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# Asymptotic stability and explosion of solutions for a class of hyperbolic problem with delay time.

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بسم الله الرحمن الرحيم

# الإهداء:

الى نفسي ...

إلى تلك الطموحة التي لم تستسلم , المثابرة التي تعثرت لكنها نهضت,

الحالمة التي آمنت أن البدايات الصعبة تصنع النهايات الجميلة.

احلم بغد أجمل ابدع فيه و اشرق كالضياء

فآنا لها وإن أبت رغما عنها أتيت بها .

و بكل حب أهدي تخرجي :

الى أبي الغالي ...

الذي كان دائما مصدر دعمي وقوتي . يامن غرست في قلبي العزيمة و قيمة

الاجتهاد و الصبر و كنت السند الذي لايميل .

الى أمي الحبيبة...

من كانت لي النور في عتمة الايام، قلب الحياة و روحها الدافئة،

يا دعاء لا ينقطع و وطنا يحتويني في كل حال .

لا يعلو فضل على فضلكم حفظكم الله لي من كل مكروه.

الى أخواتي العزيزات...

يا من كن زهرات العمر، ورفيقات القلب، و بهجة الدرب

"خلود ، حكيمة"

الى القلوب الطاهرة و ملائكة البيت الأحفاد:

"عدي، ميار، مسك"

# شكر و عرفان:

لك الحمد يا من علمت الإنسان ما لم يعلم , وبك نستعين على دروب العلم والمعرفة و نسألك ان تبارك لنا في هذا الدرب .

في هذه اللحظة التي أقطف فيها ثمرة سنوات من الجد و الاجتهاد , لا يسعني إلا أن أرفع كلمات الشكر و الامتنان إلى كل من كان له فضل و لو بسيط في مسيرتي العلمية .

أعبر عن بالغ تقديري و امتناني لأستاذتي المشرفة الدكتورة طوالبية سارة التي لم تبخل عليا بعلمها و خبرتها , و رافقتني بخطاها الثابتة , و نصائحها الحكيمة , فكانت خير الموجه و الداعم.

كما اتقدم بالشكر إلى كافة أساتذتي الكرام في هذا الصرح العلمي، الذين غرسوا فينا قيم البحث, و حب المعرفة , و نور العلم.

ولا أنسى أن أخص بالشكر و الدعاء عائلتي العزيزة سندي الأول و الدائم, مصدر الحب و الدعم و الدعاء و التي كانت دوما النور الذي أضاء لي الطريق و المأوى وقت التعب و الضيق.

أسأل الله أن يوفقني و يوفق الجميع لما فيه الخير و الصلاح، وأن يجعل هذا العمل نافعا، خالصا لوجهه الكريم.

# Abstract:

In this work, we investigated the stability of solutions using the energy method, and finite time blow-up of solutions through a direct approach based on Lyapunov functionals, for certain classes of nonlinear evolution equations.

First, the theoretical concepts and methodological tools introduced in this study provide a foundational framework for the detailed results and analyses presented in the subsequent chapters.

Next, we established the asymptotic stability of the solution to a Kirchhoff type viscoelastic wave equation that includes Balakrishnan Taylor damping and a time delay term. By constructing an appropriate Lyapunov functional, employing differential inequalities, we proved that the total energy of the system decays to zero over time, despite the additional complexity introduced by the memory and delay effects.

Finally, we examined the finite time blow up of solutions for a problem involving a time delay term. Our objective was to determine sufficient conditions under which the solution either decays to zero as time approaches infinity or blows up in finite time, depending on the interplay between the source and damping terms in the evolution equation.

From our literature review, we observed that classical models often incorporate nonlinear damping and first order perturbation terms, both of which have been extensively studied.

***Keywords*** : stability, Lyapunov functional, explosion.

# Résumé :

Dans ce travail, nous avons étudié la stabilité des solutions en utilisant la méthode de l'énergie, ainsi que le phénomène d'explosion en temps fini des solutions à travers une approche directe basée sur des fonctionnelles de Lyapunov, pour certaines classes d'équations d'évolution non linéaires.

Tout d'abord, les concepts théoriques et les outils méthodologiques présentés dans cette étude fournissent un cadre fondamental pour les résultats détaillés et les analyses développées dans les chapitres suivants.

Ensuite, nous avons établi la stabilité asymptotique de la solution d'une équation d'onde viscoélastique de type Kirchhoff, incluant un terme d'amortissement de Balakrishnan-Taylor ainsi qu'un terme de retard temporel.En construisant une fonctionnelle de Lyapunov appropriée et en utilisant des inégalités différentielles, nous avons démontré que l'énergie totale du système décroît jusqu'à s'annuler avec le temps, malgré la complexité introduite par les effets de mémoire et de retard.

Enfin, nous avons examiné l'explosion en temps finie des solutions d'un problème comportant un terme de retard. Notre objectif était de déterminer les conditions suffisantes permettant à la solution soit de tendre vers zéro lorsque le temps fini, selon l'interaction entre le terme source et le terme d'amortissement dans l'équation évolution.

D'après notre revue de la littérature, nous avons observé que les modèles classiques incluent souvent un amortissement non linéaire et des termes de perturbation du premier ordre, qui ont été largement étudiés.

***Mots clés*** : stabilité, fonctionnelle de Lyapunov, explosion.

# ملخص :

قمنا في هذا العمل بدراسة استقرار الحلول باستخدام طريقة الطاقة، و تفجير الحلول في الزمن المنته من خلال نهج مباشر يعتمد على دوال ليابونوف لفئات معينة من معادلات التطور غير الخطية.

اولا، توفر المفاهيم النظرية و الأدوات المنهجية المقدمة في هذه الدراسة إطارا تأسيسيا للنتائج و التحليلات التفصيلية المقدمة في الفصول اللاحقة.

بعد ذلك، أثبتنا الاستقرار التقاربي لحل معادلة موجية لزجة مرنة من نوع كيرشوف تتضمن تخميد بالاكريشنان تايلور و حد تأخير زمني. من خلال بناء دالة ليابونوف مناسبة و استخدام المتباينات التفاضلية، اثبتنا ان الطاقة الكلية للنظام تتضاءل الى الصفر بمرور الوقت، على الرغم من التعقيد الاضافي الذي أدخلته تأثيرات الذاكرة و التأخير.

وأخيرًا، قمنا بفحص تفجير الحلول في الوقت المحدود لمشكلة تتضمن حد التأخير الزمني. كان هدفنا هو تحديد الشروط الكافية التي بموجبها إما أن يتحلل الحل إلى الصفر مع اقتراب الزمن من اللانهاية أو ينفجر في زمن محدود، اعتمادًا على التفاعل بين حد المصدر والتخميد في معادلة التطور.

من خلال مراجعتنا للأدبيات، لاحظنا أن النماذج الكلاسيكية غالبًا ما تتضمن شروط التخميد غير الخطية وشروط الاضطراب من الدرجة الأولى، وكلاهما تمت دراسته على نطاق واسع.



***الكلمات المفتاحية*** : الاستقرار، دالة ليابونوف، الانفجار.

# Contents


**1 Recalls and preliminaries** **7**

1.1 Notations and general notions . . . . . 7

1.2 Some functional spaces . . . . . 8

1.2.1 Normed space . . . . . 8

1.2.2 Complete space . . . . . 8

1.2.3 Banach space . . . . . 9

1.2.4 Hilbert space . . . . . 10

1.2.5 Lebesgue space . . . . . 11

1.2.6 Sobolev space . . . . . 13

1.3 Operators . . . . . 15

1.4 Some fundamental inequalities and formulae . . . . . 16

1.5 Notion of local and global existence . . . . . 17

1.5.1 Local and global existence and explosion . . . . . 17

1.5.2 Lyabunov functional . . . . . 18

**2 Asymptotic stability of solution of the elastic membrane equation with Balakrishnan-Taylor damping and a time delay** **19**

2.1 Introduction . . . . . 19

2.2 Preliminaries . . . . . 20

2.2.1 Equivalent problem . . . . . 22

2.2.2 Unique solution . . . . . 22

**3 Blow up of solution of the elastic membrane equation with Balakrishnan-Taylor damping and time delay** **36**

3.1 preliminaries . . . . . 37
3.2 Finite-time explosion . . . . . 38

## General Introduction

This dissertation is devoted to the study of a nonlinear evolution problem involving wave equations with memory effects and time delays, focusing on both the stability and blow-up behavior of its solutions. Such problems commonly arise in various fields, including plasma physics, thermoelasticity, heat conduction, and chemical engineering. In recent years, significant research attention has been directed toward partial differential equations (PDEs) that incorporate memory and delay terms, owing to their wide-ranging applications in physics, biology, materials science, and engineering.

In particular, viscoelastic systems which are characterized by their natural damping properties and their dependence on the deformation history serve as fundamental examples where memory and delay terms play crucial roles in governing the system's dynamic behavior. These systems exhibit a mix of elasticity and viscosity, meaning they not only respond to current stimuli but also retain influence from past states, a phenomenon that is captured mathematically by integral memory terms. Meanwhile, time delay terms model the presence of delayed feedback or signal transmission lags that are commonly encountered in real-world control systems.

This research focuses specifically on the asymptotic stability and finite-time blow-up behavior of solutions to a class of nonlinear Kirchhoff-type viscoelastic wave equations. The model under investigation includes:

- A Balakrishnan–Taylor damping term, representing a nonlinear structural damping dependent on the gradient of the solution and its rate of change,
- A memory term modeled by a convolution with a positive relaxation function,
- A time-delayed damping term in the velocity component.

It is well established in the literature that the presence of time delay may severely impact the stability of a dynamical system. Even a small delay, if not properly countered by damping mechanisms, can destabilize a system that is otherwise stable in the absence of delay. Therefore, understanding the interplay between nonlinear damping , memory, and delay is vital for predicting the system's long-term behavior and ensuring robustness in practical applications such as aerospace structures, where such equations model the motion of elastic panels in supersonic flows.

The contributions of this work are organized into three main chapters:

**Chapter 1**, titled : Recalls and preliminaries.

In this chapter, we recall the fundamental tools and concepts that will be used in the subsequent chapters.

**Chapter 2**, titled : Asymptotic stability of solution of the elastic membrane equation with Balakrishnan-Taylor damping and a time delay.

The delay problem investigated here is motivated by the panel flutter equation encountered in aerodynamic experiments at supersonic speeds , where the interaction between fluid flow and a thin elastic structure leads to complex oscillatory behavior. In this context, the Balakrishnan–Taylor damping was initially introduced to stabilize the system, but the effects of combining this damping mechanism with time-delayed feedback term remain largely unexplored in the existing literature.

Previous studies have addressed the energy decay and stability properties of viscoelastic equations with memory , including delay terms under certain assumptions. Notably, it was shown that if the delayed damping coefficient is sufficiently small compared to the instantaneous damping, exponential stability may still be achieved. However, when the delay dominates, the solution may exhibit instability. Motivated by these findings, the main objective of this work is to establish the asymptotic stability of the solution to a Kirchhoff-type viscoelastic wave equation with Balakrishnan Taylor damping and a time delay, extending the analysis beyond what has been previously treated in the literature.

**Chapter 3**, titled : Blow up of solution of the elastic membrane equation with Balakrishnan-Taylor damping and time delay.

In this chapter, we are concerned with the finite-time blow-up (explosion) of solutions to a nonlinear viscoelastic wave equation with delay, which models the dynamic behavior of an elastic membrane subjected to internal damping, external forces, and time-delayed feedback. The equation under consideration incorporates both nonlinear structural damping and a memory kernel, alongside a nonlinear source term and a time-delay term in the velocity. These components reflect the complexity of real materials and control systems, especially in aerodynamic applications such as flutter panel experiments conducted in supersonic wind tunnels.

The delay term plays a significant role in the system's stability properties. While damping tends to stabilize solutions, the presence of a nonlinear source term and time delay may induce instabilities that lead to the blow-up of solutions in finite time. Understanding the interaction between dissipative and destabilizing effects is crucial in predicting system behavior and ensuring safe design in engineering applications.

The primary goal of this chapter is to analyze the conditions under which solutions to the proposed delayed viscoelastic equation cease to exist in finite time, i.e., exhibit blow-up behavior.

The analysis is conducted by constructing appropriate energy functionals and employing

analytical techniques inspired by methods developed for nonlinear wave equations with delay.

In this chapter, we focus on the study of the following problem.

$$\begin{cases} u_{tt} - \left[a + b\left\|\nabla u\right\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t))\right]\Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_0 u_t(x,t) \\ \qquad +\mu_1 u_t(x,t-\tau) = |u|^{p-2}\,u & , x \in \Omega, t > 0 \\ u(x,t) = 0 & , \ x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, t > 0, \\ u_t(x,t-\tau) = f_0(x,t), & x \in \Omega, t > 0, t \in (0,\tau). \end{cases}$$

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ (with $n \geq 2$) with a smooth boundary $\partial\Omega$. The constants $a, b, \sigma, \mu_0, \mu_1$ are all positive, and $g(t) > 0, \tau > 0$ represents the time delay. The initial data $u_0, u_1, f_0$ are functions belonging to appropriate functional spaces. The structural viscoelastic damping terms $\left[a + b\left\|\nabla u\right\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t))\right]$ represent the nonlinear rigidity of the membrane.

The first equation of problem is related to the flutter panel equation with control mechanisms involving both memory and time delay. Viscoelastic materials exhibit natural damping due to their distinctive property of retaining the influence of past deformations a phenomenon that arises from the material's memory behavior.

The memory of their past. In particular, this equation arises in wind tunnel experiments at supersonic speeds (see [26]). The model with Balakrishnan–Taylor damping [$(\sigma \neq 0)$ and $g = 0, \mu_0 = 0, \mu_1 = 0$] was initially proposed by Balakrishnan and Taylor [3], and by Bass and Zes [7]. Kirane and Said-Houari [14] considered the viscoelastic wave equation of the form:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_0 u_t(x,t) + \mu_1 u_t(x,t-\tau) = 0,$$

for $x \in \Omega, t > 0$ together with initial and Dirichlet boundary conditions. They established general energy decay results
under the condition that $\mu_1 \leq \mu_0$.
T.A. Apalara, S.A. Messaoudi, and M.I. Mustafa [2] considered type II thermoelasticity with a viscoelastic damping and a delay.

$$\begin{aligned} & u_{tt}(x,t) - \mu_0 u(x,t) - (\mu + \lambda)\nabla(\operatorname{div} u(x,t)) + \beta\nabla\theta(x,t) \\ & + \int_0^t g(s)\Delta u(x,t-s)ds + \mu_0 u_t(x,t) + \mu_1 u_t(x,t-\tau) = 0 \\ & \theta_{tt}(x,t) - \kappa\Delta\theta(x,t) - \delta\Delta\theta_t(x,t) + \beta \operatorname{div} u_{tt}(x,t) = 0, \end{aligned}$$

for $x \in \Omega, t > 0$ together with initial and Dirichlet boundary conditions. They investigated the energy decay in thermoelasticity type III with a viscoelastic damping and a delay term. Recently, Wu [24] studied the nonlinear viscoelastic equation with a linear damping and a delay term of the form

$$|u_t|^p u_{tt} - \Delta u(x,t) - \Delta u_{tt}(x,t) + \int_0^t g(t-s)\Delta u(s)ds + \mu_0 u_t(x,t) + \mu_1 u_t(x,t-\tau) = 0,$$

for $x \in \Omega, t > 0, \rho > 0$ together with initial and Dirichlet boundary conditions. He proved the asymptotic behavior for a
viscoelastic wave equation with a delay for $\mu_1 \leq \mu_0$.
Mi, J.L., Jong, Y.P., and Han Kang, Y. [25] studied the asymptotic stability of the problem.

By addressing both the asymptotic decay and blow-up behavior, this work contributes to the deeper theoretical understanding of nonlinear viscoelastic wave equations with delay, and offers implications for the design and analysis of real-world systems where memory and time-lag effects cannot be ignored.

# Chapter 1

# Recalls and preliminaries

In this chapter, we recall some definitions, some basic functional spaces (Banach space, Hilbert space....), and some fundamental theorems and formulas, as well as the notion of local and global existence.

The aim of this chapter is to use the preliminary results in later chapters.

## 1.1 Notations and general notions

In this section we have defined some of the symbols and operators used in our work.
The gradient of a function $u$ is defined by

$$\operatorname{grad} u = \nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}), \text{ then } |\nabla u|^2 = \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2. \tag{1.1}$$

The divergence of a function is also defined by

$$\operatorname{div} u = \nabla . u = \frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} + ... + \frac{\partial u}{\partial x_n} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}. \tag{1.2}$$

The Laplacian of $u$

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}. \tag{1.3}$$

$\frac{\partial u}{\partial \eta}$ denotes the normal derivative of $u$ outside $\partial\Omega$, i.e.: $\frac{\partial u}{\partial \eta} = \nabla u . \vec{\eta}$, where $\vec{\eta}$ is the unit vector of the normal outside $\Gamma$ [8].
$C^K(\Omega).0 \leq K \leq \infty$ denotes the vector space of functions whose partial derivatives of order less than or equal to $k$ exist and are continuous in $\Omega$ [8].

$C_0^K(\Omega).0 \leq K \leq \infty$ denotes the vector subspace of functions of $C^K(\Omega)$, with compact support in $\Omega$ [19].
The space $C_0^K(\Omega)$ we will also denote $D(\Omega)$, is called the space of test functions on $\Omega$ [19].
$\mathcal{L}(E,F)$ denotes the space of linear and continuous (or bounded) operators from $E$ into $F$ with norm.

## 1.2 Some functional spaces

### 1.2.1 Normed space

**Definition 1.1** *[1] Let $E$ be a vector space over the field $\mathbb{k}$, whith $\mathbb{k} = \mathbb{R}$ or $\mathbb{k} = \mathbb{C}$. a norm on $E$ is an application $\|.\|$ of $E$ in $[0, +\infty[$ satisfying*

1)$\|x\| = 0 \Leftrightarrow x = 0$.
2) $\forall x, y \in E$, we have $\|x + y\| \leq \|x\| + \|y\|$.
3) $\forall \lambda \in \mathbb{k}, x \in E$, we have $\|\lambda x\| = |\lambda| \, \|x\|$.

**Remark 1.1** *$(E, \|.\|_E)$ is a normed space.*

**Definition 1.2** *[17] ( Equivalence of norms )*

Let $E$ be a normed vector space, and the two norms $\|.\|_1$ and$\|.\|_2$ are equivalents.

$$\Leftrightarrow \exists \alpha > 0, \exists \beta > 0, \alpha \, \|.\|_1 \leq \|.\|_2 \leq \beta \, \|.\|_1 .$$

**Definition 1.3** *[10] (Cauchy sequence)*

Let $(E, \|.\|)$ be a normed space, and $(u_n)_{n\in\mathbb{N}^*}$ be a sequence of eliments of $E$ , we say that the sequence $(u_n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N_\varepsilon, \forall n, m \geq N_\varepsilon \Rightarrow \|u_n - u_m\| < \varepsilon.$$

### 1.2.2 Complete space

**Definition 1.4** *[1] A metric space $E$ is complete, if every Cauchy sequence$(u_n)_{n\in\mathbb{N}^*}$ of the space $E$ has a limit in $E$ .*

### 1.2.3 Banach space

Let $(E, \|.\|)$ be a normed space,

**Definition 1.5** *[17] A complete normed vector space is called a Banach space.*

**Definition 1.6** *$E'$ is the linear space of all continuous linear functions*

$$f : E \to \mathbb{R},$$

and is called the dual space of $E$.

**Proposition 1.1** *[7] The space $E$ with norm $\|.\|_E$ defines*

$$\|f\|_E = \sup\{|f(u)| : \|u\| \leq 1\},$$

is also a Banach space.

**Definition 1.7** *Let $(E, \|.\|_E)$ and $(F, \|.\|_F)$ be two Banach spaces such that there exists an injective linear application of $E$ in $F$, this application allows $E$ to be considered as a sub vector space of $F$ and we denote $E \hookrightarrow F$ or $E \subset F$.*

This inclusion is said to be

1. **Continuous**: If there is a constant $C > 0$, such that $\|u\|_F \leq C \|u\|_F$ . We denote $E \hookrightarrow_{continuous} F$ .
2. **Compact**: If for any bounded sequence in $E$ (for the norm of $E$), we can extract a subsequence which converges in $F$ (for the norm of $F$), and we denote $E \hookrightarrow_{compact} F$.
3. **Dense**: If for any $u \in E$ there exists a sequence $(u_n)_{n\in\mathbb{N}^*} \subset E$; such that $\lim_{n\to\infty} u_n = u$ (convergence being for the norm of $F$).

*Weak and strong topology*

**Definition 1.8** *[17] Let $E$ be a Banach space, and $(u_n)_{n\in\mathbb{N}}$ a sequence in $E$, then $u_n$ converges strongly to $u$ in $E$ if and only if*

$$\forall \varepsilon > 0, \exists N > 0 \quad \|u_n - u\|_E \leq \varepsilon.$$

**Definition 1.9** *[17] Let $E$ be a Banach space and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $E$, then $u_n$ converges weakly to $u$ in $E$ if and only if*

$$\forall f \in L_c(E,\mathbb{R}), f(u_n) \to f(u).$$

$$\forall f \in L_c(E,\mathbb{R}), \forall \varepsilon > 0, \exists N_\varepsilon > 0, \forall n \geq N_\varepsilon \quad |f(u_n) - f(u)| \leq \varepsilon.$$

**Proposition 1.2** *[17] If $E$ is of finite dimension, then*

$$\text{Strong convergence} \Longrightarrow \text{weak convergence},$$

but the converse is false.

### 1.2.4 Hilbert space

**Definition 1.10** *[8] Let $E$ be a vector space, the scalar product defined by $\langle u, v\rangle$ is a symmetric, positive definite bilinear form $E \times E$ in $\mathbb{R}$ ,symmetric, positive definite(ie $\langle u, u\rangle \geq 0\ \forall u \in E$ and $\langle u, u\rangle > 0$ if $u \neq 0$).*

**Definition 1.11** *[17] E is a Hilbert space if it is a Banach space for the associated norm*

$$\|u\|_E = \langle u, u\rangle^{1/2} \quad \text{(i.e)} \quad \|u\|_E^2 = \langle u, u\rangle.$$

**Definition 1.12** *A normed vector space with a scalar product is called a pre-Hilbert space.*

**Definition 1.13** *[8] A complete pre-Hilbert space is called a Hilbert space.*

**Definition 1.14** *[8] (Hilbert basis)*

Let $E$ be a Hilbert space for the scalar product $\langle ., .\rangle$. We call a (countable) Hilbertian basis of $E$ is a countable family $\{e_n\}_{n\leq 1}$ of elements of $E$ such that

**(i)** $|e_n| = 1 \forall n, (e_n, e_m) = 0 \forall m, n, m \neq n.$

**(ii)** The vector space generated by this family is dense in $E$.

**Definition 1.15** *[8] (Separable space)*

A metric space is said to be separable if there exists a countable and dense subset $D \subset E$.

**Example 1.1** *The spaces $\mathbb{R}$ and $\mathbb{C}$ are separable (for example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$).*

**Theorem 1.1** *[8] Every separable Hilbert space has a Hilbertian basis.*

**Proposition 1.3** *Let E be a Hilbert space for the scalar product $\langle .,. \rangle$, and let $(e_n)_{n\geq 1}$ be a Hilbertian basis of $E$. There exists a unique sequence $(u_n)_{n\geq 1}$ defined by $u_n = (u, e_n)$, such that the partial sum $\sum_{n=1}^{P} u_n e_n$ converges to $u$ when $p$ tends to infinity.*

Furthermore, we have

$$\|u\|^2 = \langle u, u \rangle = \sum_{n\geq 1} \|\langle u, e_n \rangle\|^2 .$$

Then we write

$$u = \sum_{n\geq 1} (u, e_n) e_n.$$

### 1.2.5 Lebesgue space

**Definition 1.16** *Let $E$ be a Banach space, $1 \leq p \leq \infty$ and $[0, T]$ an interval of $\mathbb{R}$. We call the **Lebesgue** space with values in $E$ and denote by $L^p((0, T), E)$ the space of functions $u : ]0, T[ \to E$ that are measurable and satisfy*

i) if $1 \leq p \leq \infty, \|u\|_{L^p((0,T),E)} = \left( \int_0^T |u(x)|^p \, dx \right)^{1/p} < \infty.$

ii) if $p = \infty, \|u\|_{L^\infty((0,T),X)} = ess \sup_{x\in]0,T[} |u(x)| < \infty.$

**Lemma 1.1** *[15] Let $u \in L^p((0, T), E)$ and $\frac{\partial u}{\partial t} \in L^p((0, T), E)$ where $(1 \leq p \leq \infty)$, then the function $u$ be continuous from $[0, T]$ in $E$ $(i, e)$ $u \in C^1((0, T), E)$.*

1) For $1 \leq p \leq \infty$, $L^p((0, T), E)$ is a Banach space and in particular $L^2((0, T), E)$ is a Hilbert space, when $E$ a Hilbert's space.

2) For $1 \leq p \leq \infty$, if $E$ reflexif, then $L^p((0, T), E)$ is also reflexive.

3) For $1 \leq p \leq \infty$, if $E$ separable, then $L^p((0, T), E)$ is also separable.

**Proposition 1.4** *[9] The space $L^p((0, T), E)$ associated with the norm $\|.\|_{L^p((0,T),X)}$, is a Banach space.*

**Proposition 1.5** *[12] Let $E$ and $F$ be tow Banach spaces, $E \subset F$ with continuous injection then*

$$L^p((0, T), E) \subset L^p((0, T), F),$$

with continuous injection.

**The space $L^p(\Omega)$**

**Definition 1.17** *[8] Let $p \in \mathbb{R}$avec $1 \leq p \leq \infty$, let*

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R}, u \text{ is measurable and } \int_\Omega |u(x)|^p \, dx < +\infty \right\}.$$

Note

$$\|u\|_p = \left( \int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

We will check later that $\|.\|_{L^P}$ is a norm.

**Definition 1.18** *[8] If $p = \infty$, we have*

$$L^p(\Omega) = \left\{ \begin{array}{c} u : \Omega \to \mathbb{R}, u \text{ u is measurable, there exists a positive constant } C \text{ such that} \\ |u(x)| \leq C \quad \text{p.p on } \Omega \end{array} \right\}.$$

We note

$$\|u\|_{L^\infty(\Omega)} = \inf \{C, |u(x)| \leq C \quad \text{p.p on } \Omega\}.$$

**Definition 1.19** *[19] Let be $\Omega$ an open of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. We define the space $L^p_{loc}(\Omega)$ by the set of $f : \Omega$ such that $f \in L^p(\Omega')$, for all $\Omega'$ and whose closure is compact in $\mathbb{R}^n$ .*

**Proposition 1.6** *[22] $L^p$ given its norm $\|.\|_{L^p}$, is a **Banach** space for all $1 \leq p \leq \infty$.*

**Theorem 1.2** *[19] $L^p$ is reflexive if and only if $1 \leq p \leq \infty$.*

**Theorem 1.3** *[19] $L^p$ is separable for $1 \leq p \leq \infty$ .*

**Proposition 1.7** *[22] $L^\infty$ is a **non-separable Banach** space.*

**Lemma 1.2** *[7] If $\mu(\Omega) < \infty$, $1 \leq p \leq \infty$, then $L^q(\Omega) \hookrightarrow L^p(\Omega)$, and*

$$\|u\|_{L^p(\Omega)} \leq \mu(\Omega)^{1/p - 1/q} \|u\|_{L^q(\Omega)}.$$

**Theorem 1.4** *[8] Let $(u_n)$ be a sequence of $L^p$ and $u \in L^p$, such that $\|u_n - u\|_{L^p(\Omega)} \to 0$.*

Then there exists an extracted sub-sequence $(u_{nk})$ such that

$$u_{nk}(x) \to u(x) \text{ p.p on } \Omega.$$

**Remark 1.2** *Suppose $(1 \leq p \leq \infty)$ and $E$ is a reflexive Banach space then weak star convergence is equivalent to weak convergence.*

**Theorem 1.5** *[19] Let be $\Omega$ a bounded open space of $\mathbb{R}^n$.*

1. If $u_n \rightharpoonup^* u$ in $L^\infty(\Omega)$, then $u_n \rightharpoonup u$ in $L^p(\Omega)$, $\forall p \geq 1$.
2. If $u_n \rightharpoonup u$ in $L^p(\Omega)$, then $\|u_n\|_{L^p(\Omega)} \rightharpoonup \|u\|_{L^p(\Omega)}$ in $\mathbb{R}$, for all $1 \leq p \leq \infty$.
3. If $1 \leq p \leq \infty$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$, then $\exists K > 0$ such that $\|u_n\|_{L^p(\Omega)} \leq K$ and

   $$\|u\|_{L^p(\Omega)} \leq \lim_{n\to\infty} \inf \|u_n\|_{L^p(\Omega)}.$$

   The result is also true if $p = \infty$ and $u_n \rightharpoonup^* u$ in $L^\infty(\Omega)$.
4. If $1 \leq p \leq \infty$ and if $\exists K > 0$ such that $\|u_n\|_{L^p(\Omega)} \leq K$, then there exists a sub-sequence $u_{n_i}$, and $u \in L^p(\Omega)$ such that $u_{n_i} \to u \in L^p(\Omega)$ The result is also true if $p = \infty$ and then we have $u_n \rightharpoonup^* u$ in $L^\infty(\Omega)$ .
5. If $1 \leq p \leq \infty$ and $u_n \rightharpoonup u$ in $L^p(\Omega)$, then there exists a sub-suite $u_{n_i}$ such that $u_{n_i} \rightharpoonup u$ p.p and $|u_{n_i}| \leq h$ p.p with $h \in L^p(\Omega)$ .

**Lemma 1.3** *[15] Let $\Omega$ a bounded open of $\mathbb{R}^n \times \mathbb{R}$, $u_m$ and $u$ be functions of $L^p(\Omega)$, $1 < p < \infty$, such that*

$$\|u\|_{L^p(\Omega)} \leq C, u_m \to u \text{ p.p in } \Omega.$$

Then $u_m \to u$ in $L^p(\Omega)$ weakly.

### 1.2.6 Sobolev space

#### Weak derivative

**Definition 1.20** *[19] Let $\Omega$ be an open of $\mathbb{R}^n$ and $u \in L^1_{loc}(\Omega)$, we said that the exist the function $v \in L^1_{loc}(\Omega)$ if $f_i \in L^1_{loc}(\Omega)$ is the weak partial derivative of $u$ with respect to $x_i$ if*

$$\int_\Omega u(x)\varphi(x)dx = -\int_\Omega u(x)\frac{\partial\varphi}{\partial x_i}(x)dx, \forall\varphi \in C_0^\infty(\Omega).$$

By using $v = \frac{\partial u}{\partial x_i}$ or $u = u_{x_i}$.

**Space** $W^{1,p}(\Omega)$

**Definition 1.21** *[8] For any open space of* $\mathbb{R}^n$ *and* $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, *the space* $W^{1,p}(\Omega)$ *is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega), \exists f \in L^p(\Omega) \text{ such that } \int_\Omega u(x)\varphi'(x)dx = -\int_\Omega f(x)\varphi(x)dx, \right\}.$$

We pose

$$H(\Omega) = W^{1,2}(\Omega).$$

**Space** $W^{m,p}(\Omega)$

**Definition 1.22** *[8] Let* $\Omega$ *an open space of* $\mathbb{R}^n$, $m \geq 2$ *and* $p \in \mathbb{R}$, $1 \leq p \leq +\infty$, *the space* $W^{m,p}(\Omega)$ *is defined by*

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega); \begin{array}{c} \forall\alpha \text{ with } |\alpha| \leq m \exists f_\alpha \in L^p(\Omega) \text{ such that} \\ \int_\Omega u D^\alpha\varphi dx = (-1)^{|\alpha|}\int_\Omega f_\alpha\varphi dx, \forall\varphi \in C_0^\infty(\Omega) \end{array} \right\}.$$

where $\alpha \in \mathbb{N}^n, |\alpha| = \sum_{i=1}^{N}\alpha_I$, and $D^\alpha\varphi = \frac{\partial^{\alpha_1+\alpha_2+....+\alpha_N}}{\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}...\partial_{x_N}^{\alpha_N}}\varphi$.

The space $W^{m,p}(\Omega)$ with norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0<|\alpha|\leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

**Definition 1.23** *[8] If* $p = 2$, *we denote by* $W^{m,2}(\Omega) = H^m$ *and* $W_0^{m,2}(\Omega) = H_0^m(\Omega)$, *provided by the norm*

$$\|u\|_{H^m(\Omega)} = \left( \sum_{|\alpha|\leq m} \left( \|\partial^\alpha u\|_{L^2(\Omega)} \right)^2 \right)^{1/2},$$

such that $H^m(\Omega)$ Hilbert space, with the scalar product

$$\langle u,v\rangle_{H^m(\Omega)}=\sum_{|\alpha|\leq m}\langle D^\alpha u,D^\alpha v\rangle_{L^2}=\sum_{|\alpha|\leq m}\int_\Omega \partial^\alpha u\partial^\alpha v dx,\ \text{for all } u,v\in H^m(\Omega).$$

**Proposition 1.8** *[21]*

1) The spaces $W^{m,p}(\Omega)$ are Banach spaces.
2) If $m\geq m^{'}$, $H^m(\Omega)\hookrightarrow H^{m'}(\Omega)$, with continuous injection.
3) if $m=0$ we have $W^{0,p}(\Omega)=L^p(\Omega)$.

## 1.3 Operators

Let $H$ and $H^{'}$ be two Hilbert spaces over $C$.

**Definition 1.24** *[12]Any linear and continuous application $T:H\rightarrow H^{'}$ is called an operator.*

**Definition 1.25** *([12] **Bounded operators** )*

Let $A:X\rightarrow Y$ be an operator, $A$ is said a bounded if there exists a constant $C>0$ such that

$$\|Au\|\leq C\|u\|,\forall u\in X.$$

**Definition 1.26** *(**Symmetric operators** [11])*

The operator $A:H\rightarrow H$ is said a symmetric if $\forall u,v\in D(A)\subset H$.

$$(Au,v)=(v,Au).$$

**Definition 1.27** *(**adjoint operators** [6])*

Let $A:D(A)\subset E\rightarrow F$ A non-bounded operator with dense domain. There exists a unique operator $A^*:D(A^*)\subset F^{'}\rightarrow E^{'}$ called the adjoint of $A$ which satisfies

$$\forall u\in D(A),\forall v\in D(A^*),(v,Au)_{F^{'},F}=(A^*v,u)_{E^{'},E},$$

where

$$D(A^*)=\left\{v\in F^{'},\exists c\geq 0,|(v,Au)|<c|u|_E\,\forall u,v\in D(A)\right\}.$$

**Remark 1.3** *$A$ is self-adjoint if*

$$A^* = A.$$

This implies $D(A) = D(A^*)$.

**Definition 1.28** ***(Monotone operator [8])***

Let $A : D(A) \subset H \to H$ be a non-bounded linear operator. A is said a monotone (some authors say that $A$ is **accretive** or that $-A$ is **dissipative**) if

$$(Av, v) \geq 0, \forall v \in D(A).$$

$A$ is **maximal monotone** if moreover $R(I + A) = H$ ie.

$$\forall f \in H, \exists u \in D(A) \text{ such that } u + Au = f.$$

## 1.4 Some fundamental inequalities and formulae

**Corollary 1.1** ***([18] Green's formula)***

Let be $(\Omega)$ a bounded open of class $C^1$. Then for all functions $u \in C^2(\overline{\Omega})$ and $v \in C^2(\overline{\Omega})$ we have

$$\int_\Omega \Delta u(x)v(x)dx = \int_\Gamma \frac{\partial u}{\partial \eta}(x)v(x)d\Gamma - \int_\Omega \nabla u(x).\nabla v(x)dx, \tag{1.4}$$

Where $\frac{\partial u}{\partial \eta} = \nabla u(x).\eta(x)$ (normal derivative of $u$).

**Lemma 1.4** ***([8] Hölder inequality)***

Let $u \in L^P(\Omega)$ and $v \in L^{P'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $1 \leq p < \infty$, $1 < p' \leq \infty$ Then

$$\int_\Omega |uv|\, dx \leq \|u\|_{L^p(\Omega)} \|v\|_{L^{p'}(\Omega)}.$$

**Corollary 1.2** ***([8] Poincare's inequality)***

Assume that $\Omega$ is a bounded open. Then there is a constant $C$ (depending on $\Omega$ and $p$) such that

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p}, \forall u \in W_0^{1,p}(\Omega) \text{ and } 1 \leq p \leq \infty.$$

**Lemma 1.5** ***([16] Young's inequality)***

Let $u, v \geq 0$, and for all $\delta > 0$, we have

$$uv \leq \frac{\delta^p}{p}u^p + \frac{\delta^{-q}}{q}v^q, \text{ and } \frac{1}{p} + \frac{1}{q} = 1.$$

**Lemma 1.6** *([21] Young's inequality with $\varepsilon$)*

Let $u \geq 0$, $v \geq 0$ and $p$, $q$ be two positive numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. For all $\varepsilon > 0$, we have

$$uv \leq \varepsilon u^p + c(\varepsilon)v^q.$$

**Theorem 1.6** *([10] Cauchy-Shwartz inequality)*

Let $\langle u, v\rangle$ be a scalar product on the real vector space $V$ with norm $\|v\|^2 = \langle u, v\rangle$, then

$$\langle u, v\rangle^2 \leq \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)},$$

for all vectors u and $v \in V$, equality occurring if $u$ and $v$ are linearly dependent.

## 1.5 Notion of local and global existence

### 1.5.1 Local and global existence and explosion

The study of local existence and solutions's uniquenes for the problem of partial differential equations is based on the theory of existence for abstract semi-linear differential equations (see A. Pazy [20]).

Let $(X, \|.\|)$ be a Banach space and let $A : D(A) \subset X \to X$ be a linear operator, $f : X \to X$. Given $u_0 \in X$.
Consider the problem.

$$\begin{cases} \frac{du}{dt} - Au = f, t > 0, \\ u(0) = u_0. \end{cases} \tag{1.5}$$

**Definition 1.29** *[4] A function $u$ of the real variable $t \geq 0$ with values in $X$, is said a local solution of the problem $(1.5)$ ,if there exists $u$ defined on the maximal interval $[0, T^*)$ wich for $t < T^*$ is the unique solution of $(1.5)$ in $C^1([0, T^*), X)$. In particular, one of the following two eventualities occurs.*

*i)* $T^* = \infty$.
*ii)* $T^* < \infty$ and $\lim_{t \to T^*} \|u\| = \infty$.

**Remark 1.4** *[13] if property $(i)$ is satisfied, we say that the solution $u$ is global .*

If $(ii)$ is satisfied , we say that u is an exploding solution in finite time .

### 1.5.2 Lyabunov functional

**Definition 1.30** *[22] We call a Lyapunov functional associated with an evolution system formed by $m$ equations, any function*

$$L : \mathbb{R}^+ \to \mathbb{R}^+,$$

such that

$$\frac{d}{dt}(L(u_1(t,.), ...u_m(t,.))) \leq 0,$$

for all $t > 0$, and any solution $(u_1(t,.), ...u_m(t,.))$ of the system .

# Chapter 2

# Asymptotic stability of solution of the elastic membrane equation with Balakrishnan-Taylor damping and a time delay

## 2.1 Introduction

A delay is used to separate the occurrence between two events, particularly in a mechanical or electronic device. because, in most cases, physical, chemical, biological, thermal and economic phenomena do not depend naturally and solely on the present state, in many natural and engineered systems, the evolution of a process depends not only on its current state but also on its history. This dependency is often captured through delay differential equations , which incorporate time delays to model memory effects inherent in the system.

These delays can arise from various factors such as transmission times, maturation periods, or reaction times, and they play a crucial role in accurately describing the dynamics of the system.

**Real world examples of delay effects:**

**Epidemiology**: In the modeling of infectious diseases, delays are introduced to account for incubation periods and the time it takes for individuals to become infectious. For instance, during the Covid-19 pandemic, models were employed to simulate the spread of the virus, incorporating delays to reflect the time between exposure and the onset of symptoms. These models provided more accurate predictions of the epidemic's trajectory compared to traditional models.

**Biology**: In ecological systems, delays can represent gestation periods or maturation times.

A notable example is the modeling of predator-prey dynamics, where the response of predators to changes in prey population is delayed due to the time required for reproduction and maturation. Such models help in understanding the stability and oscillatory behaviors observed in natural populations.

**Economics**: Economic systems often exhibit delays due to the time it takes for policies to have an effect or for information to be disseminated. For example, in modeling the impact of fiscal policies, delays are incorporated to reflect the time between policy implementation and its effect on the economy. These delays can significantly influence the stability and performance of economic systems.

**Engineering**: In control systems, delays can occur due to the time required for sensors to detect changes and for actuators to respond. These delays can affect the stability and performance of the system. For example, in the design of aircraft autopilot systems, delays are considered to ensure smooth and stable control of the aircraft.

We consider the following initial–boundary value problem with a delay term :

$$\begin{cases} u_{tt}(x,t) - (a+b)\|\nabla u(t)\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t))\Delta u(x,t) \\ + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_0 u_t(x,t) + \mu_1 u_t(x,t-\tau) = 0, x \in \Omega, t > 0 \\ u(x,t) = 0 \qquad\qquad , \ x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \qquad\qquad x \in \Omega \\ u_t(x,t-\tau) = f_0(x,t), \qquad\qquad x \in \Omega, t > 0, t \in (0,\tau). \end{cases} \tag{2.1}$$

Where $\Omega$ is a bounded domain of $\mathbb{R}^n (n \geq 2)$ with a regular boundary $\partial\Omega, u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, the viscoelastic structural damping terms $(a + b \parallel \nabla u(t) \parallel^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t)))$ are denoted by $a, b, \sigma$ is the nonlinear stiffness of the membrane, and $a$ is an in-plane tensile load. $a$, $b$, $\sigma$, $\mu_0$, $\mu_1$ are fixed positive constants. Here the relaxation function $g$ is a positive satisfying the condition given in $(H1)$. $\tau > 0$ represents the time delay and $u_0$, $u_1$, $f_0$ are given functions belonging to suitable spaces. Eq. (2.1) is related to the flutter panel equation with memory term and time delay control.

## 2.2 Preliminaries

**(H1)** $g : [0; \infty) \to (0; \infty)$ is a non-increasing function of class $C^1$,such that

$$a - \int_0^\infty g(s)ds := t > 0, \tag{2.2}$$

and

$$\int_0^\infty g(s)ds < \frac{1}{1 + \frac{1}{(p(1-\beta)^2+2\beta(1-\beta))(p-2)}}, 0 < \beta < 1 \tag{2.3}$$

(**H2**) There exists a non-increasing positive differentiable function $\xi(t)$ , such that

$$g^{'}(t) \leq -\xi(t)g(t), \forall t \geq 0, \tag{2.4}$$

and

$$\int_0^\infty \xi(t)dt = \infty.$$

**(H3)** Let $\delta$ is a positive constant such that

$$\begin{aligned} \tau\mu_1 &< \delta < \tau(2\mu_0 - \mu_1) \text{ if } \mu_1 < \mu_0 \\ \delta &= \mu_0 \quad \text{if} \quad \mu_1 = \mu_0. \end{aligned} \tag{2.5}$$

**Lemma 2.1** : *For any function $g \in C^1(\mathbb{R})$ and $u \in H_0^1(0, T, L^1(\Omega))$, we have*

$$\begin{aligned} \int_\Omega \nabla u_t(t) \int_0^t g(t-s)\nabla u(s)dsdx &= \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}\frac{d}{dt}(g \circ \nabla u)(t) - \frac{1}{2}g(t)\int_\Omega |\nabla u(t)|^2 dx \\ &\quad + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(t)ds \int_\Omega |\nabla u(s)|^2 dx\right), \end{aligned} \tag{2.6}$$

*or*

$$(g \circ u)(t) := \int_0^t g(t-s) \int_\Omega |u(t) - u(s)|^2 dxds.$$

**Lemma 2.2** : *For $u \in L^2(0, T, H_0^1(\Omega))$,we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t) - u(s))ds\right)^2 dx \leq (a-l)C_*^2(g \circ u)(t), \tag{2.7}$$

*where $C_*$ is the Poincaré constant and $l$ is given by* (2.2).

### 2.2.1 Equivalent problem

We introduce the following new dependent variable

$$z(x,\rho,t) = u_t(x,t-\tau\rho), x \in \Omega, \rho \in (0,1),\ t > 0. \tag{2.8}$$

Consequently, we have

$$\tau z_t(x,\rho,t) + z\rho(x,\rho,t) = 0, \text{in } \Omega \times (0,1) \times (0,\infty). \tag{2.9}$$

Therefore, problem (2.1) is equivalent to:

$$\begin{cases} u_{tt}(x,t) - \left(a + b\left\|\nabla u(t)\right\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t))\right)\Delta u(x,t) \\ + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_0 u_t(x,t) + \mu_1 z(x,1,t) = 0, x \in \Omega, t > 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, & x \in \Omega, \rho \in (0,1), t > 0 \\ z(x,0,t) = u_t(x,t), & x \in \Omega, t > 0, \\ u(x,t) = 0 \quad , & x \in \partial\Omega, t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega \\ z(x,\rho,0) = f_0(x,\tau\rho), & x \in \Omega, \rho \in (0,1). \end{cases} \tag{2.10}$$

First, let $\delta$ be a positive constant such that

$$\tau\mu_1 < \delta < \tau(2\mu_0 - \mu_1). \tag{2.11}$$

### 2.2.2 Unique solution

**Theorem 2.1** *Assume that $\mu_0 \leq \mu_1$. Then given $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, $f_0 \in L^2(\Omega \times (0,1))$ and $T > 0$, there exists an unique weak solution $(u,z)$ of the problem (2.10) on $(0,T)$ such that*

$$\begin{aligned} u &\in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)), \\ u_t &\in L^2(0,T; H_0^1(\Omega)) \cap L^2((0,T) \times \Omega). \end{aligned}$$

**Proof.** The proof of this theorem is based of Faedo–Galerkin methods and using the formula (23) of [14] with (2.12), we can prove the existence of the unique solution of problem (2.10) .
By (2.6) and inspired by [14], we define the modified energy functional of problem (2.10) as

follows:

$$E(t) = \frac{1}{2}\left(\| u_t(t) \|_2^2 + \frac{b}{2} \| \nabla u(t) \|^4\right) + \frac{1}{2}\left(a - \int_0^t g(s)ds\right) \| \nabla u(t) \|^2 \tag{2.12}$$
$$+\frac{1}{2}(g \circ \nabla u)(t) + \frac{\delta}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx.$$

Where $\delta$ is a positive constant satisfying (2.11) at $\tau = 1$. ■
Our main result is as follows:

**Theorem 2.2** *Suppose that Hypotheses (H1)–(H3) hold and $\mu_0 > \mu_1$. Then there exist two positive constants $k$, $\nu$ such that, for any solution of problem (2.1), the energy functional satisfies*

$$E(t) \leq k e^{-\nu \int_{t_0}^t \xi(s)ds}, \forall t \geq t_0. \tag{2.13}$$

**Proof.** In this section, we prove our main result. We define the Lyapunov functional

$$L(t) := NE(t) + \varepsilon_1 \Psi(t) + \Phi(t) + \varepsilon_2 I(t) \tag{2.14}$$

where $N, \varepsilon_1, \varepsilon_2$ are positive real numbers which will be chosen later and

$$\Psi(t) = \int_\Omega u(t)u_t(t)dx + \frac{\sigma}{4} \| \nabla u(t) \|^4, \tag{2.15}$$

$$\Phi(t) = \int_\Omega u_t(t) \int_0^t g(t-s)(u(t) - u(s))dsdx, \tag{2.16}$$

$$I(t) = \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho dx. \tag{2.17}$$

We first deduce that, for $\varepsilon_1$, $\varepsilon_2$ small enough and by choosing $N$ large enough, the Lyapunov functional $L(t)$ and the energy $E(t)$ are equivalent in the sense that there exist two positive constants $\beta_1$ and $\beta_2$ such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \forall t \geq 0. \tag{2.18}$$

In fact, we consider the functional

$$H(t) = \varepsilon_1 \Psi(t) + \Phi(t) + \varepsilon_2 I(t),$$

and show that

$$|H(t)| \leq CE(t), \quad \text{for some constant } C. \tag{2.19}$$

Using Young's inequality, Poincaré's inequality and (2.7), we have

$$\begin{aligned}
\Phi(t) &\leq \left| -\int_\Omega u_t(t) \int_0^t g(t-s)(u(t)-u(s))dsdx \right| \\
&\leq \frac{1}{2}\int_\Omega |u_t(t)|^2 + \frac{1}{2}\int_\Omega \left( \int_0^t g(t-s)(u(t)-u(s))ds \right)^2 dx \\
&\leq \frac{1}{2}\int_\Omega |u_t(t)|^2 + \frac{1}{2}(a-l)C_*^2(g \circ \nabla u)(t).
\end{aligned} \tag{2.20}$$

And

$$\begin{aligned}
\varepsilon_1\Psi(t) + \varepsilon_2 I(t) &= \varepsilon_1 \int_\Omega u(t)u_t(t)dx + \frac{\varepsilon_1\sigma}{4}\|\nabla u(t)\|^4 + \varepsilon_2 \int_\Omega \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)d\rho dx. \\
&\leq \frac{\varepsilon_1}{2}\int_\Omega |u_t(t)|^2 dx + \frac{\varepsilon_1 C_*^2}{2}\int_\Omega |\nabla u(t)|^2 dx + \frac{\varepsilon_1\sigma}{4}\|\nabla u(t)\|^4 + \varepsilon_2\widetilde{C}\int_\Omega \int_0^1 z^2(x,\rho,t)d\rho dx.
\end{aligned} \tag{2.21}$$

For some positive constant $\widetilde{C}$ . By using (2.12), (2.20) and (2.21), we get (2.19) for some positive constant $C$. Thus, it is obvious that from (2.14),(2.19) and by choosing $N$ large enough, our result proved. Now, we will estimate the derivative of $L(t)$ according to the following steps.

**Step 1**: Estimate of the derivative of $E(t)$.

Multiplying the first equation in (2.10) by $u_t(t)$, then we have

$$\begin{aligned}
&\int_\Omega u_{tt}u_t dx - \left(a + b\|\nabla u(t)\|^2 + \sigma\langle \nabla u, \nabla u_t\rangle\right)\int_\Omega \Delta u.u_t dx \\
&+ \int_\Omega \int_0^t g(t,s)\Delta u(x,s)u_t(x,t)dxds + \mu_0\int_\Omega u_t u_t dx + \mu_1\int_\Omega z(x,1,t)u_t(x,t)dx \\
= &\frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 - \left(a + b\|\nabla u(t)\|^2 + \sigma\int_\Omega \nabla u.\nabla u_t dx\right)\int_\Omega \Delta u.u_t dx \\
&+ \int_\Omega \int_0^t g(t-s)\Delta u(x,s)u_t(x,t)dxds + \mu_0\int_\Omega u_t u_t dx + \mu_1\int_\Omega z(x,1,t)u_t(x,t)dx.
\end{aligned}$$

By using Green's theorem

$$= \frac{1}{2}\frac{d}{dt}\|u_t(t)\|^2 - \left(a + b\|\nabla u(t)\|^2 + \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)\right)\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2$$
$$-\int_\Omega\int_0^t g(t-s)\nabla u(x,s)\nabla u_t(x,t)dxds + \mu_0\|u_t(t)\|^2 + \mu_1\int_\Omega z(x,1,t)u_t(x,t)dx,$$

then we have

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2 + \left(a + \frac{b}{2}\|\nabla u(t)\|^2\right)\|\nabla u(t)\|^2\right\} + \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2$$
$$+\mu_0\|u_t(t)\|^2 + \mu_1\int_\Omega z(x,1,t)u_t(x,t)dx$$
$$= \int_\Omega \nabla u_t(t) + \int_0^t g(t-s)\nabla u(s)dsdx. \tag{2.22}$$

Now, we use (2.6), the term on the right-hand side of (2.22) can be rewritten as follows:

$$\int_\Omega \nabla u_t(t)\int_0^t g(t-s)\nabla u(s)dsdx = \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2$$
$$-\frac{1}{2}\frac{d}{dt}(g\circ\nabla u)(t) + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2$$
$$= \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\,\|\nabla u(t)\|^2 - (g\circ\nabla u)(t)\right]$$
$$+\frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2. \tag{2.23}$$

Consequently, from (2.22) and (2.23) we derive

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2 + \left(a + \frac{b}{2}\|\nabla u(t)\|^2 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2\right\} + \sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 + \mu_0\|u_t(t)\|^2$$
$$+\mu_1\int_\Omega z(x,1,t)u_t(x,t)dx$$
$$= \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\,\|\nabla u(t)\|^2 - (g\circ\nabla u)(t)\right) + \frac{1}{2}(g'\circ\nabla u)(t) - \frac{1}{2}g(t)\|\nabla u(t)\|^2,$$

so

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^2+\left(a+\frac{b}{2}\|\nabla u(t)\|^2-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2+(g\circ\nabla u)(t)\right\}\\ =\;&-\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2+\mu_0\|u_t(t)\|^2+\mu_1\int_\Omega z(x,1,t)u_t(x,t)dx-\frac{1}{2}g(t)\|\nabla u(t)\|^2+\frac{1}{2}(g'\circ\nabla u)(t).\end{aligned} \tag{2.24}$$

We multiply the second equation in (2.10) by $\delta z$ and integrate the result over $\Omega\times(0,1)$, we see that

$$\begin{aligned}\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)&=0\\ &\Leftrightarrow\ \tau z_t(x,\rho,t)=-z_\rho(x,\rho,t)\\ &\Leftrightarrow\ \tau\delta\int_\Omega\int_0^1 zz_t(x,\rho,t)d\rho dx=-\delta\int_\Omega\int_0^1 zz_\rho(x,\rho,t)d\rho dx\\ &\Leftrightarrow\ \frac{\tau}{2}\delta\int_\Omega\int_0^1\frac{d}{dt}z^2(x,\rho,t)d\rho dx=\frac{-\delta}{2}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx\\ &\Leftrightarrow\ \frac{\delta}{2}\int_\Omega\int_0^1\frac{d}{dt}z^2(x,\rho,t)d\rho dx=\frac{-\delta}{2\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx,\end{aligned}$$

so

$$\begin{aligned}\frac{\delta}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx&=\delta\int_\Omega\int_0^1 z_t(x,\rho,t)z(x,\rho,t)d\rho dx\\ &=-\frac{\delta}{2\tau}\int_\Omega\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx\\ &=-\frac{\delta}{2\tau}\int_\Omega\left[z^2(x,1,t)-z^1(x,0,t)\right]dx.\end{aligned} \tag{2.25}$$

From (2.8), (2.11), (2.12), (2.24), (2.25) and Young's inequality, we obtain

$$\begin{aligned}\frac{dE(t)}{dt}&\le-\left(\mu_0-\frac{\delta}{2\tau}-\frac{\mu_1}{2}\right)\left\|u'(t)\right\|^2-\left(\frac{\delta}{2\tau}-\frac{\mu_1}{2}\right)\int_\Omega z^2(x,1,t)dx\\ &\quad-\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2-\frac{1}{2}g(t)\|\nabla u(t)\|^2+\frac{1}{2}(g'\circ\nabla u)(t).\\ &\le-m_0\left\{\left\|u'(t)\right\|^2+\int_\Omega z^2(x,1,t)dx\right\}-\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2-\frac{1}{2}g(t)\|\nabla u(t)\|^2+\frac{1}{2}(g'\circ\nabla u)(t)\\ &<0.\end{aligned} \tag{2.26}$$

Where $m_0 = min\{\mu_0 - \frac{\delta}{2\tau} - \frac{\mu_1}{2}, \frac{\delta}{2\tau} - \frac{\mu_1}{2}\} > 0$.

**Step 2**: Estimate of the derivative of $\Psi(t)$.

Using the first equation in (2.10), $u$ direct computation leads to the following identity:

$$\begin{aligned}
\Psi'(t) &= \left(\int_\Omega u(t)u_t(t)dx + \frac{\sigma}{4}\|\nabla u(t)\|^4\right)' \\
&= \int_\Omega u_{tt}(t)u(t)dx + \int_\Omega |u_t(t)|^2 dx + \sigma\|\nabla u(t)\|^2(\nabla u(t), \nabla u_t(t)) \\
&= \|u_t(t)\|^2 - \left(a + b\|\nabla u(t)\|^2\right)\|\nabla u(t)\|^2 + \int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dxdt \\
&\quad + \int_0^t g(s)ds\|\nabla u(t)\|^2 - \mu_0\int_\Omega u_t(t)u(t)dx - \mu_1\int_\Omega z(x,1,t)u(t)dx. \qquad (2.27)
\end{aligned}$$

Now, by using Young's inequality and (2.7), we deduce

$$\begin{aligned}
&\int_\Omega \nabla u(t)\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))dsdx \\
&\leq \eta\|\nabla u(t)\|^2 + \frac{1}{4\eta}\int_\Omega \left(\int_0^t g(t-s)(\nabla u(s) - \nabla u(t))ds\right)^2 dx \\
&\leq \eta\|\nabla u(t)\|^2 + \frac{(a-l)}{4\eta}(g\circ\nabla u)(t), \forall \eta > 0. \qquad (2.28)
\end{aligned}$$

Thus, from (2.27) and (2.28), we obtain

$$\begin{aligned}
\Psi'(t) &\leq \|u_t(t)\|^2 - \left(a + b\|\nabla u(t)\|^2\right)\|\nabla u(t)\|^2 + \eta\|\nabla u(t)\|^2 \\
&\quad + \frac{(a-l)}{4\eta}(g\circ\nabla u)(t) + \int_0^t g(s)ds\|\nabla u(t)\|^2 - \mu_0\int_\Omega u_t(t)u(t)dx - \mu_1\int_\Omega z(x,1,t)u(t)dx. \qquad (2.29)
\end{aligned}$$

Next, using Young's inequality and Poincaré's inequality, we have, for any $\eta > 0$,

$$\int_\Omega u_t(t)u(t)dx \leq \eta C_*^2\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{4\eta}\int_\Omega |u_t(t)|^2 dx,$$

and

$$\int_\Omega z(x,1,t)u(t)dx \leq \eta C_*^2\int_\Omega |\nabla u(t)|^2 dx + \frac{1}{4\eta}|z(x,1,t)|^2 dx.$$

Consequently, we get

$$\begin{aligned}\Psi'(t) \leq & \left(1+\frac{\mu_0}{4\eta}\right)\|u_t(t)\|^2-\left(a+b\|\nabla u(t)\|^2-\eta+\int_0^t g(s)ds-\eta(\mu_0+\mu_1)C_*^2\right)\|\nabla u(t)\|^2\\ &+\frac{(a-l)}{4\eta}(g\circ\nabla u)(t)+\frac{\mu_1}{4\eta}\int_\Omega |z(x,1,t)|^2\,dx. \end{aligned} \tag{2.30}$$

**Step 3**: Estimate of the derivative of $\Phi(t)$.
Differentiating (2.16) with respect to $t$ and using of the first equation in (2.10), we have

$$\begin{aligned}\Phi'(t) &= \left(\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx\right)'\\ &= -\int_\Omega u_{tt}(t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &\quad -\int_\Omega u_t(t)\left(\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_\Omega u_t(t)\int_0^t g(t-s)u_t(t)dsdx\right)\\ &= -\int_\Omega u_{tt}(t)\int_0^t g(t-s)(u(t)-u(s))dsdx-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx\\ &\quad -\int_\Omega u_t(t)\int_0^t g(t-s)u_t(t)dsdx,\end{aligned}$$

replacing The value of $u_{tt}$ from (2.10) in $\Phi'(t)$, we get

$$\begin{aligned}\Phi'(t) = & -\int_\Omega\left\{\left(a+b\|\nabla u(t)\|^2+\sigma(\nabla u(t),\nabla u_t(t))\right)\Delta u(t)-\int_0^t g(t-s)\Delta u(s)ds-\mu_0u_t(x,t)-\mu_1z(x,1,t)\right.\\ &\left.\times\int_0^t g(t-s)(u(t)-u(s))dsdx-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_0^t g(s)ds\,\|u_t(t)\|^2,\right.\end{aligned}$$

by using Green's theorem,we get

$$\begin{aligned}
\Phi^{'}(t) &= \int_{\Omega}\nabla u(t)\left(a+b\left\|\nabla u(t)\right\|^{2}\right)\times\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&+\int_{\Omega}\nabla u(t)\sigma(\int_{\Omega}\nabla u(t).\nabla u_{t}(t)).\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&+\int_{0}^{t}g(t-s)\nabla u(s)ds\times\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&-\mu_{0}\int_{\Omega}u_{t}(t)\int_{0}^{t}g(t-s)(u(t)-u(s))dsdx-\mu_{1}\int_{\Omega}z(x,1,t)\int_{0}^{t}g(t-s)(u(t)-u(s))dsdx \\
&-\int_{\Omega}u_{t}(t)\int_{0}^{t}g^{'}(t-s)(u(t)-u(s))dsdx-\int_{0}^{t}g(s)ds\left\|u_{t}(t)\right\|^{2} \\
&= \left(a+b\left\|\nabla u(t)\right\|^{2}\right)\int_{\Omega}\nabla u(t).\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&+\int_{0}^{t}g(t-s)\left|\nabla u(t)-\nabla u(s)\right|dsdx\times\int_{0}^{t}g(t-s)\left|\nabla u(t)-\nabla u(s)\right|dsdx \\
&-\int_{\Omega}\int_{0}^{t}g(s)\nabla u(t)\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&+\sigma\left(\int_{\Omega}\nabla u(t).\nabla u_{t}(t)dx\right)\int_{\Omega}\nabla u(t).\int_{0}^{t}g(t-s)(\nabla u(t)-\nabla u(s))dsdx \\
&+\mu_{0}\int_{\Omega}u_{t}(t)\int_{0}^{t}g(t-s)(u(t)-u(s))dsdx+\mu_{1}\int_{\Omega}z(x,1,t)\int_{0}^{t}g(t-s)(u(t)-u(s))dsdx \\
&-\int_{\Omega}u_{t}(t)\int_{0}^{t}g^{'}(t-s)(u(t)-u(s))dsdx-\int_{0}^{t}g(s)ds\left\|u_{t}(t)\right\|^{2},
\end{aligned}$$

so

$$\begin{aligned}\Phi'(t) &= \left(a+b\|\nabla u(t)\|^2-\int_0^t g(s)ds\right)\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\ &+\int_\Omega\left(\int_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,ds\right)^2 dx\\ &+\sigma\left(\int_\Omega \nabla u(t).\nabla u_t(t)dx\right)\int_\Omega \nabla u(t).\int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx\\ &+\mu_0\int_\Omega u_t(t)\int_0^t g(t-s)(u(t)-u(s))dsdx+\mu_1\int_\Omega z(x,1,t)\int_0^t g(t-s)(u(t)-u(s))dsdx\\ &-\int_\Omega u_t(t)\int_0^t g'(t-s)(u(t)-u(s))dsdx-\int_0^t g(s)ds\,\|u_t(t)\|^2\\ &= I_1+I_2+I_3+I_4+I_5+I_6-\int_0^t g(s)ds\,\|u_t(t)\|^2.\end{aligned}$$

Now,we now estimate the right side of (2.31). By using (2.2), (2.7), (2.12), (2.26) and Young's inequality, we derive

$$I_1\le\left(a+b\|\nabla u(t)\|^2-\int_0^t g(s)ds\right)\int_\Omega|\nabla u(t)|\int_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,dsdx. \tag{2.31}$$

Since $\left(a-\int_0^t g(s)ds\right)\|\nabla u(t)\|^2\le 2E(t)\le 2E(0)$and by (2.2), (2.12), (2.26), we see that

$$\|\nabla u(t)\|^2\le\frac{2}{l}E(0).$$

Thus from (2.32), we obtain

$$\begin{aligned}I_1 &\le \left(a+\frac{2b}{l}E(0)\right)\int_\Omega|\nabla u(t)|\int_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,dsdx\\ &\le \eta\|\nabla u(t)\|^2+\frac{1}{4\eta}\left(a+\frac{2b}{l}E(0)\right)(a-l)(g\circ\nabla u)(t).\end{aligned}$$

Next, we obtain the results as follows:

$$I_2=\int_\Omega\left(\int_0^t g(t-s)\,|\nabla u(t)-\nabla u(s)|\,ds\right)^2 dx\le(a-l)(g\circ\nabla u)(t),$$

$$\begin{aligned} I_3 &\leq \sigma \left| \int_\Omega \nabla u(t).\nabla u_t(t)dx \int_\Omega \nabla u(t). \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))dsdx \right| \\ &\leq \sigma^2 \left( \int_\Omega \nabla u(t)\nabla u_t(t)dx \right)^2 \frac{1}{2}\|\nabla u(t)\|^2 + \frac{1}{2l}\int_\Omega \left( \int_0^t g(t-s)(\nabla u(t)-\nabla u(s))ds \right)^2 dx \\ &\leq \sigma^2 E(0) \left( \frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2 \right)^2 + \frac{(a-l)}{2l}(g\circ\nabla u)(t). \end{aligned}$$

And

$$\begin{aligned} I_4 &\leq \mu_0 \left| \int_\Omega u_t(t) \int_0^t g(t-s)(u(t)-u(s))dsdx \right| \\ &\leq \eta\mu_0^2\|u_t(t)\|^2 + \frac{1}{4\eta}(a-l)C_*^2(g\circ\nabla u)(t) \text{ for any } \eta>0, \end{aligned}$$

also

$$\begin{aligned} I_5 &\leq \mu_1 \left| \int_\Omega z(x,1,t) \int_0^t g(t-s)(u(t)-u(s))dsdx \right| \\ &\leq \eta\mu_1^2 \int_\Omega z(x,1,t)dx + \frac{C_*^2}{4\eta}(a-l)(g\circ\nabla u)(t) \text{ for any } \eta>0, \end{aligned}$$

finaly

$$\begin{aligned} I_6 &= \left| -\int_\Omega u_t(t) \int_0^t g'(t-s)(u(t)-u(s))dsdx \right| \\ &\leq \eta\mu_0^2\|u_t(t)\|^2 - C_*^2\left(\frac{g(0)}{4\eta}\right)(g'\circ\nabla u)(t) \text{ for any } \eta>0. \end{aligned}$$

Combining these estimates with (2.31), we get

$$\begin{aligned}
\Phi^{'}(t) \leq\ & \eta \|\nabla u(t)\|^2 + \frac{1}{4\eta}\left(a + \frac{2bE(0)}{l}\right)^2 (a-l)(g\circ\nabla u)(t) \\
& +(a-l)(g\circ\nabla u)(t) + \sigma^2 E(0)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 \\
& +\frac{a-l}{2l}(g\circ\nabla u)(t) + \eta\mu_0^2\|u_t(t)\|^2 + \frac{1}{4\eta}(a-l)C_*^2(g\circ\nabla u)(t) \\
& +\eta\mu_1^2\int_\Omega z^2(x,1,t)dx + \frac{C_*^2}{4\eta}(a-l)(g\circ l)(g\circ\nabla u)(t) \\
& +\eta\|u_t(t)\|^2 - \frac{g(0)}{4\eta}C_*^2(g^{'}\circ\nabla u)(t) - \int_0^t g(s)ds\,\|u_t(t)\|^2 \\
=\ & \sigma^2 E(0)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 + \eta(\mu_0^2+1)\|u_t(t)\|^2 - \int_0^t g(s)ds\,\|u_t(t)\|^2 \\
& +\eta\|\nabla u(t)\|^2 + \left[\frac{1}{4\eta}\left(a+\frac{2bE(0)}{l}\right)^2 + \frac{1}{2l} + 1 + \frac{C_*^2}{2\eta}\right](a-l)(g\circ\nabla u)(t) \\
& -C_*^2\left(\frac{g(0)}{4\eta}\right)(g^{'}\circ\nabla u)(t) + \eta\mu_1^2\int_\Omega z^2(x,1,t)dx \\
\leq\ & \sigma^2 E(0)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 - \left[\int_0^t g(s)ds - \eta(\mu_0^2+1)\right]\|u_t(t)\|^2 \\
& +\eta\|\nabla u(t)\|^2 + \left[\frac{1}{4\eta}\left(a+\frac{2bE(0)}{l}\right)^2 + \frac{1}{2l} + 1 + \frac{C_*^2}{2\eta}\right](a-l)(g\circ\nabla u)(t) \\
& -C_*^2\left(\frac{g(0)}{4\eta}\right)(g^{'}\circ\nabla u)(t) + \eta\mu_1^2\int_\Omega z^2(x,1,t)dx. \qquad (2.32)
\end{aligned}$$

**Step 4**: Estimate of the derivative of $I(t)$.

Differentiating (2.17) with respect to $t$ and using $[eq(2.9)]$,we have

$$\begin{aligned}
\frac{d}{dt}I(t) &= \frac{d}{dt}\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx \\
&= 2\int_\Omega\int_0^1 e^{-2\tau\rho}zz_t(x,\rho,t)d\rho dx \\
&= -\frac{2}{\tau}\int_\Omega\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)d\rho dx \\
&= -2\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx \\
&\quad -\frac{1}{\tau}\left(\int_\Omega\int_0^1 -2\tau e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx - 2\int_\Omega\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)d\rho dx\right) \\
&= -2\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx - \frac{1}{\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx \\
&= -2\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx - \frac{1}{\tau}\int_\Omega \left[e^{-2\tau\rho}z^2(x,\rho,t)\right]_0^1 dx \\
&= -2\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx - \frac{1}{\tau}\int_\Omega \left[e^{-2\tau}z^2(x,1,t) - z^2(x,0,t)\right]dx \ ,
\end{aligned}$$

so

$$\begin{aligned}
\frac{d}{dt}I(t) &= -2\int_\Omega\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)d\rho dx - \frac{C}{\tau}\int_\Omega z^2(x,\rho,t)dx + \frac{1}{\tau}\int_\Omega (u_t(x,t))^2dx \\
&= -2I(t) + \frac{1}{\tau}\int_\Omega (u_t(x,t))^2dx - \frac{C}{\tau}\int_\Omega z^2(x,\rho,t)dx, \qquad (2.33)
\end{aligned}$$

where $C$ is a positive constant.

**Step 5**: Estimate of the derivative of $L(t)$.

Let there exist $t_0 > 0$ such that $0 < g_0 = \int_0^{t_0} g(s)ds \leq \int_0^t g(s)ds$ for all $t \geq t_0$.

By using (2.14), (2.26), (2.30), (2.33) and (2.34), a series of computations yields, for $t \geq t_0$,

$$\begin{aligned}
L^{'}(t) \leq\ & -Nm_0 \parallel u_t(t) \parallel^2 -Nm_0 \int_\Omega z^2(x,1,t)dx - N\sigma \left(\frac{1}{2}\frac{d}{dt} \parallel \nabla u(t) \parallel^2\right)^2 \\
& -\frac{N}{2}g(t)\|\nabla u(t)\|^2 + \frac{N}{2}(g' \circ \nabla u)(t) + \varepsilon_1 \left(1+\frac{\mu_0^2}{4\eta}\right)\|u_t(t)\|^2 \\
& -\varepsilon_1 \left[(a+b\|\nabla u(t)\|^2) - \eta - g_0 - \eta(\mu_0+\mu_1)C_*^2\right]\|\nabla u(t)\|^2 \\
& +\frac{\varepsilon_1(a-l)}{4\eta}(g\circ\nabla u)(t) + \frac{\varepsilon_1\mu_1}{4\eta}\int_\Omega |z(x,1,t)|^2\,dx \\
& +\sigma^2 E(0)\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right) - \left[g_0 - \eta(\mu_0^2+1)\right]\|u_t(t)\|^2 \\
& +\eta\|\nabla u(t)\|^2 + \left[\frac{1}{4\eta}\left(a+\frac{2bE(0)}{l}\right)^2 + \frac{1}{2l} + 1 + \frac{C_*^2}{2\eta}\right](a-l)(g\circ\nabla u)(t) \\
& -C_*^2\left(\frac{g(0)}{4\eta}\right)(g'\circ\nabla u)(t) + \eta\mu_1^2\int_\Omega z^2(x,1,t)dx \\
& +\varepsilon_2\left(-2I(t) + \frac{1}{\tau}\|u_t(t)\|^2 - \frac{C}{\tau}\int_\Omega z^2(x,1,t)dx\right) \\
=\ & -\left[Nm_0 - \varepsilon_1\left(1+\frac{\mu_0}{4\eta}\right) + g_0 - \eta(\mu_0^2+1) - \frac{\varepsilon_2 C}{\tau}\right]\|u_t(t)\|^2 \\
& -\left\{\frac{N}{2}g(t) + \varepsilon_1\left[a+b\|\nabla u(t)\|^2 - \eta - g_0 - \eta(\mu_0+\mu_1)C_*^2\right] - \eta\right\}\|\nabla u(t)\|^2 \\
& +\left[\frac{\varepsilon_1}{4\eta} + \frac{1}{4\eta}\left(a+\frac{2bE(0)}{l}\right)^2 + \frac{1}{2l} + 1 + \frac{C_*^2}{2\eta}\right](a-l)(g\circ\nabla u)(t) \\
& +\left(\frac{N}{2} - \frac{g(0)C_*^2}{4\eta}\right)(g'\circ\nabla u)(t) - \left(Nm_0 - \eta\mu_1^2 - \frac{\varepsilon_2 C}{\tau}\right)\int_\Omega z^2(x,1,t)dx \\
& -[N-\sigma E(0)]\sigma\left(\frac{1}{2}\frac{d}{dt}\|\nabla u(t)\|^2\right)^2 - 2\varepsilon_2 I(t)\ \text{, for any } t \geq t_0.
\end{aligned}$$

Now, choosing $\varepsilon_1, \varepsilon_2, \eta$ small enough and $N > 0$ large enough, we obtain

$$K_1 = Nm_0 - \varepsilon_1\left(1+\frac{\mu_0}{4\eta}\right) + g_0 - \eta(\mu_0^2+1) - \frac{\varepsilon_1 C}{\tau} > 0,$$

$$K_2 = \frac{N}{2}g(t) + \varepsilon_1[a + b \parallel \nabla u(t) \parallel^2 - \eta - g_0 - \eta(\mu_0+\mu_1)C_*^2] - \eta > 0,$$

$$K_3 = \left[\frac{\varepsilon_1}{4\eta} + \frac{1}{4\eta}\left(a+\frac{2bE(0)}{l}\right)^2 + \frac{1}{2l} + 1 + \frac{C_*^2}{2\eta}\right](a-l) > 0,$$

$$K_4 = \frac{N}{2} - \frac{g(0)C_*^2}{4\eta} > 0,$$

$$K_5 = Nm_0 - \eta\mu_1^2 + \frac{\varepsilon_2 C}{\tau} > 0,$$

and

$$K_6 = (N - \sigma E(0))\sigma > 0.$$

Then we arrive at

$$\begin{aligned} L^{'}(t) \leq & -K_1 \parallel u_t(t) \parallel^2 - K_2 \parallel \nabla u(t) \parallel^2 + K_3(g \circ \nabla u)(t) + K_4(g^{'} \circ \nabla u)(t) \\ & -K_5 \int_\Omega z^2(x,1,t)dx - K_6 \left(\frac{1}{2}\frac{d}{dt} \parallel \nabla u(t) \parallel^2\right)^2 - 2\varepsilon_2 I(t) \text{ for any } t \geq t_0. \end{aligned}$$

Since $I(t) \geq 0$ and by (2.4) there exist two positive constants $\gamma_1$ and $\gamma_2$ such that

$$L^{'}(t) \leq -\gamma_1 E(t) + \gamma_2(g \circ \nabla)(t) \ \ \forall t \geq t_0. \tag{2.34}$$

By multiplying equation (2.35) in $\xi(t)$, and considering that $\xi(t)$ is non-negative, along with using equations (2.4) and (2.26), we can observe that.

$$\begin{aligned} \xi(t)L^{'}(t) &\leq -\gamma_1\xi(t)E(t) + \gamma_2\xi(t)(g \circ \nabla u)(t) \\ &\leq -\gamma_1\xi(t)E(t) - \gamma_2(g^{'} \circ \nabla u)(t) \\ &\leq -\gamma_1\xi(t)E(t) - 2\gamma_2 E^{'}(t), \forall t \geq t_0. \end{aligned}$$

Using $\xi^{'}(t) \leq 0$ and (2.18) we get

$$\frac{d}{dt}(\xi(t)L(t) + 2\gamma_2 E(t)) \leq -\gamma_1\xi(t)E(t), \forall t \geq t_0. \tag{2.35}$$

Now, we define

$$F(t) = \xi(t)L(t) + 2\gamma_2 E(t).$$

Since $\xi(t)$ is a non-increasing positive function and by (2.14), (2.18), (2.19) we can easily observe that $F(t)$ is equivalent to $E(t)$.
Thus (2.36) implies that

$$\frac{d}{dt}F(t) \leq -\nu\xi(t)F(t) \text{ for some } \ \nu > 0 \text{ and } \forall t \geq t_0. \tag{2.36}$$

A simple integration of (2.37) over $(t_0, t)$ leads to

$$F(t) \leq F(t_0)e^{-\nu\int_{t_0}^{t}\xi(s)ds} \quad \forall t \geq t_0.$$

As a result, the equivalent relations of $F(t), L(t)$ and $E(t)$ yields the result in Theorem 2.2. ■

# Chapter 3

# Blow up of solution of the elastic membrane equation with Balakrishnan-Taylor damping and time delay

In this chapter, we extend the analytical framework introduced in Chapter 2 by presenting a new problem formulation (3.1). This problem is designed to study the blow-up of solutions to the nonlinear viscoelastic equation with delay, defined by:

$$\begin{cases} u_{tt} - \left[a + b\left\|\nabla u\right\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t))\right]\Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds + \mu_0 u_t(x,t) \\ \qquad +\mu_1 u_t(x,t-\tau) = |u|^{p-2}\,u, & x \in \Omega, t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, t > 0, \\ u_t(x,t-\tau) = f_0(x,t), & x \in \Omega, t > 0, t \in (0,\tau). \end{cases} \tag{3.1}$$

Where $\Omega$ is a bounded domain of $\mathbb{R}^n (n \geq 2)$ with a regular boundary $\partial\Omega, a, b, \sigma, \mu_0,\ \mu_1$ are positive constants and $g(t)$ is a positive function representing the kernel of the memory term. memory satisfies the condition given in **H1**. $\tau > 0$ represents a delayed time, $u_0$, $u_1$, $f_0$ are functions belonging to appropriate spaces.

The equation in $(3.1)_1$ is related to the equation of the flutter panel with a memory term and control of the time delay. Viscoelastic materials exhibit natural damping, because of the particular

property of these memory of their past history. In particular, this equation arises in wind tunnel experiment at supersonic speeds.

## 3.1 preliminaries

Problem (3.1) adheres to the same foundational hypotheses (H1), (H2), and (H3), ensuring consistency with the theoretical structure previously developed.

Additionally, the formulation respects the equalities and inequalities detailed in Lemmas 2.1, 2.2, and 2.3 from Chapter 2, in order to maintain coherence with the established mathematical framework.

We also need the following lemmas :

**Lemma 3.1** *[24]:Let $p \geq 2$, then there exists a positive constant $c_s$, such that*

$$\|u\|_p \leq c_s \|\nabla u\|_2, u \in H_0^1(\Omega).$$

**Lemma 3.2** *[24]:Suppose $p \geq 2$, then there exists a positive constant $C$, such that*

$$\|u\|_p^s \leq C \left( \|\nabla u\|_2^2 + \|u\|_p^p \right), \tag{3.2}$$

*for any $u \in H_0^1(\Omega)$ and $2 \leq s \leq p$.*

By using the same variable introduce by (2.9) and (2.10) in chapter 02, the problem (3.1) is equivalent to:

$$\begin{cases} u_{tt} - \left[a + b\|\nabla u\|_2^2 + \sigma(\nabla u(x,t), \nabla u_t(x,t)\right] \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds \\ \qquad +\mu_0 u_t(x,t) + \mu_1 z(x,1,t) = |u|^{p-2}u, & x \in \Omega, t > 0 \\ \tau z_t(x,\rho,t) + z\rho(x,\rho,t) = 0, & x \in \Omega, t > 0, \rho \in (0,1) \\ z(x,0,t) = u_t(x,t), & x \in \Omega, t > 0 \\ u(x,t) = 0, & x \in \partial\Omega, t \geq 0, \\ u(x,t) = u_0(x), u_t(x,0) = u_1(x), & x \in \Omega, \\ z(x,\rho,0) = f_0(x,t,\rho), & x \in \Omega, t > 0, \rho \in (0,1). \end{cases} \tag{3.3}$$

Let $\delta$ be a positive constant such that

$$\tau\mu_1 < \delta < \tau(2\mu_0 - \mu_1). \tag{3.4}$$

**Theorem 3.1** *[24] : Assume that* $\mu_0 < \mu_1$ *Let* $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega), f_0 \in L^2(\Omega \times (0,1))$ *ant* $T > 0$ *there is a unique weak solution to problem (3.3) on the interval* $(0,T)$ *such that*

$$\begin{aligned} u &\in C([0,T]; H_0^1(\Omega)) \cap C^1([0,T]; L^2(\Omega)). \\ u_t &\in L^2(0,T; H_0^1(\Omega)) \cap L^2((0,T) \times \Omega). \end{aligned}$$

## 3.2 Finite-time explosion

In this section, we study the finite-time explosion of the solution to problem (3.3), with the initial energy positivity condition.
First, we define the energy functional of problem (3.3) as follows

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}\left[\frac{b}{2}\|\nabla u\|_2^2 + \left(a - \int_0^t g(s)ds\right)\right]\|\nabla u\|_2^2 + \frac{\delta}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx \\ &\quad + \frac{1}{2}(g \circ u)(t) - \frac{1}{p}\|u\|_p^p. \end{aligned} \tag{3.5}$$

Where $\delta$ is a positive constant satisfying (3.4) with $\tau = 1$.

**Lemma 3.3** *$E(t)$ is a non-increasing function and*

$$E'(t) \leq -\alpha\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx\right), \text{for } t \geq 0, \tag{3.6}$$

*where $\alpha$ is a positive constant.*

**Proof.** : Multiplying equation $(3.3)_1$ by $u_t$ and integrating over $\Omega$, then by using the Green's formula and formula [(2.6) presented in chapter 02], we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|u_t\|_2^2 + \left(a - \int_0^t g(t-s)ds\right)\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4 + (g \circ \nabla u)\right] + \frac{1}{p}\frac{d}{dt}\|u\|_p^p \\ &= -\frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 - \frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g' \circ \nabla u) - \mu_0\|u_t\|_2^2 + \mu_1\int_\Omega z(x,1,t)u_t dx, \end{aligned} \tag{3.7}$$

so, multiplying equation $(3.3)_2$ by $\frac{\delta}{\tau}z$ and then integrating over $(0,1) \times \Omega$ , we find

$$\frac{\delta}{2}\frac{d}{dt}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx = -\frac{\delta}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\delta}{2\tau}\int_\Omega z^2(x,0,t)dx, \tag{3.8}$$

by summing (3.7) and (3.8), we get

$$\begin{aligned} E'(t) &= -\frac{\sigma}{4}\left(\frac{d}{dt}\|\nabla u\|_2^2\right)^2 - \frac{g(t)}{2}\|\nabla u\|_2^2 + \frac{1}{2}(g'\circ\nabla u) - \mu_0\|u_t\|_2^2 - \mu_1\int_\Omega z(x,1,t)u_t dx \\ &\quad -\frac{\delta}{2\tau}\int_\Omega z^2(x,1,t)dx + \frac{\delta}{2\tau}\int_\Omega z^2(x,0,t)dx, \end{aligned} \tag{3.9}$$

then we can write

$$E'(t) \le -\left(\mu_0 - \frac{\delta}{2\tau}\right)\|u_t\|_2^2 - \frac{\delta}{2\tau}\int_\Omega z^2(x,1,t)dx - \mu_1\int_\Omega z(x,1,t)u_t dx, \tag{3.10}$$

we have

$$-\mu_1\int_\Omega z(x,1,t)u_t dx \le \frac{\mu_1}{2}\int_\Omega z^2(x,1,t)dx + \frac{\mu_1}{2}\|u_t\|_2^2\, dx, \tag{3.11}$$

replacing (3.10) in (3.11), we obtain

$$\begin{aligned} E'(t) &\le -\left(\mu_0 - \frac{\delta}{2\tau} - \frac{\mu_1}{2}\right)\|u_t\|_2^2 - \left(\frac{\delta}{2\tau} - \frac{\mu_1}{2}\right)\int_\Omega z^2(x,1,t)dx, \\ &\le -\alpha_1\left(\|u_t\|_2^2 + \int_\Omega z^2(x,1,t)dx\right), \text{ for all } t \ge 0. \end{aligned} \tag{3.12}$$

$\alpha_1 = \min\left\{\mu_0 - \frac{\delta}{2\tau} - \frac{\mu_1}{2}, \frac{\delta}{2\tau} - \frac{\mu_1}{2}\right\} > 0$. ■

Then, from (3.5) and Lemma 3.1, we have

$$\begin{aligned} E(t) &\ge \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u) - \frac{1}{p}\|u\|_p^p \\ &\ge \frac{1}{2}l\|\nabla u\|_2^2 + \frac{1}{2}(g\circ\nabla u) - \frac{B_1^p}{p}\left(l^{\frac{1}{2}}\|\nabla u\|_2\right)^p \\ &\ge F\left(\sqrt{l\|\nabla u\|_2^2 + (g\circ\nabla u)}\right), \end{aligned} \tag{3.13}$$

where $B_1 = \frac{C_s^p}{l^{\frac{p}{2}}}$, and

$$F(x) = \frac{1}{2}x^2 - \frac{B_1^p}{p}x^p, x > 0. \tag{3.14}$$

**Remark 3.1** *We can verify that the function $F$ is increasing on $(0,\lambda_1)$ ,decreasing on $(\lambda_1,\infty)$, and $F$ has a maximum $\lambda_1 = B_1^{-\frac{p}{p-2}}$ with the maximum value $E_1 = F(\lambda_1) = \frac{p-2}{2p}\lambda_1^2$.*

**Proof.** Derive (3.14) with respect to $t$ and obtain

$$F'(x) = x - B_1^p x^{p-1}, \tag{3.15}$$

let $\lambda_1$ be a positive constant, such that

$$F'(\lambda_1) = 0,$$

from $(3.15)$, we have

$$\lambda_1 = B_1^{-\frac{p}{p-2}},$$

this implies

$$\begin{aligned} F(\lambda_1) &= \frac{1}{2}\lambda_1^2 - \frac{B_1^p}{p}\lambda_1^p, \lambda_1 > 0 \\ &= \frac{1}{2}B_1^{-\frac{2p}{p-2}} - \frac{B_1^p}{p}\left(B_1^{-\frac{p^2}{p-2}}\right) \\ &= \frac{1}{2}B_1^{-\frac{2p}{p-2}} - \frac{1}{p}B_1^{-\frac{2p}{p-2}} \\ &= \frac{p-2}{2p}B_1^{-\frac{2p}{p-2}} \\ &= \frac{p-2}{2p}\lambda_1^2. \end{aligned}$$

Then, the function $F$ is increasing on $(0, \lambda_1)$, and decreasing on $(\lambda_1, \infty)$ . ■

**Lemma 3.4** *[25] : Assume that $p \geq 2, l\|\nabla u\|_2^2 \geq \lambda_1^2$, and **(H1)-(H3)** are verified with $E(0) < E_1$ then there exists $\lambda_2 > \lambda_1$such that, for all $t \in [0, T)$ ,*

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2. \tag{3.16}$$

*and*

$$\|u\|_p^p \geq \frac{B_1^p}{p}\lambda_2^p. \tag{3.17}$$

**Proof.** : If $E(t)$ is non-increasing, then

$$E(t) \leq E(0) < E_1,$$

for all $t > 0$. From the property of $F$, there exists $\lambda_2^{'} < \lambda_1 < \lambda_2$ such that $F(\lambda_2^{'}) = F(\lambda_2) = E(0)$. Then, since $F(0) = 0$,$l\|\nabla u_0\|_2^2 > \lambda_1^2$ and the continuity of $F(\sqrt{l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)})$ , we deduce that

$$l\|\nabla u\|_2^2 + (g \circ \nabla u)(t) \geq \lambda_2^2. \forall t \in [0, T).$$

Moreover, from the definition of $E(t)$ by $(3.5), (3.16)$, and the definition of $F$, we have

$$\|u\|_p^p \geq \frac{1}{2}(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)) - E(0) \geq \frac{1}{2}\lambda_2^2 - F(\lambda_2) = \frac{B_1^p}{p}\lambda_2^p.$$

So we complete the proof. ■

**Theorem 3.2** *Let **(H1)** and **(H2)** be satisfied. Assume $p > 4$; and $u_0, u_1 \in H_0^1(\Omega)$ with $l\|\nabla u\|_2^2 > \lambda_1^2$, $E(0) < \beta E_1$,Then the solution $u$ of problem $(3.3)$ is exploded in finite time $T^*$ estimated by*

$$T^* \leq \frac{(1-\alpha)}{\alpha\Gamma\left[L(0)\right]^{\frac{\alpha}{\alpha-1}}}.$$

*Where $\Gamma$ is a positive constant.*

**Proof.** : By the contradiction, we assume that the solution of problem (3.3) is global then, there exists $K_1 > 0$ such that

$$\|u_t\|_2^2 + \|\nabla u\|_2^4 + \|\nabla u\|_2^2 + \|u\|_p^p \leq K_1.\forall t \geq 0. \tag{3.18}$$

We define

$$H(t) = E_2 - E(t), \tag{3.19}$$

where $E_2 \in (E(0), \beta E_1)$ .
By Lemma 3.3, and $(3.17)$, we have

$$H(t) \geq H(0) = E_2 - E(t) > 0, \tag{3.20}$$

and by $E_1 = \frac{p-1}{2p}\lambda_1^2$ , and $0 < \beta < 1$, we obtain

$$\begin{aligned} H(t) &\leq E_2 - \frac{1}{2}(l\|\nabla u\|_2^2 + (g \circ \nabla u)(t)) + \frac{1}{p}\|u\|_p^p \\ &\leq \beta E_1 - \frac{1}{2}\lambda_2^2 + \frac{1}{p}\|u\|_p^p \\ &\leq E_1 - \frac{1}{2}\lambda_1^2 + \frac{1}{p}\|u\|_p^p \\ &\leq -\frac{1}{2}\lambda_1^2 + \frac{1}{p}\|u\|_p^p \\ &\leq \frac{1}{p}\|u\|_p^p. \end{aligned} \tag{3.21}$$

We therefore define the following functional

$$L(t) = H^{1-\alpha}(t) + \varepsilon\int_\Omega u_t u dx + \frac{\varepsilon}{2}\mu_0\int_\Omega u^2 dx + \varepsilon\frac{\sigma}{4}\left(\int_\Omega |\nabla u|^2 dx\right)^2. \tag{3.22}$$

Where $0 < \varepsilon < 1$ to be determined later and

$$0 < \alpha < \min\left(\frac{p-2}{2p}, \frac{1}{2}\right). \tag{3.23}$$

Deriving (3.22), we obtain

$$\begin{aligned} L^{'}(t) &= (1-\alpha)H^{-\alpha}(t)H^{'}(t)+\varepsilon\left\|u_t\right\|_2^2+\varepsilon\int_\Omega u_{tt}udx \\ &\quad +\varepsilon\mu_0\int_\Omega u_tudx+\varepsilon\frac{\sigma}{2}\left(\frac{d}{dt}\left\|\nabla u\right\|^2\right)\left(\left\|\nabla u\right\|^2\right). \end{aligned} \tag{3.24}$$

using $(3.3)_1$ and Green's formula, we find

$$\begin{aligned} \varepsilon\int_\Omega u_{tt}udx &= -\varepsilon a\left\|\nabla u\right\|_2^2-\varepsilon b\left\|\nabla u\right\|_2^4+\varepsilon\left\|u\right\|_p^p-\mu_1\varepsilon\int_\Omega z(x,1,t)udx-\varepsilon\mu_0\int_\Omega u_tudx \\ &\quad +\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx-\frac{\sigma}{2}\varepsilon\left(\frac{d}{dt}\left\|\nabla u\right\|^2\right)\left(\left\|\nabla u\right\|_2^2\right). \end{aligned} \tag{3.25}$$

Replacing (3.25) in (3.24), we obtain

$$\begin{aligned} L^{'}(t) &= (1-\alpha)H^{-\alpha}(t)H^{'}(t)+\varepsilon\left\|u_t\right\|_2^2-\varepsilon a\left\|\nabla u\right\|_2^2-\varepsilon b\left\|\nabla u\right\|_2^4+\varepsilon\left\|u\right\|_p^p \\ &\quad -\mu_1\varepsilon\int_\Omega z(x,1,t)udx+\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx. \end{aligned} \tag{3.26}$$

We use Hölder's inequality and Young's inequality, for $\delta_1,\eta>0$,

$$-\mu_1\varepsilon\int_\Omega z(x,1,t)udx\geq-\delta_1\mu_1\varepsilon\left\|u\right\|_2^2-\frac{\mu_1}{4\delta_1}\int_\Omega z^2(x,1,t)dx, \tag{3.27}$$

and

$$\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx=\varepsilon\left\|\nabla u\right\|_2^2\int_0^t g(s)ds+\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\left|\nabla u(s)-\nabla u(t)\right|dsdx, \tag{3.28}$$

we can see that

$$\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\left|\nabla u(s)-\nabla u(t)\right|dsdx\geq-\frac{\varepsilon}{4\eta}\left\|\nabla u\right\|_2^2\int_0^t g(s)ds-\eta\varepsilon\left(g\circ\nabla u\right)(t), \tag{3.29}$$

from (3.28) and (3.29)

$$\varepsilon\int_\Omega\nabla u(t)\int_0^t g(t-s)\nabla u(s)dsdx\geq-\eta\varepsilon\left(g\circ\nabla u\right)(t)+\varepsilon\left(1-\frac{1}{4\eta}\right)\int_0^t g(s)ds\left\|\nabla u\right\|_2^2. \tag{3.30}$$

Substituting (3.27) and (3.30) into (3.26), we obtain

$$\begin{aligned} L^{'}(t) \quad \geq \quad & (1-\alpha)H^{-\alpha}(t)H^{'}(t)+\varepsilon\left\|u_t\right\|_2^2+\varepsilon\left[-a-\left(\frac{1}{4\eta}-1\right)\int_0^t g(s)ds\right]\left\|\nabla u\right\|_2^2-\varepsilon b\left\|\nabla u\right\|_2^4 \\ & +\varepsilon\left\|u\right\|_p^p-\eta\varepsilon\left(g\circ\nabla u\right)(t)-\delta_1\mu_1\varepsilon\left\|u\right\|_2^2-\frac{\mu_1}{4\delta_1}\int_\Omega z^2(x,1,t)dx, \end{aligned} \tag{3.31}$$

we choose $\delta_1=\frac{\mu_1}{4\alpha_1 k}H^{\alpha}(t)$ such that $k$ is a large positive constant to be determined later,we obtain

$$-\delta_1\mu_1\varepsilon\left\|u\right\|_2^2=-\frac{\mu_1^2}{4\alpha_1 k}H^{\alpha}(t)\left\|u\right\|_2^2, \tag{3.32}$$

and

$$-\frac{\mu_1}{4\delta_1}\varepsilon\int_\Omega z^2(x,1,t)dx=-\varepsilon\alpha_1 kH^{-\alpha}(t)\int_\Omega z^2(x,1,t)dx, \tag{3.33}$$

now, by (3.6) and (3.19) we have

$$-\int_\Omega z^2(x,1,t)dx\geq-\frac{1}{\alpha_1}H^{'}(t), \tag{3.34}$$

where $\alpha_1$ defines (3.12)
Therefore (3.33) and (3.34) follow

$$-\varepsilon\alpha_1 kH^{-\alpha}(t)\int_\Omega z^2(x,1,t)udx\geq-\varepsilon\alpha_1 kH^{-\alpha}(t)H^{'}(t). \tag{3.35}$$

If we replace (3.32) and (3.35) in (3.31), we see that

$$\begin{aligned} L^{'}(t) \quad \geq \quad & (1-\alpha-\varepsilon k)H^{-\alpha}(t)H^{'}(t)+\varepsilon\left[-a-\left(\frac{1}{4\eta}-1\right)\int_0^t g(s)ds\right]\left\|\nabla u\right\|_2^2 \\ & -\varepsilon b\left\|\nabla u\right\|_2^4-\eta\varepsilon\left(g\circ\nabla u\right)(t)-\frac{\mu_1^2}{4\alpha_1 k}H^{\alpha}(t)\left\|u\right\|_2^2+\varepsilon\left\|u\right\|_p^p, \end{aligned} \tag{3.36}$$

add the term $\varepsilon p(H(t)-E_2+E(t))$ , then use the definition of $E(t)$ to obtain

$$\begin{aligned} L^{'}(t) \quad \geq \quad & (1-\alpha-\varepsilon k)H^{-\alpha}(t)H^{'}(t)+\varepsilon\left(\frac{p+2}{2}\right)\left\|u_t\right\|_2^2+\varepsilon b\left(\frac{p}{4}-1\right)\left\|\nabla u\right\|_2^4+\varepsilon pH(t) \\ & +\varepsilon\left[\frac{(p-2)}{2}a-\left(\frac{(p-2)}{2}+\frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\left\|\nabla u\right\|_2^2+\varepsilon\left(\frac{p}{2}-\eta\right)\left(g\circ\nabla u\right) \\ & -\frac{\mu_1^2}{4\alpha_1 k}H^{\alpha}(t)\left\|u\right\|_2^2+\varepsilon\frac{\delta_1 p}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx-\varepsilon pE_2. \end{aligned} \tag{3.37}$$

Now we take who $\eta$ satisfied

$$\frac{a-l}{2(1-\beta)l(p-2)} < \eta < \frac{p(1-\beta)}{2} + \beta, \tag{3.38}$$

which is possible thanks to $(2.3)$(from chapter 02),
by $(3.38)$, we have

$$\left[\frac{(p-2)}{2}a - \left(\frac{(p-2)}{2} + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\|\nabla u\|_2^2 \geq \beta\frac{(p-2)}{2}l\|\nabla u\|_2^2, \tag{3.39}$$

and

$$\left(\frac{p}{2} - \eta\right)(g\circ\nabla u) \geq \beta\frac{(p-2)}{2}(g\circ\nabla u), \tag{3.40}$$

then, by $(3.39)$ and $(3.40)$, we use the fact that $l\|\nabla u\|_2^2 + (g\circ\nabla u)(t) \geq \lambda_2^2.$, we obtain

$$\begin{aligned}
&\left[\frac{(p-2)}{2}a - \left(\frac{(p-2)}{2} + \frac{1}{4\eta}\right)\int_0^t g(s)ds\right]\|\nabla u\|_2^2 + \left(\frac{p}{2} - \eta\right)(g\circ\nabla u) - pE_2 \\
\geq\ & \beta\frac{(p-2)}{2}\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)\right) - pE_2 \\
=\ & \beta\frac{(p-2)}{2}\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)\right) + \beta\frac{(p-2)}{2}\frac{\lambda_1^2}{\lambda_2^2}\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)\right) - pE_2 \\
\geq\ & C_1\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)\right) + C_2.
\end{aligned} \tag{3.41}$$

where $C_1 = \beta\frac{(p-2)}{2}\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2}$, $C_2 = \beta\frac{(p-2)}{2}\lambda_1^2 - pE_2.$ in addition, from $E_2 < \beta E_1$ and $E_1 = \frac{(p-2)}{2}\lambda_1^2$,we observe that $\beta\frac{\lambda_2^2-\lambda_1^2}{\lambda_2^2} > 0$, and $C_2 = \beta\frac{(p-2)}{2}\lambda_1^2 - pE_2 > \beta\left(\frac{(p-2)}{2}\lambda_1^2 - pE_2\right)\frac{(p-2)}{2}\lambda_1^2 - pE_2 = 0$
So $(3.37)$ becomes

$$\begin{aligned}
L'(t) \geq\ & (1-\alpha-\varepsilon k)H^{-\alpha}(t)H'(t) + \varepsilon C_1\left(l\|\nabla u\|_2^2 + (g\circ\nabla u)\right) - \frac{\mu_1^2}{4\alpha_1 k}\varepsilon H^{\alpha}(t)\|u\|_2^2 \\
&+\varepsilon\left(\frac{p+2}{2}\right)\|u_t\|_2^2 + \varepsilon b\left(\frac{p}{4}-1\right)\|\nabla u\|_2^4 + \varepsilon pH(t) + \varepsilon\frac{\delta_1 p}{2}\int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx.
\end{aligned} \tag{3.42}$$

Using $(3.21)$, we find

$$H^{\alpha}(t) \leq \left(\frac{1}{p}\right)^{\alpha}\|u\|_p^{\alpha p},$$

it means that

$$H^{\alpha}(t)\|u\|_2^2 \leq \left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\|u\|_p^{2+\alpha p}, \tag{3.43}$$

by Lemma 3.2, for $2+\alpha p \leq p$ and (3.43), we obtain

$$H^{\alpha}(t)\left\|u\right\|_2^2 \leq C_s\left[\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\left\|u\right\|_p^p+\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\left\|\nabla u\right\|_2^2\right], \tag{3.44}$$

by decomposing $pH(t)$

$$pH(t)=\left(2a_1+(p-2a_1)\right)H(t), \tag{3.45}$$

where

$$a_1<\min\left(\frac{p-4}{2},C_1\right).$$

We then use the definition of $2\alpha_1 H(t)$, replacing (3.44) and (3.45) in (3.42), we obtain

$$\begin{aligned}
L^{'}(t) \;\geq\; & \left[1-\alpha-\varepsilon k\right]H^{-\alpha}(t)H^{'}(t)+\varepsilon\left(\frac{p+2}{2}-a_1\right)\left\|u_t\right\|_2^2 \\
& +\varepsilon\left(C_1 l-C_s\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\frac{\mu_1^2}{4\alpha_1 k}-a_1 l\right)\left\|\nabla u\right\|_2^2+\varepsilon\left(C_1-a_1\right)\left(g\circ\nabla u\right) \\
& +\varepsilon b\left(\frac{p}{4}-1-\frac{a_1}{2}\right)\left\|\nabla u\right\|_2^4+\varepsilon\left(p-2a_1\right)H(t)+\varepsilon\delta\left(\frac{p}{2}-a_1\right)\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx \\
& +\varepsilon\left(\frac{2a_1}{p}-C_s\frac{\mu_1^2}{4\alpha_1 k}\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\right)\left\|u\right\|_p^p.
\end{aligned} \tag{3.46}$$

We choose the constant $k$, such that

$$\left(C_1-a_1\right)l-c_s\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}\frac{\mu_1^2}{4\alpha_1 k}>0, \tag{3.47}$$

and

$$\frac{2a_1}{p}-c_s\frac{\mu_1^2}{4\alpha_1 k}\left(\frac{1}{p}\right)^{\alpha}|\Omega|^{\frac{p-2}{2}}>0, \tag{3.48}$$

by (3.47) and (3.48), we obtain

$$k>\frac{c_s}{4\alpha_1}p^{-\alpha}|\Omega|^{\frac{p-2}{2}}\max\left(\frac{\mu_1^2}{l\left(C_1-a_1\right)},\frac{p}{2a_1}\right), \tag{3.49}$$

and $\varepsilon$ very small that

$$\varepsilon<\frac{(1-\alpha)}{k}, \tag{3.50}$$

and

$$L(0)=H^{1-\alpha}(0)+\varepsilon\int_{\Omega}u_1u_0dx+\frac{\varepsilon}{2}\mu_0\int_{\Omega}u_0^2dx+\sigma\varepsilon\left\|\nabla u_0\right\|_2^4>0. \tag{3.51}$$

As well as, from (3.46), (3.49) and (3.50) there are $\xi>0$, such as

$$L^{'}(t) \;\geq\; \xi\left[H(t)+\left\|\nabla u\right\|_2^2+\left(g\circ\nabla u\right)+\left\|\nabla u\right\|_2^4+\int_{\Omega}\int_0^1 z^2(x,\rho,t)d\rho dx+\left\|u\right\|_p^p+\left\|u_t\right\|_2^2\right]. \tag{3.52}$$

So, (3.52) implies

$$L(t) \geq L(0) > 0 \text{ for } t \geq 0.$$

The following estimate is

$$\begin{aligned} \left| \int_\Omega u_1 u dx \right| &\leq \|u\|_2 \|u_t\|_2 \\ &\leq C \|u\|_p \|u_t\|_2 , \end{aligned} \tag{3.53}$$

this implies

$$\left| \int_\Omega u_1 u dx \right|^{\frac{1}{1-\alpha}} \leq C \|u\|_p^{\frac{1}{1-\alpha}} \|u_t\|_2^{\frac{1}{1-\alpha}} . \tag{3.54}$$

Still we apply Young's inequality, we get

$$\left| \int_\Omega u_1 u dx \right|^{\frac{1}{1-\alpha}} \leq C_3 \left[ \|u\|_p^{\frac{\mu}{1-\alpha}} + \|u_t\|_2^{\frac{\theta}{1-\alpha}} \right] . \tag{3.55}$$

For $\frac{1}{\mu} + \frac{1}{\theta} = 1$, and $C_3 = C\frac{1-2\alpha}{2(1-\alpha)}$, we choose $\theta = 2(1-\alpha)$, we obtain $\mu = \frac{2(1-\alpha)}{1-2\alpha}$

$$\left| \int_\Omega u_1 u dx \right|^{\frac{1}{1-\alpha}} \leq C_3 \left[ \|u\|_p^s + \|u_t\|_2^2 \right] , \tag{3.56}$$

by (3.23), we have $2 \leq s = \frac{2}{1-2\alpha} \leq p$, we use Lemma 3.2, we get

$$\left| \int_\Omega u_1 u dx \right|^{\frac{1}{1-\alpha}} \leq C_3 \left[ \|u\|_p^s + \|u_t\|_2^2 + \|\nabla u\|_2^2 \right] . \tag{3.57}$$

Therefore, (3.57), we have

$$\begin{aligned} L(t)^{\frac{1}{1-\alpha}} &= \left( H^{1-\alpha}(t) + \varepsilon \int_\Omega u_1 u dx + \frac{\varepsilon}{2}\mu_0 \int_\Omega u^2 dx + \frac{\sigma}{4}\varepsilon \|\nabla u\|_2^4 \right)^{\frac{1}{1-\alpha}} \\ &\leq C_4 \left( H(t) + \|u\|_p^p + \|u_t\|_2^2 + \|u\|_2^{\frac{2}{1-\alpha}} + \|\nabla u\|_2^{\frac{4}{1-\alpha}} \right) , \end{aligned} \tag{3.58}$$

For $t \geq 0$ and $C_4 > 0$,
by (3.18) and (3.20), we have

$$\|\nabla u\|_2^{\frac{4}{1-\alpha}} \leq K_1^{\frac{1}{1-\alpha}} \leq K^{\frac{1}{1-\alpha}} \frac{H(t)}{H(0)}, \text{and } \|u\|_2^{\frac{2}{1-\alpha}} \leq K^{\frac{1}{1-\alpha}} \frac{H(t)}{H(0)}. \tag{3.59}$$

Of $(3.59)$, we obtain

$$L(t)^{\frac{1}{1-\alpha}} \leq C_5 \left( H(t) + \|u\|_p^p + \|u_t\|_2^2 \right), t \geq 0. \tag{3.60}$$

With $C_5 > 0$. By combining $(3.52)$ with $(3.61)$, we conclude that

$$L^{'}(t) \geq \Gamma L(t)^{\frac{1}{1-\alpha}}, t \geq 0. \quad \Gamma > 0, \tag{3.61}$$

this implies

$$\frac{\alpha - 1}{\alpha} \left( L^{\frac{\alpha}{1-\alpha}}(t) \right)^{'} \geq \Gamma,$$

So

$$\left( L^{\frac{\alpha}{1-\alpha}}(t) \right)^{'} \leq \gamma, \quad \gamma = \frac{\alpha - 1}{\alpha} \Gamma. \tag{3.62}$$

By a simple integration of $(3.62)$ on $(0, t)$, we deduce that

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{\alpha}{1-\alpha}}(0) - \Gamma \frac{t\alpha}{1-\alpha}}. \tag{3.63}$$

Since $L(0) > 0, L(t)$ is exploded in finite time, the proof is finished. ■

## Conclusion

This dissertation has explored the dynamic behavior of nonlinear viscoelastic wave equations of Kirchhoff type, focusing on the dual phenomena of asymptotic stability and finite-time blow-up in the presence of memory effects and delayed feedback. The analysis presented highlights the delicate balance between stabilizing mechanisms such as structural damping and memory and destabilizing influences like time delays and nonlinear source terms.

By establishing rigorous conditions for both stability and blow-up, this work contributes to the theoretical foundation necessary for understanding the long-term evolution of viscoelastic systems under realistic physical constraints. It also reinforces the importance of incorporating delay and memory effects into models used in engineering and applied physics, where such features are inherent to material response and control systems.

Looking forward, several avenues remain open for further research. One particularly promising direction is the incorporation of spatial heterogeneity or time-varying coefficients to better model non-uniform or evolving media. Another is the study of feedback control strategies designed to counteract the destabilizing effects of delay, especially in systems governed by high-dimensional PDEs.Another promising avenue would be to study numerical schemes and simulations for such equations, aiming to verify theoretical predictions and provide computational tools for engineers and scientists. Additionally, extending the analysis to include fractional-order derivatives could yield richer models with broader applicability.

Such extensions not only deepen the theoretical understanding of wave phenomena in complex media but also pave the way for more robust and predictive modeling in fields ranging from aerospace to biomedical engineering.

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