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** 【 إِهْدَاء 】 **

۞ بِسْمِ اللَّهِ الرَّحْمٰنِ الرَّحِيم *۞*

۞ الْحَمْدُ لِلَّهِ وَكَفَىٰ ۞

وَالصَّلَاةُ وَالسَّلَامُ عَلَى الْحَبِيبِ الْمُصْطَفَىٰ

۞ أَمَّا بَعْدُ ۞

:فَأُهْدِي هَٰذَا الْعَمَلَ

✧ إِلَى ابْنِي الْحَبِيبِ وَقُرَّةِ عَيْنِي شَاهِين ✧

✧ وَإِلَى زَوْجَتِي الْغَالِيَةِ ✧

✧ وَإِلَى صَدِيقِي وَرَفِيقِ الدَّرْبِ أَيْمَن ✧

۞ وَالسَّلَامُ عَلَى مَنِ اتَّبَعَ الْهُدَى *۞*





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Abstract

In this work, we show a new way of using the averaging theory of first order for studying families of periodic orbits in the extension of the Van der Pol oscillator to a Hamiltonian system of the form

$$\begin{aligned}\dot{x} &= p_y,\\ \dot{y} &= p_x - \mu(1-x^2)y,\\ \dot{p}_x &= -y - \mu 2xyp_y,\\ \dot{p}_y &= -x + \mu(1-x^2)p_y,\end{aligned}$$

where p_x and p_y are the conjugate momenta of the variables x and y, respectively and μ is a small parameter.

In the second part of this work, we provide the maximum number of limit cycles for continuous and discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated either by one or two parallel straight lines.

Keywords: Hamiltonian system, Van der Pol oscillator, Periodic orbit, Differential system, Averaging theory.

Résumé

Dans ce travail, nous présentons une nouvelle méthode d'application de la théorie de moyennisation du premier ordre pour l'étude des orbites périodiques dans une extension de l'oscillateur de Van der Pol à un système Hamiltonien de la forme

$$\begin{aligned}\dot{x} &= p_y,\\ \dot{y} &= p_x - \mu(1-x^2)y,\\ \dot{p}_x &= -y - \mu 2xyp_y,\\ \dot{p}_y &= -x + \mu(1-x^2)p_y,\end{aligned}$$

où p_x et p_y sont les moments conjugués des variables x et y, respectivement, et μ est un petit paramètre.

Dans la deuxième partie de ce travail, nous fournissons le nombre maximum de cycles limites pour les systèmes différentiels continus et discontinus formés par des selles Hamiltoniennes linéaires séparées par une ou deux droites parallèles.

Mots clés: Système Hamiltonien, Oscillateur de Van der Pol, Orbite périodique, Système différentiel, Théorie de moyennisation.

ملخص:

في هذا العمل، نُقدّم طريقةً جديدةً لاستخدام نظرية المتوســـــطات من الرتبة الأولى لدراسة عائلات من المدارات الدورية لامتداد مُذبذب فان دير بول إلى نظام هاميلتون من الشكل

$$\begin{aligned}
\dot{x} &= p_y, \\
\dot{y} &= p_x - \mu(1-x^2)y, \\
\dot{p}_x &= -y - \mu 2xyp_y, \\
\dot{p}_y &= -x + \mu(1-x^2)p_y,
\end{aligned}$$

حيث p_x و p_y هما الزخمان المُترافقان للمتغيرات x و y ، على التوالي، و μ وسيط حقيقي صغيرجدا.

في الجزء الثاني من هذا العمل، ندرس الحد الأقصى لعدد الدورات الحـــــدية لبعض الأنظمة التفاضلية المستوية متعددة التـــــــعريف، سواء كانت مستمرة أو غير مستمرة. تتكون هذه الأنظمة من سروج هاميلتونية خطية، ويفصل بينها إما خط مستقيم واحد أو خطان مستقيمان متوازيان.

الكلمات المفتاحية:
نظام هاميلتون،مذبذب فانديربول،مدارالدوري،نظام تفاضلي،نظريةالمتوسطات.

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General introduction

In the qualitative theory of planar vector fields, closely related to the second part of the well-known Hilbert's 16th problem, the analysis of the existence, number, and the distribution of limit cycles has turned out to be the land of great promise for planar polynomial differential systems of the form

$$\begin{cases} \dfrac{dx}{dt} = P(x, y), \\ \dfrac{dy}{dt} = Q(x, y), \end{cases} \tag{1}$$

where P and Q are polynomials of degree lower than or equal to n.

Let $M(n)$ be the maximal number of small amplitude limit cycles surrounding either an elementary focus or center. On the one hand, estimating $M(n)$ is purely algebraic and algorithmically solvable. Nevertheless, it is still complicated.

In the first place, let us recall some classic literature with respect to limit cycles of elementary singularities. The best-known result is $M(2) = 3$ obtained by [3] in 1952. Around an isolated focus, a number of good results have been obtained for $n = 3$. James and Lloyd [24] described a class of cubic systems with eight limit cycles in 1991. Yu and Corless [54] employed both symbolic and numerical computations to present a cubic system which exhibits nine limit cycles in 2009. Later on, Chen et al. [10] reinvestigated this system and found all real solutions by applying modular computations based on regular chains in 2013. Lloyd and Pearson [36] presented another example of a nonlinear differential system of degree three with nine limit cycles by means of purely symbolic computation in 2012.

Around an isolated center, a lot of excellent conclusions have also been gained for $n = 3$. Żoładek [57] proposed a rational Darboux integral and claimed the existence of 11 limit cycles in 1995, which was reconsidered and proved that this system can actually have only nine limit cycles using up to second-order Melnikov functions. More than a decade later, another two cubic-order systems admitting 11 limit cycles were constructed [5, 12]. Very recently, the system considered in [12] was restudied by Yu and Tian [55] to show the existence of 12 limit cycles, which is the best result obtained so far for cubic polynomial systems with all limit cycles generated from a single singular point to the best of our knowledge. For $n \geq 4$, there are very few results due to the tedious and intricate calculations and simplifications

when computing focal values. Huang et al. [21] gave an example of a quartic system with eight limit cycles bifurcated from a fine focus in 2013. Important results on the center problem for quintic systems were obtained in [14]. Next, let us come back to some classic literature regarding limit cycle bifurcation of nilpotent singularities. Liu and Li [28] proved that eight limit cycles can be created from the eight-order weakened focus of a cubic nilpotent system. Qiu and Li [44] proved that, using two different perturbations, 11 or 12 limit cycles with different distributions could be created from the third-order nilpotent critical point in a class of quartic systems. Li and Jin [26] proved that, using two different perturbations, 14 or 15 limit cycles with different distributions could be created from the third-order nilpotent critical point in a class of quintic systems. Compared with elementary singular points, more limit cycles may be produced from nilpotent singular points of vector fields with the same degree. Therefore, it is more challenging to obtain more limit cycles around an elementary singular point.

On the other hand, studying global bifurcation of limit cycles for planar differential systems is much more difficult. Let $H(n)$ denote the Hilbert number, the upper bound on the number of limit cycles that planar polynomial vector fields can have. Chen and Wang [11] and Shi [46] separately proved $H(2) \geq 4$. The best results known so far for $n = 3$ is $H(3) \geq 13$. Based on counting the number of zeros of some Abelian integrals, Li et al. [27] constructed a planar cubic system and demonstrated that it has at least 13 limit cycles. Motivated by [27], Yang et al. [53] considered an equivalent system and proved its possession of 13 limit cycles, which can be regarded as an improvement of [27] in terms of both simpler and more concrete conditions for the existence of 13 limit cycles and simpler and more direct proof methods. Liu and Li [28, 29] obtained a sufficient condition for the existence of 13 limit cycles in centrally symmetric cubic systems, having the distribution of one large limit cycle bifurcated from the equator that surrounds 12 small limit cycles bifurcated from two symmetric foci. The best results reported so far for quartic-order systems is $H(4) \geq 16$.

This work is organized into four chapters:

The first chapter provides a review of the essential concepts related to the study of dynamical systems that we will need throughout this work.

In the second chapter, we introduce the averaging theory to investigate limit cycles in differential systems, with an example.

In the third chapter, we study the periodic orbits of the extension of the Van der Pol oscillator to a Hamiltonian system of the form

$$\begin{aligned}
\dot{x} &= p_y, \\
\dot{y} &= p_x - \mu(1-x^2)y, \\
\dot{p}_x &= -y - \mu 2xyp_y, \\
\dot{p}_y &= -x + \mu(1-x^2)p_y,
\end{aligned}$$

where p_x, p_y are the conjugate momenta of the variables x and y, respectively and μ is a small parameter.

In the last chapter, we provide the maximum number of limit cycles for continuous and discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated either by one or two parallel straight lines, We show that when these piecewise differential systems are either continuous or discontinuous and are separated by one straight line, or are continuous and are separated by two parallel straight lines, they do not have limit cycles. On the other hand, when these systems are discontinuous and separated by two parallel straight lines, we prove that the maximum number of limit cycles that they can have is one and that this maximum is reached by providing an example of such a system with one limit cycle. When the line of discontinuity of the piecewise differential system is formed by one straight line, the symmetry of the problem allows to take this straight line without loss of generality as the line $x = 0$. Similarly, when the line of discontinuity of the piecewise differential system is formed by two parallel straight lines due to the symmetry of the problem, we can assume without loss of generality that these two straight lines are $x = \pm 1$.

[2]

Chapter 1

Preliminaries

This chapter primarily intends to recall several definitions and results related to differential equations and dynamical systems, most of which will be essential throughout this work.

1.1 Dynamical systems

Definition 1.1.1. *A dynamical system on $\mathbb{R}^n$ is a map $\mathcal{F} : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that*

(1) $\mathcal{F}(., x) : \mathbb{R}^n \to \mathbb{R}^n$ *is continuous,*
(2) $\mathcal{F}(t, .) : \mathbb{R} \to \mathbb{R}^n$ *is continuous,*
(3) $\mathcal{F}(0, x) = x$,
(4) $\mathcal{F}(t + s, x) = \mathcal{F}(t, \mathcal{F}(s, x)) \ \ \forall t, s \in \mathbb{R}, \forall x \in \mathbb{R}^n$.

1.2 Equilibrium point and hyperbolic equilibrium point

Definition 1.2.1. *Consider the nonlinear differential system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

a point $x_0 \in \mathbb{R}^n$ is called an equilibrium point or critical point of (1.1) if $f(x_0) = 0$.

Definition 1.2.2. *Consider the linear system*

$$\dot{x} = Ax \text{ with } A = Df(x_0)$$

this system is called the linearization of (1.1) at x_0.

Definition 1.2.3. *An equilibrium point x_0 is called a hyperbolic equilibrium point of (1.1) if none of the eigenvalues of the matrix $Df(x_0)$ have zero real part.*

1.3 Degenerate and nondegenerate critical points

Definition 1.3.1. *Critical point of the system*

$$\dot{x} = f(x), \tag{1.2}$$

at which $Df(x_0)$ has no zero eigenvalues is called a nondegenemte critical point of the system, otherwise, it is called a degenerate critical point of the system.

1.4 Flow of the nonlinear differential system

Definition 1.4.1. *Consider the nonlinear differential system*

$$\dot{x} = f(x), \tag{1.3}$$

and the initial value problem

$$\dot{x} = f(x), \text{ with } x(0) = x_0, \tag{1.4}$$

with $x \in \mathbb{R}^n$, E an open subset of $\mathbb{R}^n$ and $f \in C^1(E)$. For $x_0 \in E$, let $\phi(t, x_0)$ be the solution of the initial value problem (1.4) defined on its maximal interval of existence $I(x_0)$. Then for $t \in I(x_0)$, the set of mappings ϕ_t defined by

$$\phi_t(x_0) = \phi(t, x_0),$$

is called the flow of the differential system (1.3) or the flow defined by the differential equation (1.3).

1.5 Flow of the linear differential system

Definition 1.5.1. *Consider the linear differential system*

$$\dot{x} = Ax \tag{1.5}$$

and the initial value problem

$$\dot{x} = Ax, \ x(0) = x_0 \text{ with } x \in \mathbb{R}^n, \tag{1.6}$$

where A is a constant matrix $(n \times n)$, the solution of the problem (1.6) is $x(t) = e^{At}x_0$, the set of mappings $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ can be considered as describing the movement of points $x_0 \in \mathbb{R}^n$ along the trajectories of (1.5). This set of mappings is called the flow of the linear system (1.5).

1.6 Locally and globally Lipschitz function

Definition 1.6.1. *Let E be an open subset of $\mathbb{R}^n$. A function $f : E \to \mathbb{R}^n$ is said to satisfy a Lipschitz condition on E if there is a positive constant k such that for all $x, y \in E$*

$$\|f(x) - f(y)\| \leq k \|x - y\|,$$

the function f is said to be locally Lipschitz on E if for each point $x_0 \in E$ there is an ε-neighborhood of x_0, $N_\varepsilon(x_0) \subset E$ and a constant $k_0 > 0$ such that for all $x, y \in N_\varepsilon(x_0)$

$$\|f(x) - f(y)\| \leq k_0 \|x - y\|,$$

by an ε-neighborhood of a point $x_0 \in \mathbb{R}^n$, we mean an open ball of positive radius ε; i.e.,

$$N_\varepsilon(x_0) = \{x_0 \in \mathbb{R}^n, \|x - x_0\| < \varepsilon\}.$$

1.7 Phase plane and phase portrait

Definition 1.7.1. *Consider the planar system*

$$\begin{cases} \dot{x} = P(x, y), \\ \dot{y} = Q(x, y), \end{cases} \tag{1.7}$$

a phase portrait is the set of trajectories in phase space. In particular, for autonomous systems of differential equations in two variables. The solutions $(x(t), y(t))$ of the system (1.7) represent curves called orbits in the plane (x, y). The critical points of this system are constant solutions and the complete figure of the orbits of this system as well as these critical points represent the phase portrait and the plane (x, y) is the phase plane.

1.8 Periodic solution

Definition 1.8.1. *Any solution $\Psi_t(x)$ of the non-linear differential system (1.1) is periodic if there exists a number $T > 0$ such that*

$$\Psi(t + T, x) = \Psi(t, x) \quad \text{for all } t \in \mathbb{R}.$$

The smallest positive real number $T > 0$ satisfying the above formula is called the period.

1.9 Limite cycle

Definition 1.9.1. *A limit cycle of a differential system is a periodic orbit isolated in the set of all periodic orbits of the system.*

1.10 Hamiltonian system

Definition 1.10.1. *Let E be an open subset of $\mathbb{R}^{2n}$ and let $H \in \mathcal{C}^2(E)$ where $H = H(x, y)$ with $x, y \in \mathbb{R}^n$.*
A system of the form

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}, \\ \dot{y} = -\dfrac{\partial H}{\partial x}, \end{cases} \tag{1.8}$$

where

$$\frac{\partial H}{\partial x} = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n}\right)^T,$$
$$\frac{\partial H}{\partial y} = \left(\frac{\partial H}{\partial y_1}, \ldots, \frac{\partial H}{\partial y_n}\right)^T.$$

is called a Hamiltonian system with n degrees of freedom on E.

1.11 Gradient system

Definition 1.11.1. *Let E be an open subset of $\mathbb{R}^n$ and let $V \in \mathcal{C}^2(E)$. A system of the form*

$$\dot{x} = -\operatorname{grad} V(x),$$

where

$$\operatorname{grad} V(x) = \left(\frac{\partial V}{\partial x_1}, \ldots\ldots, \frac{\partial V}{\partial x_n}\right)^T,$$

is called a gradient system on E.

1.12 Conservation of energy

Theorem 1.12.1 (Conservation of energy)**.** *The total energy $H(x, y)$ of the Hamiltonian system (1.8) remains constant along trajectories of (1.8).*

1.13 First integral of the differential system

Definition 1.13.1. *We consider the autonomous differential system*

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

where $f : U \to \mathbb{R}^n$ is $\mathcal{C}^2$, $x \in U$ is an open subset of $\mathbb{R}^n$.
Let $F : U \to \mathbb{R}$ be a non-constant function of class $\mathcal{C}^1$ such that

$$\nabla F(x) \cdot f(x) = 0,$$

where

$$\nabla F(x) = \left(\frac{\partial F(x)}{\partial x_1}, ..., \frac{\partial F(x)}{\partial x_n} \right).$$

Then F is called a first integral of the system $\dot{x} = f(x)$, because F is constant on the solutions this system.

1.14 Van der Pol equation

Definition 1.14.1. *The Van der Pol equation is an ordinary differential equation of the form*

$$\ddot{x} - \mu \left(1 - x^2\right) \dot{x} + x = 0,$$

where μ is a real parameter. The Van der Pol equation can be written in its two-dimensional form

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \mu (1 - x^2) y - x. \end{cases}$$

1.15 Poisson brackets

Definition 1.15.1. *Let us consider two functions $f_1(p, q, t)$ and $f_2(p, q, t)$. The Poisson bracket is defined as*

$$[f_1, f_2] = \sum_{i=1}^{n} \left(\frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i} \right)$$

with $p, q \in U$ is an open subset of $\mathbb{R}^n$, and $t \geq 0$.

Example 1.15.1. *Let $f_1(p, q) = p - q$ and $f_2(p, q, t) = \sin q$. Then the Poisson bracket is given by*

$$[f_1, f_2] = \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial q} = -\cos q.$$

1.16 The phase plane for linear systems of differential equations

We consider the linear system

$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}, \quad \text{imposing that } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

Assuming that det $A \neq 0$; Clearly the origin is the only singular point of this system. The characteristic equation this system at the origin is

$$\det(A - \lambda I) = (\lambda - \lambda_1)(\lambda - \lambda_2),$$

where $\lambda_1$and λ_2 are the eigenvalues (i.e., the roots of the characteristic equation).
Real eigenvalues
Case 1: λ_1,λ_2 **Real, Distinct, and Positive (unstable improper node)**

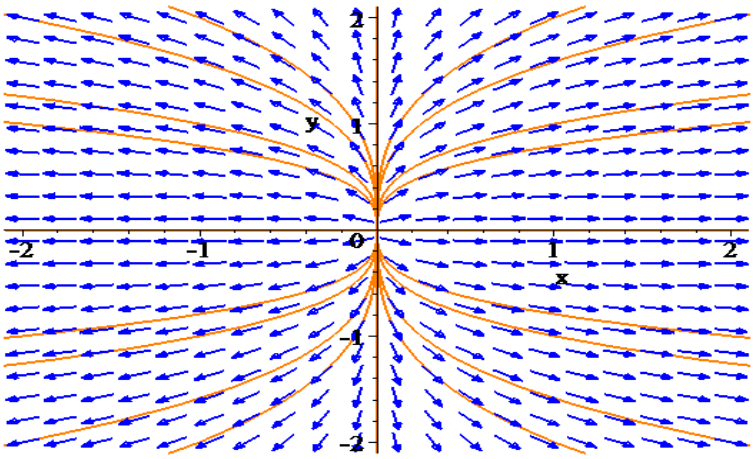


Figure 1.1: Phase plane for an unstable improper node

Case 2: λ_1,λ_2 **Real, Distinct, and Negative (stable improper node)**

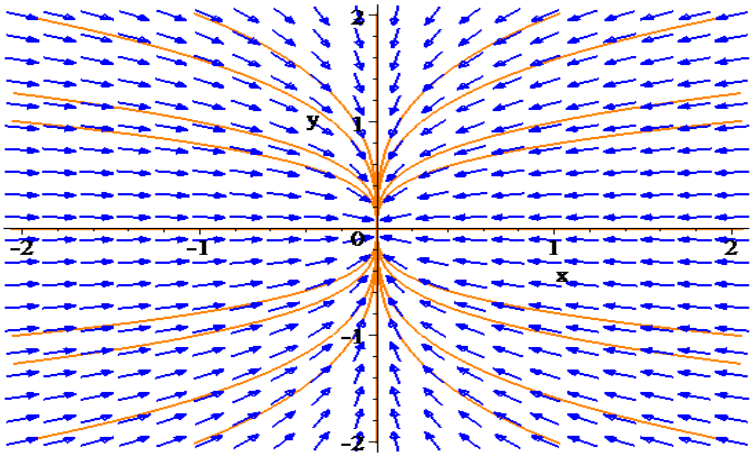


Figure 1.2: Phase plane for a stable improper node

Case 3: λ_1,λ_2 **Real, Opposite Signs (saddle point)**

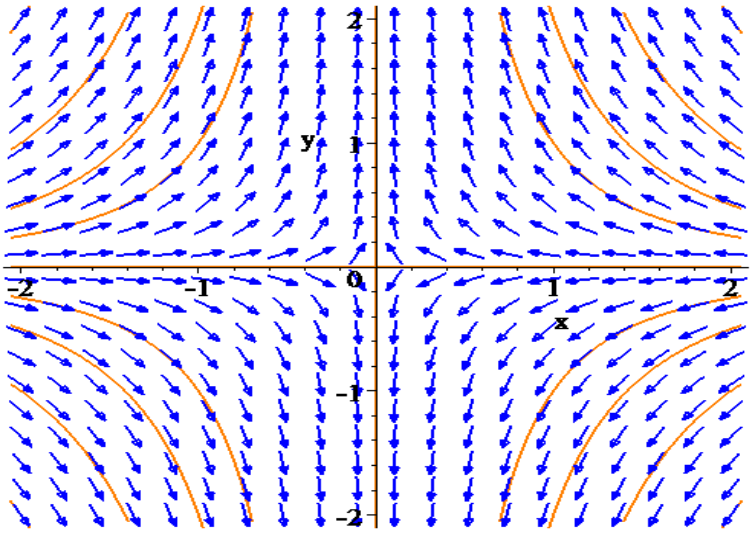


Figure 1.3: Phase plane for a saddle point

Case 4: Equal Eigenvalues $\lambda_1 = \lambda_2 = \lambda$ Two cases are possible
(**a**) A is diagonalizable. If $\lambda > 0$ (**unstable proper node**), if $\lambda < 0$ (**stable proper node**)

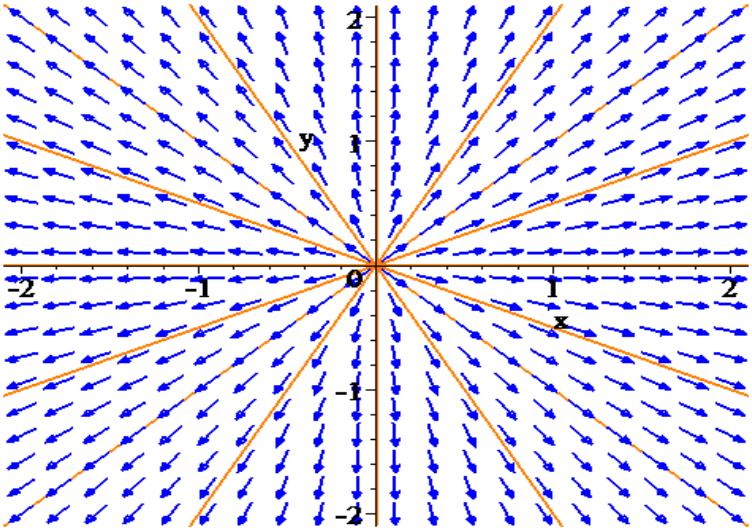


Figure 1.4: Phase plane for an unstable proper node

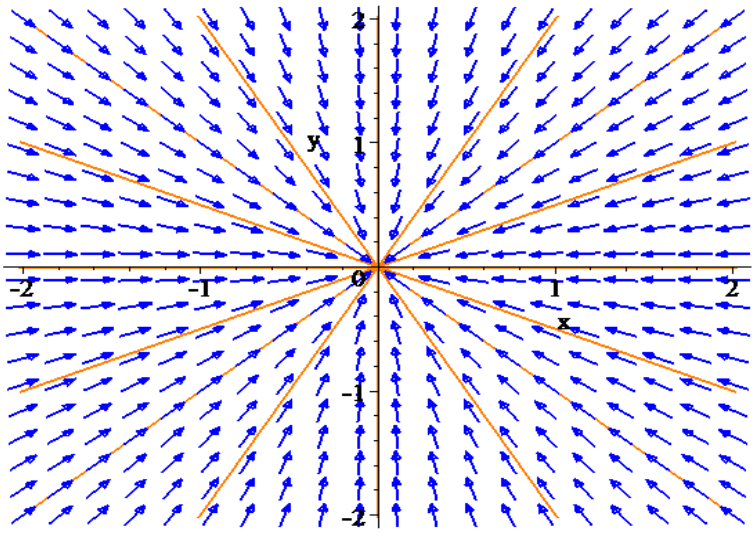


Figure 1.5: Phase plane for a stable proper node

(**b**) A is non-diagonalizable. If $\lambda > 0$ (**unstable degenerate node**), if $\lambda < 0$ (**stable degenerate node**)

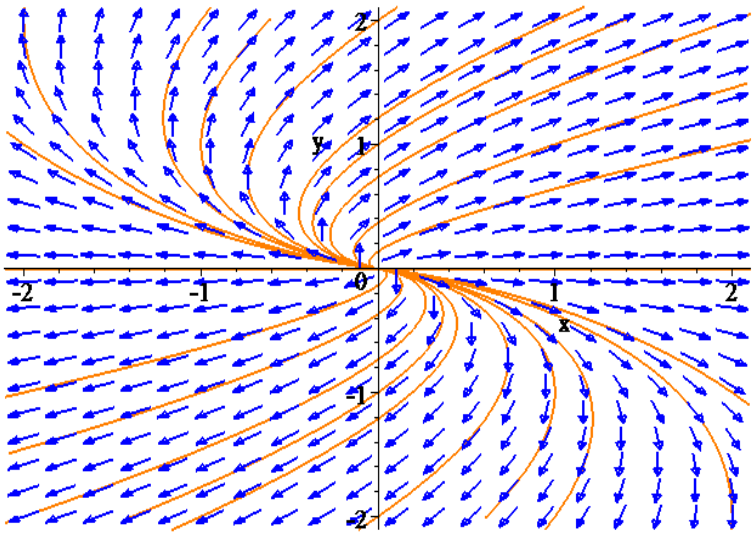


Figure 1.6: Phase plane for an unstable degenerate node

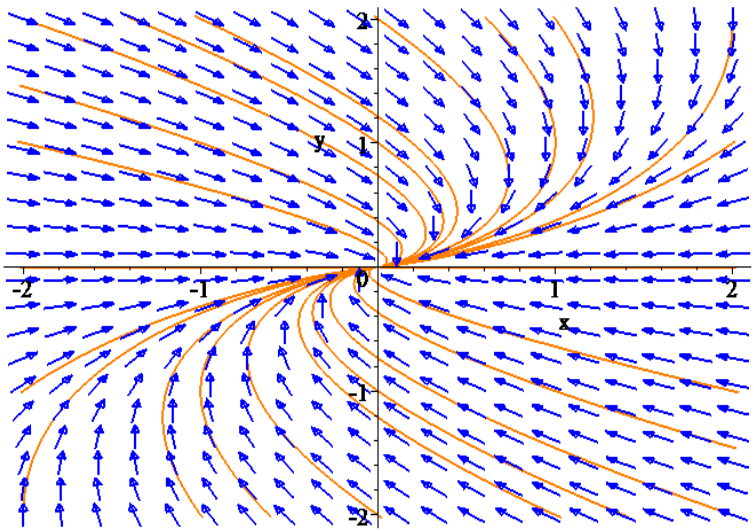


Figure 1.7: Phase plane for a stable degenerate node

Complex eigenvalues
Case 5: Complex eigenvalues with nonzero real part $\lambda_{1,2} = \alpha \pm i\beta$. If $\alpha > 0$ (**unstable spirals**), if $\alpha < 0$ (**stable spirals**)

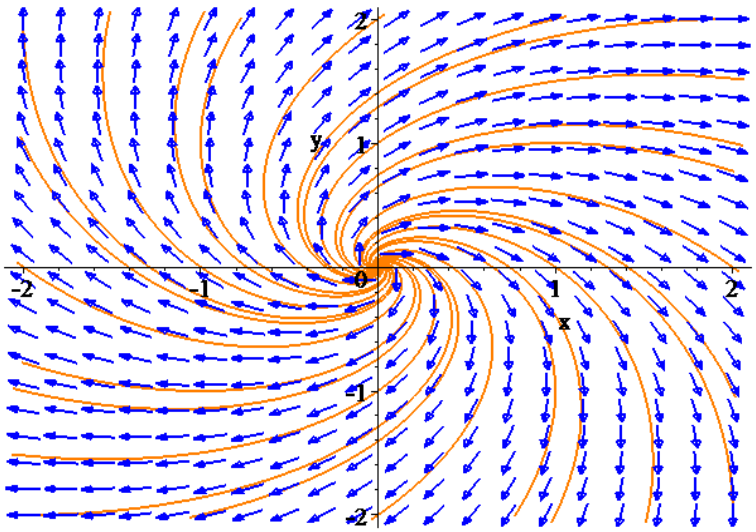


Figure 1.8: Phase plane for an unstable spirals

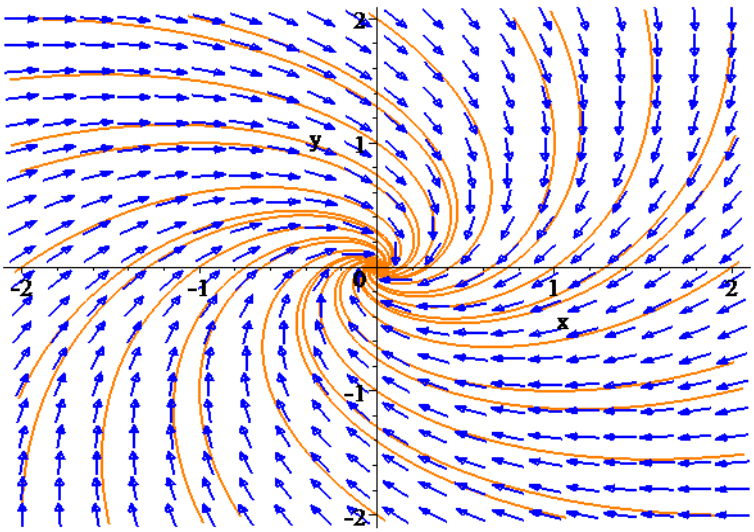


Figure 1.9: Phase plane for a stable spirals

Case 6: Purely Imaginary Eigenvalues $\lambda_{1,2} = \pm i\beta$ (center)

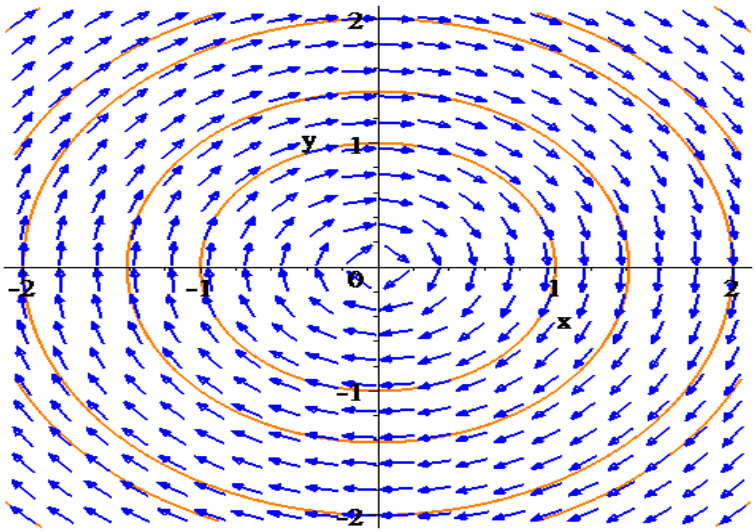


Figure 1.10: Phase plane for a center

1.17 Classification of equilibrium points of nonlinear systems

Definition 1.17.1. *An equilibrium point x_0 of (1.1) is called a sink if all of the eigenvalues of the matrix $Df(x_0)$ have negative real part; it is called a source if all*

of the eigenvalues of $Df(x_0)$ have positive real part; and it is called a saddle if it is a hyperbolic equilibrium point and $Df(x_0)$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

Chapter 2

The averaging theory

Here we summarize the basic results from averaging theory for computing periodic orbits that we shall use for proving Theorem 3.1.1 in the chapter 3, for a proof see Theorem 11.5 of [52], or Theorem 1.2.1 of [31].

2.1 First order averaging theory

We consider the initial value problem

$$\dot{x} = \mu F(\theta, x) + \mu^2 R(\theta, x, \mu), \quad x(0) = x_0 \tag{2.1}$$

with $x \in U$ where U is an open subset of $\mathbb{R}^n$, and $\theta \geq 0$. We assume that $F(\theta, x)$ and $R(\theta, x, \mu)$ are periodic functions in θ of period T. Of course the dot in (2.1) denotes derivative with respect to the variable θ.

Second we consider in the open set U the following initial value problem for the averaged differential system

$$\dot{y} = \mu f(y), \quad y(0) = x_0 \tag{2.2}$$

where $f(x)$ is the averaged function of the function $F(\theta, x)$, i.e.

$$f(x) = \frac{1}{T} \int_0^T F(\theta, x) d\theta \tag{2.3}$$

Theorem 2.1.1. *For the two initial value problems (2.1) and (2.2) we assume:*
(i) the function F is $\mathcal{C}^2$ and bounded by a constant which does not depend on μ in $[0, \infty) \times U$ for all $\mu \in (0, \mu_0]$.
(ii) the functions F and R are periodic in θ of period T, and T is independent of μ.

(iii) the solution $y(\theta)$ of (2.2) belongs to U for all $\theta \in [0, 1/\mu]$.
Then for each equilibrium point p of the averaged system (2.2) satisfying

$$\det\left(\frac{\partial f}{\partial y}\right)\Big|_{y=p} \neq 0 \tag{2.4}$$

there is a periodic solution $x(\theta, \mu)$ of period T for the system (2.1) verifying $x(\theta, \mu) \to p$ as $\mu \to 0$.
The stability of the periodic solution $x(\theta, \mu)$ is given by the stability of the singular point.

Example 2.1.1. *Consider the differential system*

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + \mu\,(7\,x + y - 7\,x^2y + 8\,xy^2 + 6\,x^3y + 7\,xy^3 + 3\,x^4 \\ \quad +4\,y^4 + 3\,x^5 + y^5)\,. \end{cases} \tag{2.5}$$

Using the the polar change of variables $(x, y) = (r\cos\theta, r\sin\theta)$ where $r > 0$, system (2.5) writes

$$\begin{cases} \dot{r} = \mu r\sin(\theta)\left(-8\,r^3(\cos(\theta))^2 - 8\,(\cos(\theta))^3 r^2 + 3\,r^4(\cos(\theta))^5\right. \\ \quad +7\,r^3(\cos(\theta))^4 + 7\,\cos(\theta) + 8\,r^2\cos(\theta) + 4\,r^3 + \sin(\theta)\,r^4(\cos(\theta))^4 \\ \quad -r^3(\cos(\theta))^3\sin(\theta) - 2\,\sin(\theta)\,r^4(\cos(\theta))^2 - 7\,r^2(\cos(\theta))^2\sin(\theta) \\ \quad \left.+7\,\sin(\theta)\,r^3\cos(\theta) + \sin(\theta) + \sin(\theta)\,r^4\right), \\ \dot{\theta} = -1 + \Big(\sin(\theta)\,r^4(\cos(\theta))^5 + 7\,r^3(\cos(\theta))^2\sin(\theta) + \sin(\theta)\,r^4\cos(\theta) \\ \quad -2\,r^4(\cos(\theta))^3\sin(\theta) + \cos(\theta)\sin(\theta) - 7\,r^2(\cos(\theta))^3\sin(\theta) \\ \quad -8\,r^3(\cos(\theta))^3 - 8\,(\cos(\theta))^4 r^2 + 8\,r^2(\cos(\theta))^2 + 3\,r^4(\cos(\theta))^6 \\ \quad \left.+7\,r^3(\cos(\theta))^5 + 7\,(\cos(\theta))^2 - r^3(\cos(\theta))^4\sin(\theta) + 4\,\cos(\theta)\,r^3\right)\mu, \end{cases}$$

this differential system can be reduced to the normal form for applying the averaging theory. Taking θ as the new independent variable we obtain the differential equation

$$\begin{aligned} \frac{dr}{d\theta} &= -\mu r\sin(\theta)\left(-8\,r^3(\cos(\theta))^2 - 8\,(\cos(\theta))^3 r^2 + 3\,r^4(\cos(\theta))^5\right. \\ &\quad + 7\,r^3(\cos(\theta))^4 + 7\,\cos(\theta) + 8\,r^2\cos(\theta) + 4\,r^3 + \sin(\theta)\,r^4(\cos(\theta))^4 \\ &\quad - r^3(\cos(\theta))^3\sin(\theta) - 2\,\sin(\theta)\,r^4(\cos(\theta))^2 - 7\,r^2(\cos(\theta))^2\sin(\theta) \\ &\quad \left.+7\,\sin(\theta)\,r^3\cos(\theta) + \sin(\theta) + \sin(\theta)\,r^4\right) + \mathcal{O}(\mu^2). \\ &= \mu F(\theta, r) + \mathcal{O}(\mu^2). \end{aligned}$$

Now, we compute the averaged function $f(r)$ defined in the Theorem 2.1.1 and we get

$$\begin{aligned} f(r) &= \frac{1}{2\pi}\int_0^{2\pi} F(\theta, r)d\theta, \\ &= -\frac{r\,(5\,r^4 + 8 - 14\,r^2)}{16}, \end{aligned}$$

The positive roots of $f(r)$ *are* $\sqrt{2}$ *and* $\dfrac{2\sqrt{5}}{5}$ *. Since*

$$\frac{df}{dr}\left(\sqrt{2}\right) = -\frac{3}{2} < 0.$$

it follows that the corresponding limit cycle is asymptotically stable.
On the other hand,

$$\frac{df}{dr}\left(\frac{2\sqrt{5}}{5}\right) = \frac{3}{5} > 0.$$

which implies that this limit cycle is unstable.
Therefore, from Theorem 2.1.1, it follows that for $|\mu| \neq 0$ *sufficiently small, system (2.5) has a two limit cycles. See Figure 2.1*

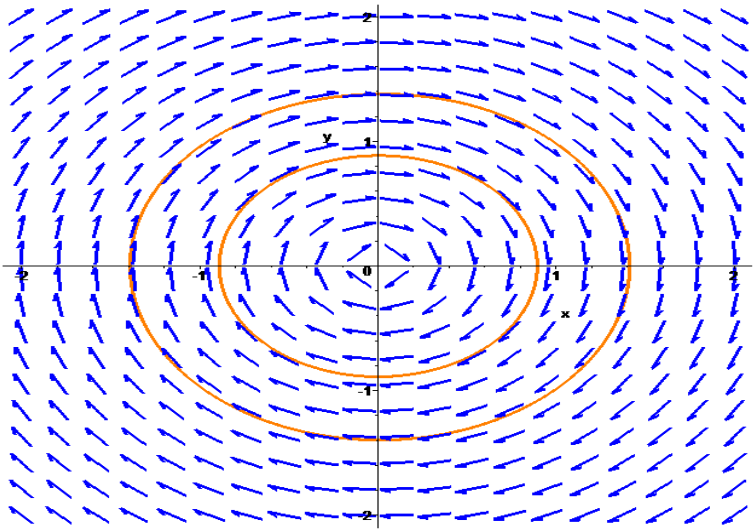


Figure 2.1: Two limit cycles for $\varepsilon = 0.001$

Chapter 3

A family of periodic orbits for the extended Hamiltonian system of the Van der Pol oscillator

In this chapter, we show a new way of using the averaging theory for studying families of periodic orbits of a Hamiltonian system. We do this study computing a new family of periodic orbits of the extension of the Van der Pol oscillator to a Hamiltonian system of two degrees of freedom.

3.1 Introduction and statement of the main results

The classical Van der Pol oscillator is modeled by the second-order differential equation

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \tag{3.1}$$

where $x = x(t)$ is the position coordinate at time t, μ is a parameter, and the dot denotes derivative with respect to the time t, see [49, 50, 51].

Initially the differential equation (3.1) allowed to Van der Pol to explain the stable oscillations observed in electrical circuits employing vacuum tubes. Later on this differential equation has been used for explaining different phenomena in biology, physics, ...Thus in biology was utilised as a model for studying the action potentials of neurons, see for instance [16, 19, 41]. While in physics equation (3.1) was also used in phonation to model the right and left vocal fold oscillators (see [37]), and in seismology to model two plates in a geological fault (see [8]).

More recently in 2015, see equations (9) of [45], the Van der Pol oscillator was

written in the Hamiltonian formalism by extending it to a four-dimensional autonomous differential as follows

$$\begin{aligned} \ddot{x} - \mu(1-x^2)\dot{x} + x = 0, \\ \ddot{y} + \mu(1-x^2)\dot{y} + y = 0. \end{aligned} \tag{3.2}$$

Observe that the motion of the Van der Pol oscillator (3.1) is not affected because the evolution of the variable $x(t)$ is independent of the variable $y(t)$.

Defining $p_y = \dot{x}$ and $p_x = \dot{y}+\mu(1-x^2)y$ from (3.2) we obtain that this differential system of two differential equations of second order admits the following Hamiltonian formalism

$$\begin{aligned} \dot{x} &= p_y, \\ \dot{y} &= p_x - \mu(1-x^2)y, \\ \dot{p}_x &= -y - \mu 2xyp_y, \\ \dot{p}_y &= -x + \mu(1-x^2)p_y, \end{aligned} \tag{3.3}$$

with the Hamiltonian

$$H = p_x p_y + xy - \mu(1-x^2)yp_y. \tag{3.4}$$

Of course p_x and p_y are the conjugate momenta of the variables x and y, respectively. In [2] it is shown that the Hamiltonian system (3.3) establishes a connection between the phase of the limit cycle of the Van der Pol system with the Hannay angle of the Hamiltonian system.

In this work our objective is to study analytically the periodic orbits of the Hamiltonian system (3.3) using the averaging theory. In general the periodic orbits of the Hamiltonian systems are studied numerically because its analytical study is in general difficult and sometimes impossible. Here the technique used with the averaging theory can be extended to many other Hamiltonian systems.

Our main result is the following.

Theorem 3.1.1. *For μ sufficiently small the following statements hold for the Hamiltonian system (3.3).*

(a) For each $h \in \mathbb{R}$ there exists a periodic orbit

$$\gamma_h(t) = \Big(x(t,\mu,h), y(t,\mu,h), p_x(t,\mu,h), p_y(t,\mu,h)\Big)$$

such that when the parameter $\mu \to 0$, $\gamma_h(t)$ tends to the periodic orbit

$$\begin{aligned} &\left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right),\right. \\ &\left. -\mu^2 - \frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right) \end{aligned}$$

of the Hamiltonian system (3.3) with $\mu = 0$.

(b) The periodic orbit $\gamma_h(t)$ live on the energy level $H = h$, and when h varies in R we get a family of periodic orbits.

Theorem 3.1.1 is proved in section 3.3.

In section 3.2 we present the algorithm that we follow for studying the families of periodic orbits of the Hamiltonian systems.

Since the differential system formed by the equations $\dot{x}$ and $\dot{p}_y$ is the first order differential system associated to the Van der Pol oscillator (1.1), and this differential system does not depend on the variables y and p_y, it follows immediately from Theorem 3.1.1 the following corollary.

Corollary 3.1.1. *For μ sufficiently small and for each $h \in \mathbb{R}$, the Van der Pol oscillator has a periodic orbit*

$$\gamma_h(t) = \Big(x(t, \mu, h), p_y(t, \mu, h)\Big)$$

such that when the parameter $\mu \to 0$, $\gamma_h(t)$ tends to the periodic orbit

$$\left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right),$$

solution of the Van der Pol oscillator with $\mu = 0$.

Of course, for a fixed value of μ sufficiently small, the periodic orbit $\gamma_h(t)$ of Corollary 3.1.1 is the classical limit cycle of the Van der Pol oscillator, because it is known that such a limit cycle is unique for a given value of μ.

3.2 The algorithm

The algorithm that we will follow for studying the families of periodic orbits of a Hamiltonian system has the following steps.

1. We select an equilibrium point of the Hamiltonian system having at least one pair of complex conjugate eigenvalues and translate it to the origin of coordinates.

2. We do a change of variables which writes the matrix of the linear part of the Hamiltonian system at the origin of coordinates in its real Jordan normal form. Then, in general the obtained differential system have lost its Hamiltonian structure.

3. The coordinates of the differential system whose linear part at the origin of coordinates produce complex eigenvalues are changed to polar Coordinates.

More precisely, if the real Jordan normal form of the linear part at the origin of coordinates has the block

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix},$$

associated to the coordinates x and y with α non identically zero. Then we do the change of variables $(x, y) \to (r, \theta)$, where $x = r\cos\theta$ and $y = r\sin\theta$. We shall take α as the small parameter μ necessary for applying the averaging theory, see the chapter 2. But if there are more than one block associated to complex eigenvalues, for all the other blocks different from the first we do the following change of variables. Take one of these blocks, for instance

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix},$$

associated to the coordinates u and v and do the change of variables $(u, v) \to (R, \varphi)$, where

$$x = R\cos(\varphi + k\theta), y = R\sin(\varphi + k\theta),$$

and the constant k is chosen in such a way that the differential system in the new coordinates must satisfy $\dot{\varphi} = \mathcal{O}(\alpha)$, with α sufficiently small. This step will become more clear when we apply the algorithm to the Hamiltonian system (3) in the next section.

4. Now we take the angle variable θ as the new independent variable.

5. Since generically the periodic orbits in the Hamiltonian systems appear in families of periodic orbits parametrized by h, being h the different values of the energy levels $H = h$ of the Hamiltonian H, and the averaging theory only detects isolated orbits, we must apply the averaging theory in the energy levels $H = h$, eliminating one variable using the relation $H = h$. If the Hamiltonian H is the unique independent first integral of the Hamiltonian system, the differential system obtained on the energy level $H = h$ is in the normal form (2.1) of the appendix, ready for applying to it the averaging theory for studying its periodic orbits.

6. In case that the Hamiltonian system has additional independent first integrals F_k, also independent with the Hamiltonian H of the Hamiltonian system, we need to fix them, i.e., $F_k = f_k$, and work on these invariant subspaces in order that we can apply the averaging theory and study the periodic orbits of the Hamiltonian system.

3.3 Proof of theorem 3.1.1.

Now we apply the algorithm described in section 3.2 to the Hamiltonian system (3.3) for proving Theorem 3.1.1. We will indicate all the steps of the algorithm using its

numbers.

1. It is easy to verify that the Hamiltonian system (3.3) has a unique equilibrium point localized at the origin of coordinates, and that its four eigenvalues are

$$\frac{\pm\mu \pm \sqrt{\mu^2-4}}{2}.$$

2. Now we write the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -\mu & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & \mu \end{pmatrix}$$

of linear part of the Hamiltonian system 3.1.1 into its real Jordan normal form

$$\begin{pmatrix} \frac{\mu}{2} & -\frac{\sqrt{4-\mu^2}}{2} & 0 & 0 \\ \frac{\sqrt{4-\mu^2}}{2} & \frac{\mu}{2} & 0 & 0 \\ 0 & 0 & -\frac{\mu}{2} & -\frac{\sqrt{4-\mu^2}}{2} \\ 0 & 0 & \frac{\sqrt{4-\mu^2}}{2} & -\frac{\mu}{2} \end{pmatrix}.$$

doing the change of variables $(x, y, p_x, p_y) \to (X, Y, Z, V)$, where

$$(x, y, p_x, p_y) = \left(X,\ Z,\ \frac{1}{2}\left(\mu Z - \sqrt{4-\mu^2}V\right),\ \frac{1}{2}\left(\mu X - \sqrt{4-\mu^2}Y\right)\right).$$

Then the Hamiltonian system (3.3) becomes

$$\begin{aligned} \dot{X} &= \frac{\mu}{2}X - \frac{\sqrt{4-\mu^2}}{2}Y, \\ \dot{Y} &= \frac{\sqrt{4-\mu^2}}{2}X + \frac{\mu}{2}Y + \frac{\mu^2}{\sqrt{4-\mu^2}}X^3 - \mu X^2 Y, \\ \dot{Z} &= -\frac{\sqrt{4-\mu^2}}{2}V - \frac{\mu}{2}Z + \mu X^2 Z, \\ \dot{V} &= -\frac{\mu}{2}V + \frac{\sqrt{4-\mu^2}}{2}Z + \frac{3\mu^2}{\sqrt{4-\mu^2}}X^2 Z - 2\mu X Y Z, \end{aligned} \tag{3.5}$$

and the Hamiltonian (3.4) becomes the first integral

$$\begin{aligned} H = &\frac{1}{4}\left(\mu Z - \sqrt{4-\mu^2}V\right)\left(\mu X - \sqrt{4-\mu^2}Y\right) \\ &- \frac{1}{2}\mu\left(1-X^2\right)Z\left(\mu X - \sqrt{4-\mu^2}Y\right) + XZ, \end{aligned} \tag{3.6}$$

of the differential system (3.5).

3. We do the change of variables $(X, Y, Z, V) \to (r, \theta, R, \varphi)$ where

$$(X, Y, Z, V) = \Big(r\cos\theta,\ r\sin\theta,\ R\cos(\varphi + k\theta),\ R\sin(\varphi + k\theta)\Big),$$

and the differential system (3.5) writes in the new variables as

$$\begin{aligned}
\dot r &= \mu\left(\frac{1}{2}r + \sin\theta\cos^2\theta\left(\frac{\mu\cos\theta}{\sqrt{4-\mu^2}} - \sin\theta\right)r^3\right),\\
\dot\theta &= \frac{\sqrt{4-\mu^2}}{2} + \mu\cos^3\theta\left(\frac{\mu\cos\theta}{\sqrt{4-\mu^2}} - \sin\theta\right)r^2,\\
\dot R &= \mu\left(-\frac{1}{2}R + \frac{1}{2}\cos\theta\cos(\varphi+\theta)\left(\frac{6\mu\cos\theta\sin(\varphi+\theta)}{\sqrt{4-\mu^2}}\right.\right.\\
&\quad \left.\left. +3\cos(\varphi+2\theta) - \cos\varphi\right)r^2R\right),\\
\dot\varphi &= \mu\cos\theta\left(\cos\theta\cos(\varphi+\theta)\left(\frac{3\mu\cos(\varphi+\theta)}{\sqrt{4-\mu^2}} - \sin(\varphi+\theta)\right)\right.\\
&\quad \left. -\frac{\mu\cos^3\theta}{\sqrt{4-\mu^2}} - 2\sin\theta\cos^2(\varphi+\theta) + \sin\theta\cos^2\theta\right)r^2,
\end{aligned} \tag{3.7}$$

where we have take $k = 1$ in order that $\dot\varphi = \mathcal{O}(\mu)$. The first integral H given in (3.6) in the variables (r, θ, R, φ) writes

$$\begin{aligned}
H = {} & rR\cos\theta\cos(\varphi+\theta) - \frac{1}{2}\mu R\cos(\varphi+\theta)\left(1 - r^2\cos^2\theta\right)\\
& \left(\mu r\cos\theta - \sqrt{4-\mu^2}r\sin\theta\right) + \frac{\mu r\cos\theta - \sqrt{4-\mu^2}r\sin\theta}{4}\\
& \left(\mu R\cos(\varphi+\theta) - \sqrt{4-\mu^2}R\sin(\varphi+\theta)\right).
\end{aligned} \tag{3.8}$$

4. We take now the variable θ as the new independent variable and the differential system (3.7) becomes

$$\begin{aligned}
r' &= \frac{\mu}{2}\left(r - 2r^3\sin^2\theta\cos^2\theta\right) + \mathcal{O}(\mu^2),\\
R' &= -\frac{\mu}{2}\left(1 + r^2\cos\theta\cos(\varphi+\theta)(\cos\varphi - 3\cos(\varphi+2\theta))\right)R + \mathcal{O}(\mu^2),\\
\varphi' &= \frac{\mu r^2}{4}\cos\theta\left(\sin(2\varphi+\theta) - 3\sin(2\varphi+3\theta) - 3\sin\theta + \sin(3\theta)\right) + \mathcal{O}(\mu^2),
\end{aligned} \tag{3.9}$$

where the prime denotes derivative with respect to the variable θ.

5. From (3.8) and $H = h$ we obtain that

$$\varphi = \arccos\left(\frac{h}{rR}\right) + \mathcal{O}(\mu). \tag{3.10}$$

Substituting this φ given in (3.10) we obtain the following expression for the differential system (3.9) restricted to the energy level $H = h$

$$\begin{aligned} r' &= \mu\left(\frac{r}{2} - r^3\sin^2\theta\cos^2\theta\right) + \mathcal{O}(\mu^2) \\ &= \mu F_{11}(\theta, r, R) + \mathcal{O}(\mu^2), \\ R' &= \frac{\mu}{2}\left(-hr\cos\theta\cos\left(\arccos\left(\frac{h}{rR}\right) + \theta\right) - R\right) \\ &\quad + 3r^2R\cos\theta\cos\left(\arccos\left(\frac{h}{rR}\right) + \theta\right)\cos\left(\arccos\left(\frac{h}{rR}\right) + 2\theta\right) \\ &\quad + \mathcal{O}(\mu^2) \\ &= \mu F_{12}(\theta, r, R) + \mathcal{O}(\mu^2). \end{aligned} \tag{3.11}$$

The differential system (3.11) is already written in the normal form (2.1) of the chapter 2 for applying the averaging theory. Here using the notation of the appendix we have that $F = (F_{11}, F_{12})$, $x = (r, R)$ and $T = 2\pi$ then we compute the averaged function $f(x) = (f_{11}(r, R), f_{12}(r, R))$ defined in (2.2) of the chapter 2, and we get

$$\begin{aligned} f_{11}(r, R) &= \frac{1}{2}\left(r - \frac{r^3}{4}\right), \\ f_{12}(r, R) &= \frac{1}{8}\left(3r^2 - 4\right)R - \frac{h^2}{4R}. \end{aligned} \tag{3.12}$$

The real solutions of the system $f_{11}(r, R) = 0$ and $f_{12}(r, R) = 0$ with r and R positive are $r = 2$ and $R = |h|/2$. Moreover, on this solution the Jacobian (2.3) for our differential system (3.12) is

$$\det\left(\begin{array}{cc} \frac{1}{8}\left(4 - 3r^2\right) & 0 \\ \frac{3rR}{4} & \frac{1}{8}\left(\frac{2h^2}{R^2} + 3r^2 - 4\right) \end{array}\right)\Bigg|_{(r,R)=(2,|h|/2)} = -8\pi^2 \neq 0. \tag{3.13}$$

From Theorem 3 of the chapter 2 we get for μ sufficiently small that for each $h \in \mathbb{R}$ we have the periodic solution

$$\gamma_h^1(\theta, \mu, h) = \Big(r(\theta, \mu, h), R(\theta, \mu, h)\Big)$$

of the differential system (3.12) such that $\gamma_h^1(\theta;\mu,h) \to (2,|h|/2)$, when $\mu \to 0$. The periodic orbit $\gamma_h^1(\theta,\mu,h)$ provides the following periodic orbit of the differential system (3.9)

$$\gamma_h^2(\theta,\mu,h) = \Big(r(\theta,\mu,h), R(\theta,\mu,h), \varphi(\theta,\mu,h)\Big),$$

such that $\gamma_h^2(\theta;\mu,h) \to (2,|h|/2,\varphi^*)$, when $\mu \to 0$, where $\varphi^* = 0$ if $h > 0$ and $\varphi^* = \pi$ if $h < 0$.
The periodic orbit $\gamma_h^2(\theta;\mu,h)$ provides the following periodic orbit of the differential system (3.7)

$$\gamma_h^3(t,\mu,h) = \Big(r(t,\mu,h), \theta(t,\mu,h), R(t,\mu,h), \varphi(t,\mu,h)\Big),$$

such that

$$\gamma_h^3(t;\mu,h) \to \left(2, \sqrt{4-\mu^2}\, t/2, |h|/2, \varphi^*\right),$$

when $\mu \to 0$.
The periodic orbit $\gamma_h^3(t;\mu,h)$ provides the following periodic orbit of the differential system (3.5)

$$\gamma_h^4(t;\mu,h) = \Big(X(t;\mu,h), Y(t;\mu,h), Z(t;\mu,h), V(t;\mu,h)\Big),$$

such that

$$\gamma_h^4(t;\mu,h) \to \left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), 2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right), \right.$$
$$\left. \frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right)\right),$$

when $\mu \to 0$.
The periodic orbit $\gamma_h^4(t;\mu,h)$ provides the following periodic orbit of the differential system (3.3)

$$\gamma_h(t;\mu,h) = \Big(x(t,\mu,h), y(t,\mu,h), p_x(t,\mu,h), p_y(t,\mu,h)\Big),$$

such that

$$\gamma_h(t;\mu,h) \to \left(2\cos\left(\frac{\sqrt{4-\mu^2}}{2}t\right), \frac{|h|}{2}\cos\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), \right.$$
$$\left. -\frac{|h|}{2}\sin\left(\varphi^* + \frac{\sqrt{4-\mu^2}}{2}t\right), -2\sin\left(\frac{\sqrt{4-\mu^2}}{2}t\right)\right),$$

when $\mu \to 0$. This completes the proof of Theorem 3.1.1.

Chapter 4

Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles

In this chapter, we provide the maximum number of limit cycles for continuous and discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated either by one or two parallel straight lines.

4.1 Introduction

The study of limit cycles of differential systems in $\mathbb{R}^2$ (i.e., periodic orbits of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system) goes back essentially to Poincaré [43] at the end of the nineteenth century and their existence became important in application due to their relation with real world phenomena, see for instance the limit cycle of van der Pol equation [49, 50], or the one of the Belousov-Zhavotinskii model [4, 56].

Continuous piecewise linear differential systems separated by straight lines appear naturally in control theory (see, for instance, [2, 20, 23, 25, 35, 42]). The easiest continuous piecewise linear differential systems are the ones formed by two linear differential systems separated by a straight line and for such systems it is well known that one is the upper bound on the number of limit cycles that they can have and that this upper bound is reached (see, for instance, [17, 32, 38, 39] and the references therein).

The unique linear differential systems which are Hamiltonian are linear centers and linear saddles. In [34] the authors obtained the maximum number of limit cycles of continuous and discontinuous piecewise differential systems formed by linear centers and separated by either one or two parallel straight lines. In the present paper we do a symmetric study for continuous and discontinuous piecewise differential

systems formed by linear Hamiltonian saddles and separated by either one or two parallel straight lines.

In the following two theorems, we prove that if the continuous differential systems are formed by linear Hamiltonian saddles, then if they are separated by either one straight line (Theorem 4.1.1) or by two parallel straight lines (Theorem 4.1.2), they do not have limit cycles.

Theorem 4.1.1. *A continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles does not have limit cycles.*

Theorem 4.1.2. *A continuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles does not have limit cycles.*

The study of discontinuous piecewise linear differential systems separated by straight lines goes back to Andronov et al. [1]. Nowadays, they have attracted the attention of many authors mainly because these systems appear in mechanics, electrical circuits, economy, etc. (see, for instance, the books [13, 47], the surveys [40, 48] and the references therein). To provide an explicit example, consider a planar Coulomb friction damping vibration system of the form

$$m\ddot{x} + kx + \mu mg\,\mathrm{sgn}(\dot{x}) = 0$$

where x is the displacement of the oscillator mass m, k is the stiffness coefficient of spring, μ is the coefficient Coulomb friction, g is the gravitational acceleration, and $\mathrm{sgn}(\dot{x})$ is the sign function of relative sliding speed $\dot{x} = y$ for $\dot{x} > 0$ and $\dot{x} < 0$. Note that this model is equivalent to

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = -\frac{k}{m}x - \mu g \quad \text{if } y > 0 \\ \dot{x} &= y, \quad \dot{y} = -\frac{k}{m}x + \mu g \quad \text{if } y < 0, \end{aligned}$$

which is a discontinuous planar piecewise linear differential system separated by the straight line $y = 0$.

In planar discontinuous piecewise linear differential systems, we can have two kinds of limit cycles: The sliding limit cycle and the crossing limit cycle. A sliding limit cycle contains a segment of the discontinuity lines, and a crossing limit cycle only contains isolated points of the discontinuity lines. In the present paper, we only study crossing limit cycles, but for simplicity, we shall call them limit cycles instead of crossing limit cycles.

The easiest discontinuous piecewise differential systems are the ones formed by two linear differential systems and separated by a straight line. There are examples of such systems with three limit cycles, but it is not known if there is the maximum number of limit cycles that such systems can exhibit (see [6, 7, 18, 22, 30, 33]).

We first show, as for the continuous systems, that discontinuous piecewise differential systems formed by linear Hamiltonian saddles and separated by one straight line do not have limit cycles.

Theorem 4.1.3. *A discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles does not have limit cycles.*

It can be extended to discontinuous systems separated by two parallel straight lines, but in this case, the upper bound on the number of limit cycles is one, and this upper bound is reached. Thus, we have proved the extension of the fifth Hilbert problem on the maximum number of limit cycles for the polynomial differential systems of a given degree to the class of discontinuous piecewise differential systems formed by three linear Hamiltonian saddles separated by two parallel straight lines.

Theorem 4.1.4. *A discontinuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles can have at most one limit cycle. Moreover, there are systems in this class with one limit cycle, see Figure 4.1.*

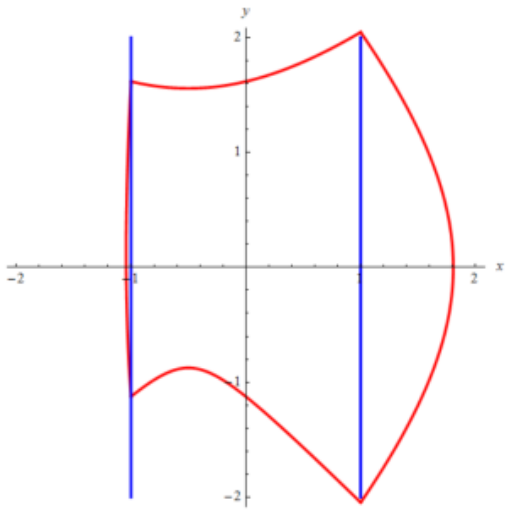


Figure 4.1: The limit cycle of the discontinuous piecewise differential system formed by the three linear Hamiltonian saddles (4.19), (4.20) and (4.21). This limit cycle has travelled in a counterclockwise sense.

4.2 Preliminaries

4.2.1 Piecewise differential systems

A piecewise differential system on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The line of separation Σ of the piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function with 0 as a regular value. Note that Σ is the separating boundary of the regions

$$\Sigma^+ = \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) > 0\} \quad \text{and} \quad \Sigma^- = \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) < 0\}.$$

So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x, y) = \begin{cases} X(x, y), & \text{if } h(x, y) \geq 0, \\ Y(x, y), & \text{if } h(x, y) \leq 0. \end{cases} \tag{4.1}$$

As usual, system (4.1) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$ when the separation line Σ is well understood.

When the piecewise differential system, with the vector field $Z = (X, Y)$ given in (4.1), satisfies $X(x, y) = Y(x, y)$ at all the points (x, y) such that $h(x, y) = 0$, we say that we have a *continuous* piecewise differential system. Otherwise, we say that we have a *discontinuous* piecewise differential system.

In order to establish a definition for the trajectories of a discontinuous piecewise differential system Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X, i.e.,

$$Xh(p) = \langle \nabla h(p), X(p) \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [15]. We can divide the line of discontinuity Σ into the following sets:

1. **Crossing set**: $\Sigma^c = \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
2. **Escaping set**: $\Sigma^e = \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
3. **Sliding set**: $\Sigma^s = \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The escaping Σ^e or sliding Σ^s regions are, respectively, defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ, while the interior of its complement in Σ defines the crossing region Σ^c (see Figure 4.2). The complementary of the union of these regions is the set formed by the tangency points between X or Y with Σ.

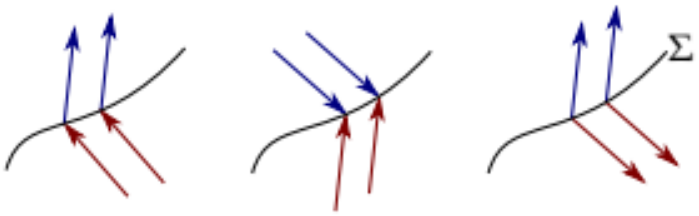


Figure 4.2: Crossing, sliding, and escaping regions, respectively.

4.2.2 Linear Hamiltonian saddles

A linear differential system which is Hamiltonian and has a saddle will be called simply a linear Hamiltonian saddle.

The following lemma provides a normal form for an arbitrary differential system with linear Hamiltonian saddles.

Proposition 4.2.1. *Differential systems with a linear Hamiltonian saddle can be written as*

$$\begin{aligned} \dot{x} &= -bx - \delta y + d, \\ \dot{y} &= ax + by + c, \end{aligned} \tag{4.2}$$

with $\alpha \in \{0, 1\}$, $b, \delta, d, c \in R$. *Moreover, if* $\alpha = 0$ *then* $c = 0$, *and if* $\alpha = 1$ *then* $\delta = b^2 - \omega^2$ *with* $\omega \neq 0$.

Proof of proposition 4.2.1. A general linear differential system in $\mathbb{R}^2$ can be written as

$$\begin{aligned} \dot{x} &= a_0 x + b_0 y + c_0, \\ \dot{y} &= a_1 x + b_1 y + c_1. \end{aligned}$$

Consider that it has a Hamiltonian saddle. The eigenvalues of this system are

$$\frac{a_1 + b_0 \pm \sqrt{4a_0 b_1 + (a_1 - b_0)^2}}{2}.$$

This system has a Hamiltonian saddle if $a_1 + b_0 = 0$ and $4a_0 b_1 + (a_1 - b_0)^2 = \omega^2$ for some $\omega \neq 0$. Hence, $a_1 = -b_0$ and $4a_0 b_1 + 4b_0^2 = \omega^2$ for some $\omega \neq 0$. If $a_0 \neq 0$, we have $b_1 = -\dfrac{4b_0^2 - \omega^2}{4a_0}$, and if $a_0 = 0$, then $b_0 \neq 0$. Therefore, any linear differential system with a Hamiltonian saddle can be written as

$$\begin{aligned} \dot{x} &= -bx - \delta y + d, \\ \dot{y} &= ax + by + c, \end{aligned} \tag{4.3}$$

where $\delta \in \mathbb{R}$ if $a = 0$, and if $a \neq 0$ then $\delta = \dfrac{4b^2 - \omega^2}{4a}$ with $\omega \neq 0$. It is possible to do a rescaling of the independent variable because it does not change the orbits, and so it does not change the number of crossing limit cycles. After a rescaling of the independent variable of the form $\tau = at$ if $a \neq 0$ we can assume that Equation (4.3) can be written as in (4.2) where $\delta \in \mathbb{R}$ and $\alpha = 0$ if $a = 0$. If $a \neq 0$, then $\delta = b^2 - \omega^2$ for some $\omega \neq 0$ and $\alpha = 1$. So $\alpha \in \{0, 1\}$ if $\alpha = 0$, then $b \neq 0$, and setting $Y = y + \dfrac{c}{b}$ we get

$$\begin{aligned} \dot{x} &= -bx - \delta Y + d_2, \\ \dot{Y} &= bY, \end{aligned} \tag{4.4}$$

with $d_2 = d + \dfrac{\delta c}{b}$. This completes the proof of the proposition. □

A first integral of system (4.2) is

$$H(x, y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{4.5}$$

4.3 Proof of the main theorems

Proof of theorem 4.1.1. Take a continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality (and due to symmetry), assume the straight line of continuity is $x = 0$. By Proposition 4.2.1, the systems in $x < 0$ and $x > 0$ can be written in the form (4.2).
We have the system

$$\begin{aligned} \dot{x} &= -b_1 x - \delta_1 y + d_1, \\ \dot{y} &= \alpha_1 x + b_1 y + c_1, \end{aligned} \tag{4.6}$$

for $x < 0$, with first integral

$$H_1 = -\frac{\alpha_1}{2}x^2 - b_1 xy - \frac{\delta_1}{2}y^2 - c_1 x + d_1 y, \tag{4.7}$$

and the system

$$\begin{aligned} \dot{x} &= -b_2 x - \delta_2 y + d_2, \\ \dot{y} &= \alpha_2 x + b_2 y + c_2, \end{aligned} \tag{4.8}$$

for $x > 0$, with first integral

$$H_2 = -\frac{\alpha_2}{2}x^2 - b_2 xy - \frac{\delta_2}{2}y^2 - c_2 x + d_2 y. \tag{4.9}$$

Continuity of the piecewise system at $x = 0$ implies $\delta_1 = \delta_2$, $d_1 = d_2$, $b_1 = b_2$, and $c_1 = c_2$.
If the system has a limit cycle, it must intersect $x = 0$ at two points $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are first integrals, we have:

$$H_1(0, y_1) = H_1(0, y_2) \quad \text{and} \quad H_2(0, y_1) = H_2(0, y_2), \tag{4.10}$$

which simplifies to

$$(2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0.$$

Thus, all periodic orbits belong to a continuum of periodic orbits, proving no isolated limit cycles exist. This completes the proof of Theorem 4.1.1. □

Proof of theorem 4.1.2. Assume that we have a continuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem we can assume that the straight lines of discontinuity are $x = -1$ and $x = 1$. It follows from Proposition 4.2.1 that we can assume that the systems in $x < -1$, $-1 < x < 1$, and $x > 1$ are written as in (4.2).
We have system (4.6) with first integral (4.7) in $x < -1$, system (4.8) with first integral (4.9) in $-1 < x < 1$, and system

$$\begin{aligned} \dot{x} &= -b_3 x - \delta_3 y + d_3, \\ \dot{y} &= \alpha_3 x + b_3 y + c_3, \end{aligned} \tag{4.11}$$

with first integral

$$H_3 = -\frac{\alpha_3}{2}x^2 - b_3 xy - \frac{\delta_3}{2}y^2 - c_3 x + d_3 y. \tag{4.12}$$

in $x > 1$.
Since the piecewise differential system is continuous, systems (4.6) and (4.8) must coincide at $x = -1$, and systems (4.8) and (4.11) must coincide at $x = 1$. This yields

$$b_1 = b_2 = b_3, \quad d_1 = d_2 = d_3, \quad \delta_1 = \delta_2 = \delta_3, \quad c_1 = c_3 + a_1 - 2a_2 + a_3, \quad c_2 = c_3 - a_2 + a_3.$$

If the system has a limit cycle, it must intersect $x = \pm 1$ at four points: $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$, and $(1, y_4)$ with $y_1 > y_2$ and $y_3 < y_4$. Using the first integrals H_1, H_2, and H_3, we derive

$$\begin{aligned} &H_1(-1, y_1) - H_1(-1, y_2) = 0, \\ &H_2(-1, y_2) - H_2(1, y_3) = 0, \\ &H_3(1, y_3) - H_3(1, y_4) = 0, \\ &H_2(1, y_4) - H_2(-1, y_1) = 0, \end{aligned} \tag{4.13}$$

which simplify to

$$\begin{aligned}
&(2b_3 + 2d_3 - \delta_3(y_1 + y_2))(y_1 - y_2) = 0,\\
&4c_3 - 4a_2 + 4a_3 + 2(b_3 + d_3)y_2 + 2(b_3 - d_3)y_3 - \delta_3(y_2^2 - y_3^2) = 0,\\
&(2b_3 - 2d_3 + \delta_3(y_3 + y_4))(y_3 - y_4) = 0,\\
&4c_3 - 4a_2 + 4a_3 + 2(b_3 + d_3)y_1 + 2(b_3 - d_3)y_4 - \delta_3(y_1^2 - y_4^2) = 0.
\end{aligned}$$

The solutions (y_1, y_2, y_3, y_4) satisfying $y_1 < y_2$ and $y_3 < y_4$ are

$$y_2 = \frac{2(b_3 + d_3)}{\delta_3} - y_1, \quad y_3 = \frac{d_3 - b_3}{\delta_3} \pm \frac{\sqrt{\Delta}}{\delta_3}, \quad y_4 = \frac{d_3 - b_3}{\delta_3} \mp \frac{\sqrt{\Delta}}{\delta_3},$$

where $\Delta = (b_3 - d_3)^2 - 4(c_3 - a_2 + a_3)\delta_3 - 2(b_3 + d_3)\delta_3 y_1 + \delta_3^2 y_1^2$. Only two solutions exist (depending on the choice of $\pm$ signs), implying all periodic orbits belong to a continuum. Thus, no isolated limit cycles exist. This completes the proof of Theorem 4.1.2. □

Proof of theorem 4.1.3. Assume that we have a discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem, we can assume that the straight line of continuity is $x = 0$. It follows from Proposition 1 that we can assume that the systems in $x < 0$ and $x > 0$ are written in the form (4.2).

We have system (4.6) with first integral (4.7) in $x < 0$, and system (4.8) with first integral (4.9) in $x > 0$.

Note that if the discontinuous piecewise differential system has a limit cycle, taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting the discontinuity line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals, we have that (4.10) must be satisfied, that is

$$\begin{aligned}
&(2d_1 - \delta_1(y_1 + y_2))(y_1 - y_2) = 0,\\
&(2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0.
\end{aligned} \tag{4.14}$$

The solutions (y_1, y_2) of (4.14) satisfying the condition $y_1 < y_2$ either do not exist if $d_1/\delta_1 \neq d_2/\delta_2$, or there is a continuum of solutions. Thus, the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits, and consequently, this differential system has no limit cycles. This completes the proof of the theorem. □

Proof of Theorem 4.1.4. Take a discontinuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem, we can assume that the straight lines of discontinuity are $x = -1$ and $x = 1$. It follows from

Proposition 4.2.1 that the systems in $x < -1$, $-1 < x < 1$, and $x > 1$ are written as in (4.2).
We have system (4.6) with first integral (4.7) in $x < -1$, system (4.8) with first integral (4.9) in $-1 < x < 1$, and system (4.11) with Hamiltonian (4.12) in $x > 1$. Note that if the discontinuous piecewise differential system has a limit cycle, taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting each discontinuity line $x = \pm 1$ in exactly two points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and H_3 are three first integrals, we have that system (4.13) must be satisfied. Doing so we get

$$\begin{aligned} &(2(b_1 + d_1) - \delta_1(y_1 + y_2))(y_1 - y_2) = 0, \\ &4c_2 + 2(b_2 + d_2)y_2 + 2(b_2 - d_2)y_3 - \delta_2(y_2^2 - y_3^2) = 0, \\ &(2(b_3 - d_3) + \delta_3(y_3 + y_4))(y_3 - y_4) = 0, \\ &4c_2 + 2(b_2 + d_2)y_1 + 2(b_2 - d_2)y_4 - \delta_2(y_1^2 - y_4^2) = 0. \end{aligned} \tag{4.15}$$

Assume first that $\delta_1 = \delta_3 = 0$ the solutions of Equation (4.15) are $d_1 = c_1 - b_1$, $d_3 = b_\varepsilon + c_3$ and $y_3 = f_1(y_2)$, $y_4 = f_2(y_1)$ being f_1, f_2 functions in the variables y_2 and y_1, respectively. In this case all periodic orbits of the these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.
Assume now that $\delta_1 = 0$ and $\delta_3 \neq 0$ the solutions of Equation (4.15) are $d_1 = -b_1$,

$$y_3 = \frac{2(d_3 - b_3)}{\delta_3} - y_4 \tag{4.16}$$

and $y_2 = f_1(y_4)$, $y_1 = f_2(y_4)$ being f_1, f_2 functions in the variable y_4, respectively. In this case all periodic orbits of these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.
Assume now that $\delta_1 \neq 0$ and $\delta_3 = 0$, the solutions of Equation (4.15) are $d_1 = -b_1$,

$$y_1 = \frac{2(b_1 + d_1)}{\delta_1} - y_2, \tag{4.17}$$

$d_3 = b_3$, and $y_2 = f_1(y_2)$, $y_1 = f_2(y_2)$ being f_1, f_2 functions in the variable y_2, respectively. In this case, all periodic orbits of the these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.
Finally, assume that $\delta_1\delta_3 \neq 0$. The solution of the first and third equations are (4.17) and (4.16). Introducing these solutions into the second and fourth equations in (4.15) we get

$$\begin{aligned} e_1 = 4((b_3 - d_3)^2\delta_2 - (b_2 - d_2)(b_3 - d_3)\delta_3 + c_2\delta_3^2) + 2(b_2 + d_2)\delta_3^2 y_2 \\ +2\delta_3(2(b_3 - d_3)\delta_2 - (b_2 - d_2)\delta_3)y_4 - \delta_2\delta_3^2(y_2^2 - y_4^2) = 0 \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} e_2 \;=\; & 4(\delta_1((b_1 + d_1)(b_2 + d_2) + c_2\delta_1) + (b_1 + d_1)^2\delta_2) \\ & -2\delta_1((b_2 + d_2)\delta_1 - 2(b_1 + d_1)\delta_2)y_2 \end{aligned}$$

$$+2(b_2-d_2)\delta_1^2 y_4 - \delta_1^2\delta_2(y_2^2-y_4^2) = 0.$$

Taking

$$e_3 = \delta_1^2 e_1 - \delta_2^2 e_2 = 0$$

and solving in y_4 we get

$$y_4 = \frac{A_0}{A_1} + \frac{A_2}{A_1} y_2$$

where

$$\begin{aligned} A_0 &= (b_3-d_3)^2\delta_1^2\delta_2 - (b_2-d_2)(b_3-d_3)\delta_1^2\delta_3 \\ &\quad +(b_1+d_1)((b_1+d_1)\delta_2 - (b_2+d_2)\delta_1)\delta_3^2, \end{aligned}$$

$$A_1 = \delta_1^2\delta_3((d_3-b_3)\delta_2 + (b_2-d_2)\delta_3),$$

$$A_2 = \delta_1((b_2+d_2)\delta_1 - (b_1+d_1)\delta_2)\delta_3^2,$$

whenever $A_1 \neq 0$. The case with $A_1 = 0$ yields $d_2 = b_2 + (d_3-b_3)\delta_2/\delta_3$. Introducing this into $e_3 = 0$ and solving in y_2, we obtain $y_2 = y_1 = (b_1 - c_1 + d_1)/\delta_1$, which is not possible. So, we assume that $A_1 \neq 0$. Now introducing y_4 into the first equation in (4.18) and solving in y_2 we get

$$y_2 = y_{2\pm} = \frac{(b_1+d_1)}{\delta_1} \pm \frac{\sqrt{\Delta}}{A_4}$$

Where

$$\begin{aligned} A_4 &= \delta_1^2\delta_2\delta_3^2\Big(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 + 2d_2\delta_1\delta_3 \\ &\quad - b_1\delta_2\delta_3 - d_1\delta_2\delta_3\Big)\Big(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 \\ &\quad - 2b_2\delta_1\delta_3 + b_1\delta_2\delta_3 + d_1\delta_2\delta_3\Big), \\ \Delta &= 4\delta_1^4\delta_2\delta_3^2\Big((b_3-d_3)\delta_2 + (-b_2+d_2)\delta_3\Big)^2 \\ &\quad \times \Big((b_3-d_3)\delta_1\delta_2 + 2d_2\delta_1\hat{\rho}_3 - (b_1+d_1)\delta_2\delta_3\Big) \\ &\quad \times \Big((b_3-d_3)\delta_1\delta_2 - 2b_2\delta_1\delta_3 + (b_1+d_1)\delta_2\delta_3\Big) \\ &\quad \times \Big((b_3-d_3)^2\delta_1^2\delta_2 - 2(b_2-d_2)(b_3-d_3)\delta_1^2\delta_3 \\ &\quad + \Big(2\delta_1((b_1+d_1)(b_2+d_2) + 2c_2\delta_1) - (b_1+d_1)^2\delta_2\Big)\delta_3^2\Big), \end{aligned}$$

whenever $A_4 \neq 0$, and if $A_4 = 0$ then there is at most one solution y_2. When $A_4 \neq 0$, since

$$y_1 = y_{1\pm} = \frac{2(d_1+b_1)}{\delta_1} - y_{2\pm} = \frac{d_1+b_1}{\delta_1} \mp \frac{\sqrt{\Delta}}{A_4} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, an upper bound for the number of limit cycles is one.

To complete the proof of Theorem 4.1.4, we provide an example of a system in this class with one limit cycle. The Hamiltonians of the three linear Hamiltonian systems with a saddle are

$$\begin{aligned} H_1(x,y) &= 16x + 2y + x^2 - \frac{309 - \sqrt{157881}}{48}xy + \frac{65\left(\sqrt{157881} - 405\right)}{1536}y^2, \\ H_2(x,y) &= x - y + x^2 - y^2, \\ H_3(x,y) &= 8x - x^2 + y^2, \end{aligned}$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\begin{aligned} \dot{x} &= 2 - \frac{1}{48}(309 - \sqrt{157881})x - \frac{65}{768}(405 - \sqrt{157881})y, \\ \dot{y} &= -16 - 2x + \tfrac{1}{48}(309 - \sqrt{157881})y, \end{aligned} \tag{4.19}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\begin{aligned} \dot{x} &= -1 - 2y, \\ \dot{y} &= -1 - 2x, \end{aligned} \tag{4.20}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\begin{aligned} \dot{x} &= 2y, \\ \dot{y} &= -8 + 2x. \end{aligned} \tag{4.21}$$

These three linear differential systems are saddles because the determinants of their linear part are $-\dfrac{8569}{48} + \dfrac{7\sqrt{157881}}{16} < 0, -4$ and -4, respectively.
The discontinuous piecewise differential system formed by the three linear differential systems (4.19)-(4.21) has one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points; these points must satisfy system (4.13), and this system has a unique solution satisfying $y_1 > y_2$ and $y_3 < y_4$, namely

$$(y_1, y_2, y_3, y_4) = \left(\frac{16}{65} + \frac{\sqrt{4873}}{36\sqrt{2}}, \frac{16}{65} - \frac{\sqrt{4873}}{36\sqrt{2}}, \frac{97\sqrt{4873}}{2340\sqrt{2}}, \frac{97\sqrt{4873}}{2340\sqrt{2}}\right).$$

Drawing the corresponding limit cycle associated with this solution yields the limit cycle shown in Figure 4.1 □

General Conclusion

The averaging theory is an important perturbation method in the analysis of dynamical systems, widely used to detect the existence of periodic solutions in ordinary differential equations under certain conditions. It excels in simplifying complex systems and capturing their essential periodic behavior, making it an indispensable tool for studying nonlinear dynamics. The effectiveness of this method has been demonstrated across numerous studies in the scientific literature, particularly in the fields of physics and engineering. Consequently, averaging theory is regarded as a central technique in the qualitative theory of differential systems.

In this work, we studied the periodic orbits of a Hamiltonian system resulting from the Van Der Pol equations using the averaging theory, yielding significant insights into the periodic solutions. The technique employed with the averaging theorem can be extended to many other Hamiltonian systems.

Moreover, we investigated continuous and discontinuous planar piecewise differential systems formed solely by linear Hamiltonian saddles separated by one or two parallel straight lines. When these systems are continuous and separated by either one or two parallel lines, we proved that they have no limit cycles. However, for the discontinuous piecewise differential systems separated by two parallel straight lines, we demonstrated that they can have at most one limit cycle, with some systems exhibiting either zero or one limit cycle. In the case of discontinuous systems separated by only one straight line, we established that they cannot have limit cycles.

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