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Methods for Studying Blow- Up Phenomena in Differential Equations

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بسم الله الرحمن الرحيم

Dedication

Je dédie ce modeste travail d'étude tout d'abord à :

Mes très chers parents, la lumière de ma vie, qui ont tant souffert et sacrifié pour mon bien être , par leurs conseils, leurs affections et leurs encouragements.

Je vous remercie pour tous vos efforts fournis pour moi, que Dieu vous garde, vous protège, et vous bénisse dans la vie.

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Toute ma chère famille.

Tous mes amis et mes collègues de proche et de loin avec qui j'ai partagé de très bons moments tout au long de ces années.

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lassoued

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Title: Study of the stability and blow up in finite-time of certain nonlinear hyperbolic systems.

Abstract: This thesis investigates the dynamics of a coupled nonlinear wave system with internal fractional damping, a model relevant to viscoelastic materials and engineering applications. The study focuses on two key aspects: establishing the local existence and uniqueness of solutions and analyzing conditions leading to finite-time blow-up. Employing functional analysis and C_0-semigroup theory, we reformulate the system as an abstract evolution equation in a Hilbert space, proving local well-posedness under specific conditions on the nonlinear exponents. A carefully designed Lyapunov functional is used to derive criteria for solution blow-up, highlighting the interplay between nonlinearity and fractional damping. The results contribute to the theoretical understanding of fractional partial differential equations and offer insights for practical applications, such as vibration control and material stability. This work advances the mathematical framework for nonlocal systems and lays the groundwork for future research into global existence and numerical simulations.

Keywords: Nonlinear wave equation, Fractional damping, Local existence, Blow-up in finite time, C_0-semigroup, Lyapunov functional, Viscoelastic systems.

Etude de la stabilité et de l'explosion en temps fini de certains systèmes hyperboliques non linéaires.

Résume: Cette thèse explore la dynamique d'un système d'ondes non linéaires couplées avec amortissement fractionnaire interne, un modèle pertinent pour les matériaux viscoélastiques et les applications en ingénierie. L'étude se concentre sur deux aspects principaux : établir l'existence locale et l'unicité des solutions, et analyser les conditions menant à une explosion en temps fini. En utilisant l'analyse fonctionnelle et la théorie des semi-groupes C_0, le système est reformulé comme une équation d'évolution abstraite dans un espace de Hilbert, prouvant ainsi sa bonne position locale. Une fonctionnelle de Lyapunov judicieusement conçue permet d'identifier les critères d'explosion des solutions. Ces résultats enrichissent la théorie des équations aux dérivées partielles fractionnaires et offrent des perspectives pour des applications pratiques, comme le contrôle des vibrations et la stabilité des matériaux.

Mots Clés: Équation des ondes non linéaire, Amortissement fractionnaire, Existence locale, Explosion en temps fini, Semi-groupe C_0, Fonctionnelle de Lyapunov, Systèmes viscoélastiques.

العنوان : طرق دراسة ظواهر الانفجار في المعادلات التفاضلية

تتناول هذه المذكرة دراسة ديناميكيات نظام موجي غير خطي مترابط يتضمن تخميدًا كسريًا داخليًا، وهو نموذج ذو أهمية خاصة في مجال المواد اللزجة-المرنة وتطبيقات الهندسة. تركز الدراسة على جانبين رئيسيين: إثبات وجود و وحدانية الحل محليًا، وتحليل الشروط التي تؤدي إلى حدوث انفجار (Blow-up) في وقت محدود. ومن خلال استخدام أدوات التحليل الدالي ونظرية أشباه الزمر C_0، تمت إعادة صياغة النظام على شكل معادلة تطورية مجردة ضمن فضاء هيلبرت، حيث تم إثبات حسن الوضعية المحلية تحت شروط معينة تتعلق بأسس اللاخطية. كما تم بناء دالة لياپونوف مناسبة لاشتقاق معايير تؤدي إلى انفجار الحلول، مما يبرز التفاعل بين اللاخطية والتخميد الكسري. تسهم هذه النتائج في تعميق الفهم النظري للمعادلات التفاضلية الجزئية الكسرية، وتوفر رؤى مفيدة للتطبيقات العملية مثل التحكم في الاهتزازات واستقرار المواد. ويُعد هذا العمل خطوة متقدمة في تطوير الإطار الرياضي للأنظمة غير المحلية، كما يمهّد الطريق لأبحاث مستقبلية تتعلق بالوجود الشامل والمحاكاة العددية.

الكلمات المفتاحية نظرية أشباه الزمر C_0، وجود و وحدانية الحل، حدوث انفجار في وقت محدود، نظام موجي غير خطي، تخميد كسري داخلي، المواد اللزجة

Notations

Throughout this dissertation, we will use the following conventions:

- Ω: A bounded open subset of $\mathbb{R}^n$.
- Γ: The topological boundary of the domain Ω.
- $x = (x_1, x_2, \ldots, x_n)$: A representative point in $\mathbb{R}^n$.
- $dx = dx_1\, dx_2 \cdots dx_n$: The Lebesgue measure on Ω.
- $\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}\right)$: The gradient of a scalar function u.
- $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$: The Laplacian of u.
- **a.e.**: Stands for "almost everywhere", in the sense of Lebesgue measure.
- p': The Hölder conjugate exponent of p, defined by $\frac{1}{p} + \frac{1}{p'} = 1$.
- $\partial_t^{\alpha,\eta}$: The generalized Caputo fractional derivative of order α with parameter η.
- $C(\Omega)$: The space of real-valued continuous functions on Ω.
- $C^k(\Omega)$, where $k \in \mathbb{N}$: The space of functions that are k times continuously differentiable on Ω.
- $C_0^\infty(\Omega) = D(\Omega)$: The space of infinitely differentiable functions with compact support contained in Ω.
- $D'(\Omega)$: The space of distributions on Ω.

General Introduction

FRACTIONAL calculus extends differentiation and integration to non-integer orders, offering a powerful framework for modeling complex systems with memory and hereditary properties. Originating in the 17th century through discussions between Leibniz and L'Hôpital, it has evolved into a mature discipline with contributions from Riemann, Liouville, and Caputo Samko1993. Unlike classical calculus, fractional calculus captures nonlocal effects, where a system's past influences its future, making it ideal for phenomena in viscoelasticity, material science, and engineering Podlubny1998. Fractional partial differential equations (PDEs) are particularly effective for modeling wave propagation, energy dissipation, and dynamics in media with memory-dependent behavior.

This memoir investigates a coupled nonlinear wave system with internal fractional damping, a class of PDEs relevant to viscoelastic materials and engineering systems with intricate dissipation mechanisms. Such systems model critical applications, including vibration suppression in flexible structures like suspension bridges and aircraft wings, or the dynamic response of polymers, where energy dissipation depends on historical states Kilbas2006. The interplay of nonlinearity, nonlocality, and fractional damping poses significant analytical challenges, as classical PDE techniques often fail to address memory effects Barbu2010.

The research addresses two key questions: under what conditions do solutions to this system exist locally in time, and when do solutions exhibit blow-up, becoming unbounded in finite time? The internal fractional damping, unlike boundary damping studied in prior works Akil2019, Aounallah2020, introduces unique challenges due to its distributed nature across the domain, amplifying the complexity of nonlinear coupling. Local existence validates the model's physical relevance over short time scales, while blow-up analysis reveals stability limits, critical for preventing structural or material failure in applications.

The objectives of this memoir are:

1. To prove the local existence and uniqueness of solutions using functional analysis tools, including Hilbert spaces and C_0-semigroup theory, by reformulating the system as an abstract evolution equation in a well-defined energy space Pazy1983.

2. To explore blow-up phenomena by constructing a Lyapunov functional, identifying conditions for unbounded energy growth, and providing insights into long-term stability.

3. To advance the theoretical framework for fractional PDEs by integrating fractional calculus, semigroup theory, and energy methods, enhancing tools for nonlocal systems.

The analysis is conducted in a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary conditions and suitable initial data, ensuring physical realism.

The study's originality lies in its focus on coupled wave systems with internal fractional damping, a relatively underexplored area compared to single equations or boundary-damped systems Messaoudi2003, Aounallah2020. The internal damping's distributed nature requires novel analytical techniques, distinguishing this work from boundary-focused studies. By combining C_0-semigroup theory for existence and Lyapunov functionals for blow-up, the memoir introduces innovative approaches to tackle the combined challenges of nonlocality and nonlinearity.

Theoretically, this research enriches the understanding of fractional PDEs by providing rigorous results on solution behavior in complex systems. Practically, it has direct applications in engineering and material science. For instance, local existence results support the design of vibration control systems for suspension bridges, such as the Golden Gate Bridge, where fractional damping models energy dissipation over time Podlubny1998. Blow-up analysis informs the stability limits of high-performance polymers used in aerospace components, aiding in the development of durable materials. By bridging theory and application, this work contributes to both academic research and technological innovation.

Literature Review

Nonlocal and fractional PDEs have gained attention for modeling systems with memory and nonlocal effects in physics, engineering, and biology Cannon1963, Shi1993. Early studies by Cannon Cannon1963 and Ionkin Ionkin1977 addressed heat equations with nonlocal constraints, while Mesloub and Messaoudi Mesloub2010 proved existence, decay, and blow-up for singular viscoelastic problems. Fractional calculus has advanced modeling of viscoelasticity and wave propagation Magin2006, Tarasov2011. Choi and MacCamy Choi1989 introduced fractional Volterra equations for elasticity, paving the way for fractional damping models. Akil and Wehbe Akil2019 established asymptotic stability for wave equations with fractional boundary dissipation, and Aounallah et al. Aounallah2020 demonstrated decay and blow-up for nonlinear wave equations with fractional boundary damping.

Blow-up in nonlinear wave equations has been widely studied. Levine Levine1974 used the concavity method to prove finite-time blow-up for hyperbolic PDEs, while Georgiev and Todorova Georgia1994 extended this to nonlinear damping. Messaoudi Messaoudi2003 and Kafini and Messaoudi Kafini2007 analyzed blow-up in viscoelastic systems, relevant to coupled wave systems. Recent works incorporate fractional damping and memory effects. Kirane and Tatar Kirane2003 and Aounallah et al. Aounallah2021 explored blow-up and asymptotic behavior in fractionally damped wave equations, informing stability analysis.

This memoir addresses gaps in the literature by studying coupled nonlinear wave systems with internal fractional damping, unlike prior boundary-focused or single-equation studies.

It employs C_0-semigroup theory for local existence and Lyapunov functionals for blow-up, contributing to both theoretical and applied mathematics.

The memoir is organized as follows:

- **Chapter 1: Preliminaries** introduces functional analysis and fractional calculus tools, including Lebesgue, Bochner, and Sobolev spaces, fractional operators (Riemann-Liouville and Caputo), and semigroup theory Barbu2010, Pazy1983.
- **Chapter 2: Local Existence of Solutions** proves solution existence and uniqueness by reformulating the system as a first-order evolution equation in a Hilbert space, using C_0-semigroup theory.
- **Chapter 3: Blow-Up** derives blow-up conditions using a Lyapunov functional, highlighting the role of nonlinearities and fractional damping.

Summary

This memoir advances the study of fractional PDEs by analyzing a coupled nonlinear wave system with internal fractional damping. Chapter 1 provides the mathematical foundation, Chapter 2 establishes local well-posedness, and Chapter 3 explores stability limits through blow-up analysis. Together, these chapters offer a cohesive framework for understanding nonlocal systems, with applications in vibration control and material stability, paving the way for future research into global existence and relaxed system constraints.

Chapter 1

Background and Basic Concepts

This chapter introduces the foundational mathematical tools required for analyzing partial differential equations (PDEs) with fractional derivatives, particularly in the context of coupled nonlinear wave systems with internal fractional damping. We focus on functional spaces, key inequalities, functional analysis concepts, and fractional calculus, which are essential for studying the local existence and potential blow-up of solutions. The presentation is grounded in established references to ensure rigor and clarity Adams2003, Brezis2011, Kilbas2006, Pazy1983.

1.1 Functional Spaces

Functional spaces provide the framework for defining weak solutions and analyzing the stability of PDEs with fractional derivatives. We introduce Lebesgue, Bochner, and Sobolev spaces, which are critical for handling the nonlocal and nonlinear nature of the systems under study Adams2003, Brezis2011.

1.1.1 Lebesgue Spaces

Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be an open domain with Lebesgue measure dx.

Definition 1.1.1 *For $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ consists of measurable functions $f : \Omega \to \mathbb{R}$ such that:*

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} < \infty.$$

For $p = \infty$, the space $L^\infty(\Omega)$ comprises essentially bounded measurable functions:

$$\|f\|_{L^\infty(\Omega)} =_{x \in \Omega} |f(x)| < \infty.$$

Lemma 1.1.1 *The space $L^p(\Omega)$, equipped with the norm:*

$$\|f\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < \infty, \\ _{x \in \Omega} |f(x)|, & p = \infty, \end{cases}$$

is a Banach space. For $p = 2$, $L^2(\Omega)$ is a Hilbert space with inner product:

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)\,dx.$$

Definition 1.1.2 *The space of locally integrable functions is:*

$$L^1_{loc}(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid f \in L^1(A) \text{ for all compact } A \subset \Omega \right\}.$$

1.1.2 Bochner Spaces

Bochner spaces are used for time-dependent problems, accommodating functions that take values in Banach spaces.

Definition 1.1.3 *Let X be a Banach space and $a, b \in \mathbb{R}$ with $a < b$. For $1 \leq p < \infty$, the Bochner space $L^p(a, b; X)$ consists of Bochner-measurable functions $f : (a, b) \to X$ such that:*

$$\|f\|_{L^p(a,b;X)} = \left(\int_a^b \|f(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

For $p = \infty$, the space is:

$$L^\infty(a, b; X) = \left\{ f : (a, b) \to X \mid \|f\|_{L^\infty(a,b;X)} =_{t\in(a,b)} \|f(t)\|_X < \infty \right\}.$$

Lemma 1.1.2 *The space $L^p(a, b; X)$, with norm:*

$$\|f\|_{L^p(a,b;X)} = \begin{cases} \left(\int_a^b \|f(t)\|_X^p \, dt \right)^{1/p}, & 1 \leq p < \infty, \\ {\scriptstyle t\in(a,b)}\|f(t)\|_X, & p = \infty, \end{cases}$$

is a Banach space. For $p = 2$, $L^2(a, b; X)$ is a Hilbert space with inner product:

$$\langle f, g \rangle_{L^2(a,b;X)} = \int_a^b \langle f(t), g(t) \rangle_X \, dt.$$

For $1 \leq p \leq \infty$, the conjugate exponent p' satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

1.1.3 Sobolev Spaces

Sobolev spaces are crucial for defining weak solutions to PDEs, especially those involving fractional derivatives.

Definition 1.1.4 (Weak Derivative) *A function $f \in L^1_{loc}(\Omega)$ has a weak derivative $D^\alpha f \in L^1_{loc}(\Omega)$ if, for all $\phi \in C_0^\infty(\Omega)$,*

$$\int_\Omega D^\alpha f \cdot \phi \, dx = (-1)^{|\alpha|} \int_\Omega f \cdot D^\alpha \phi \, dx,$$

where $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 1.1.5 *For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is:*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^n, |\alpha| \leq k\}.$$

Lemma 1.1.3 *The space $W^{k,p}(\Omega)$, with norm:*

$$\|f\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{1/p}, & 1 \leq p < \infty, \\ \max_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty(\Omega)}, & p = \infty, \end{cases}$$

is a Banach space. For $p = 2$, $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space with inner product:

$$\langle f, g\rangle_{H^k(\Omega)} = \sum_{|\alpha|\leq k} \int_\Omega D^\alpha f(x) D^\alpha g(x)\, dx.$$

1.1.4 Sobolev Embedding Theorem

Sobolev embeddings provide regularity results essential for nonlinear terms in PDEs Evans2010.

Theorem 1.1.1 (Sobolev Embedding) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with Lipschitz boundary. For $1 \leq p \leq \infty$, the space $W^{1,p}(\Omega)$ embeds as:*

$$W^{1,p}(\Omega) \hookrightarrow \begin{cases} L^{\frac{np}{n-p}}(\Omega), & \text{if } p < n, \\ L^q(\Omega), \quad q \in [p, \infty), & \text{if } p = n, \\ L^\infty(\Omega) \cap C^{0,\alpha}(\overline{\Omega}), \quad \alpha = \frac{p-n}{p}, & \text{if } p > n. \end{cases}$$

There exists a constant $C = C(n, p, \Omega)$ such that:

$$\begin{cases} \|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, & \text{for } p < n, \\ \|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}, & \text{for } p > n. \end{cases}$$

For general $k \in \mathbb{N}$, if $kp < n$, then:

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \quad q \leq \frac{np}{n - kp},$$

and if $kp > n$, then:

$$W^{k,p}(\Omega) \hookrightarrow C^{m,\alpha}(\overline{\Omega}),$$

for suitable $m \in \mathbb{N}$ and $\alpha \in (0, 1)$.

1.1.5 Sobolev Spaces with Banach Space Values

For time-dependent PDEs, we define Sobolev spaces with Banach space-valued functions.

Definition 1.1.6 *Let X be a Banach space, $a, b \in \mathbb{R}$ with $a < b$, $k \in \mathbb{N}$, and $1 \leq p \leq \infty$. The space $W^{k,p}(a,b;X)$ is:*

$$W^{k,p}(a,b;X) = \left\{ v \in L^p(a,b;X) \mid \frac{d^i v}{dt^i} \in L^p(a,b;X) \text{ for all } i \leq k \right\},$$

where $\frac{d^i v}{dt^i}$ is the weak derivative.

Lemma 1.1.4 *The space $W^{k,p}(a,b;X)$, with norm:*

$$\|f\|_{W^{k,p}(a,b;X)} = \begin{cases} \left(\sum_{i=0}^{k} \left\| \frac{d^i f}{dt^i} \right\|_{L^p(a,b;X)}^p \right)^{1/p}, & 1 \leq p < \infty, \\ \max_{i=0,\ldots,k} \left\| \frac{d^i f}{dt^i} \right\|_{L^\infty(a,b;X)}, & p = \infty, \end{cases}$$

is a Banach space. For $p = 2$, $H^k(a,b;X) = W^{k,2}(a,b;X)$ is a Hilbert space with inner product:

$$\langle f, g \rangle_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left\langle \frac{d^i f}{dt^i}(t), \frac{d^i g}{dt^i}(t) \right\rangle_X dt.$$

1.2 Known Inequalities

Inequalities are indispensable for establishing estimates in fractional PDEs. We present key results used in subsequent chapters Adams2003, Brezis2011.

1.2.1 Sobolev-Poincaré Inequality

Theorem 1.2.1 (Sobolev-Poincaré Inequality) *Let $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$, be a bounded open set in some direction, and let p satisfy:*

- $2 \leq p < \infty$ *if* $n = 1, 2$,
- $2 \leq p \leq \frac{2n}{n-2}$ *if* $n \geq 3$.

There exists a constant $C = C(\Omega, p, n) > 0$ such that for all $u \in H_0^1(\Omega)$:

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^2(\Omega)}.$$

Definition 1.2.1 *An open set $\Omega \subset \mathbb{R}^n$ is bounded in a direction if there exists a unit vector $e \in \mathbb{R}^n$ and a constant $A > 0$ such that:*

$$\Omega \subset \{x \in \mathbb{R}^n \mid -A \leq x \cdot e \leq A\}.$$

Corollary 1.2.1 *For a bounded open set $\Omega \subset \mathbb{R}^n$, the norm:*

$$\|u\|_{H_0^1(\Omega)} = \|\nabla u\|_{L^2(\Omega)}$$

is equivalent to the standard $H^1(\Omega)$ norm on $H_0^1(\Omega)$.

1.2.2 Hölder's Inequality

Theorem 1.2.2 (Hölder's Inequality) *Let $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For $f \in L^p(\Omega)$, $g \in L^q(\Omega)$:*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

1.2.3 Young's Inequality

Theorem 1.2.3 (Young's Inequality) *Let $1 \leq p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$. For all $a, b \in \mathbb{R}$:*

$$|ab| \leq \frac{|a|^p}{p} + \frac{|b|^q}{q}.$$

1.2.4 Green's Formula

Theorem 1.2.4 (Green's Formula) *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with a C^1 boundary. For $u \in H^2(\Omega)$, $v \in H^1(\Omega)$:*

$$\int_\Omega \Delta u(x) v(x)\, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(x) v(x)\, d\sigma - \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx.$$

1.2.5 Grönwall's Inequality

Theorem 1.2.5 (Grönwall's Integral Inequality) *Let $\phi, \phi_2 : [0, T] \to \mathbb{R}$ be continuous, $\phi(t) \geq 0$, $\phi_2(t) \geq 0$, and:*

$$\phi(t) \leq C_1 + C_2 \int_0^t \phi(s)\phi_2(s)\, ds, \quad \forall t \in [0, T].$$

Then:

$$\phi(t) \leq C_1 \exp\left(C_2 \int_0^t \phi_2(s)\, ds \right).$$

1.3 Lebesgue's Dominated Convergence Theorem

Theorem 1.3.1 (Lebesgue's Dominated Convergence Theorem) *Let $(E, \mathcal{A}, \mu)$ be a measure space, and let $(f_n)_{n\in\mathbb{N}}$ be measurable functions from E to $\mathbb{R}$ or $\mathbb{C}$. Suppose:*

1. *$f_n \to f$ pointwise on E.*
2. *There exists $g \in L^1(E, \mu)$, $g \geq 0$, such that $|f_n(x)| \leq g(x)$.*

Then:

1. *$f \in L^1(E, \mu)$.*

2. $\lim_{n\to\infty} \int_E |f_n - f|\, d\mu = 0.$

3. $\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu.$

Proof See Rudin1987. ■

In the context of fractional PDEs, such as:

$$\partial_t^\alpha u + (-\Delta)^s u = f \quad \text{in } \Omega \times (0,T),$$

Lebesgue's theorem ensures the convergence of Galerkin approximations to weak solutions Kilbas2006.

1.4 Functional Analysis

Functional analysis provides the tools to reformulate PDEs as abstract evolution equations, particularly through operators and semigroups Barbu2010, Pazy1983.

1.4.1 Maximal Monotone Operators

Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\| \cdot \|_{\mathcal{H}}$.

Definition 1.4.1 *An operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is monotone if:*

$$\langle Av, v \rangle_{\mathcal{H}} \geq 0, \quad \forall v \in D(A).$$

Definition 1.4.2 *An operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is maximal monotone if it is monotone and for every $f \in \mathcal{H}$, there exists $u \in D(A)$ such that:*

$$u + Au = f.$$

Proposition 1.4.1 *If $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is maximal monotone, then:*

1. *$D(A)$ is dense in $\mathcal{H}$.*

2. *A is closed.*

3. *For $\lambda > 0$, $(I + \lambda A) : D(A) \to \mathcal{H}$ is bijective, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq 1$.*

See Barbu2010.

1.4.2 Semigroups

Semigroups are used to solve evolution equations of the form $\partial_t u + Au = f$.

Definition 1.4.3 *A family $\{S(t)\}_{t\geq 0} \subset \mathcal{L}(\mathcal{H})$ is a semigroup if:*

1. $S(0) = I$.
2. $S(t+s) = S(t)S(s)$, *for all* $t, s \geq 0$.

It is a C_0-semigroup (strongly continuous) if:

$$\lim_{t\to 0} S(t)x = x, \quad \forall x \in \mathcal{H}.$$

The generator A is defined by:

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}, \quad x \in D(A),$$

where $D(A)$ is the set of $x \in \mathcal{H}$ for which the limit exists.

Theorem 1.4.1 (Hille-Yosida, Lumer-Phillips) *An operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ is maximal monotone if and only if $-A$ generates a C_0-semigroup of contractions.*

Corollary 1.4.1 *For a maximal monotone operator A, the fractional evolution equation:*

$$\partial_t^\alpha u + Au = 0, \quad t > 0, \quad 0 < \alpha < 1, \quad u(0) = u_0,$$

has a unique solution $u(t) = S(t)u_0$, where $S(t)$ is the semigroup generated by $-A$ K best-sellingBarbu2010.

1.4.3 Lax-Milgram Theorem

Theorem 1.4.2 (Lax-Milgram) *Let H be a Hilbert space, and let $a : H \times H \to \mathbb{R}$ be a continuous and coercive bilinear form. For any $L \in H'$, there exists a unique $u \in H$ such that:*

$$a(u, v) = L(v), \quad \forall v \in H.$$

1.5 Introduction to Fractional Calculus

Fractional calculus extends differentiation and integration to non-integer orders, making it ideal for modeling systems with memory and hereditary properties Podlubny1998.

1.5.1 Historical Development

Fractional calculus originated in 1695 through discussions between Leibniz and L'Hôpital, with significant contributions from Euler, Fourier, Abel, Riemann, and Liouville. Modern developments by Caputo and others have solidified its applications in physics and engineering Samko1993.

1.5.2 Fractional Operators

We define the Riemann-Liouville and Caputo fractional operators, which are central to our analysis.

Definition 1.5.1 (Riemann-Liouville Fractional Integral) *For $\alpha > 0$ and a function $f \in L^1(a, b)$, the left-sided Riemann-Liouville fractional integral of order α is:*

$$({}_aI^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,$$

where $\Gamma(\alpha)$ is the Gamma function.

Definition 1.5.2 (Riemann-Liouville Fractional Derivative) *For $\alpha \geq 0$, let $n = \lceil \alpha \rceil$. The Riemann-Liouville fractional derivative of order α is:*

$$({}_aD^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^n ({}_aI^{n-\alpha} f)(x).$$

Definition 1.5.3 (Caputo Fractional Derivative) *For $\alpha \geq 0$, let $n = \lceil \alpha \rceil$. The Caputo fractional derivative of order α is:*

$$({}_aD^{\alpha} f)(x) = ({}_aI^{n-\alpha} f^{(n)})(x),$$

provided $f^{(n)} \in L^1(a, b)$.

Definition 1.5.4 (Generalized Caputo Fractional Derivative) *For $0 < \alpha < 1$ and $\eta \geq 0$, the generalized Caputo fractional derivative is:*

$$(D^{\alpha,\eta} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} e^{-\eta(x-t)} f'(t)\, dt.$$

Definition 1.5.5 (Generalized Fractional Integral) *For $0 < \alpha < 1$ and $\eta \geq 0$, the generalized fractional integral is:*

$$(I^{\alpha,\eta} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} e^{-\eta(x-t)} f(t)\, dt.$$

Remark 1.5.1 *The operators $D^{\alpha,\eta}$ and $I^{\alpha,\eta}$ incorporate an exponential weight, making them suitable for modeling damping with memory effects Mbodje2006.*

Lemma 1.5.1 *Let $\delta > 0$ and $B(t) \in C^2(0, \infty)$ be nonnegative, satisfying:*

$$B''(t) - 4(\delta+1)B'(t) + 4(\delta+1)B(t) \geq 0.$$

If:

$$B'(0) > r_2 B(0) + l_0,$$

then:

$$B'(t) > l_0, \quad t > 0,$$

where $r_2 = 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$. See LiTsai2003.

Lemma 1.5.2 *Let* $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$, $\xi \in \mathbb{R}$, *for* $0 < \alpha < 1$. *Consider the system:*

$$\partial_t \phi(\xi, t) + (\xi^2 + \eta)\phi(\xi, t) = U(t)\mu(\xi),$$

with $\phi(\xi, 0) = 0$, *and output:*

$$O(t) = \varrho \int_{\mathbb{R}} \phi(\xi, t)\mu(\xi)\, d\xi.$$

Then:

$$O(t) = (I_\eta^{1-\alpha} U)(t),$$

where $I_\eta^{1-\alpha}$ *is the generalized fractional integral of order* $1-\alpha$ *with parameter* $\eta > 0$. *See Mbodje2006.*

Lemma 1.5.3 *LiTsai2003. If* $J(t)$ *is a non-creasing function on* $[t_0, \infty)$, $t_0 \geq 0$ *and satisfies the differential inequality*

$$J'(t)^2 \geq a + bJ(t)^{2+\frac{1}{\delta}} \quad \textit{for} \; t \geq t_0, \tag{1.1}$$

where $a > 0$, $b \in \mathbb{R}$, *then there exist a finite time* T^* *such that*

$$\lim_{t \to T^{*-}} J(t) = 0,$$

and the upper bound of T^* *is estimated, respectively, by the following cases :*

i) *if* $b < 0$ *and* $J(t_0) < \min\left\{1, \sqrt{a/(-b)}\right\}$ *then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{a}{-b}}}{\sqrt{\frac{a}{-b}} - J(t_0)}. \tag{1.2}$$

ii) *If* $b = 0$, *then*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{a}}. \tag{1.3}$$

iii) *If* $b > 0$, *then*

$$T^* \leq \frac{J(t_0)}{\sqrt{a}}$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{a}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right), \tag{1.4}$$

where

$$c = \left(\frac{b}{a}\right)^{\delta/(2+\delta)}.$$

Chapter 2

Local Existence of Solutions

NONLINEAR wave systems with internal fractional damping model complex physical phenomena, such as wave propagation in viscoelastic materials and energy dissipation in memory-dependent media. These systems, characterized by nonlocal damping terms that account for historical effects, pose significant analytical challenges due to their nonlinearity and nonlocality. Establishing the local existence of solutions is essential for validating the mathematical model and understanding short-term dynamics, with applications in engineering (e.g., vibration control in suspension bridges) and material science (e.g., polymer dynamics). This chapter proves the local existence of solutions to a coupled nonlinear wave system with fractional damping, using Hilbert spaces and C_0-semigroup theory to reformulate the problem as an abstract evolution equation. The chapter is organized as follows: Section 2.1 presents the problem and necessary assumptions, Section 2.2 proves local existence via semigroup theory, and Section 2.3 discusses the results and future directions.

2.1 Assumptions and Problem Setup

This section introduces the coupled nonlinear wave system, reformulates it to incorporate fractional damping, and defines the energy functional critical for the analysis.

2.1.1 Problem Description

We consider the following nonlinear wave system in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$:

$$(p)\begin{cases} u_{tt} - \Delta u(t) + au_t + b\,\partial_t^{\alpha,\beta} u(t) = |v|^{q+1}|u|^{p-1}u, & x \in \Omega,\ t > 0, \\ v_{tt} - \Delta v(t) + av_t + b\,\partial_t^{\alpha,\beta} v(t) = |u|^{p+1}|v|^{q-1}v, & x \in \Omega,\ t > 0, \\ u = 0, \quad v = 0 & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \end{cases} \tag{2.1}$$

where $a, b > 0$, $0 < \alpha < 1$, $\beta \geq 0$, and $\partial_t^{\alpha,\beta}$ denotes the generalized Caputo fractional derivative with parameter β. The nonlinear terms $|v|^{q+1}|u|^{p-1}u$ and $|u|^{p+1}|v|^{q-1}v$ model coupled interactions, and $p, q > 0$ are exponents satisfying specific conditions.

To handle the fractional damping, we reformulate system (P) ausing a lemma1.5.2, yielding the equivalent augmented system:

$$(p')\begin{cases} u_{tt} - \Delta u(t) + au_t + b_1 \int_{-\infty}^{+\infty} \psi_1(x,\xi,t)\mu(\xi)d\xi = |v|^{q+1}|u|^{p-1}u, & x \in \Omega, t > 0, \\ v_{tt} - \Delta v(t) + av_t + b_1 \int_{-\infty}^{+\infty} \psi_2(x,\xi,t)\mu(\xi)d\xi = |u|^{p+1}|v|^{q-1}v, & x \in \Omega, t > 0, \\ \partial_t\psi_1(x,\xi,t) + (\xi^2+\beta)\psi_1(x,\xi,t) - u_t(x,t)\mu(\xi) = 0 & x \in \Omega, \xi \in \mathbb{R}, t > 0 \\ \partial_t\psi_2(x,\xi,t) + (\xi^2+\beta)\psi_2(x,\xi,t) - v_t(x,t)\mu(\xi) = 0, & x \in \Omega, \xi \in \mathbb{R}, t > 0 \\ u = 0, \quad v = 0, & x \in \partial\Omega, t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega, \\ \psi_1(x,\xi,0) = 0, \quad \psi_2(x,\xi,0) = 0, \quad x \in \Omega, \xi \in \mathbb{R}, \end{cases}$$

where we define $b_1 = b\frac{\sin(\alpha\pi)}{\pi}$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$.

2.1.2 Assumptions

The analysis assumes the following:

1. The domain $\Omega \subset \mathbb{R}^n$ is bounded with a smooth boundary.

2. The exponents p, q satisfy:

$$\begin{cases} p, q > 2, & \text{if } n = 1, 2, \\ 2 < p, q < \frac{2n}{n-2}, & \text{if } n \geq 3, \end{cases} \tag{2.2}$$

ensuring the nonlinear terms $|v|^{q+1}|u|^{p-1}u$ and $|u|^{p+1}|v|^{q-1}v$ belong to $L^2(\Omega)$, leveraging Sobolev embeddings. For $n \geq 3$, the embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ requires $p, q < \frac{2n}{n-2}$ to maintain integrability of the nonlinearities. For $n = 1, 2$, the embedding $H_0^1(\Omega) \hookrightarrow L^r(\Omega)$ for any $r < \infty$ (or all r when $n = 1$) allows more flexibility, but $p, q > 2$ ensures local integrability and compatibility with the energy space $\mathbf{W}$.

3. Initial data satisfy $U_0 = (u_0, u_1, 0, v_0, v_1, 0)^T \in \mathbf{W}$, where:

$$\mathbf{W} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R}) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R}). \tag{2.3}$$

2.1.3 Energy Functional

We define the energy functional for system (P') as:

$$\begin{aligned} E(t) =& \frac{p+1}{2}\|u_t(t)\|_2^2 + \frac{p+1}{2}\|\nabla u(t)\|_2^2 + \frac{q+1}{2}\|v_t(t)\|_2^2 \\ &+ \frac{q+1}{2}\|\nabla v(t)\|_2^2 - \int_\Omega |v|^{q+1}|u|^{p+1}dx \\ &+ \frac{b_1(p+1)}{2}\|\psi_1\|_{L^2(\Omega\times\mathbb{R})}^2 + \frac{b_1(q+1)}{2}\|\psi_2\|_{L^2(\Omega\times\mathbb{R})}^2. \end{aligned} \tag{2.4}$$

Lemma 2.1.1 *Let (u, v, ψ_1, ψ_2) be a solution of the problem (P'). Then, the energy functional defined by (2.4) satisfies*

$$\begin{aligned} \frac{dE(t)}{dt} =& -a(p+1)\|u_t\|_2^2 - a(q+1)\|v_t\|_2^2 \\ &- b_1(p+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1^2(x,\xi,t)d\xi dx \\ &- b_1(q+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2^2(x,\xi,t)d\xi dx \\ &\leq 0. \end{aligned} \tag{2.5}$$

Proof Multiplying the first equation in (P') by $(p+1)u_t$, integrating over Ω, and using integration by parts, we obtain:

$$\begin{aligned} &\frac{p+1}{2}\frac{d}{dt}\left[\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2\right] - \int_\Omega |v|^{q+1}\frac{d}{dt}|u|^{p+1}\,dx \\ &+ a(p+1)\|u_t(t)\|_2^2 + b_1(p+1)\int_\Omega u_t\int_{-\infty}^{+\infty}\psi_1(x,\xi,t)\mu(\xi)\,d\xi\,dx = 0. \end{aligned} \tag{2.6}$$

Multiplying the second equation in (P') by $(q+1)v_t$, integrating over Ω, and using integration by parts, we get:

$$\begin{aligned}&\frac{q+1}{2}\frac{d}{dt}\left[\|v_t(t)\|_2^2+\|\nabla v(t)\|_2^2\right]-\int_\Omega |u|^{p+1}\frac{d}{dt}|v|^{q+1}\,dx\\&+a(q+1)\|v_t(t)\|_2^2+b_1(q+1)\int_\Omega v_t\int_{-\infty}^{+\infty}\psi_2(x,\xi,t)\mu(\xi)\,d\xi\,dx=0.\end{aligned}\tag{2.7}$$

Multiplying the third equation in (P') by $b_1(p+1)\psi_1$, integrating over $\Omega\times(-\infty,+\infty)$, we obtain:

$$\begin{aligned}&\frac{b_1(p+1)}{2}\frac{d}{dt}\int_\Omega\int_{-\infty}^{+\infty}|\psi_1|^2\,d\xi\,dx+b_1(p+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\psi_1|^2\,d\xi\,dx\\&-b_1(p+1)\int_\Omega u_t\int_{-\infty}^{+\infty}\psi_1(x,\xi,t)\mu(\xi)\,d\xi\,dx=0.\end{aligned}\tag{2.8}$$

Multiplying the fourth equation in (P') by $b_1(q+1)\psi_2$, integrating over $\Omega\times(-\infty,+\infty)$, we get:

$$\begin{aligned}&\frac{b_1(q+1)}{2}\frac{d}{dt}\int_\Omega\int_{-\infty}^{+\infty}|\psi_2|^2\,d\xi\,dx+b_1(q+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)|\psi_2|^2\,d\xi\,dx\\&-b_1(q+1)\int_\Omega v_t\int_{-\infty}^{+\infty}\psi_2(x,\xi,t)\mu(\xi)\,d\xi\,dx=0.\end{aligned}\tag{2.9}$$

Combining identities (2.6), (2.7), (2.8), and (2.9), we complete the proof of the lemma. ■

2.2 Local Existence via Semigroup Theory

This section proves the local existence of solutions to system (P') by reformulating it as an abstract evolution equation and applying C_0-semigroup theory.

2.2.1 Reformulation as an Evolution Equation

Introduce $z=u_t$, $w=v_t$, and define the state vector $U=(u,z,\psi_1,v,w,\psi_2)^T$. The nonlinear term is:

$$F(U(t))=(0,|v|^{q+1}|u|^{p-1}u,0,0,|u|^{p+1}|v|^{q-1}v,0)^T.$$

System (P') is rewritten as:

$$(P'')\begin{cases}U_t(t)+AU(t)=F(U(t)),\\U(0)=U_0.\end{cases}$$

where the operator $A : D(A) \to \mathbf{W}$ is defined by

$$AU = \begin{pmatrix} -z \\ -\Delta u + az + b_1 \int_{-\infty}^{+\infty} \psi_1(x,\xi,t)\mu(\xi)d\xi \\ (\xi^2+\beta)\psi_1(x,\xi,t) - z\mu(\xi) \\ -w \\ -\Delta v + aw + b_1 \int_{-\infty}^{+\infty} \psi_2(x,\xi,t)\mu(\xi)d\xi \\ (\xi^2+\beta)\psi_2(x,\xi,t) - w\mu(\xi) \end{pmatrix},$$

The energy space is given by

$$\mathbf{W} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R}) \times H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega \times \mathbb{R}).$$

For any $U = (u, z, \psi_1, v, w, \psi_2)^T \in \mathbf{W}$ and $\bar{U} = (\bar{u}, \bar{z}, \bar{\psi_1}, \bar{v}, \bar{w}, \bar{\psi_2})^T \in \mathbf{W}$, the inner product is defined as

$$\begin{aligned} \langle U, \bar{U} \rangle_{\mathbf{W}} = &\int_\Omega [\nabla u \cdot \nabla \bar{u} + z\bar{z}]dx + b_1 \int_\Omega \int_{-\infty}^{+\infty} \psi_1 \bar{\psi_1} d\xi dx \\ &\int_\Omega [\nabla v \cdot \nabla \bar{v} + w\bar{w}]dx + b_1 \int_\Omega \int_{-\infty}^{+\infty} \psi_2 \bar{\psi_2} d\xi dx \end{aligned} \tag{2.10}$$

The domain of the operator A is given by

$$D(A) = \left\{ \begin{array}{l} U = (u, z, \psi_1, v, w, \psi_2)^T \in \mathbf{W};\ u, v \in H^2(\Omega); w, z \in H_0^1(\Omega); \\ \\ (\xi^2+\beta)\psi_1 - z\mu(\xi) \in L^2(\Omega \times \mathbb{R}); (\xi^2+\beta)\psi_2 - w\mu(\xi) \in L^2(\Omega \times \mathbb{R}); \\ \\ |\xi|\psi_1 \in L^2(\Omega \times \mathbb{R}); |\xi|\psi_2 \in L^2(\Omega \times \mathbb{R}); \end{array} \right\}.$$

Remark 2.2.1 *The condition $|\xi|\psi_i \in L^2(\Omega \times \mathbb{R})$ ensures the damping terms decay sufficiently as $\xi \to \infty$, critical for boundedness.*

2.2.2 Main Result

Theorem 2.2.1 *Suppose the exponents p, q satisfy (2.2), and the initial data $U_0 \in \mathbf{W}$. Then, system (P'') admits a unique local solution:*

$$U \in C([0,T), \mathbf{W}),$$

for some $T > 0$.

Proof Following Pazy1983, we prove that A is a maximal monotone operator and F is locally Lipschitz continuous in $\mathbf{W}$.

Step 1: Density and Monotonicity of A. To apply the Lumer-Phillips theorem, we first verify that $D(A)$ is dense in $\mathbf{W}$. Let $U = (u, z, \psi_1, v, w, \psi_2)^T \in \mathbf{W}$. Since $C_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and $L^2(\Omega)$, there exist sequences $\{u_n\}, \{v_n\}, \{z_n\}, \{w_n\} \subset C_c^\infty(\Omega)$ such that $u_n \to u$, $v_n \to v$ in $H_0^1(\Omega)$, and $z_n \to z$, $w_n \to w$ in $L^2(\Omega)$. Similarly, $C_c^\infty(\Omega \times \mathbb{R})$ is dense in $L^2(\Omega \times \mathbb{R})$, so there exist $\{\psi_{1,n}\}, \{\psi_{2,n}\} \subset C_c^\infty(\Omega \times \mathbb{R})$ such that $\psi_{i,n} \to \psi_i$ in $L^2(\Omega \times \mathbb{R})$. For each n, the functions $\psi_{i,n}$ have compact support in ξ, ensuring $|\xi|\psi_{i,n} \in L^2(\Omega \times \mathbb{R})$. Moreover, since $z_n, w_n \in C_c^\infty(\Omega)$ and $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$ is locally bounded for $0 < \alpha < 1$, the term $(\xi^2 + \beta)\psi_{i,n} - z_{i,n}\mu(\xi) \in L^2(\Omega \times \mathbb{R})$. Thus, $U_n = (u_n, z_n, \psi_{1,n}, v_n, w_n, \psi_{2,n})^T \in D(A)$, and $U_n \to U$ in $\mathbf{W}$, proving density.

Next, for $U \in D(A)$, compute:

$$\begin{aligned}\langle AU, U\rangle_{\mathbf{W}} =& a\|z\|_2^2 + a\|w\|_2^2 \\ &+ b_1 \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)\psi_1^2(x, \xi, t)\, d\xi\, dx \\ &+ b_1 \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)\psi_2^2(x, \xi, t)\, d\xi\, dx \geq 0,\end{aligned}$$

since $a, b_1, \beta \geq 0$. Thus, A is monotone.

Step 2: Surjectivity of $I + A$. We will show the subjectivity of the operator (I + A). For any $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathbf{W}$, we will show that

$$(I + A)U = F$$

is satisfied for some $U \in D(A)$; that is,

$$\begin{cases} u - z = f_1 \in H_0^1(\Omega), \\ \\ z - \Delta u + az + b_1 \displaystyle\int_{-\infty}^{+\infty} \psi_1(x, \xi, t)\mu(\xi)d\xi = f_2 \in L^2(\Omega), \\ \\ \psi_1 + (\xi^2 + \beta)\psi_1(x, \xi, t) - z(x, t)\mu(\xi) = f_3(x, \xi) \in L^2(\Omega \times \mathbb{R}), \\ \\ v - w = f_4 \in H_0^1(\Omega), \\ \\ w - \Delta v + aw + b_1 \displaystyle\int_{-\infty}^{+\infty} \psi_2(x, \xi, t)\mu(\xi)d\xi = f_5 \in L^2(\Omega), \\ \\ \psi_2 + (\xi^2 + \beta)\psi_2(x, \xi, t) - w(x, t)\mu(\xi) = f_6(x, \xi) \in L^2(\Omega \times \mathbb{R}), \end{cases} \tag{2.11}$$

By solving the third and sixth equations in (2.11), we obtain

$$\psi_1 = \frac{f_3 + z(x, t)\mu(\xi)}{(1 + \xi^2 + \beta)} \tag{2.12}$$

and

$$\psi_2 = \frac{f_6 + w(x,t)\mu(\xi)}{(1+\xi^2+\beta)} \tag{2.13}$$

Inserting $z = u - f_1$, and (2.12) in the second equation in (2.11), we obtain

$$\sigma u - \Delta u = G_1. \tag{2.14}$$

where

$$\begin{aligned}
&\sigma = (1 + a + b_2),\\
&G_1 = f_2 + (1 + a + b_2)f_1 - b_1 \int_{-\infty}^{+\infty} \frac{f_3\mu(\xi)}{\xi^2+\beta+1} d\xi,\\
&b_2 = \frac{b_1\pi}{sin(\alpha\pi)}(1+\beta)^{\alpha-1}.
\end{aligned}$$

Similarly, for the fifth equation:

$$\sigma v - \Delta v = G_2. \tag{2.15}$$

where

$$G_2 = f_5 + (1 + a + b_2)f_4 - b_1 \int_{-\infty}^{+\infty} \frac{f_6\mu(\xi)}{\xi^2+\beta+1} d\xi$$

Using the Lax–Milgram theorem, the variational formulations

$$B_1(u,\bar{u}) = L_1(\bar{u}), \quad B_2(v,\bar{v}) = L_2(\bar{v}),$$

with

$$B_i(u,\bar{u}) = \sigma \int_\Omega u\bar{u}\, dx + \int_\Omega \nabla u \cdot \nabla \bar{u}\, dx, \quad L_i(\bar{u}) = \int_\Omega G_i \bar{u}\, dx, \quad i = 1, 2,$$

admit unique solutions $u, v \in H_0^1(\Omega)$. By elliptic regularity, we deduce $u, v \in H^2(\Omega)$. Substituting these into the first and fourth equations yields $z, w \in H_0^1(\Omega)$. Using the third and sixth equations, we then obtain $\psi_1, \psi_2 \in L^2(\Omega \times \mathbb{R})$, satisfying all the requirements of the domain $D(A)$. Therefore, $I + A$ is surjective, and A is maximal monotone.

Step 3: Local Lipschitz Continuity of F. Let:

$$U = (u_1, z_1, \psi_1, v_1, w_1, \psi_2), \quad \bar{U} = (u_2, z_2, \bar{\psi_1}, v_2, w_2, \bar{\psi_2}) \in \mathbf{W}.$$

We have:

$$\|F(U) - F(\bar{U})\|_{\mathbf{W}}^2 = \left\||v_1|^{q+1}|u_1|^{p-1}u_1 - |v_2|^{q+1}|u_2|^{p-1}u_2\right\|_{L^2}^2 + \left\||u_1|^{p+1}|v_1|^{q-1}v_1 - |u_2|^{p+1}|v_2|^{q-1}v_2\right\|_{L^2}^2.$$

Estimate the first term:

$$\left\||v_1|^{q+1}(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2) + (|v_1|^{q+1} - |v_2|^{q+1})|u_2|^{p-1}u_2\right\|_{L^2}.$$

Using Sobolev embeddings $H_0^1(\Omega) \hookrightarrow L^{\frac{2n}{n-2}}(\Omega)$ for $n \geq 3$, and given $p < \frac{2n}{n-2}$, we have $|u_1|^{p-1}u_1 \in L^{\frac{2n}{(n-2)(p-1)+2}}(\Omega)$, and:

$$\left\||u_1|^{p-1}u_1 - |u_2|^{p-1}u_2\right\|_{L^{\frac{2n}{n+2}}} \leq C_p \left(\|u_1\|_{L^{\frac{2n}{n-2}}}^{p-1} + \|u_2\|_{L^{\frac{2n}{n-2}}}^{p-1}\right) \|u_1 - u_2\|_{L^{\frac{2n}{n-2}}}.$$

By Hölder's inequality:

$$\left\| |v_1|^{q+1} \left(|u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 \right) \right\|_{L^2} \leq \left\| |v_1|^{q+1} \right\|_{L^{\frac{2n}{n-2}}} \left\| |u_1|^{p-1}u_1 - |u_2|^{p-1}u_2 \right\|_{L^{\frac{2n}{n+2}}}.$$

Since $v_1 \in H_0^1(\Omega)$:

$$\left\| |v_1|^{q+1} \right\|_{L^{\frac{2n}{n-2}}} = \|v_1\|^{q+1}_{L^{\frac{2n(q+1)}{n-2}}} \leq C_\Omega^{q+1} \|v_1\|^{q+1}_{H_0^1}.$$

Thus, the first term is bounded by:

$$C\|v_1\|^{q+1}_{H_0^1} \left(\|u_1\|^{p-1}_{H_0^1} + \|u_2\|^{p-1}_{H_0^1} \right) \|u_1 - u_2\|_{H_0^1}.$$

For the second term, using:

$$\left| |v_1|^{q+1} - |v_2|^{q+1} \right| \leq (q+1) \left(|v_1|^q + |v_2|^q \right) |v_1 - v_2|,$$

we obtain:

$$\left\| \left(|v_1|^{q+1} - |v_2|^{q+1} \right) |u_2|^{p-1}u_2 \right\|_{L^2} \leq (q+1) \left\| (|v_1|^q + |v_2|^q) |v_1 - v_2| |u_2|^{p-1}u_2 \right\|_{L^2}.$$

Using Hölder's inequality:

$$\| |v_1|^q |v_1 - v_2| |u_2|^{p-1}u_2 \|_{L^2} \leq \|v_1\|^q_{L^{\frac{2n}{n-2}}} \|v_1 - v_2\|_{L^{\frac{2n}{n-2}}} \|u_2\|^{p-1}_{L^{\frac{2n}{n-2}}},$$

and thus:

$$\left\| \left(|v_1|^{q+1} - |v_2|^{q+1} \right) |u_2|^{p-1}u_2 \right\|_{L^2} \leq C \left(\|v_1\|^q_{H_0^1} + \|v_2\|^q_{H_0^1} \right) \|v_1 - v_2\|_{H_0^1} \|u_2\|^{p-1}_{H_0^1}.$$

A similar argument applies to the second component. Consequently:

$$\|F(U) - F(\bar{U})\|_{\mathbf{W}} \leq C \cdot \Phi \left(\|u_1\|_{H_0^1}, \|u_2\|_{H_0^1}, \|v_1\|_{H_0^1}, \|v_2\|_{H_0^1} \right) \cdot \left(\|u_1 - u_2\|_{H_0^1} + \|v_1 - v_2\|_{H_0^1} \right),$$

where Φ is continuous, proving F is locally Lipschitz.

Step 4: Existence. Since $-A$ generates a C_0-semigroup of contractions (by the Lumer-Phillips theorem) and F is locally Lipschitz, standard results Pazy1983 guarantee a unique local solution $U \in C([0,T), \mathbf{W})$ for some $T > 0$. ■

Remark 2.2.2 *The proof implicitly uses the contraction mapping principle in the fixed-point argument underlying semigroup theory for nonlinear problems, aligning with Barbu2010.*

2.3 Discussion and Future Directions

The local existence result establishes that, under conditions (2.2), system (P) admits a unique solution over a short time interval. This validates the mathematical model for short-term dynamics in viscoelastic materials, where fractional damping captures energy dissipation over time, as seen in applications like vibration control in structures (e.g., suspension bridges) or polymer dynamics in aerospace components Podlubny1998. Theoretically, the result lays the foundation for studying global existence, stability, or blow-up phenomena, which are critical for understanding long-term behavior. Future research could explore:

- Extending solutions to global existence under weaker conditions on p, q.
- Investigating blow-up phenomena using Lyapunov functionals, as planned for Chapter 3.
- Relaxing boundary conditions or incorporating more general damping mechanisms.

Chapter 3

BLOW-UP

NONLINEAR wave systems with fractional damping, as studied in Chapter 2, exhibit complex dynamics where solutions may not remain bounded over time. Under certain conditions, the energy of the system can grow unbounded, leading to finite-time blow-up, a phenomenon critical for understanding stability limits in applications like vibration control in structures or material dynamics in viscoelastic media. This chapter investigates the blow-up of solutions to the coupled nonlinear wave system (P) introduced in Chapter 2, using a Lyapunov functional to establish conditions under which solutions become unbounded in finite time. The chapter is organized as follows: Section 3.1 presents the Lyapunov functional and auxiliary results, Section 3.2 proves the main blow-up theorem, and Section 3.3 discusses the results and future research directions.

3.1 Lyapunov Functional and Auxiliary Results

Definition 3.1.1 *A solution* $\mathbf{u} = (u, v, \psi_1, \psi_2)$ *of the system (P′) is said to blow up in finite time if there exists a time* $T^* < \infty$ *such that:*

$$\lim_{t \to T^{*-}} \int_\Omega (u^2 + v^2)\, dx = +\infty. \tag{3.1}$$

Remark 3.1.1 *A simple integration of (2.5) over (0, t) leads to*

$$\begin{aligned} E(t) =& E(0) - a(p+1) \int_0^t \|u_s\|_2^2 ds - a(q+1) \int_0^t \|v_s\|_2^2 ds \\ & - b_1(p+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_1^2(x, \xi, s) d\xi dx ds \\ & - b_1(q+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_2^2(x, \xi, s) d\xi dx ds. \end{aligned} \tag{3.2}$$

Now, introduce the functional L(t) defined as follows:

$$L(t) = (p+1)\|u\|_2^2 + (q+1)\|v\|_2^2 + a(p+1) \int_0^t \|u\|_2^2 ds + a(q+1) \int_0^t \|v\|_2^2 ds + b_1 H(t) \tag{3.3}$$

where

$$\begin{aligned} H(t) =&(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\bigg(\int_0^s \psi_1(x,\xi,z)dz\bigg)^2 d\xi dx ds \\ &+ (q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\bigg(\int_0^s \psi_2(x,\xi,z)dz\bigg)^2 d\xi dx ds. \end{aligned} \tag{3.4}$$

Lemma 3.1.1 *Let (u,ϕ) be a regular solution of the problem (21). Then the following equality holds:*

$$\int_{\Gamma_0}\int_{-\infty}^{+\infty} (\xi^2+\eta)\,\phi(\xi,t)\left(\int_0^t \phi(\xi,s)\,ds\right)d\xi\,d\rho = \int_{\Gamma_0} u(x,t)\left(\int_{-\infty}^{+\infty}\phi(\xi,t)\,\mu(\xi)\,d\xi\right)d\rho - \int_{\Gamma_0}\int_{-\infty}^{+\infty} |\phi(\xi,t)|^2$$

[3].

Lemma 3.1.2 (Boundedness of $H(t)$) *Suppose that $\|\nabla u\|_2^2$ and $\|\nabla v\|_2^2$ remain bounded on $[0,T)$. Then, the functional $H(t)$ defined in (3.4) satisfies:*

$$H(t) \le C < +\infty, \tag{3.5}$$

where:

$$H(t) \le \frac{1}{2C_1 B q e^{-\eta C_2}}\left[C_2\alpha - \frac{1}{2}\alpha + C_3 - \alpha\eta\right]\Gamma(\alpha)T^4, \tag{3.6}$$

and:

$$C_1 = \sup_{t\in[0,T)}\left\{\|\nabla u\|_2^2, \|\nabla v\|_2^2, 1\right\}, \tag{3.7}$$

with constants B, η, C_2, C_3 as established in [3].

Proof The proof follows [3], leveraging the boundedness of $\|\nabla u\|_2^2$ and $\|\nabla v\|_2^2$ to control the growth of ψ_1, ψ_2 via the equations for ψ_i in (P'). The explicit bound accounts for the fractional damping effects through $\Gamma(\alpha)$ and the time horizon T. ■

Lemma 3.1.3 *Assume that condition (2.2) holds. Then, the second derivative of the functional $L(t)$ satisfies:*

$$\begin{aligned} L''(t) \ge&(\theta+2)\bigg[(p+1)\|u_t\|_2^2 + (q+1)\|v_t\|_2^2\bigg] \\ &+ 2\theta\bigg[-E(0) + a(p+1)\int_0^t \|u_s\|_2^2\,ds + a(q+1)\int_0^t \|v_s\|_2^2\,ds \\ &+ b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1^2(x,\xi,s)\,d\xi\,dx\,ds \\ &+ b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2^2(x,\xi,s)\,d\xi\,dx\,ds\bigg], \end{aligned} \tag{3.8}$$

where $\theta = p+q+2$.

Proof Form (3.3), the first derivative is:

$$\begin{aligned}L'(t) =&2(p+1)\int_\Omega u_t u\,dx + 2(q+1)\int_\Omega v_t v\,dx + a(p+1)\|u\|_2^2 + a(q+1)\|v\|_2^2\\ &+ 2b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1(x,\xi,s)\int_0^s \psi_1(x,\xi,z)\,dz\,d\xi\,dxds\\ &+ 2b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2(x,\xi,s)\int_0^s \psi_2(x,\xi,z)\,dzd\xi\,dxds.\end{aligned}$$

Differentiating again, using the equations in (P') and the divergence theorem:

$$\begin{aligned}L''(t) =& -2(p+1)\|\nabla u\|_2^2 - 2(q+1)\|\nabla v\|_2^2 + 2(p+1)\|u_t\|_2^2 + 2(q+1)\|v_t\|_2^2\\ &+ 2\theta\int_\Omega |v|^{q+1}|u|^{p+1}\,dx\\ &+ 2b_1(p+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1(x,\xi,t)\left(\int_0^t\psi_1(x,\xi,s)\,ds\right)d\xi\,dx\\ &+ 2b_1(q+1)\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2(x,\xi,t)\left(\int_0^t\psi_2(x,\xi,s)\,ds\right)d\xi\,dx\\ &- 2b_1(p+1)\int_\Omega u\int_{-\infty}^{+\infty}\psi_1(x,\xi,t)\mu(\xi)\,d\xi\,dx\\ &- 2b_1(q+1)\int_\Omega v\int_{-\infty}^{+\infty}\psi_2(x,\xi,t)\mu(\xi)\,d\xi\,dx.\end{aligned}$$

where $\theta = p+q+2$. Using the energy functional (2.5) and integrating (3.2), we express:

$$\begin{aligned}2\int_\Omega |v|^{q+1}|u|^{p+1}\,dx =& -2E(0) + 2a(p+1)\int_0^t\|u_s\|_2^2\,ds\\ &+ 2a(q+1)\int_0^t\|v_s\|_2^2\,ds + (q+1)\|v_t\|_2^2 + (p+1)\|\nabla u\|_2^2\\ &+ (q+1)\|\nabla v\|_2^2 + 2b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1^2(x,\xi,s)\,d\xi\,dx\,ds\\ &+ 2b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2^2(x,\xi,s)\,d\xi\,dx\,ds\\ &+ (p+1)\|u_t\|_2^2 + b_1(p+1)\|\psi_1\|^2_{L^2(\Omega\times\mathbb{R})} + b_1(q+1)\|\psi_2\|^2_{L^2(\Omega\times\mathbb{R})}.\end{aligned} \tag{3.9}$$

Using lemma 3.1.1 we get:

$$\begin{aligned}\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1(x,\xi,t)\left(\int_0^t\psi_1(x,\xi,s)\,ds\right)d\xi\,dx =& \int_\Omega u\int_{-\infty}^{+\infty}\psi_1(x,\xi,t)\mu(\xi)\,d\xi\,dx\\ &- \|\psi_1\|^2_{L^2(\Omega\times\mathbb{R})},\end{aligned} \tag{3.10}$$

$$\int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2(x,\xi,t)\left(\int_0^t \psi_2(x,\xi,s)\,ds\right)\,d\xi\,dx = \int_\Omega v\int_{-\infty}^{+\infty}\psi_2(x,\xi,t)\mu(\xi)\,d\xi\,dx - \|\psi_2\|^2_{L^2(\Omega\times\mathbb{R})}, \tag{3.11}$$

Substituting into (3.9) and simplifying using (3.10) and (3.11), we obtain:

$$\begin{aligned}
L''(t) =& -2\theta E(0) + 2a\theta(p+1)\int_0^t \|u_s\|_2^2\,ds + 2a\theta(q+1)\int_0^t \|v_s\|_2^2\,ds \\
&+ (p+1)(\theta+2)\|u_t\|_2^2 + (q+1)(\theta+2)\|v_t\|_2^2 + (p+1)(p+q)\|\nabla u\|_2^2 + (q+1)(p+q)\|\nabla v\|_2^2 \\
&+ 2b_1\theta(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1^2(x,\xi,s)\,d\xi\,dx\,ds \\
&+ 2b_1\theta(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2^2(x,\xi,s)\,d\xi\,dx\,ds \\
&+ b_1(p+1)(p+q)\|\psi_1\|^2_{L^2(\Omega\times\mathbb{R})} + b_1(q+1)(p+q)\|\psi_2\|^2_{L^2(\Omega\times\mathbb{R})}.
\end{aligned}$$

Dropping non-negative terms, we get 3.8 which completes the proof. ■
Now, we prove the following lemma.

Lemma 3.1.4 *Suppose that (2.2) holds and that either one the following conditions is satisfied*

(i) $E(0) < 0$.

(ii) $E(0) = 0$, *and*

$$L'(0) > a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2. \tag{3.12}$$

(iii) $E(0) > 0$, *and*

$$L'(0) > r\left[L(0) + \frac{l_0}{\theta+2}\right] + a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2, \tag{3.13}$$

where

$$r = \frac{\theta+2-\sqrt{\theta^2-4\theta}}{2}$$

and

$$l_0 = a(\theta+2)\left[(p+1)\|u_0\|_2^2 + (q+1)\|v_0\|_2^2\right] + 2\theta E(0). \tag{3.14}$$

Then $L'(t) > a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2$, *for* $t > t_0$, *where*

$$t^* > \max\left\{0, \frac{L'(0) - a(p+1)\|u_0\|_2^2 - a(q+1)\|v_0\|_2^2}{2\theta E(0)}\right\}, \tag{3.15}$$

where $t_0 = t^*$ *in case(i), and* $t_0 = 0$ *in case(ii) and (iii)*

Proof (i) If $E(0) < 0$, then from (3.8), we get

$$L^{''}(t) \geq -2\theta E(0),$$

we easily obtain :

$$L^{'}(t) \geq L^{'}(0) - 2\theta E(0)t.$$

Then

$$L^{'}(t) > a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2, \quad \forall t \geq t^*,$$

where t^*, is defined in (3.15).

(ii) If $E(0) = 0$ then from (3.8) we fined

$$L''(t) \geq 0, \quad \forall t \geq 0.$$

We easily obtain

$$L'(t) \geq L'(0), \quad \forall t \geq 0.$$

Using (3.12) we have

$$L^{'}(t) > a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2, \quad \forall t \geq 0.$$

(iii) For the last case of $E(0) > 0$ apply Young's inequality to 3.1:

$$\begin{aligned} &\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)\psi_1(\xi, s) \int_0^s \psi_1(\xi, z) dz d\xi dx ds \\ &\leq \tfrac{1}{2} \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta)|\psi_1(\xi, s)|^2 d\xi dx ds \\ &+ \tfrac{1}{2} \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \left(\int_0^s \psi_1(\xi, z) dz \right)^2 d\xi dx ds \end{aligned} \tag{3.16}$$

and similarly for ψ_2. Using:

$$\|u\|_2^2 \leq \int_0^t \|u_s\|_2^2 \, ds + \int_0^t \|u\|_2^2 \, ds + \|u_0\|_2^2,$$

and we note that

$$2 \int_0^t \int_\Omega u_s u dx ds = \int_0^t \frac{d}{ds} \|u_s\|_2^2 ds = \|u\|_2^2 - \|u_0\|_2^2.$$

We bound:

$$\begin{aligned}
L'(t) \leq & (p+1)\|u_t\|_2^2 + (p+1)\|u\|_2^2 + (q+1)\|v_t\|_2^2 + (q+1)\|v\|_2^2 \\
& + a(p+1)\left(\int_0^t \|u_s\|_2^2\, ds + \int_0^t \|u\|_2^2\, ds + \|u_0\|_2^2\right) \\
& + a(q+1)\left(\int_0^t \|v_s\|_2^2\, ds + \int_0^t \|v\|_2^2\, ds + \|v_0\|_2^2\right) \\
& + b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1^2(x,\xi,s)\, d\xi\, dx\, ds \\
& + b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\left(\int_0^s \psi_1(x,\xi,z)\, dz\right)^2 d\xi\, dx\, ds \\
& + b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2^2(x,\xi,s)\, d\xi\, dx\, ds \\
& + b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\left(\int_0^s \psi_2(x,\xi,z)\, dz\right)^2 d\xi\, dx\, ds.
\end{aligned} \tag{3.17}$$

Combination of (3.3) and (3.17), shows that

$$\begin{aligned}
L'(t) \leq & L(t) + (p+1)\left[\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\, ds + a\|u_0\|_2^2\right] \\
& + b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1^2(x,\xi,s)\, d\xi\, dx\, ds \\
& + (q+1)\left[\|v_t\|_2^2 + a\int_0^t \|v_s\|_2^2\, ds + a\|v_0\|_2^2\right] \\
& + b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2^2(x,\xi,s)\, d\xi\, dx\, ds.
\end{aligned}$$

Using(3.8),we obtain

$$\begin{aligned}
(\theta+2)\Big(L'(t) - L(t)\Big) \leq & L''(t) + 2\theta E(0) + a(\theta+2)\left[(p+1)\|u_0\|_2^2 + (q+1)\|v_0\|_2^2\right] \\
& - (p+q)\Big[(p+1)a\int_0^t \|u_s\|_2^2\, ds + (q+1)a\int_0^t \|v_s\|_2^2\, ds \\
& + b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1^2(x,\xi,s)\, d\xi\, dx\, ds \\
& + b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2^2(x,\xi,s)\, d\xi\, dx\, ds,\Big].
\end{aligned}$$

That is

$$F''(t) - (\theta+2)F'(t) + (\theta+2)F(t) + l_0 \geq 0,$$

where l_0,is defined in (3.14). Now let

$$B(t) = F(t) + \frac{l_0}{\theta + 2}.$$

Then $B(t)$ satisfies

$$B''(t) - (\theta + 2)B'(t) + (\theta + 2)B(t) \geq 0. \tag{3.18}$$

Using Lemma 1.5.1 in (3.18), then if

$$B'(0) > \left(\frac{\theta + 2 - \sqrt{\theta^2 - 4\theta}}{2} \right) B(0) + a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2.$$

Then

$$F'(t) > a(p+1)\|u_0\|_2^2 + a(q+1)\|v_0\|_2^2 \quad \forall t \geq 0.$$

■

3.2 Main Blow-Up Result

Theorem 3.2.1 *Suppose that (2.2) holds and that either one of the following conditions is satisfied*
(i)$E(0) < 0$.
(ii) $E(0) = 0$ *and (3.12) holds.*
(iii) $0 < E(0) < \frac{\left(\frac{p+q-2}{4}\right)^2 \left\{F'(t^*) - (p+1)\|u_0\|_2^2 - (q+1)\|v_0\|_2^2\right\}^2}{((p+q)^2-4)\left(\frac{p+q-2}{2(p+q)}\right) J^{\frac{-1}{\gamma_1}}(t^*)}$ *and (3.13) holds. Then the solutions* (u, v, ψ_1, ψ_2) *blows up in finite time* T^* *in the sense of (3.1).*
In case (i):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{m_1}{-m_2}}\right\}$, *then we have*

$$T^* \leq t_0 + \frac{1}{\sqrt{-m_2}} \ln \frac{\sqrt{\frac{m_1}{-m_2}}}{\sqrt{\frac{m_1}{-m_2}} - J(t_0)}.$$

In case (ii):

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)},$$

or

$$T^* \leq t_0 + \frac{J(t_0)}{J'(t_0)}.$$

In case (iii):

$$T^* \leq \frac{J(t_0)}{\sqrt{\sigma}},$$

or

$$T^* \leq t_0 + 2^{\frac{3\gamma_1+1}{2\gamma_1}} \frac{\gamma_1 c}{\sqrt{m_1}} \{1 - [1 - cJ(t_0)]^{\frac{1}{2\gamma_1}}\},$$

where $c = (\frac{m_2}{m_1})^{\frac{\gamma_1}{2+\gamma_1}}$, $\gamma_1 = \frac{p+q-2}{4}$, *and* $J(t)$, m_1 *and* m_1 *are given in (3.31) and (3.32) respectively. Note that in case (i),* $t_0 = t^*$ *is given in (3.15) and* $t_0 = 0$ *in case (ii) and (iii).*

Proof Define:

$$J(t) = \left[L(t) + a(p+1)(T-t)\|u_0\|_2^2 + a(q+1)(T-t)\|v_0\|_2^2\right]^{-\gamma_1}, \quad t \in [t_0, T]. \tag{3.19}$$

Differentiating:

$$J^{'}(t) = -\gamma_1 J(t)^{1+\frac{1}{\gamma_1}} \left[L'(t) - a(p+1)\|u_0\|_2^2 - a(q+1)\|v_0\|_2^2\right]$$

By Lemma 3.1.4, $J'(t) < 0$. The second derivative is:

$$J^{''}(t) = -\gamma_1 J(t)^{1+\frac{2}{\gamma_1}} G(t), \tag{3.20}$$

where

$$\begin{aligned} G(t) =& L^{''}(t)\left[L(t) + a(p+1)(T-t)\|u_0\|_2^2 + a(q+1)(T-t)\|v_0\|_2^2\right] \\ &- (1+\gamma_1)\left\{L^{'}(t) - a(p+1)\|u_0\|_2^2 - a(q+1)\|v_0\|_2^2\right\}^2. \end{aligned} \tag{3.21}$$

We get, from (3.8)

$$\begin{aligned} L''(t) \geq& (\theta+2)\left[(p+1)\|u_t\|_2^2 + (q+1)\|v_t\|_2^2\right] \\ &+ 2\theta\Big[-E(0) + a(p+1)\int_0^t \|u_s\|_2^2\, ds + a(q+1)\int_0^t \|v_s\|_2^2\, ds \\ &+ b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1^2(x,\xi,s)\, d\xi\, dx\, ds \\ &+ b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2^2(x,\xi,s)\, d\xi\, dx\, ds\Big] \end{aligned} \tag{3.22}$$

Therefore, we find

$$\begin{aligned} L''(t) \geq & -2\theta E(0) \\ & + \theta\Bigg[(p+1)\Bigg\{\|u_t\|_2^2 + a\int_0^t \|u_s\|_2^2\, ds\Bigg\} + (q+1)\Bigg\{\|v_t\|_2^2 + a\int_0^t \|v_s\|_2^2\, ds\Bigg\} \\ & + b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1^2(x,\xi,s)\, d\xi\, dx\, ds \\ & + b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2^2(x,\xi,s)\, d\xi\, dx\, ds\Bigg] \end{aligned} \tag{3.23}$$

It follows from (3.1) that

$$\begin{aligned} & L'(t) - a(p+1)\|u_0\|_2^2 - a(q+1)\|v_0\|_2^2 = 2(p+1)\int_\Omega u_t u\, dx + 2(q+1)\int_\Omega v_t v\, dx \\ & + 2b_1(p+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_1(x,\xi,s)\int_0^s \psi_1(x,\xi,z)\, dz\, d\xi\, dxds \\ & + 2b_1(q+1)\int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2+\beta)\psi_2(x,\xi,s)\int_0^s \psi_2(x,\xi,z)\, dzd\xi\, dxds. \\ & + 2a(p+1)\int_0^t \int_\Omega u_s u dxds + 2a(q+1)\int_0^t \int_\Omega v_s v dxds \end{aligned}$$

since

$$\|u\|_2^2 - \|u_0\|_2^2 = 2\int_0^t \int_\Omega u_s u dxds$$

and

$$\|v\|_2^2 - \|v_0\|_2^2 = 2\int_0^t \int_\Omega v_s v dxds$$

It then follows that

$$\begin{aligned} G(t) \geq & L''(t)\Bigg[L(t) + a(p+1)(T-t)\|u_0\|_2^2 + a(q+1)(T-t)\|v_0\|_2^2\Bigg] \\ & - (1+\gamma_1)\Big\{L'(t) - a(p+1)\|u_0\|_2^2 - a(q+1)\|v_0\|_2^2\Big\}^2. \end{aligned} \tag{3.24}$$

It then follows that

$$
\begin{aligned}
G(t) \geq & -2\theta E(0) J^{\frac{-1}{\gamma}}(t) \\
& + \theta \Bigg[(p+1) \Bigg\{ \|u_t\|_2^2 + a \int_0^t \|u_s\|_2^2 \, ds \Bigg\} + (q+1) \Bigg\{ \|v_t\|_2^2 + a \int_0^t \|v_s\|_2^2 \, ds \Bigg\} \\
& + b_1 (p+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_1^2(x, \xi, s) \, d\xi \, dx \, ds \\
& + b_1 (q+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_2^2(x, \xi, s) \, d\xi \, dx \, ds \Bigg] \\
& \times \Bigg[(p+1) \|u\|_2^2 + (q+1) \|v\|_2^2 + a(p+1) \int_0^t \|u\|_2^2 ds + a(q+1) \int_0^t \|v\|_2^2 ds \\
& + b_1 (p+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \Bigg(\int_0^s \psi_1(x, \xi, z) dz \Bigg)^2 d\xi dx ds \\
& + b_1 (q+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \Bigg(\int_0^s \psi_2(x, \xi, z) dz \Bigg)^2 d\xi dx ds \Bigg] \\
& - 4(1+\gamma) \Bigg[(p+1) \int_\Omega u_t u \, dx + (q+1) \int_\Omega v_t v \, dx \\
& + b_1 (p+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_1(x, \xi, s) \int_0^s \psi_1(x, \xi, z) \, dz \, d\xi \, dx ds \\
& + b_1 (q+1) \int_0^t \int_\Omega \int_{-\infty}^{+\infty} (\xi^2 + \beta) \psi_2(x, \xi, s) \int_0^s \psi_2(x, \xi, z) \, dz d\xi \, dx ds. \\
& + a(p+1) \int_0^t \int_\Omega u_s u dx ds + a(q+1) \int_0^t \int_\Omega v_s v dx ds \Bigg]^2 .
\end{aligned} \tag{3.25}
$$

For simplicity of calculations, we make the notations

$$\begin{aligned}
A =&(p+1)\|u\|_2^2+(q+1)\|v\|_2^2+a(p+1)\int_0^t\|u\|_2^2ds+a(q+1)\int_0^t\|v\|_2^2ds\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\int_0^s\psi_1(x,\xi,z)dz\bigg)^2d\xi dxds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\int_0^s\psi_2(x,\xi,z)dz\bigg)^2d\xi dxds
\end{aligned}$$

$$\begin{aligned}
B =&(p+1)\int_\Omega u_tu\,dx+(q+1)\int_\Omega v_tv\,dx\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1(x,\xi,s)\int_0^s\psi_1(x,\xi,z)\,dz\,d\xi\,dxds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2(x,\xi,s)\int_0^s\psi_2(x,\xi,z)\,dzd\xi\,dxds.\\
&+a(p+1)\int_0^t\int_\Omega u_sudxds+a(q+1)\int_0^t\int_\Omega v_svdxds
\end{aligned}$$

$$\begin{aligned}
C =&(p+1)\bigg\{\|u_t\|_2^2+a\int_0^t\|u_s\|_2^2\,ds\bigg\}+(q+1)\bigg\{\|v_t\|_2^2+a\int_0^t\|v_s\|_2^2\,ds\bigg\}\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_1^2(x,\xi,s)\,d\xi\,dx\,ds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\psi_2^2(x,\xi,s)\,d\xi\,dx\,ds
\end{aligned}$$

Thus,we get

$$G(t)\geq-2\theta E(0)J^{\frac{-1}{\gamma}}(t)+\theta\bigg\{AC-B^2\bigg\}. \tag{3.26}$$

Now we observe that,for all $(\rho,\eta)\in\mathbb{R}^2$ and $t>0$,

$$
\begin{aligned}
&A\rho^2+2B\rho\eta+C\eta^2=\\
&(p+1)\rho^2\|u\|_2^2+(q+1)\rho^2\|v\|_2^2+a(p+1)\int_0^t\rho^2\|u\|_2^2ds+a(q+1)\int_0^t\rho^2\|v\|_2^2ds\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\int_0^s\rho\psi_1(x,\xi,z)dz\bigg)^2d\xi dxds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\int_0^s\rho\psi_2(x,\xi,z)dz\bigg)^2d\xi dxds\\
&2(p+1)\int_\Omega\eta u_t\rho u\,dx+(q+1)\int_\Omega\eta v_t\rho v\,dx\\
&+2b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\eta\psi_1(x,\xi,s)\int_0^s\rho\psi_1(x,\xi,z)\,dz\,d\xi\,dxds\\
&+2b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\eta\psi_2(x,\xi,s)\int_0^s\rho\psi_2(x,\xi,z)\,dzd\xi\,dxds.\\
&+2a(p+1)\int_0^t\int_\Omega\eta u_s\rho udxds+2a(q+1)\int_0^t\int_\Omega\eta v_s\rho vdxds\\
&(p+1)\bigg\{\eta^2\|u_t\|_2^2+a\int_0^t\eta^2\|u_s\|_2^2\,ds\bigg\}+(q+1)\bigg\{\eta^2\|v_t\|_2^2+a\int_0^t\eta^2\|v_s\|_2^2\,ds\bigg\}\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\eta^2\psi_1^2(x,\xi,s)\,d\xi\,dx\,ds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\eta^2\psi_2^2(x,\xi,s)\,d\xi\,dx\,ds
\end{aligned}
$$

Then

$$
\begin{aligned}
&A\rho^2+2B\rho\eta+C\eta^2=(p+1)\|\rho u+\eta u_t\|_2^+(q+1)\|\rho v+\eta v_t\|_2^2\\
&+a(p+1)\int_0^t\|\rho u+\eta u_s\|_2^2ds+a(q+1)\int_0^t\|\rho v+\eta v_t\|_2^2ds\\
&+b_1(p+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\eta\psi_1(x,\xi,s)+\int_0^s\rho\psi_1(x,\xi,z)dz\bigg)^2d\xi dxds\\
&+b_1(q+1)\int_0^t\int_\Omega\int_{-\infty}^{+\infty}(\xi^2+\beta)\bigg(\eta\psi_2(x,\xi,s)+\int_0^s\rho\psi_2(x,\xi,z)dz\bigg)^2d\xi dxds
\end{aligned}
$$

It is easy to see that

$$A\rho^2+2B\rho\eta+C\eta^2\geq 0$$

and

$$B^2-AC\leq 0. \tag{3.27}$$

Hence,we infer from (3.27) that

$$G(t) \geq -2\theta E(0) J^{\frac{-1}{\gamma}}(t) \quad \forall t \geq t^*. \tag{3.28}$$

Therefore, by(3.19),(3.20),and(3.28),weget

$$J^{''}(t) \leq \left(\frac{(p+q)^2-4}{2} E(0)\right) J(t)^{1+\frac{1}{\gamma_1}}, \tag{3.29}$$

Note that by Lemma 3.1.4,$J'(t) < 0$ for $t' \geq t^*0$. Multiplying (3.29)by $J'(t)$and integrating it from t^* to t, we obtain

$$J'^2(t) \geq m_1 + m_2 J(t)^{2+\frac{1}{\gamma_1}}, \tag{3.30}$$

where

$$\begin{aligned} m_1 =& \left[\left(\frac{p+q-2}{4}\right)^2 \left\{F'(t^*) - (p+1)\|u_0\|_2^2 - (q+1)\|v_0\|_2^2\right\}^2 \right. \\ & \left. - \left((p+q)^2-4\right)\left(\frac{p+q-2}{2(p+q)}\right) E(0) J^{\frac{-1}{\gamma_1}}(t^*)\right] \end{aligned} \tag{3.31}$$

$$m_2 = \left((p+q)^2-4\right)\left(\frac{p+q-2}{2(p+q)}\right) E(0) \tag{3.32}$$

We observe that $m_1 > 0$,if and only if

$$E(0) < \frac{\left(\frac{p+q-2}{4}\right)^2 \left\{F'(t^*) - (p+1)\|u_0\|_2^2 - (q+1)\|v_0\|_2^2\right\}^2}{\left((p+q)^2-4\right)\left(\frac{p+q-2}{2(p+q)}\right) J^{\frac{-1}{\gamma_1}}(t^*)}$$

then by Lemma 1.5.3 there exists a finite time T^* such that This will imply that

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + a\int_0^t \|u\|_2^2\, ds + \|v\|_2^2 + a\int_0^t \|v\|_2^2\, ds + b_1 H(t)\right]^{-1} = 0. \tag{110}$$

That is,

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + a\int_0^t \|u\|_2^2\, ds + \|v\|_2^2 + a\int_0^t \|v\|_2^2\, ds + b_1 H(t)\right] = +\infty. \tag{111}$$

Thus, there exists a T such that $t_0 < T \leq T_0$ and

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + \|v\|_2^2 + b_1 H(t)\right] = +\infty. \tag{3.33}$$

Indeed, if $\|u\|_2^2 \to +\infty$ as $t \to T^-$, and $\|v\|_2^2 \to +\infty$ as $t \to T^-$, then clearly,

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + \|v\|_2^2 + b_1 H(t)\right] = +\infty.$$

On the other hand, if $\|\nabla u\|_2^2$ and $\|\nabla v\|_2^2$ remained bounded on $[t_0, T)$, by Lemma 3.1.2, we get

$$\lim_{t \to T^{*-}} \left[\|u\|_2^2 + \|v\|_2^2 + b_1 H(t)\right] = C < +\infty.$$

Hence, $\|\nabla u\|_2^2 \to +\infty$ as $t \to T^-$ and $\|\nabla v\|_2^2 \to +\infty$ as $t \to T^-$. ■

3.3 Discussion and Future Directions

The blow-up result confirms that under negative or critical initial energy conditions, the nonlinear wave system (P) exhibits solutions that become unbounded in finite time, driven by the nonlinear coupling terms. This behavior is significant for applications, such as modeling material failure in viscoelastic structures or instability in control systems Podlubny1998. The Lyapunov functional approach highlights the role of fractional damping in modulating energy growth. Future research could explore:

- **Global Behavior**: Investigating conditions for global existence versus blow-up to delineate the stability boundary.
- **Numerical Simulations**: Developing computational methods to validate blow-up times and energy growth.
- **Generalized Models**: Extending the analysis to variable α or non-homogeneous boundary conditions.

Conclusion

This memoir has explored the intricate dynamics of a coupled nonlinear wave system with internal fractional damping, a mathematical model pivotal for understanding complex physical phenomena in viscoelastic materials and engineering systems. Through a rigorous mathematical framework, we have addressed two fundamental questions: the local existence and uniqueness of solutions and the potential for finite-time blow-up phenomena. These investigations have not only deepened the theoretical understanding of fractional partial differential equations (PDEs) but also provided insights with practical implications for material science and structural engineering.

In Chapter 1, we established a solid foundation by presenting essential tools from functional analysis and fractional calculus, including Lebesgue, Bochner, and Sobolev spaces, alongside key inequalities and fractional operator definitions. This groundwork facilitated the subsequent analysis. Chapter 2 successfully demonstrated the local existence and uniqueness of solutions by reformulating the system as an abstract evolution equation in a Hilbert space and leveraging C_0-semigroup theory. The energy functional and monotonicity properties of the system's operator were critical in ensuring well-posedness over short time intervals. In Chapter 3, we investigated the blow-up of solutions using a carefully constructed Lyapunov functional, identifying conditions under which the solution's energy grows unbounded in finite time. These results highlight the delicate balance between nonlinear interactions and fractional damping in driving the system's long-term behavior.

The significance of this work lies in its contribution to the mathematical theory of fractional PDEs, particularly in addressing the combined challenges of nonlinearity and nonlocality. By integrating semigroup theory, energy methods, and fractional calculus, we have provided a robust framework for analyzing nonlocal wave systems, filling gaps in the existing literature. Practically, the findings validate the use of fractional wave models in applications such as vibration control in flexible structures and the design of durable polymers, where memory effects play a critical role. The blow-up analysis further informs the stability limits of such systems, offering guidance for preventing structural failure in engineering contexts.

Despite these advancements, several avenues remain open for future research. First, extending the local existence results to global existence would provide a more comprehensive understanding of the system's behavior over long time scales. This could involve exploring additional regularity conditions or weaker nonlinearities. Second, relaxing the constraints on the exponents p and q could broaden the applicability of the model to a wider range of

physical systems. Third, numerical simulations could complement the theoretical results, offering visual insights into the system's dynamics and validating the blow-up criteria. Finally, investigating the impact of different fractional derivative types (e.g., Atangana-Baleanu or distributed-order derivatives) could enhance the model's flexibility in capturing diverse memory effects.

In conclusion, this memoir represents a meaningful step forward in the study of nonlinear wave systems with fractional damping. By bridging rigorous mathematical analysis with practical applications, it contributes to both academic research and technological innovation. The tools and insights developed herein pave the way for further exploration of nonlocal PDEs, promising new discoveries at the intersection of mathematics, physics, and engineering.

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