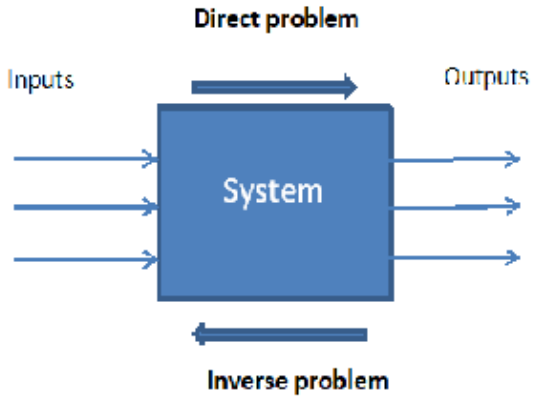


FIGURE 1.2 – Direct Problem and Inverse Problem .

1.1.1 Inverse Problems Examples

Example 1.1 : *Find a polynomial p of degree n with zeros $x_1, ..., x_n$. This is the inverse problem to the direct problem : Given a polynomial p find its zeros $x_1, ..., x_n$. In this example the inverse problem is easier to solve . The solution is $p(x) = c(x - x_1)...(x - x_n)$ with an arbitrary constant c .*

Example 1.2 ***(Backwards heat equation) :***

Consider the one-dimensional heat equation :

$$\begin{cases} \frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} & (x,t) \in (0,\pi) \times \mathbb{R} > 0 \ . \\ z(0,t) = z(\pi,t) = 0 & t \geq 0 \ . \\ z(x,0) = z_0(x) & 0 \leq x \leq \pi \ . \end{cases}$$

The separation of variables leads to the (formal) solution :

$$z(x,t) = \sum_{n=1}^{\infty} b_n \exp(-n^2 t) \sin(nx) \quad \text{with} \quad b_n = \frac{2}{\pi} \int_0^{\pi} u_0(y) \sin(ny)\, dy$$

The direct problem is to solve for $z(.,T)$ given initial temperature z_0 and final time T. In the inverse problem, we use $z(.,T)$ to find earlier temperatures like $z(.,0)$, requiring us to derive z_0 from the integral equation :

$$z(x,T) = \frac{2}{\pi} \int_0^{\pi} k(x,y) z_0(y) dy \quad , \quad 0 \leq x \leq \pi \quad ,$$

where :

$$k(x,y) := \sum_{n=1}^{\infty} \exp\left(-n^2 T\right) \sin\left(nx\right) \sin\left(ny\right) \ .$$

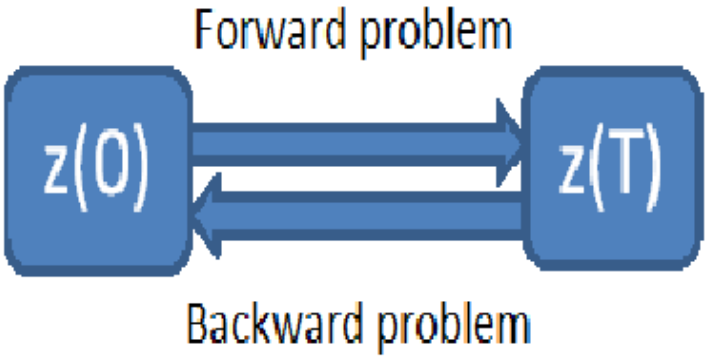


FIGURE 1.3 – Forwards and Backwards Heat Equation .

Example 1.3 ***(Diffusion in an inhomogeneous medium) :***

The equation of diffusion in an inhomogeneous medium (now in two dimensions)
is described by the equation :

$$\frac{\partial y(x,t)}{\partial t} = \frac{1}{c} \operatorname{div}\left(\gamma(x) \nabla y(x,t)\right) \quad , \quad x \in \Omega \quad , \quad t > 0 \ .$$

where c is a constant and $\gamma = \gamma(x)$ is a parameter describing the medium .
In the stationary case, this reduces to :
$\operatorname{div}(\gamma \nabla y) = 0$ in Ω , $\Omega \subset \mathbb{R}^2$ such that $\mathbb{R}^2$ is Bounded linear space .
The direct problem is find a solution y on $\partial\Omega$ with given values and a function γ .
The inverse problem is take a value of y and flux $\gamma \partial y / \partial v$ on $\partial\Omega$ to find the unknown γ in Ω .

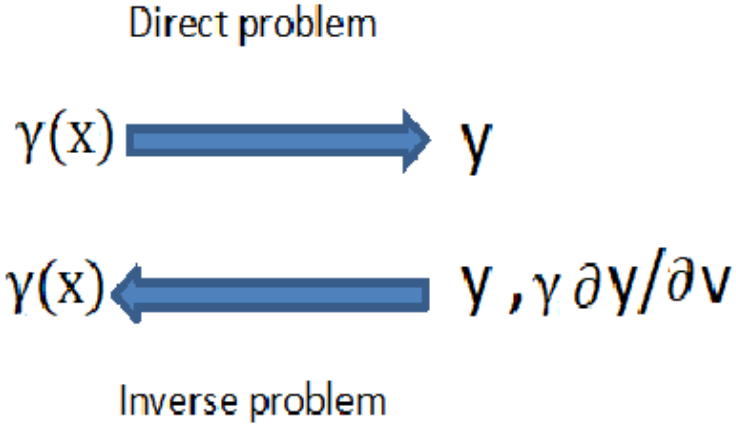


FIGURE 1.4 – Inverse Problem Modeling .

Example 1.4 ***(Inverse Stefan problem) :***

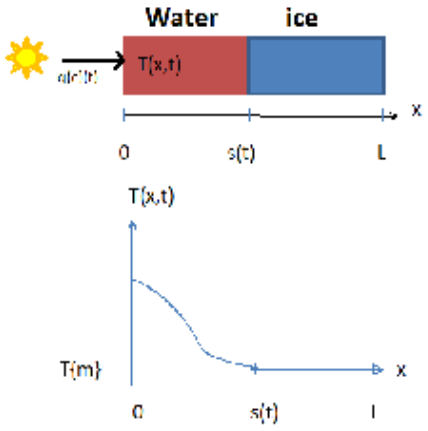
The physicist Stefan modeled summer melting of arctic ice using a one-dimensional approach . At time $t = 0$, a block of ice occupies the region $x \geq \ell$. Heating the left end causes melting , resulting in water between $x = 0$ and $x = s(t)$ for $s(t) > 0$, while $x \geq s(t)$ remains ice .
Let $u(x,t)$ be the temperature for $0 < x < s(t)$ at time t , satisfying the one-dimensional heat equation .

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{in} \quad D := \{(x,t) \in \mathbb{R}^2 : 0 < x < s(t), t > 0\} .$$

Subject to boundary conditions $\frac{\partial}{\partial x}u(0,t) = f(t)$ and $u(s(t),t) = 0$ for $t \in [0,T]$
and initial condition $u(x,0) = u_0(x)$ for $0 \leq x \leq \ell$.
u_0 represents the initial temperature and $f(t)$ the heat flux at $x = 0$. The interface's movement between water and ice is proportional to the heat flux, as described by the Stefan condition :

$$\frac{ds(t)}{dt} = -\frac{\partial u(s(t),t)}{\partial x} \quad \text{for } t \in [0,T] .$$

The direct problem computes the curve s from boundary data f and u_0 , while the inverse problem reconstructs u and f from a curve s . We express the direct problem as evaluating an operator K on a known model x in space X , and the inverse problem as solving $K(x) = y$. Direct problem : evaluate $K(x)$ given x . Inverse problem : solve $K(x) = y$ given y .
To define an inverse problem, we must specify the operator K , including its domain and range which helps identify linear and nonlinear issues . Evaluating $K(x)$ usually involves solving a boundary value problem or an integral .



$FIGURE\ 1.5 -$ Inverse Stefan Problem For Ice Heating .

1.1.2 Well-posed and ill-posed problems

The mathematical concept of a well-posed problem comes from a definition by Jacques Salomon Hadamard $[1865 - 1963]$ (French mathematician) who formulated that mathematical models of physical phenomena should, have Dynamic properties .
1. A solution exists .
2. The solution is unique .
3. Continuous dependence on imposed boundary conditions (stability) .

Definition 1.4 *(**Well-posedness**) : Let X and Y be normed spaces, and $T : X \to Y$ a linear operator . The equation $Tx = y$ is called properly posed or well-posed if the following holds :*

1. **Existence** : For all $y \in Y$, there is (at least one) $x \in X$ such that $Tx = y$.
2. **Uniqueness** : For all $y \in Y$, there is at most one $x \in X$ with $Tx = y$.
3. **Stability** : The solution x depends continuously on y , that is, for every sequence (x_n) in X with $Tx_n \to Tx (n \to \infty)$, it follows that $x_n \to x\ (n \to \infty)$.

Definition 1.5 *(**Ill-posed problem**) : If at least one of the properties (Existence , Uniqueness and Stability) is not verified, the problem is said to be ill-posed .*

1.1.3 Ill-posed Problems Examples

Example 1.5 *Let $A \in M_{n,n}(\mathbb{R})$ an invertible matrix and $b \in \mathbb{R}^n$ a column vector defined as follows :*

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & 3 \\ 0 & 5 & 5 \end{pmatrix} \quad , b = \begin{pmatrix} 3 \\ -6 \\ 5 \end{pmatrix}$$

The system $Ax = b$ has no solution $\det(A) = 0$ then the first condition is not satisfied, so the problem is ill-posed .

Example 1.6 *(**Polynomial problem**) :*

The number of real roots of the polynomial $p(x) = -x^2 + 2x - m$ varies discontinuously when m varies continuously on $\mathbb{R}$, $\Delta = 4 - 4m$ In fact, there are two real roots if $m < 1\ (\Delta > 0)$ a double root if $m = 1(\Delta = 0)$, and no root if $m > 1(\Delta < 0)$ so the problem is ill-posed .

1.2 Control Problems

Control theory is primarily concerned with ensuring the stability of systems against disturbances, but also includes optimizing these systems according to standards and establishing finite relationships .
In this section, we will provide a precise definition of control and then illustrate this definition with examples .

Definition 1.6 *(**control problem**): A control problem is a dynamic problem whose behavior can be modified over time we act on such a problem by means of a control, a general nonlinear problem can be defined by the equation :*

$$\begin{cases} \frac{dx}{dt}(t) = f(x(t), u(t), t) \ , \quad t \in [0, T] \\ x(0) = x_0 \in D(f) \end{cases}$$

where :
$x(t)$: is system state belongs to $\mathbb{R}^n$.
$u(t)$: is control input to $\mathbb{R}^p$.
x_0 : is a given initial state .
$D(f)$: The domain of definition of the function f . ■
In the infinite dimensional setting a control problem by a problem of a partial differential equations .

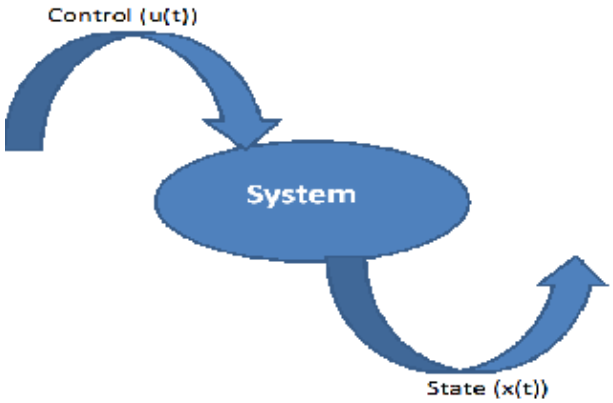


FIGURE 1.6 – A Control Problem

Definition 1.7 ***(Controllability problem)*** *:*

Consider the following system with infinite dimensional :

$$\begin{cases} \frac{\partial x}{\partial t}(t) = Ax(t) + Bu(t)\ ,\ t \in [0,T] \\ x(0) = x_0 \in D(A) \end{cases} \tag{1.1}$$

where :
$(A, D(A))$ is the infinitesimal generator of C_0 semi-group $\{S(t)\}_{t\geq 0}$ in a Hilbert space X .
X : is the Hilbert space .
U : is the control space .
B : is an operator in $\mathcal{L}(U, X)$.
x_0 : is the initial data .
The function $x(t) \in X$, we assume that $u \in L^2(0,T,U)$ is the control function .
we know that the solution of (1.1) is :

$$x(t) = S(t)\, x_0 + \int_0^t S(t-s)\, Bu(s)\, ds \ , \quad t \in (0,T) \ .$$

then :

$$x(T) = S(T)\, x_0 + \int_0^T S(T-t)\, Bu(t)\, dt \ .$$

the following operator is introduced :

$$\mathcal{L}_T : L^2(0,T;U) \longrightarrow X \quad \text{with} \quad \mathcal{L}_T u = \int_0^T S(T-t)\, Bu(t)\, dt \ .$$

Definition 1.8 ***(Exact controllability) :***

The system $(1.1)$is said to be Exactly controllable in H on $[0,T]$ if :

$$\forall x_d \in H, \exists u(.) \in L^2(0,T;U) \text{ such that the solution of } (1.1) \text{ satisfies : } x(T) = x_d$$

Definition 1.9 ***(Weak controllability) :***

The system $(1.1)$is said to be weakly controllable or (approximately controllable) in H on $[0,T]$ if :
$\forall \varepsilon > 0\ , \forall x_0 \in H$, $\exists u(.) \in L^2(0,T;U)$ such that the solution of (1.1) satisfies :

$$\| x(T) - x_d \|_H < \varepsilon$$

Definition 1.10 ***(Optimal control problem) :***

Let'sY and U be Hilbert space and let's consider the following optimal control problem infinite dimension written in the abstract :

$$\inf J(u, y)$$

where the constant :

$$Ay = Bu$$

where :
$u \in \mathcal{U}_{ad} \subset U$, $\mathcal{U}_{ad}$ is the set of admissible controls is closed convex non empty set of the control space U .
Y : is the state space .
A : is a partial differential operator on y .
$B \in \mathcal{L}(U, Y)$: is the control operator .
J : is the cost function strictly convex function from $U \times Y$ to $\mathbb{R}_+ \cup \{+\infty\}$.

1.2.1 Control Problem Examples

Example 1.7 ***(Cruise control)*** *:*

The goal cruise control is to regulate the throttle to maintain a desired despite external disturbances like inclines declines, and wind resistance .
input (control variable) : engine power .
output (controlled variable) : vehicle speed .
disturbance : road slope,wind resistance and load changes .

Example 1.8 ***(Optimal control of blood glucose levels)*** *:*

consider a model for blood glucose level, $x(t)$ (mg) at time t (s) . Without intervention, $x(t)$ decreases proportionally to its current value :

$$\dot{x}(t) = -\alpha x(t)$$

where :
α : is a given strictly positive constant .
glucose level $x(0) = a$ (mg) , target $x(t_f) = c \neq a$. Transfusion rate $u(t)$ mg/s (control variable) subject to constraint :

$$0 \leq u(t) \leq m \quad , \ t \in [0, t_f] \ .$$

where :

$m \geq 0$: is the maximum penetration speed , the evolution of the state $x(t)$ therefore follows the differential equation :

$$\dot{x}(t) = -\alpha x(t) + u(t) \quad , \quad t \in [0, t_f] \ .$$

minimize total glucose transfused :

$$J(x(.), u(.), t_f) = \int_0^{t_f} u(s)\, ds \ .$$

Given unknown $t_f \gneqq 0$, so the problem can be formalized as follows :

$$\begin{cases} mim_{t_f} \ \ J(x(.), u(.), t_f) \ , \\ \dot{x}(t) = -\alpha x(t) + u(t) \ , t \in [0, t_f] \ , \\ u(t) \in [0, m] \quad , \ \ t \in [0, t_f] \ , \\ x(0) = a \ , x(t_f) = c \ . \end{cases}$$

Chapter 2

Observability inverse problem

Observation is a fundamental property of control system analysis that describes whether the internal state of the system can be fully characterized based solely on outputs over time .

2.1 Observability problem

Definition 2.1 *(**Observability problem**) :* *Let H and Z be two Hilbert spaces and consider the following observed system :*

$$\begin{cases} \frac{\partial x}{\partial t} = Ax \\ x(0) = x_0 \\ y = Cx \end{cases} \tag{2.1}$$

where :
Z : is the observability space .
A : is operator linear .
$C \in \mathcal{L}(H, Z)$: is the observation operator .
x_0 : is value not given .
The solution of (2.1) is :

$$y(t) = CS(t)x_0 = kx_0$$

Observability is an inverse source problem .

Definition 2.2 *(**Exact observability**) : The pair (A, C) is said to be exactly observability on $(0, T)$ if y_0 can be uniquely and continuously determined knowledge of the output $y \in L^2(0, T; Z)$ i.e. the operator $K : H \longrightarrow L^2(0, T; Z)$ with $Ky = CS(T)y$.*

Definition 2.3 ***(Weak observability) :*** *The pair* (A, C)*is weakly observable on* $[0, T]$ *if* y_0 *can be uniquely and continuously determined and continuous form knowledge of the output of* $Y \in L^2(0, T; Z)$ *i.e.* $\ker y = 0 \Longrightarrow y = 0$ *on* $(0, T)$.

Proposition 2.1 ***(A observability inequality)*** *: the pair* (A, B) *is exactly controllable on* $[0, T]$ *if*

$$\exists c > 0 \forall x \in Z : \int_0^T \| B^* S^*(t) x \|_v^2 \, dt > c \| x \|_z^2 .$$

Proposition 2.2 *: The system* (2.1) *is said to be exactly observable on* Z *if and only*

if : there exists $c > 0$ such that :

$$\forall x \in Z : \int_0^T \| B^* S^*(t) x \|_v^2 \, dt > c \| x \|_z^2 .$$

Theorem 2.1 *:*

(A, C) is Exact observability on $(0, T)$ iff (A^*, C^*) is Exact controllability on $(0, T)$.

Proof. A generates a c_0 semi grope a H called $\{S(t)\}_{t \geq 0}$ then A^*generates a $\{S(t)\}_{t \geq 0}$.

$$\begin{aligned}
(Kx \, , z)_{L^2(0,T,Z)} &= (CS(t) x \, , z)_{L^2(0,T,Z)} \\
&= \int_0^T (CS(t) x \, , z(t))_Z \, dt \\
&= \int_0^T (x \, , S^*(t) C^* z(t))_H \, dt \\
&= (x \, , K^* z(t))_H \qquad , \forall x \in H
\end{aligned}$$

$$\Longrightarrow \quad K^* z = \int_0^T S^*(t) C^* z(t) dt \Longrightarrow K^* = \mathcal{L}_T \quad \text{and} \quad \mathcal{L}_T^* = K^*$$

The controllability of $(A^* \, , C^*)$ is related to :

$$\mathcal{L}_T u = \int_0^T S^*(t) C^* u(t) dt$$

■

Theorem 2.2 *:*

$(A \, , C)$ is Weak observability on $(0 \, , T)$ iff (A^*, C^*) is Weak controllability on $(0 \, , T)$.

Corollary 2.1 *: for the system* (2.1) *we have the following conditions let's define :*

$$N = kk^*$$

1) $(A\ ,C)$ is Exact observability if one of following conditions holds :
i) $\exists\gamma>0$, $\forall x\in H$, $\langle Nx,x\rangle_H\geq\gamma\|x\|_H^2$.
ii) $\exists\gamma>0$, $\forall x\in H$, $\|Kx\|^2_{L^2(0,T,Z)}=\int_0^T\|CS\left(t\right)x\|_H^2dt\geq\gamma\|x\|_H^2$ (**A observability inequality**) .
iii) $\exists\gamma>0$, $\forall x\in H$, $\int_0^T\|CS\left(t\right)x\|_H^2dt\geq\gamma\|x\|_H^2$.
iv) $\ker K=\{0\}$ on $\operatorname{Im}K$ is closed .
2) $(A\ ,C)$ is Weak observability if one of following conditions holds :
i) N is positive operator .
ii) $\ker K=\{0\}$.
iii) $CS\left(t\right)x=0\Longrightarrow x=0$

Example 2.1 ***(Approximate observability of a one-dimensional heat equation) :***

consider the observability problem of the following one-dimensional heat equation on :

$$(HO)\begin{cases}\frac{\partial y}{\partial t}=\frac{\partial^2 y}{\partial^2 x} & \text{in }]0,1[\times]0,T[\\ y\left(0,t\right)=y\left(1,t\right)=0 & \text{in }]0,T[\\ y\left(x,0\right)=y_0\left(x\right) & \text{in }]0,1[\\ \theta\left(t\right)=\int_0^1 b\left(x\right)y\left(x,t\right)dx\end{cases}$$

where :
$b,y_0\in L^2\left(0,1\right)$ and $T>0$.

the observation space $Z=L^2\left(0,T\right)$ and the state space $Y=L^2\left(0,1\right)$, i.e. the operator :

$$C:L^2\left(0,1\right)\longrightarrow L^2\left(0,T\right)\ \text{ with }\ Cy=\int_0^1 b\left(x\right)y\left(x,t\right)dx\in L^2\left(0,T\right)\ .$$

Find an adjoint C :

$$\begin{aligned}C^*\ &:\ L^2\left(0,T\right)\longrightarrow L^2\left(0,T,L^2\left(0,1\right)\right)\\ \left(Cy,z\right)_{L^2(0,T)}&=\int_0^T Cy\left(t\right)z\left(t\right)dt\\ &=\int_0^T\int_0^1 b\left(x\right)y\left(x,t\right)z\left(t\right)dxdt\\ &=\int_0^T\int_0^1 y\left(x,t\right)b\left(x\right)z\left(t\right)dxdt\\ &=\left(y,b\left(x\right)z\right)_{L^2\left(0,T,L^2(0,1)\right)}\\ &=\left(y,C^*z\right)_{L^2\left(0,T,L^2(0,1)\right)}\end{aligned}$$

So $C^* = b(x)$.

Operator definition A :

$$A : D(A) \subset L^2(0,1) \longrightarrow L^2(0,1) \quad \text{with} \quad Af = \frac{\partial^2 f}{\partial x^2}.$$

Find an adjoint A :

$$\begin{aligned} (Af, g)_{L^2(0,1)} &= \int_0^1 \frac{\partial^2 f}{\partial x^2} g dx \\ &= \int_0^1 f \frac{\partial^2 g}{\partial x} dx \\ &= (f, A^* g)_{L^2(0,1)} \end{aligned}$$

So $A = A^*$
According to the theory (2.2) :
so (HO) is equivalent to (HC)

$$\begin{cases} \frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial^2 x} + b(x)\, u(t) & \text{in }]0,1[\times]0,T[\\ z(0,t) = z(1,t) = 0 & \text{in }]0,T[\\ z(x,0) = z_0(x) & \text{in }]0,1[\end{cases}$$

2.2 Exact observability for the plate equation

The Plate Equation with a potential is a partial differential equation that describes the deformation of an elastic plate under the action of external forces, such as pressure or stress.
It is mathematically described as follows :

$$y_{tt} + \Delta^2 y + p(x)\, y = 0$$

$y(x,t)$: is the displacement of a point on the plate at point x and time t .
y_{tt} : The second derivative of displacement (acceleration) .
$\Delta^2 y$: is laplacien square .
$p(x)$: is potential is a function describing the effect of an external force .
We will study this equation based on [1] .

2.2.1 Exact observability for the plate equation

consider the exact observability of the problem with potential :

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = (-A_0 - p)y \ , & \text{in} \quad (0,\tau)\times\Omega \ , \\ y = \Delta y = 0 \ , & \text{on} \quad (0,\tau)\times\partial\Omega \ , \\ y(0,x) = y_0(x) \ , \ \frac{\partial y}{\partial t}(0,x) = y_1(x) & x \in \Omega \ , \end{cases} \tag{2.2}$$

we observe $\frac{\partial \Delta y}{\partial v}$ on $\Gamma \subset \partial\Omega$ i.e.

$$z = Cy = \frac{\partial \Delta y}{\partial v} \text{on} \ \ \Gamma \subset \partial\Omega$$

where $\tau > 0$.

This equation has a solution in $C^0([0,\tau]\,; H^3(\Omega)) \cap C^1([0,\tau]\,; H^1(\Omega))$.

Let's consider once again the operator : $A_0 : \mathcal{D}(A_0) \subset H_0^1(\Omega) \longrightarrow H_0^1(\Omega)$

$$H = H_0^1(\Omega) \ , \mathcal{D}(A_0) = \left\{ y \in H^3(\Omega) \mid y = \Delta y = \Delta^2 y = 0 \ \text{ on } \ \partial\Omega \right\} \ \text{, with } A_0 y = \Delta^2 y$$

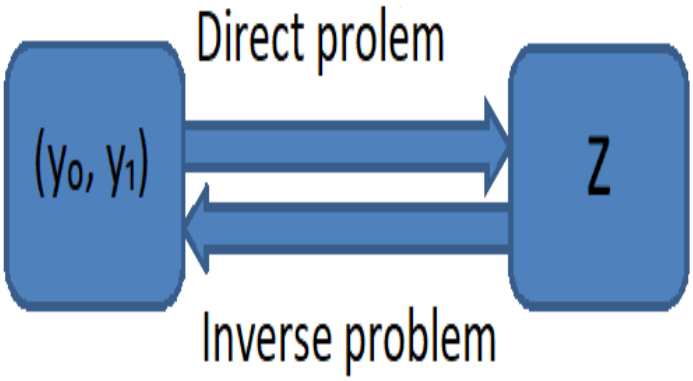


FIGURE 2.1 – Observed Inverse Problem .

According to the corollary (2.1) (1) (ii) :

$$\exists \gamma > 0, \forall x \in H, \|Kx\|^2_{L^2(0,T,Z)} \geq \gamma \|x\|^2_H.$$

$$\int_0^\tau \| \frac{\partial \Delta y}{\partial v} \|^2_{L^2(\Omega)} \, dt \geq k \left(\| \, y_0 \, \|^2_{H^3(\Omega)} + \| \, y_1 \, \|^2_{H^1(\Omega)} \right)$$

Equivalent to observability inequality :

$$\exists c > 0 \forall x \in Z : \int_0^T \| \, B^* S^*(t) \, x \, \|^2_v \, dt > c \, \| \, x \, \|^2_z$$

And then we have :

$$\mathcal{D}\left(A_0^{\frac{1}{2}}\right) = \left\{y \in H^3(\Omega) \mid y = \Delta y = 0 \text{ on } \partial\Omega\right\} \ ,$$

and if set :

$$X = L^2(\Gamma) \times L^2(\Gamma_1)$$

$$X = \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \times H \quad , \quad \mathcal{D}(A) = \mathcal{D}(A_0) \times \mathcal{D}\left(A_0^{\frac{1}{2}}\right) \ ,$$

Now we write equation (2.2) in the standard form $\frac{\partial Y}{\partial t} = (A+P)Y$, $Y(0) = (y_0, y_1)^t$.
From the system on suppose :

$$Y = \begin{pmatrix} y \\ \frac{\partial y}{\partial t} \end{pmatrix} \quad \Longrightarrow \frac{\partial Y}{\partial t} = \begin{pmatrix} \frac{\partial y}{\partial t} \\ \frac{\partial^2 y}{\partial t^2} \end{pmatrix} \ ,$$

and from it :

$$\begin{aligned} \begin{pmatrix} \frac{\partial y}{\partial t} \\ \frac{\partial^2 y}{\partial t^2} \end{pmatrix} &= \begin{pmatrix} \frac{\partial y}{\partial t} \\ -A_0 y - py \end{pmatrix} \\ &= \begin{pmatrix} \frac{\partial y}{\partial t} \\ -A_0 y \end{pmatrix} + \begin{pmatrix} 0 \\ -py \end{pmatrix} \\ &= \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix} \begin{pmatrix} y \\ \frac{\partial y}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -p & 0 \end{pmatrix} \begin{pmatrix} y \\ \frac{\partial y}{\partial t} \end{pmatrix} \end{aligned}$$

So standard form :

$$\frac{\partial Y}{\partial t} = (A+P)\,Y \quad , Y(0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} .$$

with $A = \begin{pmatrix} 0 & I \\ -A_0 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 0 \\ -p & 0 \end{pmatrix}$.
We also define the following operators :

$$p \in \mathcal{L}\left(\mathcal{D}\left(A_0^{\frac{1}{2}}\right), H\right) \ , p \in \mathcal{L}(x) \ ,$$

On suppose :

$$A_p = A + P \quad \text{with} \quad \mathcal{D}(A_p) = \mathcal{D}(A) \ .$$

From which we write a system :

$$\begin{cases} \frac{\partial Y}{\partial t} = A_p Y \\ Y(0) = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \\ z = CY = \begin{pmatrix} \frac{\partial y}{\partial v} \\ \frac{\partial \Delta y}{\partial v} \end{pmatrix} \end{cases}$$

According to the corollary (2.1) (1) (ii) :

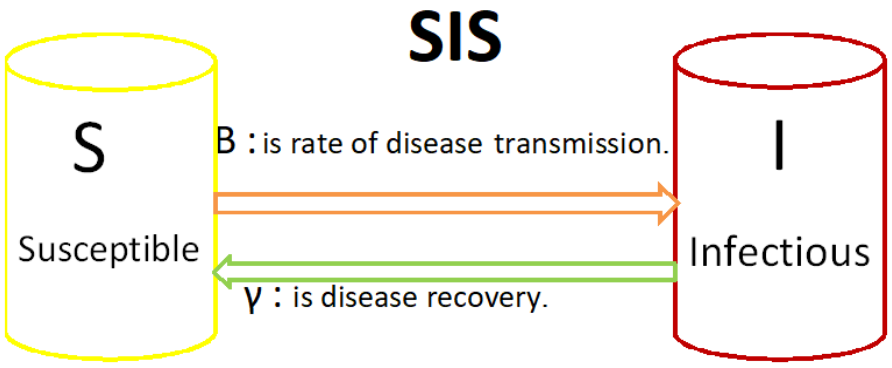
$$\int_0^\tau \left(\| \frac{\partial \Delta y}{\partial v} \|^2_{L^2(\Gamma)} + \| \frac{\partial y}{\partial v} \|^2_{L^2(\Gamma_1)} \right) dt \geq k \left(\| y_0 \|^2_{H^3(\Omega)} + \| y_1 \|^2_{H^1(\Omega)} \right)$$

Now to find k, we calculate the eigenvalues, specifically by taking the minimum eigenvalues and then doing some calculations, we find k .

Chapter 3

Inverse problem of an SIS epidemic, reaction-diffusion model by optimal control method

In mathematical modeling, especially in models of the spread of infectious diseases, (SIS) refers to a model "Susceptible-Infectious -Susceptible" . It is considered one of the basic models in mathematical epidemiology and is usually used for diseases that do not gain permanent immunity, such as some types of colds or some viruses that can come back after recovery .
Its basic formula is expressed by two differential equations that represent the change in the number of individuals in the cohort S and I over time based on transition and recovery rates β and γ .



$FIGURE$ 3.1 – Diagram of The **SIS** Model

We will study this model based on [15] .

3.1 Statement of the problem

Based on the diffusion equation system, the inverse problem in the epidemic reaction-diffusion model **SIS** is studied through the Inverse Problem Parameter (IPP) and solved using optimal control methods .

The **SIS** model is governed by the following nonlinear coupled reaction-diffusion equations with homogeneous Neumann boundary conditions :

$$\begin{cases} \frac{\partial y_1}{\partial t} - \frac{\partial^2 y_1}{\partial x^2} = -\beta(x)\frac{y_1 y_2}{y_1+y_2} + \gamma(x) y_2 , & (x,t) \in Q = (0,1)\times(0,T) , \\ \frac{\partial y_2}{\partial t} - \frac{\partial^2 y_2}{\partial t} = \beta(x)\frac{y_1 y_2}{y_1+y_2} - \gamma(x) y_2 , & (x,t) \in Q = (0,1)\times(0,T) , \\ \frac{\partial y_1}{\partial x}(0,t) = \frac{\partial y_2}{\partial x}(0,t) = 0 , & t \in (0,T) , \\ \frac{\partial y_1}{\partial x}(1,t) = \frac{\partial y_2}{\partial x}(1,t) = 0 , & t \in (0,T) . \end{cases} \tag{3.1}$$

where :

y_1 : is susceptible individuals .

y_2 : is infected individuals .

β : is rate of disease transmission .

γ : is disease recovery .

We take the control space $U = L^{\infty}(0,1)$.

Here we have an isolated spatial environment to comply with the no-flow conditions, which means that there is no entry and exit of exposed or infected individuals from the spatial environment (area isolation), i.e. no population flow across borders . This allows the exclusion of external influences .

$$y_1(x,0) = \varphi_1(x) \geq 0 \quad , \quad y_2(x,0) = \varphi_2(x) \geq 0 , \tag{3.2}$$

with :

$$\varphi_i(x) \in C^{2,\alpha}[0,1] \quad , i = 1,2 \ , \ \int_0^1 \varphi_2(x)\,dx > 0 \quad \text{and} \quad \varphi_1(x) + \varphi_2(x) \geq \varphi_0 > 0 ,$$

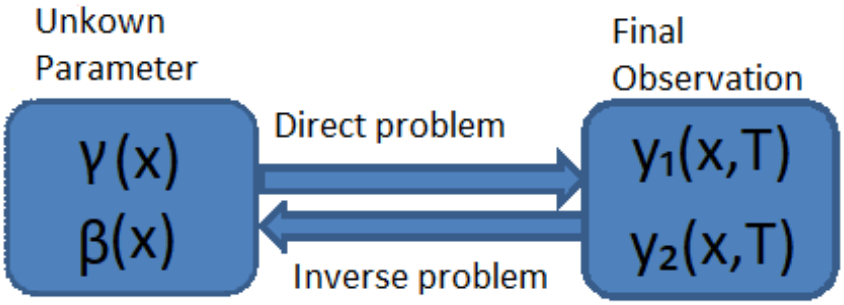
where :

$\varphi_0 > 0$, $\alpha \in (0,1)$ are all constants .

Suppose we are given additional measurements about the system, the number of susceptible individuals $y_1(x,t)$ at final moment $t = T$ and the number of infected individuals $y_2(x,t)$ at final moment $t = T$, known for all x in $(0,1)$ as shown below :

$$y_1(x,T) = f(x) \quad , \quad y_2(x,T) = g(x) \qquad \forall x \in (0,1) \ , \tag{3.3}$$

given functions $f(x)$ and $g(x)$ satisfying homogeneous Neumann boundary conditions (3.1)
As stated by **IPP**, a valuable and challenging problem involves identifying the unknown coefficients $\beta(x)$ and $\gamma(x)$ in equation (3.1) . This identification is achieved using system (3.1) , and (3.2) supplemented by data from equation (3.3) .
The objective of this study is to investigate the existence, uniqueness, and stability of the coefficients β and γ in the system of equations (3.1) using optimal control methods .



$FIGURE\ 3.2 -$ Inverse Problem for The Model .

3.2 Revaluation of SIS inverse problem into an optimal control problem

This section explains how we can turn the inverse problem into an optimal control problem given system (3.1) and (3.2) has a classical positive and unique solution .
Then , we can derive the results below , including the established Schauder's theory for parabolic equations .

Theorem 3.1 [15]: *Assume* $0 < \alpha < 1$ *.Then for any given coefficients* $\beta(x)$ *and* $\gamma(x) \in C^{\alpha}\left(\bar{I}\right)$,

$\varphi_i(x) \in C^{2,\alpha}\left(\bar{I}\right)$, $i = 1, 2$ and $\varphi_1(x) + \varphi_2(x) \geq \varphi_0 > 0$, system $(3.1) - (3.2)$ admits a unique positive classical solution, says $(y_1(x,t), y_2(x,t))$. Further, for any given constant $T \in (0, +\infty)$, $(y_1(x,t), y_2(x,t)) \in C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{Q}\right) \times C^{2+\alpha,1+\frac{\alpha}{2}}\left(\bar{Q}\right)$, and $y_1(x,t)$ and $y_2(x,t)$ are all bounded on $[0,T] \times \bar{I}$ where $I = (0,1)$.
In order to establish the corresponding control problem, we first define the admissible set by :

$$\mathcal{U}_{ad} = \left\{ \begin{array}{c} (\beta(x), \gamma(x)) : 0 < b_0 \leq \beta(x) \leq \bar{b}_0 < 1 \ , \ 0 < r_0 \leq \gamma(x) \leq \bar{r}_0 < 1 \ , \\ \forall x \in [0,1] \ , \ \nabla\beta(x), \nabla\gamma(x) \in L^2(I) \end{array} \right\}$$

Also for the additional conditions (3.3) , we assume that $f(x)$ and $g(x)$ satisfy the following conditions :

$$f(x) \in H^1(0,1) \ , \qquad g(x) \in H^1(0,1) \ .$$

Hence , we can transform the inverse problem (IPP) into the following optimal control problem : Finding $(\bar{\beta}(x), \bar{\gamma}(x)) \in \mathcal{U}_{ad}$ such that

$$J(\bar{\beta}(x), \bar{\gamma}(x)) = \inf_{(\beta,\gamma)\in\mathcal{U}_{ad}} J(\beta(x), \gamma(x)) \ . \tag{3.4}$$

where :

$$\begin{aligned} J(\beta,\gamma) \ &= \ \frac{1}{2}\int_0^1 \left[\mid y_1(x,T;\beta(x),\gamma(x)) - f(x) \mid^2 + \mid y_2(x,T;\beta(x),\gamma(x)) - g(x) \mid^2 \right] dx \\ &+ \frac{N}{2}\int_0^1 \left[\mid \nabla\beta(x) \mid^2 + \mid \nabla\gamma(x) \mid^2 \right] dx \ . \end{aligned} \tag{3.5}$$

Here (y_1, y_2) is the solution to system $(3.1) - (3.2)$ with $(\beta, \gamma) \in \mathcal{U}_{ad}$ given .
We choose the lower bound $(\bar{\beta}(x), \bar{\gamma}(x))$ in (3.4) is chosen to be the optimal definition of the coefficients $(\beta(x), \gamma(x))$ in (3.1) .

3.3 Existence and Necessary Conditions for a Minimizer's Optimality

This section focuses on proving the existence of a cost functional minimizer and deriving its optimality condition . First, we formally state the existence result as a foundation for our analysis . We begin by presenting this existence result .

Theorem 3.2 [15]*: Assume that* (y_1, y_2) *is the solution to system* $(3.1) - (3.2)$ *.Then there exists a minimizer* $(\beta, \gamma) \in \mathcal{U}_{ad}$ *such that :*

$$J(\bar{\beta}, \bar{\gamma}) = \inf_{(\beta,\gamma)\in\mathcal{U}_{ad}} J(\beta, \gamma) \ . \tag{3.6}$$

Proof. J coercive $\Longrightarrow$Exetance of minimizer

$$J(\beta,\gamma) \longrightarrow +\infty \quad \text{when} \quad \| \nabla\beta \|_{L^\infty(I)} \longrightarrow +\infty \quad \text{and} \quad \| \nabla\gamma \|_{L^\infty(I)} \longrightarrow +\infty$$

J is coercive so the existence of a minimizer of J :

$$J(\bar{\beta},\bar{\gamma}) = \inf_{(\beta,\gamma)\in\mathcal{U}_{ad}} J(\beta,\gamma)\ .$$

■

We now establish the necessary optimality condition of the minimizer to the cost functional (3.5) as follows :

Theorem 3.3 [15]: *Let $\left(\bar{\beta},\bar{\gamma}\right)$ be the solution of the optimal control problem (3.4) , $(\bar{y}_1,\bar{y}_2)$ be the solution to system (3.1)with (β,γ) replaced by $\left(\bar{\beta},\bar{\gamma}\right)$ and (p_1,p_2) be the solution to the adjoint system (3.7) . Then for any $\left(\hat{\beta},\hat{\gamma}\right)\in\mathcal{U}_{ad}$, it follows that :*

$$\int_Q \left[\left(\hat{\beta}-\bar{\beta}\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2} - \left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2\right](p_2-p_1)\,dxdt + N\int_I \left[\nabla\bar{\beta}\nabla\left(\hat{\beta}-\bar{\beta}\right)+\nabla\bar{\gamma}\nabla\left(\hat{\gamma}-\bar{\gamma}\right)\right]dx \geq 0\ . \tag{3.7}$$

Proof. For any $\left(\hat{\beta},\hat{\gamma}\right)$ and $\varepsilon\in[0,1]$, set

$$\begin{cases} \beta_\varepsilon = (1-\varepsilon)\,\bar{\beta}+\varepsilon\hat{\beta}\in\mathcal{U}_{ad} \\ \gamma_\varepsilon = (1-\varepsilon)\,\bar{\gamma}+\varepsilon\hat{\gamma}\in\mathcal{U}_{ad} \end{cases}$$

further let :

$$\begin{aligned} J(\beta_\varepsilon,\gamma_\varepsilon) &= \frac{1}{2}\int_I \left[\mid y_1^\varepsilon(x,T;\beta_\varepsilon,\gamma_\varepsilon)-f(x)\mid^2+\mid y_2(x,T;\beta_\varepsilon,\gamma_\varepsilon)-g(x)\mid^2\right]dx \\ &\quad +\frac{N}{2}\int_I\left[\mid\nabla\beta_\varepsilon\mid^2+\mid\nabla\gamma_\varepsilon\mid^2\right]dx \end{aligned}$$

$$\begin{aligned} J(\beta_\varepsilon,\gamma_\varepsilon) &= \frac{1}{2}\int_I\left[\mid y_1^\varepsilon-f(x)\mid^2+\mid y_2^\varepsilon-g(x)\mid^2\right]dx \\ &\quad +\frac{N}{2}\int_I\left[\mid\nabla\beta_\varepsilon\mid^2+\mid\nabla\gamma_\varepsilon\mid^2\right]dx\ , \\ &= \frac{1}{2}\parallel y_1^\varepsilon-f(x)\parallel^2_{L^2(I)}+\frac{1}{2}\parallel y_2^\varepsilon-g(x)\parallel^2_{L^2(I)}+\frac{N}{2}\parallel\nabla\beta_\varepsilon\parallel^2_{L^2(I)} \\ +\frac{N}{2} &\parallel \nabla\gamma_\varepsilon\parallel^2_{L^2(I)} \end{aligned}$$

where $(y_1^\varepsilon,y_2^\varepsilon)$ is the solution to system (3.1) , (3.2) with (β,γ) replaced by $(\beta_\varepsilon,\gamma_\varepsilon)$.
We take the derivative of J_ε to achieve that :

$$\begin{aligned} \frac{J_\varepsilon}{d\varepsilon}\mid_{\varepsilon=0} &= \int_I\left\{[y_1^\varepsilon-f(x)]\frac{\partial y_1^\varepsilon}{\partial\varepsilon}\mid_{\varepsilon=0}+[y_2^\varepsilon-g(x)]\frac{\partial y_2^\varepsilon}{\partial\varepsilon}\mid_{\varepsilon=0}\right\}dx \\ &\quad +N\int_I\left[\nabla\bar{\beta}\nabla(\hat{\beta}-\bar{\beta})+\nabla\bar{\gamma}\nabla(\hat{\gamma}-\bar{\gamma})\right]dx \end{aligned}$$

Since $(\bar{\beta},\bar{\gamma})$ is the optimal solution of the optimal control problem (3.4) one concludes that :

$$\frac{J_\varepsilon}{d\varepsilon}\mid_{\varepsilon=0}\geq 0$$

$$\partial J(\beta_\varepsilon,\bar{\gamma})\left(\hat{\beta}-\bar{\beta}\right)=\lim_{\varepsilon\longrightarrow 0}\frac{J(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma})-J(\bar{\beta},\bar{\gamma})}{\varepsilon}$$

$$\begin{aligned}
J(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma}) &= \frac{1}{2}\parallel y_1\left(x,T;\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma}\right)-f(x)\parallel^2_{L^2(I)}\\
&+\frac{1}{2} \parallel y_2\left(x,T;\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma}\right)-g(x)\parallel^2_{L^2(I)}\\
&+\frac{N}{2} \parallel \nabla(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right))\parallel^2_{L^2(I)}+\frac{N}{2}\parallel\nabla\bar{\gamma}\parallel^2_{L^2(I)}\\
&= \frac{1}{2}\parallel y_1(x,T;\bar{\beta},\bar{\gamma})+\varepsilon y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)-f(x)\parallel^2_{L^2(I)}\\
&+\frac{1}{2} \parallel y_2(x,T;\bar{\beta},\bar{\gamma})+\varepsilon y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)-g(x)\parallel^2_{L^2(I)}\\
&+\frac{N}{2} \parallel \nabla(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right))\parallel^2_{L^2(I)}+\frac{N}{2}\parallel\nabla\bar{\gamma}\parallel^2_{L^2(I)}
\end{aligned}$$

$\Longrightarrow$

$$\begin{aligned}
J(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma}) &= \frac{1}{2}\parallel y_1(x,T;\bar{\beta},\bar{\gamma})-f(x)\parallel^2_{L^2(I)}+\frac{1}{2}\varepsilon^2\parallel y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\parallel^2_{L^2(I)}\\
&+\varepsilon(y_1(x,T;\bar{\beta},\bar{\gamma})-f(x),y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right))_{L^2(I)}\\
&+\frac{1}{2} \parallel y_2(x,T;\bar{\beta},\bar{\gamma})-g(x)\parallel^2_{L^2(I)}+\frac{1}{2}\varepsilon^2\parallel y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\parallel^2_{L^2(I)}\\
&+\varepsilon\left(y_2(x,T;\bar{\beta},\bar{\gamma})-g(x),y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\right)_{L^2(I)}\\
&+\frac{N}{2} \parallel \nabla\bar{\gamma}\parallel^2_{L^2(I)}+\frac{N}{2}\parallel\nabla\bar{\beta}\parallel^2_{L^2(I)}+\frac{N}{2}\varepsilon^2\parallel\nabla(\hat{\beta}-\bar{\beta})\parallel^2_{L^2(I)}\\
&+N\varepsilon\left(\nabla\bar{\beta},\nabla(\hat{\beta}-\bar{\beta})\right)_{L^2(I)}
\end{aligned}$$

if

$$\begin{aligned}
\frac{J(\bar{\beta}+\varepsilon\left(\hat{\beta}-\bar{\beta}\right),\bar{\gamma})-J(\bar{\beta},\bar{\gamma})}{\varepsilon} &= \frac{1}{2}\varepsilon\parallel y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\parallel^2_{L^2(I)}+N\left(\nabla\bar{\beta},\nabla(\hat{\beta}-\bar{\beta})\right)_{L^2(I)}\\
&+\frac{1}{2}\varepsilon \parallel y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\parallel^2_{L^2(I)}+\frac{N}{2}\varepsilon\parallel\nabla(\hat{\beta}-\bar{\beta})\parallel^2\\
&+(y_1(x,T;\bar{\beta},\bar{\gamma})-f(x),y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right))_{L^2(I)}\\
&+\left(y_2(x,T;\bar{\beta},\bar{\gamma})-g(x),y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\right)_{L^2(I)}
\end{aligned}$$

Applying definition 2 : (See Appendices definition 2)

$$\begin{aligned}\Longrightarrow \quad \partial J(\beta_\varepsilon,\bar{\gamma})\left(\hat{\beta}-\bar{\beta}\right) &= \left(y_1\left(x,T;\bar{\beta},\bar{\gamma}\right)-f\left(x\right),y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\right)_{L^2(I)}\\ &+\left(y_2\left(x,T;\bar{\beta},\bar{\gamma}\right)-g\left(x\right),y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)\right)_{L^2(I)}+N\left(\nabla\bar{\beta},\nabla(\hat{\beta}-\bar{\beta})\right)_{L^2(I)}\end{aligned}$$

In the same way , we find

$$\begin{aligned}\partial J(\bar{\beta},\gamma_\varepsilon)\left(\hat{\gamma}-\bar{\gamma}\right) &= \left(y_1\left(x,T;\bar{\beta},\bar{\gamma}\right)-f\left(x\right),y_1\left(x,T;\bar{\beta},\hat{\gamma}-\bar{\gamma}\right)\right)_{L^2(I)}+N\left(\nabla\bar{\gamma},\nabla(\hat{\gamma}-\bar{\gamma})\right)_{L^2(I)}\\ &+\left(y_2\left(x,T;\bar{\beta},\bar{\gamma}\right)-g\left(x\right),y_2\left(x,T;\bar{\beta},\hat{\gamma}-\bar{\gamma}\right)\right)_{L^2(I)}\end{aligned}$$

$$\begin{aligned}\Longrightarrow\ &\partial J(\beta_\varepsilon,\bar{\gamma})\left(\hat{\beta}-\bar{\beta}\right)+\partial J(\bar{\beta},\gamma_\varepsilon)\left(\hat{\gamma}-\bar{\gamma}\right)=\\ &\int_I\left[\left(y_1\left(x,T,\bar{\beta},\bar{\gamma}\right)-f\left(x\right)\right)\left(y_1\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)+y_1\left(x,T;\bar{\beta},\hat{\gamma}-\bar{\gamma}\right)\right)\right]dx\\ &+\int_I\left[\left(y_2\left(x,T,\bar{\beta},\bar{\gamma}\right)-g\left(x\right)\right)\left(y_2\left(x,T;\hat{\beta}-\bar{\beta},\bar{\gamma}\right)+y_2\left(x,T;\bar{\beta},\hat{\gamma}-\bar{\gamma}\right)\right)\right]dx\\ &+N\int_I\left[\nabla\bar{\beta}\nabla(\hat{\beta}-\bar{\beta})+\nabla\bar{\gamma}\nabla(\hat{\gamma}-\bar{\gamma})\right]dx\end{aligned}$$

So

$$\partial J(\beta_\varepsilon,\gamma_\varepsilon)=\int_Q\left[\left(\hat{\beta}-\bar{\beta}\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}-\left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2\right]\left(p_2-p_1\right)dxdt$$

$$+N\int_I\left[\nabla\bar{\beta}\nabla\left(\hat{\beta}-\bar{\beta}\right)+\nabla\bar{\gamma}\nabla\left(\hat{\gamma}-\bar{\gamma}\right)\right]dx\geq 0$$

We now define necessary condition for optimality . Let $\left(\bar{\beta}\ ,\ \bar{\gamma}\right)$ is the solution to the optimal control problem (3.4) , $(\bar{y}_1\ ,\ \bar{y}_2)$ be the equivalent solution to system $(3.1)-(3.2)$ when $(\beta,\gamma)=\left(\bar{\beta},\bar{\gamma}\right)$, and $(p_1,\ p_2)$ is the solution to the adjoint system associated with system $(3.1)-(3.2)$. Let's define an adjoint state :

$$\begin{cases}\frac{\partial p_1}{\partial t}=-\Delta p_1+\frac{\bar{\beta}\bar{y}_2^2}{(\bar{y}_1+\bar{y}_2)^2}p_1-\frac{\bar{\beta}\bar{y}_2^2}{(\bar{y}_1+\bar{y}_2)^2}p_2\ , & (x,t)\in Q\ ,\\ \frac{\partial p_2}{\partial t}=-\Delta p_2+\left[\frac{\bar{\beta}\bar{y}_1^2}{(\bar{y}_1+\bar{y}_2)^2}-\bar{\gamma}\right]p_1-\left[\frac{\bar{\beta}\bar{y}_1^2}{(\bar{y}_1+\bar{y}_2)^2}-\bar{\gamma}\right]p_2\ , & (x,t)\in Q\ ,\\ \frac{\partial p_1}{\partial t}(0,t)=\frac{\partial p_2}{\partial t}(0,t)=0\ , & t\in(0,T)\ ,\\ \frac{\partial p_1}{\partial t}(1,t)=\frac{\partial p_2}{\partial t}(1,t)=0\ , & t\in(0,T)\ ,\\ p_1(x,T)=y_1\left(x,T,\bar{\beta},\bar{\gamma}\right)-f(x)\ , & x\in(0,1)\ ,\\ p_2(x,T)=y_2\left(x,T,\bar{\beta},\bar{\gamma}\right)-g(x)\ , & x\in(0,1)\ .\end{cases}\tag{3.8}$$

Denote $(z_1^\varepsilon, z_2^\varepsilon) = \left(\frac{y_1^\varepsilon - \bar{y}_1}{\varepsilon}, \frac{y_2^\varepsilon - \bar{y}_2}{\varepsilon}\right)$. Then taking $(\beta, \gamma) = (\beta^\varepsilon, \gamma^\varepsilon)$ in (3.1) , one has

$$\begin{cases} \frac{\partial y_1^\varepsilon}{\partial t} - \Delta y_1^\varepsilon = -\beta^\varepsilon(x)\frac{y_1^\varepsilon y_2^\varepsilon}{y_1^\varepsilon + y_2^\varepsilon} + \gamma^\varepsilon(x) y_2^\varepsilon & (x,t) \in (0,1)\times(0,T], \\ \frac{\partial y_2^\varepsilon}{\partial t} - \Delta y_2^\varepsilon = \beta^\varepsilon(x)\frac{y_1^\varepsilon y_2^\varepsilon}{y_1^\varepsilon + y_2^\varepsilon} - \gamma^\varepsilon(x) y_2^\varepsilon & (x,t) \in (0,1)\times(0,T], \\ y_1^\varepsilon(x,0) = \phi_1(x)\ , y_2^\varepsilon(x,0) = \phi_2(x) & x \in (0,1), \\ \frac{\partial y_1^\varepsilon}{\partial x}(0,t) = \frac{\partial y_2^\varepsilon}{\partial x}(0,t) = 0 & t \in (0,T), \\ \frac{\partial y_1^\varepsilon}{\partial x}(1,t) = \frac{\partial y_2^\varepsilon}{\partial x}(1,t) = 0 & t \in (0,T), \end{cases} \tag{3.9}$$

Taking $(\beta, \gamma) = \left(\bar{\beta}, \bar{\gamma}\right)$ in (3.1) we can obtain :

$$\begin{cases} \frac{\partial \bar{y}_1}{\partial t} - \Delta \bar{y}_1 = -\bar{\beta}(x)\frac{\bar{y}_1 \bar{y}_2}{\bar{y}_1 + \bar{y}_2} + \bar{\gamma}(x)\bar{y}_2 & (x,t) \in (0,1)\times(0,T], \\ \frac{\partial \bar{y}_2}{\partial t} - \Delta \bar{y}_2 = \bar{\beta}(x)\frac{\bar{y}_1 \bar{y}_2}{\bar{y}_1 + \bar{y}_2} - \bar{\gamma}(x)\bar{y}_2 & (x,t) \in (0,1)\times(0,T], \\ \bar{y}_1(x,0) = \phi_1(x)\ , \bar{y}_2(x,0) = \phi_2(x) & x \in (0,1), \\ \frac{\partial \bar{y}_1}{\partial x}(0,t) = \frac{\partial \bar{y}_2}{\partial x}(0,t) = 0 & t \in (0,T), \\ \frac{\partial \bar{y}_1}{\partial x}(1,t) = \frac{\partial \bar{y}_2}{\partial x}(1,t) = 0 & t \in (0,T), \end{cases} \tag{3.10}$$

Subtracting (3.9) in (3.10) and dividing both sides by ε , we can acquire that :

$$\begin{aligned} \frac{1}{\varepsilon}\left(\frac{\partial y_1^\varepsilon}{\partial t} - \frac{\partial \bar{y}_1}{\partial t}\right) &= \frac{\partial z_1^\varepsilon}{\partial t} \\ \frac{1}{\varepsilon}\left(\Delta y_1^\varepsilon - \Delta \bar{y}_1\right) &= \Delta z_1^\varepsilon \end{aligned}$$

as well :

$$\begin{aligned} -\beta^\varepsilon \frac{y_1^\varepsilon y_2^\varepsilon}{y_1^\varepsilon + y_2^\varepsilon} + \bar{\beta}\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1 + \bar{y}_2} &= \frac{-\beta^\varepsilon y_1^\varepsilon y_2^\varepsilon(\bar{y}_1 + \bar{y}_2) + \bar{\beta}\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} \\ &= \frac{-\beta^\varepsilon y_1^\varepsilon y_2^\varepsilon \bar{y}_1 - \beta^\varepsilon y_1^\varepsilon y_2^\varepsilon \bar{y}_2 + \bar{\beta}\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} \\ &= \frac{-\beta^\varepsilon y_1^\varepsilon \bar{y}_1[\varepsilon z_2^\varepsilon + \bar{y}_2] - \beta^\varepsilon y_2^\varepsilon \bar{y}_2[\varepsilon z_1^\varepsilon + \bar{y}_1] + \bar{\beta}\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} \\ &= \frac{-\varepsilon\beta^\varepsilon(y_1^\varepsilon \bar{y}_1 z_2^\varepsilon + y_2^\varepsilon \bar{y}_2 z_1^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} + \frac{\left(\bar{\beta} - \beta^\varepsilon\right)\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} \\ &= \frac{-\varepsilon\beta^\varepsilon(y_1^\varepsilon \bar{y}_1 z_2^\varepsilon + y_2^\varepsilon \bar{y}_2 z_1^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} + \frac{\left(\bar{\beta} - \bar{\beta} - \varepsilon\left(\hat{\beta} - \bar{\beta}\right)\right)\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} \\ &= \varepsilon\left(\frac{-\beta^\varepsilon(y_1^\varepsilon \bar{y}_1 z_2^\varepsilon + y_2^\varepsilon \bar{y}_2 z_1^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)} - \frac{\left(\hat{\beta} - \bar{\beta}\right)\bar{y}_1\bar{y}_2(y_1^\varepsilon + y_2^\varepsilon)}{(\bar{y}_1 + \bar{y}_2)(y_1^\varepsilon + y_2^\varepsilon)}\right) \end{aligned}$$

$$\begin{aligned}\gamma^{\varepsilon}y_2^{\varepsilon}-\bar{\gamma}\bar{y}_2 &= \gamma^{\varepsilon}\left[\varepsilon z_2^{\varepsilon}+\bar{y}_2\right]-\bar{\gamma}\bar{y}_2\\ &= \varepsilon\gamma^{\varepsilon}z_2^{\varepsilon}+\left(\gamma^{\varepsilon}-\bar{\gamma}\right)\bar{y}_2\\ &= \varepsilon\gamma^{\varepsilon}z_2^{\varepsilon}+\left(\ \bar{\gamma}+\varepsilon\left(\hat{\gamma}-\bar{\gamma}\right)-\bar{\gamma}\right)\bar{y}_2\\ &= \varepsilon\left(\gamma^{\varepsilon}z_2^{\varepsilon}+\left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2\right)\end{aligned}$$

In the same way , we find :

$$\beta^{\varepsilon}\frac{y_1^{\varepsilon}y_2^{\varepsilon}}{y_1^{\varepsilon}+y_2^{\varepsilon}}-\bar{\beta}\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}=\varepsilon\left(\frac{\beta^{\varepsilon}\left(y_1^{\varepsilon}\bar{y}_1z_2^{\varepsilon}+y_2^{\varepsilon}\bar{y}_2z_1^{\varepsilon}\right)}{\left(\bar{y}_1+\bar{y}_2\right)\left(y_1^{\varepsilon}+y_2^{\varepsilon}\right)}+\frac{\left(\hat{\beta}-\bar{\beta}\right)\bar{y}_1\bar{y}_2\left(y_1^{\varepsilon}+y_2^{\varepsilon}\right)}{\left(\bar{y}_1+\bar{y}_2\right)\left(y_1^{\varepsilon}+y_2^{\varepsilon}\right)}\right)$$

$-\gamma^{\varepsilon}y_2^{\varepsilon}+\bar{\gamma}\bar{y}_2=-\varepsilon\left(\gamma^{\varepsilon}z_2^{\varepsilon}+\left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2\right)$
from this , we get :

$$\begin{cases}\frac{\partial z_1^{\varepsilon}}{\partial t}-\Delta z_1^{\varepsilon}=-\frac{\beta^{\varepsilon}}{(y_1^{\varepsilon}+y_2^{\varepsilon})(\bar{y}_1+\bar{y}_2)}\left[\bar{y}_2y_2^{\varepsilon}z_1^{\varepsilon}+\bar{y}_1y_1^{\varepsilon}z_2^{\varepsilon}\right]-\left(\hat{\beta}-\bar{\beta}\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}+\gamma^{\varepsilon}z_2^{\varepsilon}+\left(\hat{\gamma}-\bar{\gamma}\right)y_2^{\varepsilon}\ , & \\ \frac{\partial z_1^{\varepsilon}}{\partial t}-\Delta z_1^{\varepsilon}=\frac{\beta^{\varepsilon}}{(y_1^{\varepsilon}+y_2^{\varepsilon})(\bar{y}_1+\bar{y}_2)}\left[\bar{y}_2y_2^{\varepsilon}z_1^{\varepsilon}+\bar{y}_1y_1^{\varepsilon}z_2^{\varepsilon}\right]+\left(\hat{\beta}-\bar{\beta}\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}-\gamma^{\varepsilon}z_2^{\varepsilon}-\left(\hat{\gamma}-\bar{\gamma}\right)y_2^{\varepsilon}\ , & \\ z_1^{\varepsilon}(x,0)=z_2^{\varepsilon}(x,0)=0\ , & x\in(0,1)\,, \\ \frac{\partial z_1^{\varepsilon}}{\partial x}(0,t)=\frac{\partial z_2^{\varepsilon}}{\partial x}(0,t)=0\,, & t\in(0,T)\,, \\ \frac{\partial z_1^{\varepsilon}}{\partial x}(1,t)=\frac{\partial z_2^{\varepsilon}}{\partial x}(1,t)=0\ , & t\in(0,T)\,,\end{cases} \tag{3.11}$$

Let $z_i=z_i^{\varepsilon}\mid_{\varepsilon=0}$. means that $\lim_{\varepsilon\longrightarrow 0}z_i^{\varepsilon}=z_i$.
in the same way :

$$\lim_{\varepsilon\longrightarrow 0}\beta^{\varepsilon}=\bar{\beta}\quad\text{and}\quad\lim_{\varepsilon\longrightarrow 0}y_i^{\varepsilon}=\lim_{\varepsilon\longrightarrow 0}\varepsilon z_i+\bar{y}_i=\bar{y}_i$$

Then z_i satisfies :

$$\begin{cases}\frac{\partial z_1}{\partial t}-\Delta z_1=-\frac{\bar{\beta}\bar{y}_2^2}{(\bar{y}_1+\bar{y}_2)^2}z_1-\left[\frac{\bar{\beta}\bar{y}_1^2}{(\bar{y}_1+\bar{y}_2)^2}-\bar{\gamma}\right]z_2-\frac{\left(\hat{\beta}-\bar{\beta}\right)\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}+\left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2 & \\ \frac{\partial z_2}{\partial t}-\Delta z_2=\frac{\bar{\beta}\bar{y}_2^2}{(\bar{y}_1+\bar{y}_2)^2}z_1+\left[\frac{\bar{\beta}\bar{y}_1^2}{(\bar{y}_1+\bar{y}_2)^2}-\bar{\gamma}\right]z_2+\frac{\left(\hat{\beta}-\bar{\beta}\right)\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2}-\left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2 & \\ z_1(x,0)=z_2(x,0)=0 & x\in(0,1)\,, \\ \frac{\partial z_1}{\partial x}(0,t)=\frac{\partial z_2}{\partial x}(0,t)=0 & t\in(0,T)\,, \\ \frac{\partial z_1}{\partial x}(1,t)=\frac{\partial z_2}{\partial x}(1,t)=0 & t\in(0,T)\,,\end{cases} \tag{3.12}$$

On the other hand :

$$\begin{aligned}\sum_{i=1}^{2}\left(p_i\frac{\partial z_i}{\partial t}+z_i\frac{\partial p_i}{\partial t}\right) &= p_1\frac{\partial z_1}{\partial t}+z_1\frac{\partial p_1}{\partial t}+p_2\frac{\partial z_2}{\partial t}+z_2\frac{\partial p_2}{\partial t}\\ &= p_1\Delta z_1-\frac{\bar\beta\bar y_2^2}{(\bar y_1+\bar y_2)^2}z_1p_1-\left[\frac{\bar\beta\bar y_1^2}{(\bar y_1+\bar y_2)^2}-\bar\gamma\right]z_2p_1-\frac{\left(\hat\beta-\bar\beta\right)\bar y_1\bar y_2p_1}{\bar y_1+\bar y_2}\\ &\quad+\left(\hat\gamma-\bar\gamma\right)\bar y_2p_1-z_1\Delta p_1+\frac{\bar\beta\bar y_2^2}{(\bar y_1+\bar y_2)^2}p_1z_1-\frac{\bar\beta\bar y_2^2}{(\bar y_1+\bar y_2)^2}p_2z_1+p_2\Delta z_2\end{aligned}$$

$$\begin{aligned}&+\frac{\bar\beta\bar y_2^2}{(\bar y_1+\bar y_2)^2}z_1p_2+\left[\frac{\bar\beta\bar y_1^2}{(\bar y_1+\bar y_2)^2}-\bar\gamma\right]z_2p_2+\frac{\left(\hat\beta-\bar\beta\right)\bar y_1\bar y_2p_2}{\bar y_1+\bar y_2}-\left(\hat\gamma-\bar\gamma\right)\bar y_2p_2\\ &-z_2\Delta p_2+\left[\frac{\bar\beta\bar y_1^2}{(\bar y_1+\bar y_2)^2}-\bar\gamma\right]p_1z_2-\left[\frac{\bar\beta\bar y_1^2}{(\bar y_1+\bar y_2)^2}-\bar\gamma\right]p_2z_2\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{2}\left(p_i\frac{\partial z_i}{\partial t}+z_i\frac{\partial p_i}{\partial t}\right) &= p_1\Delta z_1+p_2\Delta z_2-z_1\Delta p_1-z_2\Delta p_2+\left(\hat\beta-\bar\beta\right)\frac{\bar y_1\bar y_2}{\bar y_1+\bar y_2}\left(p_2-p_1\right)\\ &\quad-\left(\hat\gamma-\bar\gamma\right)\left(p_2-p_1\right)\bar y_2\end{aligned}$$

Then :

$$\begin{aligned}\sum_{i=1}^{2}\left(p_i\frac{\partial z_i}{\partial t}+z_i\frac{\partial p_i}{\partial t}\right) &= p_1\Delta z_1+p_2\Delta z_2-z_1\Delta p_1-z_2\Delta p_2\\ &\quad+\left[\left(\hat\beta-\bar\beta\right)\frac{\bar y_1\bar y_2}{\bar y_1+\bar y_2}-\left(\hat\gamma-\bar\gamma\right)\bar y_2\right]\left(p_2-p_1\right)\end{aligned}\tag{3.13}$$

Also note that :

$$\begin{aligned}\frac{\partial}{\partial t}\left(p_1z_1+p_2z_2\right) &= z_1\frac{\partial p_1}{\partial t}+p_1\frac{\partial z_1}{\partial t}+z_2\frac{\partial p_2}{\partial t}+p_2\frac{\partial z_2}{\partial t}\\ &= \sum_{i=1}^{2}\left(p_i\frac{\partial z_i}{\partial t}+z_i\frac{\partial p_i}{\partial t}\right)\end{aligned}\tag{3.14}$$

We have :

$$p_1(x,T)=y(x,T;\bar\beta,\bar\gamma)-f(x)=0 \text{ and } p_2(x,T)=y(x,T;\bar\beta,\bar\gamma)-g(x)=0$$

Consequently, from (3.12) and (3.13) and integrating on Q , one obtains that :

$$\int_I \int_0^T \frac{\partial}{\partial t}\left(p_1 z_1 + p_2 z_2\right)(x,t)\,dtdx = \int_I \int_0^T \left[\left(\hat{\beta}-\bar{\beta}\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2} - \left(\hat{\gamma}-\bar{\gamma}\right)\bar{y}_2\right](p_2-p_1)\,dtdx \quad (3.15)$$

Thereupon, it follows from (3.8) and (3.15) that :

$$\begin{aligned}&\int_I \left\{\left[y_1\left(x,T;\bar{\beta},\bar{\gamma}\right)-f(x)\right]z_1(x,T)+\left[y_2\left(x,T;\bar{\beta},\bar{\gamma}\right)-g(x)\right]z_2(x,T)\right\}dx\\ =\ &\int_I \int_0^T \left[\left(\bar{\beta}-\beta\right)\frac{\bar{y}_1\bar{y}_2}{\bar{y}_1+\bar{y}_2} - \left(\bar{\gamma}-\gamma\right)\bar{y}_2\right](p_2-p_1)\,dtdx\end{aligned}$$

■

By changing the time direction $s = T - t$ and the change of functions $p_i(x, T-s) = w_i(x,s)$ and $y_i(x,T-s) = y_i^*(x,s)$, $(s,x) \in Q$, $i = 1,2$, system (3.8) can be rewritten as :
$t = T - s \Longrightarrow \partial t = -\partial s$ So $\frac{\partial p_i}{\partial s}(x,T-s) = -\frac{\partial w_i}{\partial s}(x,s)$

$$\begin{cases}\frac{\partial w_1}{\partial s} = \Delta w_1 - \frac{\bar{\beta}\left(y_2^*\right)^2}{\left(y_1^*+y_2^*\right)^2}w_1 + \frac{\bar{\beta}\left(y_2^*\right)^2}{\left(y_1^*+y_2^*\right)^2}w_2\ , & (x,s)\in Q,\\ \frac{\partial w_2}{\partial s} = \Delta w_2 - \left[\frac{\bar{\beta}\left(y_2^*\right)^2}{\left(y_1^*+y_2^*\right)^2}-\bar{\gamma}\right]w_1 + \left[\frac{\bar{\beta}\left(y_2^*\right)^2}{\left(y_1^*+y_2^*\right)^2}-\bar{\gamma}\right]w_2\ , & (x,s)\in Q,\\ \frac{\partial w_1}{\partial s}(0,s) = \frac{\partial w_2}{\partial s}(0,s) = 0 & s\in(0,T)\,,\\ \frac{\partial w_1}{\partial s}(1,s) = \frac{\partial w_2}{\partial s}(1,s) = 0 & s\in(0,T)\,,\\ w_1(x,0) = y_1\left(x,T;\bar{\beta},\bar{\gamma}\right) - f(x)\ , & x\in(0,1)\,,\\ w_2(x,0) = y_2\left(x,T;\bar{\beta},\bar{\gamma}\right) - g(x)\ , & x\in(0,1)\,.\end{cases} \quad (3.16)$$

Lemma 3.1 [15]: *Let (p_1,p_2) be the solution to system (3.8) . Then $\| p_i \|_{L^\infty(I)}$, $i = 1,2$, is bounded on $[0,T]$.*

Proof. Let $F(u_1,u_2) = \frac{u_1u_2}{u_1+u_2}$. We derive $F(u_1,u_2)$ to get :

$$\begin{aligned}F_{u_1}(u_1,u_2) &= \frac{u_2(u_1+u_2)-u_1u_2}{(u_1+u_2)^2} \quad \text{and} \quad F_{u_2}(u_1,u_2) = \frac{u_1(u_1+u_2)-u_1u_2}{(u_1+u_2)^2}\\ &= \frac{u_2^2}{(u_1+u_2)^2} \qquad\qquad = \frac{u_1^2}{(u_1+u_2)^2}\end{aligned}$$

$\Longrightarrow F_{u_1}(y_1^*,y_2^*) = F_{u_1}\,|_{u_1=y_1^*,u_2=y_2^*}$.
Then (3.16) can be written as :

$$\begin{cases}\frac{\partial w_1}{\partial s} - \Delta w_1 + \bar{\beta}F_{u_1}(y_1^*,y_2^*)\,w_1 - \bar{\beta}F_{u_1}(y_1^*,y_2^*)\,w_2 = 0\ ,\\ \frac{\partial w_2}{\partial s} - \Delta w_2 + \left[\bar{\beta}F_{u_2}(y_1^*,y_2^*)-\bar{\gamma}\right]w_1 - \left[\bar{\beta}F_{u_2}(y_1^*,y_2^*)-\bar{\gamma}\right]w_2 = 0\ .\end{cases} \quad (3.17)$$

Multiplying the first equation of (3.17) by $w_1$and integrating on , one obtains that :

$$\int_I \frac{\partial w_1}{\partial s} w_1 dx - \int_I w_1 \Delta w_1 dx + \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) \mid w_1 \mid^2 - \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) w_2 w_1 dx = 0$$

using the Green's formula and the Cauchy inequality, we can demonstrate that :
We know that $\frac{1}{2}\frac{\partial}{\partial s}(w)^2 = w\frac{\partial w}{\partial s}$.
Applying the Green's formula : (See Appendices Theorem 2)

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_1)^2 dx - \int_\Gamma w_1 \frac{\partial w_1}{\partial \nu} d\Gamma + \int_\Omega \mid \nabla w_1 \mid^2 dx + \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) \mid w_1 \mid^2 dx \\ - \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) w_2 w_1 dx &= 0\end{aligned}$$

$$\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_1)^2 dx = \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) w_2 w_1 dx - \int_I \bar{\beta} F_{u_1}(y_1^*, y_2^*) \mid w_1 \mid^2 dx$$

We have $F_{u_1}(u_1, u_2) = \frac{u_2^2}{(u_1+u_2)^2}$ and since $u_1, u_2 > 0$, it's clear that $0 \leq F_{u_1} \leq 1$

$$\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_1)^2 dx \leq \int_I \mid \bar{\beta} \mid\mid w_2 \mid\mid w_1 \mid dx + \int_I \mid \bar{\beta} \mid\mid w_1 \mid^2 dx$$

Applying The Cauchy-Schwarz inequality : (See Appendices Theorem 1)

$$\begin{aligned}\int_I \mid \bar{\beta} \mid\mid w_2 \mid\mid w_1 \mid dx + \int_I \mid \bar{\beta} \mid\mid w_1 \mid^2 dx \leq \\ \left(\int_I \mid \bar{\beta} \mid^2 \mid w_2 \mid^2 dx\right)^{\frac{1}{2}} \left(\int_I \mid w_1 \mid^2 dx\right)^{\frac{1}{2}} + \left(\int_I \mid \bar{\beta} \mid^2 (\mid w_1 \mid^2)^2 dx\right)^{\frac{1}{2}}\end{aligned}$$

$$\bar{b}_0 \parallel w_2 \parallel_{L^2(I)} \|w_1\|_{L^2(I)} + \bar{b}_0 \parallel w_1 \parallel^2_{L^2(I)}$$

Applying $\left(ab \leq \frac{a^2}{2} + \frac{b^2}{2}\right)$ on $\bar{b}_0 \parallel w_2 \parallel_{L^2(I)} \|w_1\|_{L^2(I)}$

$$\frac{1}{2}\frac{\partial}{\partial s} \parallel w_1 \parallel_{L^2(I)} \leq \frac{1}{2}\bar{b}_0^2 \parallel w_1 \parallel^2_{L^2(I)} + \frac{1}{2} \parallel w_2 \parallel^2_{L^2(I)} + \bar{b}_0 \parallel w_1 \parallel^2_{L^2(I)}$$

$$\frac{1}{2}\frac{\partial}{\partial s} \parallel w_1 \parallel_{L^2(I)} \leq \left(\frac{\bar{b}_0^2}{2} + \bar{b}_0\right) \parallel w_1 \parallel^2_{L^2(I)} + \frac{1}{2} \parallel w_2 \parallel^2_{L^2(I)} \tag{3.18}$$

Similarly, multiplying the second equation of (3.17) byw_2 and integrating on I , one acquires :

$$\begin{aligned}\int_I \frac{\partial w_2}{\partial s} w_2 dx - \int_I w_2 \Delta w_2 dx + \int_I \left[\bar{\beta} F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_1 w_2 dx \\ - \int_I \left[\bar{\beta} F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_2^2 = 0\end{aligned}$$

Applying the Green's formula : (See Appendices Theorem 2)

$$\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_2)^2\,dx - \int_\Gamma w_2\frac{\partial w_2}{\partial \nu}d\Gamma + \int_\Omega \mid \nabla w_2 \mid^2 dx + \int_I \left[\bar{\beta}F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_1 w_2 dx$$
$$- \int_I \left[\bar{\beta}F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_2^2 = 0$$

$$\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_2)^2\,dx = \int_I \left[\bar{\beta}F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_2^2 - \int_I \left[\bar{\beta}F_{u_2}(y_1^*, y_2^*) - \bar{\gamma}\right] w_1 w_2 dx$$

We have $F_{u_2}(u_1, u_2) = \frac{u_1^2}{(u_1+u_2)^2}$ and since $u_1, u_2 > 0$, it's clear that $0 \leq F_{u_2} \leq 1$

$$\frac{1}{2}\frac{\partial}{\partial s}\int_I (w_2)^2 dx \leq \int_I \mid \bar{\beta} + \bar{\gamma} \mid\mid w_2 \mid\mid w_1 \mid dx + \int_I \mid \bar{\beta} + \bar{\gamma} \mid\mid w_2 \mid^2 dx$$

Applying Cauchy-Schwarz inequality :(See Appendices Theorem 1)

$$\int_I \mid \bar{\beta} + \bar{\gamma} \mid\mid w_2 \mid\mid w_1 \mid dx + \int_I \mid \bar{\beta} + \bar{\gamma} \mid\mid w_2 \mid^2 dx \leq$$
$$\left(\int_I \mid \bar{\beta} + \bar{\gamma} \mid^2 \mid w_2 \mid^2 \mid w_1 \mid^2 dx\right)^{\frac{1}{2}} + \left(\int_I \mid \bar{\beta} + \bar{\gamma} \mid^2 (\mid w_2 \mid^2)^2 dx\right)^{\frac{1}{2}}$$
$$\leq (\bar{b}_0 + \bar{r}_0) \parallel w_2 \parallel_{L^2(I)} \parallel w_1 \parallel_{L^2(I)} + (\bar{b}_0 + \bar{r}_0) \parallel w_2 \parallel^2_{L^2(I)}$$

Applying $\left(ab \leq \frac{a^2}{2} + \frac{b^2}{2}\right)$ on $(\bar{b}_0 + \bar{r}_0) \parallel w_2 \parallel_{L^2(I)} \parallel w_1 \parallel_{L^2(I)}$

$$\frac{1}{2}\frac{\partial}{\partial s} \parallel w_2 \parallel^2_{L^2(I)} \leq \frac{(\bar{b}_0 + \bar{r}_0)^2}{2} \parallel w_2 \parallel^2_{L^2(I)} + \frac{1}{2} \parallel w_1 \parallel^2_{L^2(I)} + (\bar{b}_0 + \bar{r}_0) \parallel w_2 \parallel^2_{L^2(I)}$$

$$\frac{1}{2}\frac{\partial}{\partial s} \parallel w_2 \parallel^2_{L^2(I)} \leq \left(\frac{(\bar{b}_0 + \bar{r}_0)^2}{2} + b_0 + r_0\right) \parallel w_2 \parallel^2_{L^2(I)} + \frac{1}{2} \parallel w_1 \parallel^2_{L^2(I)} \tag{3.19}$$

Then it is easy to see from (3.18) and (3.19) that :

$$\frac{\partial}{\partial s}\left(\parallel w_1 \parallel^2_{L^2(I)} + \parallel w_2 \parallel^2_{L^2(I)}\right) \leq (1 + \bar{b}_0^2 + \bar{b}_0) \parallel w_1 \parallel^2_{L^2(I)}$$

$$+ \left((\bar{b}_0 + \bar{r}_0)^2 + 2(\bar{b}_0 + \bar{r}_0) + 1\right) \parallel w_2 \parallel^2_{L^2(I)}$$

$$\frac{\partial}{\partial s}\left(\parallel w_1 \parallel^2_{L^2(I)} + \parallel w_2 \parallel^2_{L^2(I)}\right) \leq (\bar{b}_0 + 1)^2 \parallel w_1 \parallel^2_{L^2(I)}$$
$$+ (\bar{b}_0 + \bar{r}_0 + 1)^2 \parallel w_2 \parallel^2_{L^2(I)}$$
$$\leq \max\left\{(\bar{b}_0 + 1)^2, (\bar{b}_0 + \bar{r}_0 + 1)^2\right\}\left(\parallel w_1 \parallel^2_{L^2(I)} + \parallel w_2 \parallel^2_{L^2(I)}\right)$$

$$\leq \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 \left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} \right)$$

which implies that :

$$\frac{\partial \left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} \right)}{\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}} \leq \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 \partial s$$

$$\begin{aligned}
&\Longrightarrow \quad \ln |\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}| \leq \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \\
&\Longrightarrow \quad \| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} \leq \exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \right) \\
&\Longrightarrow \quad \| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} - \exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \right) \leq 0
\end{aligned}$$

We have :

$$\begin{aligned}
\frac{\partial}{\partial s} \left[\alpha \exp\left(-ks\right) \right] &= \partial\alpha \exp\left(-ks\right) - k\alpha \exp\left(-ks\right) \\
&= \left(\partial\alpha - k\alpha\right) \exp\left(-ks\right)
\end{aligned}$$

On suppose :

$$\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} = \alpha$$

$$\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 = k$$

$$\Longrightarrow \quad \left(\partial\alpha - k\alpha\right) \exp\left(-ks\right) \leq 0$$

$$\Longrightarrow \quad \exp\left(-ks\right) > 0 \; so \quad \partial\alpha - k\alpha \leq 0 \Longrightarrow \partial\alpha \leq k\alpha$$

$$\Longrightarrow \frac{\partial}{\partial s} \left[\alpha \exp\left(-ks\right) \right] \leq 0 \quad \text{so}$$

$$\frac{\partial}{\partial s} \left[\left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} \right) \exp\left(- \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \right) \right] \leq 0 \tag{3.20}$$

Integrating from 0 to s on both sides of (3.20) , we get :

$$\int_0^s \frac{\partial}{\partial s} \left[\left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)} \right) \exp\left(- \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \right) \right] dt \leq 0$$

$$\begin{aligned}
&\left(\| w_1(x,s) \|^2_{L^2(I)} + \| w_2(x,s) \|^2_{L^2(I)} \right) \exp\left(- \left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s \right) \\
- \quad &\| w_1(x,0) \|^2_{L^2(I)} + \| w_2(x,0) \|^2_{L^2(I)} \leq 0
\end{aligned}$$

We know :

$$w_1(x,0) = y_1\left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \quad \text{and} \quad w_2(x,0) = y_2\left(x, T; \bar{\beta}, \bar{\gamma}\right) - g(x)$$

$$\Longrightarrow \left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}\right) \exp\left(-\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s\right) \leq \| y_1\left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{L^2(I)}$$
$$+ \| y_2\left(x, T; \bar{\beta}, \bar{\gamma}\right) - g(x) \|^2_{L^2(I)}$$

We now multiply both sides by $\exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s\right)$:

$$\left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}\right) \leq \exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s\right) \left[\begin{array}{l} \| y_1\left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{L^2(I)} + \\ + \| y_2\left(x, T; \bar{\beta}, \bar{\gamma}\right) - g(x) \|^2_{L^2(I)} \end{array}\right]$$

$$\Longrightarrow \max_{s\in[0,T]} \left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}\right) \leq$$
$$\max_{s\in[0,T]} \left(\exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 s\right) \left[\begin{array}{l} \| y_1\left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{L^2(I)} + \\ + \| y_2\left(x, T; \bar{\beta}, \bar{\gamma}\right) - g(x) \|^2_{L^2(I)} \end{array}\right]\right)$$

$$\max_{s\in[0,T]} \left(\| w_1 \|^2_{L^2(I)} + \| w_2 \|^2_{L^2(I)}\right) \leq$$
$$\exp\left(\left(\bar{b}_0 + \bar{r}_0 + 1\right)^2 T\right) \left[\begin{array}{l} \| y_1\left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{L^2(I)} + \\ + \| y_2\left(x, T; \bar{\beta}, \bar{\gamma}\right) - g(x) \|^2_{L^2(I)} \end{array}\right]$$

which implies that $\| p_i(t) \|_{L^2(I)} = \| w_i \|_{L^2(I)}$, $i = 1, 2$ is bounded on $[0, T]$. Next we further give the H^1estimate of $w(x,t) = (w_1(x,t), w_2(x,t))$ to achieve our result by the standard embedding theorem . Multiplying the first equation of (3.17) by $\Delta w_1$and integrating on I , we have :

$$\int_I \frac{\partial w_1}{\partial s} \Delta w_1 dx - \int_I (\Delta w_1)^2 dx = \int_I \bar{\beta} F_{u_1}\left(y_1^*, y_2^*\right) w_2 \Delta w_1 dx - \int_I \bar{\beta} F_{u_1}\left(y_1^*, y_2^*\right) w_1 \Delta w_1 dx$$

Then we have :

Noting that $0 < \beta(x) < 1$ and $0 < F_{u_1}(y_1^*, y*) < 1$, we can conclude that :

$$\frac{1}{2}\frac{\partial}{\partial s} \| w_1 \|^2_{H_0^1(I)} + \| \Delta w_1 \|^2_{L^2(I)} \leq \int_I | w_2 | | \Delta w_1 | dx + \int_I | w_1 | | \Delta w_1 | dx$$

Applying the Cauchy-Schwarz inequality : (See Appendices Theorem 1)

$$\int_I \quad | \quad w_2 | | \Delta w_1 | dx + \int_I | w_1 | | \Delta w_1 | dx \leq \left(\int_I | w_2 |^2\right)^{\frac{1}{2}} \left(\int_I | \Delta w_1 |^2\right)^{\frac{1}{2}}$$
$$+ \left(\int_I | w_1 |^2\right)^{\frac{1}{2}} \left(\int_I | \Delta w_1 |^2\right)^{\frac{1}{2}}$$
$$\leq \quad \|w_2\|_{L^2(I)} \| \Delta w_1 \|_{L^2(I)} + \|w_2\|_{L^2(I)} \| \Delta w_1 \|_{L^2(I)}$$

Applying $\left(ab \leq \frac{a^2}{2} + \frac{b^2}{2}\right)$ on $\|w_2\|_{L^2(I)} \| \Delta w_1 \|_{L^2(I)}$and $\|w_2\|_{L^2(I)} \| \Delta w_1 \|_{L^2(I)}$

$$\frac{1}{2}\frac{\partial}{\partial s} \| w_1 \|^2_{H_0^1(I)} + \| \Delta w_1 \|^2_{L^2(I)} \leq \frac{\| w_2 \|^2_{L^2(I)}}{2} + \frac{\| \Delta w_1 \|^2_{L^2(I)}}{2} + \frac{\| w_1 \|^2_{L^2(I)}}{2} + \frac{\| \Delta w_1 \|^2_{L^2(I)}}{2}$$

$$\frac{1}{2}\frac{\partial}{\partial s} \| w_1 \|^2_{H_0^1(I)} + \| \Delta w_1 \|^2_{L^2(I)} \leq \frac{1}{2}\left(\| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}\right) + \| \Delta w_1 \|^2_{L^2(I)}$$

$$\Longrightarrow \frac{\partial}{\partial s} \| w_1 \|^2_{H_0^1(I)} \leq \| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}$$

Integrating from 0 to t in the above inequality , we obtain :

$$\int_0^t \frac{\partial}{\partial s} \| w_1 \|^2_{H_0^1(I)} \, dt \leq \int_0^t \left(\| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}\right) dt$$

$$\Longrightarrow \| w_1 (x,t) \|^2_{H_0^1(I)} - \| w_1 (x,0) \|^2_{H_0^1(I)} \leq \left(\| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}\right) t$$

$$\Longrightarrow \| w_1 (x,t) \|^2_{H_0^1(I)} \leq \left(\| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}\right) t + \| y_1 \left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{H_0^1(I)}$$

$$\Longrightarrow \| w_1 (x,t) \|^2_{H_0^1(I)} \leq$$

$$T \max_{t \in [0,T]} \left(\| w_2 \|^2_{L^2(I)} + \| w_1 \|^2_{L^2(I)}\right) + \| y_1 \left(x, T; \bar{\beta}, \bar{\gamma}\right) - f(x) \|^2_{H_0^1(I)}$$

So , $\| w_1 \|^2_{H_0^1(I)}$is bounded on $[0, T]$. Similarly , we can prove that $\| w_2 \|^2_{H_0^1(I)}$ is bounded on $[0, T]$. Therefore , the standard embedding theorem yields the desired result. ■

Appendices

Definition 1: *Let $J : U \subset X \to Y$ be an operator with Banach spaces X, Y and $U \neq \emptyset$ open. J is called directionally differentiable at $x \in U$ if the limit :*

$$dJ(x,h) = \lim_{t \to 0^+} \frac{J(x+th) - J(x)}{t} \in Y$$

exists for all $h \in X$. J is called Gâteaux differentiable at $x \in U$ if J is directionally differentiable at x and the directional derivative $J'(x) : h \in X \to dJ(x,h) \in Y$ is bounded and linear , i.e. , $J(x) \in \mathcal{L}(X,Y)$.

Theorem 1 : (Cauchy-Schwarz inequality)

Consider u and $v \in L^2(\Omega)$, then , Cauchy inequality given by :

$$\mid \int_\Omega uv dx \mid \leq \left(\int_\Omega \mid u \mid^2 dx \right)^{\frac{1}{2}} \left(\int_\Omega \mid v \mid^2 dx \right)^{\frac{1}{2}}$$

Theorem 2 : (Green Formula) Let $\Omega \subset \mathbb{R}^n$ be a bounded spatial domain with smooth boundary $\partial\Omega = \Gamma$, where ν denote the external unit normal vector .

Let u and v be any two real functions of class $C^1(\Omega)$, we have

$$\int_\Omega (\Delta u) v dx = \int_\Gamma v \frac{\partial u}{\partial \nu} d\Gamma - \int_\Omega \nabla u \nabla v dx$$

Bibliography

Bibliography

[1] Alves, C., Silvestre, A. L., Takahashi, T., & Tucsnak, M. (2009). Solving inverse source problems using observability. Applications to the Euler–Bernoulli plate equation. SIAM Journal on Control and Optimization, 48(3), 1632-1659 .

Bibliography

[1] Casti, J. (1980). On the general inverse problem of optimal control theory. Journal of Optimization Theory and Applications, 32, 491-497.

[2] Clason, C., & Kaltenbacher, B. (2020). Optimal control and inverse problems. Inverse Problems, 36(6), 060301 .

[3] Curtain, R. F., & Zwart, H. (2012). An introduction to infinite-dimensional linear systems theory (Vol. 21). Springer Science & Business Media .

[4] El Jai, A. (2006). Eléments de contrôlabilité. Presses Universitaires de Perpignan (PUP), Université de Perpignan .

[5] Favini, A., & Lorenzi, A. (Eds.). (2006). Differential equations: inverse and direct problems. CRC Press .

[6] Fern■ndez-Cara, E. (2024). Remarks on control and inverse problems for PDEs. SeMA Journal, 1-22 .

[7] Glass, O. (2012). Infinite dimensional controllability. In Mathematics of Complexity and Dynamical Systems (pp. 755-770). Springer, New York, NY.

[8] Hafdallah, A., & Ayadi, A. (2020). Optimal control of a thermoelastic body with missing initial conditions. International Journal of Control, 93(7), 1570-1576 .

[9] Hasanoğlu, A. H., & Romanov, V. G. (2021). Introduction to inverse problems for differential equations. Cham: Springer .

[10] Hoang, D. T. (2018). Controllability and observability of non autonomous evolution equations (Doctoral dissertation, Université de Bordeaux) .

[11] Kirsch, A. (2011). An introduction to the mathematical theory of inverse problems (Vol. 120). New York : Springer .

[12] Ramm, A. G. (2005). Inverse problems: mathematical and analytical techniques with applications to engineering. Springer Science & Business Media .

[13] Silverman, L. M., & Meadows, H. E. (1967). Controllability and observability in time-variable linear systems. SIAM Journal on Control, 5(1), 64-73 .

[14] Xiang, H., & Liu, B. (2015). Solving the inverse problem of an SIS epidemic reaction–diffusion model by optimal control methods. Computers & Mathematics with Applications, 70(5), 805819 .

[15] Zabczyk, J. (1997). Mathematical control theory: an introduction. Applications of Mathematics-New York, 42(1), 80 .

[16] Zabczyk, J. (2020). Mathematical control theory. Springer International Publishing .

[17] Zuazua, E. (2006). Controllability of partial differential equations (Doctoral dissertation, Optimization and Control).