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بسم الله الرحمن الرحيم



شكر وعرفان

الحمد لله حمد كثيرا حتى يبلغ الحمد منتهاه والصلاة والسلام على اشرف مخلوق اناره الله بنوره واصطفاه.

انطلاقا من باب من لم يشكر الناس

لم يشكر الله اتقدم بخالص الشكر والتقدير

للاستاذ المشرف " سالم عبد المالك " على ارشاداته وتوجيهاته التي لم يبخل بها علينا يوما, كما اتقدم بجزيل الشكر و العطاء الى كل يد رافقتنا في هذا العمل

كما اشكر اعضاء لجنة المناقشة التي شرفتني بقبولهم مناقشة مذكرتي كل من " الاستاذ بوعزيز خليفة " رئيسا

و" الاستاذ عون الله رضوان " ممتحنا

الذين لا شك انهم سيفيظون عليا بتوجيهاتهم القيمة و ملاحظاتهم السديدة وفي الاخير اشكر كل من قدم

لي يد العون و المساعدة

من قريب او بعيد ولو بكلمة طيبة

جزيل الشكر و العرفان.



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الى من اشددت عضدي بهم الى خيرة ايامي وصفوها الى قرة عيني ...اخوتي صلاح ,اسلام ,يوسف...
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... جدتي رحمها الله ...
اهديكم ثمرة نجاحي الذي لطالما تمنيته
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Abstract

The study of dynamical systems is a fundamental topic in the natural and engineering sciences, contributing to understanding the behavior of complex systems and analyzing their stability. In this study, we will analyze a dynamical system containing six equilibria, with the aim of classifying the system's parameter regions based on the presence of these equilibria and determining their stability conditions. We will use the Lyapunov method to analyze local and global stability, and evaluate the presence of a Hopf bifurcation at a particular equilibrium point in the Wilamowski-Rössler system. In addition, we will discuss the study of interaction propagation in three-dimensional systems and global asymptotic stability in the partial differential equations (PDEs) representing these systems. This study aims to achieve a deeper understanding of the system's dynamics and determine the stability conditions of its equilibria, contributing to the development of knowledge in the field of dynamical systems and their applications.

keywords:
Reaction-diffusion model , local and global stability, stabilization, WR, global asymptotic stability, Hopf .bifurcations, equilibria

Résumé

L'étude des systèmes dynamiques est un sujet fondamental en sciences naturelles et de l'ingénieur, contribuant à la compréhension du comportement des systèmes complexes et à l'analyse de leur stabilité. Dans cette étude, nous analyserons un système dynamique contenant six équilibres, afin de classer les régions de paramètres du système en fonction de la présence de ces équilibres et de déterminer leurs conditions de stabilité. Nous utiliserons la méthode de Lyapunov pour analyser la stabilité locale et globale, et évaluerons la présence d'une bifurcation de Hopf en un point d'équilibre particulier du système de Wilamowski-Rössler. De plus, nous aborderons l'étude de la propagation des interactions dans les systèmes tridimensionnels et la stabilité asymptotique globale dans les équations aux dérivées partielles (EDPs) représentant ces systèmes. Cette étude vise à approfondir la compréhension de la dynamique du système et à déterminer les conditions de stabilité de ses équilibres, contribuant ainsi au développement des connaissances dans le domaine des systèmes dynamiques et de leurs applications.

Mots-clés:

modèle de réaction-diffusion, stabilité locale et globale, stabilisation, WR, stabilité globale, bifurcations de Hopf, équilibres

الملخص

تُعدّ دراسة الأنظمة الديناميكية موضوعًا أساسيًا في العلوم الطبيعية والهندسية، إذ تُسهم في فهم سلوك الأنظمة المعقدة وتحليل استقرارها. في هذه الدراسة، سنُحلل نظامًا ديناميكيًا يحتوي على ستة توازنات لتصنيف مناطق معاملاته بناءً على وجود هذه التوازنات، وتحديد شروط استقرارها. سنستخدم طريقة ليابونوف لتحليل الاستقرار المحلي والشامل، وتقييم وجود تشعب هوبف عند نقطة توازن مُحددة لنظام ويلاموفسكي-روسلر. بالإضافة إلى ذلك، سنتناول دراسة انتشار التفاعلات في الأنظمة ثلاثية الأبعاد، والاستقرار التقاربي الشامل في المعادلات التفاضلية الجزئية التي تُمثل هذه الأنظمة. تهدف هذه الدراسة إلى تعميق (PDEs) فهمنا لديناميكيات النظام، وتحديد شروط استقرار توازناته، مما يُسهم في تطوير المعرفة في مجال الأنظمة الديناميكية وتطبيقاتها.

الكلمات المفتاحية:
نموذج التفاعل والانتشار، الاستقرار المحلي والعالمي، الاستقرار، الاستقرار العالمي، تشعبات هوبف، التوازنات WR.

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Introduction

Differential systems are used to model biological reactions, describe the motion of objects, and analyze economic changes. They are a powerful tool for understanding the dynamic behavior of complex phenomena. They have broad applications in physics, engineering, economics, and biology. They help analyze and explain complex phenomena and are fundamental to many scientific fields.

Nonlinear dynamical systems are a vital topic that has received widespread attention in various fields, including physics, chemistry, and engineering. One such system is the WR model, a mathematical model that describes complex chemical reactions and is used to understand the dynamic behavior of these reactions, especially chaotic behavior. The system consists of ordinary differential equations that describe interactions between variables. The WR system exhibits chaotic behavior, balanced multiplicity, and spiral waves. Numerical simulation and dynamic analysis have been used to study the system's behavior and identify the conditions that lead to chaotic behavior. Methods have also been proposed to control chaotic behavior and improve mixing in chemical reactions. The WR system is an important tool for understanding the dynamic behavior of chemical reactions and developing methods for controlling and optimizing them. In general, the WR system helps understand and optimize complex chemical reactions. In this paper, we discuss the Wilamowski-Rossler equation in detail, and we draw on the work of Alharthi,Nabil Barhoumi, Samir Ben Doukha, Imad Ali, Radwan Douaifia, and Salem Abdelmalek.

We encountered several difficulties with this nonlinear differential system, which affected the accuracy and understanding of its dynamic behavior. One of the most significant challenges was the system's nonlinearity, which made it difficult to solve analytically, forcing us to use numerical methods such as the Runge-Kutta method. We also faced extreme sensitivity to initial values, as even very slight changes can lead to completely different trajectories, requiring high precision in calculations and stability analysis around equilibrium points. Furthermore, the system contained multiple equilibrium points, making it difficult to predict the point to which the trajectory would gravitate without a comprehensive study of the Jacobian matrix. Furthermore, even small changes in parameters can lead to radical changes in behavior.

To overcome these difficulties, we relied on precise numerical solutions, linear stability analysis, studying the impact of coefficients through branch analysis, and using chaotic tools such as Lyapunov indices. These methods enable us to gain a deeper understanding of the system an make better modeling decisions. We have divided this work into the following chapters:

In the first chapter we provide an overview of the basic terms and concepts that form the basis of this study.

In chapter 2, we discuss a complex dynamical system containing six different equilibria. We aim to classify the regions of this system's parameters based on the presence of these equilibria. This includes identifying regions that contain all six equilibria, as well as those that lack one or more equilibria, focusing on analyzing the stability of each equilibrium in each region. We will organize the results into a comprehensive table that facilitates understanding the system's behavior and identifies regions where the equilibria can effectively coexist. The ultimate goal is to achieve a deeper understanding of the system's dynamics and identify the conditions necessary for the stability of its equilibria.

In the next section, we analyze the system's behavior near the equilibrium points, with the goal of determining whether these points are stable or unstable. This analysis will enable us to clearly identify regions where all six equilibria coexist.

In chapter 3, we examine the stability of the equilibria in the system, using the Lyapunov method to analyze local and global stability. The aim of this analysis is to identify and evaluate the necessa solutions in a specific context, seeking to prove the existence of solutions over an infinite timescale and verify the positivity of these solutry conditions for stability in a specific context. In the second part of the study, we will discuss the existence and positivity of globalions.

In chapter 4,examines the evaluation of the existence of Hopf bifurcation at a given equilibrium point in the WR system, presenting numerical results that support the main findings of the study.We will focus on specific representative cases to highlight key aspects of the system's stability, and four practical examples will be presented that demonstrate the stability of equilibria undervarious conditions.

Finally, in thelast chapter,we will discuss the study of interaction propagation in three-dimensional systems, providing a general introduction to these systems and defining them in a three-dimensional context. We will also discuss global asymptotic stability in the partial differential equations (PDEs) that represent these systems, with the aim of providing a deeper understanding of the dynamical phenomena in these systems.

Chapter 1

Mathematical Modeling and Preliminaries:

In this chapter, we present some of the necessary nomenclature and notions used throughout the thesis.

1.1 Nomenclature

- $\mathbb{R}$: Set of real numbers.
- H^1: Sobolev space.
- H^2: Hilbert space.
- Ω: Bounded domain of $\mathbb{R}^N$.
- $\partial\Omega$: Boundary of domain Ω.
- $det(J)$: Determinant of the matrix J.
- $tr(J)$: Trace of the matrix J.
- $Re(\lambda)$: Real part of the complex number λ.
- $\frac{\partial u}{\partial t}, \partial_t u$:Partial derivative with respect to t.
- $\frac{\partial u}{\partial v}$:Normal derivative of u outside of $\partial\Omega$.
- v: Outward unit normal vector of the boundary $\partial\Omega$**.**
- ODE :**ordinary differential equations.**
- PDE **:partial differential equations.**

- $\nabla^2 u, \Delta u$: **Laplacian operator of u defined by ∇^2u =Δu.**
- ∇u : Gradient of u defined by $\nabla u = (\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N})$.
- WR :**Willamowsky-Rossler.**

1.2 General Notions

1.2.1 Basic Theorems and Definitions

Definition 1.1 *[23] Let E be an equilibrium of dynamical system (1.8) and $\lambda_i, i = 1,2,3$ be the eigenvalues of the Jacobian matrix when evaluated at E. The point E is said to be:*

- *A **node** when $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\operatorname{sgn}(\lambda_1) = \operatorname{sgn}(\lambda_2) = \operatorname{sgn}(\lambda_3)$. Then, the node is stable if $\lambda_1, \lambda_2, \lambda_3 < 0$ and unstable if $\lambda_1, \lambda_2, \lambda_3 > 0$.*
- *A **saddle** when $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and $\exists i, j \in \{1,2,3\} : \lambda_i > 0 \wedge \lambda_j < 0$. Note that a saddle point is always unstable.*
- *A **focus-node** when $\lambda_1, \in \mathbb{R}$, $\lambda_2 = \overline{\lambda_3} = \alpha + i\beta$ with $\beta \neq 0$, and $\operatorname{sgn}(\lambda_1) = \operatorname{sgn}(\operatorname{Re}(\lambda_2)) = \operatorname{sgn}(\operatorname{Re}(\lambda_3))$.*

  Then, the focus node is stable if $\lambda_1, \operatorname{Re}(\lambda_2), \operatorname{Re}(\lambda_3) < 0$ and unstable if $\lambda_1, \operatorname{Re}(\lambda_2), \operatorname{Re}(\lambda_3) > 0$.
- *A **saddle-focus** $\lambda_1, \in \mathbb{R}$, $\lambda_2 = \overline{\lambda_3} = \alpha + i\beta$ with $\beta \neq 0$, and $-\operatorname{sgn}(\lambda_1) = \operatorname{sgn}(\operatorname{Re}(\lambda_2)) = \operatorname{sgn}(\operatorname{Re}(\lambda_3))$. Note that a saddle-focus is always unstable.*

Remark 1.1 *When the real part of the complex eigenvalues is equal to zero, linear analysis is not sufficient to determine the stability of the equilibrium. Nonlinear methods such as the consideration of higher-order derivatives and the centre manifold method can be used. If the equilibium is expected to cross into an oscillatory mode, a Hopf-bifurcation analysis may be useful.*

Lemma 1.1 *For a matrix $M \in \mathbb{R}^{3\times 3}$ of the form*

$$M = \begin{pmatrix} m_{11} & m_{12} & ma_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}$$

with eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$, we define

$$M^{[2]} = \begin{pmatrix} m_{11} + m_{22} & m_{23} & -m_{13} \\ m_{32} & m_{11} + m_{33} & -m_{12} \\ -m_{31} & m_{21} & m_{22} + m_{33} \end{pmatrix}.$$

If $\operatorname{tr}(M), \det(M) < 0$, the following holds:

- *When $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$:*

- *If* $\det\left(M^{[2]}\right) < 0$*, then all three eigenvalues are negative.*
- *If* $\det\left(M^{[2]}\right) \geq 0$*, then* $\lambda_1 < 0$ *and* $\lambda_2, \lambda_3 > 0$*. Also, if* $\det\left(M^{[2]}\right) = 0$*, then* $\lambda_1 = -\lambda_2$ *or* $\lambda_1 = -\lambda_3$*.*

- *When* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2, \lambda_3 \in \mathbb{C}$*, then* $\lambda_1 \leq \mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right)$ *and:*
  - *If* $\det\left(M^{[2]}\right) < 0$*, then* $\lambda_1 < 0$ *and* $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) < 0$*.*
  - *If* $\det\left(M^{[2]}\right) > 0$*, then* $\lambda_1 < 0$ *and* $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) > 0$*.*
  - *If* $\det\left(M^{[2]}\right) = 0$*, then* $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) = 0$*.*

Proof. The characteristic equation of M is a cubic polynomial with real coefficients and roots λ_1, λ_2, and λ_3. Note that $\lambda_1, \lambda_2, \lambda_3$ may be real or complex. It is easy to see that if the complex number $a + ib$ with $a, b \in \mathbb{R}$ is a root, then so is $a - ib$. Hence, if $b \neq 0$, then the third root is necessarily real.
First, let us assume that the polynomial has three real roots $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ and that $\lambda_1 \leq \lambda_2 \leq \lambda_3$. Since $\mathrm{tr}(M), \det(M) < 0$, it follows that $\lambda_1 + \lambda_2 + \lambda_3 < 0$ and $\lambda_1\lambda_2\lambda_3 < 0$. Consequently, either $\lambda_1, \lambda_2, \lambda_3 < 0$ or $\lambda_1 < 0$ and $\lambda_2, \lambda_3 > 0$. We know that

$$\det\left(M^{[2]}\right) = \left(\lambda_1 + \lambda_2\right)\left(\lambda_1 + \lambda_3\right)\left(\lambda_2 + \lambda_3\right).$$

Since $\mathrm{tr}(M) = \lambda_1 + \lambda_2 + \lambda_3 < 0$, then $\lambda_1 + \lambda_2 \leq 0$ and $\lambda_1 + \lambda_3 \leq 0$ with the condition that they can not both be equal to zero simulatneously. Therefore,

$$\mathrm{sgn}\left(\det\left(M^{[2]}\right)\right) = \mathrm{sgn}\left(\lambda_2 + \lambda_3\right).$$

It follows that if $\det\left(M^{[2]}\right) < 0$, then $\lambda_2 + \lambda_3 < 0$, which implies that $\lambda_2, \lambda_3 < 0$. Hence, $\lambda_1, \lambda_2, \lambda_3 < 0$. On the other hand, if $\det\left(M^{[2]}\right) > 0$, then $\lambda_2 + \lambda_3 > 0$, which implies that $\lambda_2, \lambda_3 > 0$, $\lambda_1 + \lambda_2 < 0$ and $\lambda_1 + \lambda_3 < 0$. Lastly, if $\det\left(M^{[2]}\right) = 0$, then $\lambda_1 = -\lambda_2$ or $\lambda_1 = -\lambda_3$, and thus $\lambda_2, \lambda_3 > 0$.
In the second case, we assume that $\lambda_1 \in \mathbb{R}$ and $\lambda_2, \lambda_3 \in \mathbb{C}$ with $\lambda_1 \leq \mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right)$. Since $\mathrm{tr}(M) < 0$, we get $\lambda_1 + 2\mathrm{Re}\left(\lambda_2\right) < 0$, which can be reduced to $\lambda_1 + \mathrm{Re}\left(\lambda_2\right) \leq 0$. We also know that

$$\begin{aligned}
\mathrm{sgn}\left(\det\left(M^{[2]}\right)\right) &= \mathrm{sgn}\left(\mathrm{Re}\left(\lambda_1 + \lambda_2\right)\mathrm{Re}\left(\lambda_1 + \lambda_3\right)\mathrm{Re}\left(\lambda_2 + \lambda_3\right)\right) \\
&= \mathrm{sgn}\left(\mathrm{Re}\left(\lambda_2 + \lambda_3\right)\right) \\
&= \mathrm{sgn}\left(\mathrm{Re}\left(\lambda_2\right)\right).
\end{aligned}$$

If $\det\left(M^{[2]}\right) < 0$, then $\mathrm{Re}\left(\lambda_2\right) < 0$, i.e. $\lambda_1, \mathrm{Re}\left(\lambda_2\right), \mathrm{Re}\left(\lambda_3\right) < 0$. Alternatively, if $\det\left(M^{[2]}\right) > 0$, then $\lambda_1 < 0$ and $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) > 0$. Finally, if $\det\left(M^{[2]}\right) = 0$, then $\mathrm{Re}\left(\lambda_2\right) = 0$, in which case eigenvalues are not sufficient to establish the stability of the system.
Note that simpler versions of this Lemma can be found in [20, 19]. ■

Theorem 1.1 *[11] Suppose the parameterized system*

$$\frac{dx}{dt}(t) = F(x,\alpha),\ X \in \mathbb{R}^3,\ \alpha \in \mathbb{R}, \tag{1.1}$$

has a fixed point at the origin for all values of the prameter α. Furthermore, suppose that two eigenvelues $\lambda_1(\alpha)$ and $\lambda_2(\alpha)$ of the α-dependent Jacobian of F at zero are purely imaginary for $\alpha = \alpha^*$. If the real part of the eigenvalue $(Re(\lambda_1(\alpha)) = Re(\lambda_2(\alpha))$since $\lambda_1 = \lambda_2$) satisfies

$$\frac{d}{d\alpha} Re(\lambda_1(\alpha))|_{\alpha=\alpha^*} > 0, \tag{1.2}$$

and the origin is asymptotically stable when $\alpha = \alpha^*$, then:
(a) $\alpha = \alpha^*$ is a bifurcation point of the system,
(b) for $\alpha \in (\alpha_1, \alpha^*)$ with some $\alpha_1 < \alpha^*$, the origin is a stable focus,
(c) for $\alpha \in (\alpha^*, \alpha_2)$ with some $\alpha_2 > \alpha^*$, the origin is an unstable focus surrounded by a stable limit cycle whose size increases with α.

Theorem 1.2 *[18] Consider the n-dimensional autonomous system*

$$\frac{dx}{dt} = F(x,\alpha), \tag{1.3}$$

where $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, and $F \in C^\infty$. Assume that (x^*, α^*) is an equilibrium point for system (1.3) and that $F(\lambda,\alpha)$ is the characteristic polynomial of the system's Jacobian matrix given by

$$F(\lambda,\alpha) = a_n(\alpha)\lambda^n + a_{n-1}(\alpha)\lambda^{n-1} + ... + a_1(\alpha)\lambda + a_0(\alpha). \tag{1.4}$$

Then, system (1.3) undergoes a Hopf bifurcation about $F(x,\alpha)$ if :
(H_1) $a_0(\alpha^*) > 0$, $D_1(\alpha^*) > 0$.....$D_{n-1}(\alpha^*) = 0$, *and*
(H_2) $\frac{\partial D_{n-1}(\alpha^*)}{\partial \alpha} \neq 0$,
where

$$D_n(\alpha) = \det \begin{pmatrix} a_1(\alpha) & a_0(\alpha) & \cdots & 0 \\ a_3(\alpha) & a_2(\alpha) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{2n-1}(\alpha) & a_{2n-2}(\alpha) & \cdots & a_n(\alpha) \end{pmatrix}. \tag{1.5}$$

$$a_k(\alpha) = 0; k > n \text{ or } k < 0.$$

Lemma 1.2 *For $n = 3$, conditions $(H_1)$and $(H_2)$of Theorem 1.2 reduce to*
(H_3) $a_0(\alpha^*) > 0$, $D_1(\alpha^*) = a_1(\alpha^*) > 0$, *and*

$$D_2(\alpha^*) = \det \begin{pmatrix} a_1(\alpha^*) & a_0(\alpha^*) \\ a_3(\alpha^*) & a_2(\alpha^*) \end{pmatrix} = 0, \tag{1.6}$$

(H_4) $\frac{\partial D_2(\alpha^*)}{\partial \alpha} \neq 0$, *respectively*

Theorem 1.3 (Green's formula) *[2]*

IN dimension n.For any $u \in H^2(\Omega)M$ *, and* $v \in H^1(\Omega)$ *we have*

$$\int_\Omega (\Delta u)v dx = \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v ds - \int_\Omega \nabla u \, \nabla v.$$

where s is the surface mesure of $\partial\Omega$ *.Note that in the case when* $u \in H^2(\Omega)$*,and* $v \in H_0^1(\Omega)$ *Green's identity reads*

$$\int_\Omega (\Delta u)v dx = -\int_\Omega \nabla u \, \nabla v.$$

Remark 1.2 *If we write this formula in dimension one,*Ω*becomes a segment* $(a; b)$ *in* R^1*and we find the integration by parts formula: So let* $u \, ; v \in C^{\,1}$ *[a; b] we have a special case of the Green's formula.*

$$\int_a^b \nabla u v \, dx = -\int_a^b u \, \nabla v dx + u \, (b)v \, (b) - u \, (a)v \, (a).$$

Now we recall some basic inequalities which we frequently use. We will start with this basic inequality in the study of elliptic and parabolic problems.

1.2.2 Willamowsky-Rössler chaotic reaction model

System Model

In this study, the considered reaction system is the one presented in the original paper of Willamowski and Rössler [27] and described by the rate equations

$$\begin{aligned}
&\text{Reaction 1} \quad A_1 + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X,\\
&\text{Reaction 2} \quad X + Y \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} 2Y,\\
&\text{Reaction 3} \quad A_5 + Y \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} A_2,\\
&\text{Reaction 4} \quad X + Z \underset{k_{-4}}{\overset{k_4}{\rightleftharpoons}} A_3,\\
&\text{Reaction 5} \quad A_4 + Z \underset{k_{-5}}{\overset{k_5}{\rightleftharpoons}} 2Z.
\end{aligned} \tag{1.7}$$

In this mechanism, (A_1, A_4, A_5) are the initial product concentrations, (X, Y, Z) are intermediate constituent concentrations, and (A_2, A_3) are the final product concentrations. Throughout this study, we assume that the intermediate chemicals are dynamic in nature while the input and output concentrations remain constant. These assumptions allow us to treat the chemical reactor as an open system, ensuring mass balance for all the species. Let us denote the concentrations of X, Y, and Z as x, y, and z, respectively. Under isothermal, constant volume, and well-stirred conditions, we derive the following dynamical system to describe the rate equations of the aforementioned chemical reaction system

$$\begin{cases}
\frac{dx}{dt} = (a_1 - k_{-1}x - y - z)\, x =: f_1(x, y, z), \quad \text{in } (0, +\infty)\\
\frac{dy}{dt} = (x - a_5)\, y =: f_2(x, y, z), \quad \text{in } (0, +\infty)\\
\frac{dz}{dt} = (a_4 - x - k_{-5}z)\, z =: f_3(x, y, z), \quad \text{in } (0, +\infty)\\
x(0) = x_0 \geq 0,\ y(0) = y_0 \geq 0,\ z(0) = z_0 \geq 0.
\end{cases} \tag{1.8}$$

with parameters a_1 through a_5 denoting the concentrations of species A_1 through A_5. Note the resulting model (1.8) assumes $k_{-2} = k_{-3} = k_{-4} = 0$ and $k_2 = k_4 = 1$. Also, for convenience, the model incorporates k_1, k_3, and k_5 into parameters a_1, a_4, and a_5. According to [12], system (1.8) exhibits a strange attractor for

$$(a_1, a_4, a_5, k_{-1}, k_{-5}) = (30, 16.5, 10, 0.415, 1.5)\,. \tag{1.9}$$

In order to simplify the notation of this paper, we will from here on drop the negative sign from the subscripts of reaction rate parameters k_{-1} and k_{-5}.

Definition of the Willamowsky-Rössler system

Chemical reaction engineering is an interdisciplinary field focused on understanding chemical reactions and designing chemical equipment effectively. The (WR) model is a mathematical model that exhibits chaotic behavior in chemical reactions. This research aims to comprehensively study the WR model, including identifying different regions where equilibria coexist completely or partially, and describing the nature of the stability of each equilibrium within them [27].

We will draw on previous research literature that has studied the WR model from various aspects, including the analysis of chaotic behavior, the identification of equilibria, and the control of chaotic behavior [4],[28]. We will identify different regions where equilibria coexist completely or partially, and describe the nature of the stability of each equilibrium within them.

Through this study, we will gain a deeper understanding of the behavior of the WR model and identify regions where certain equilibria exhibit global asymptotic stability and where chaotic states exist [12],[25].

This will enable us to improve the design of chemical equipment and analyze the dynamic behavior of the system, leading to improved chemical reaction efficiency and reduced costs [16] [8]. Our study draws on several important references, including the Wilamowski-Rössler study on chaotic behavior in chemical reactions [27], and Aguda et al.'s study on the determination of dynamical order using the Lotka-Volterra oscillator and a key [4].

We also benefited from the study of thermodynamic changes and their effects on chemical chaos in a WR reaction system [12], and the study of spiral waves in reaction-diffusion systems [14]. In addition, we studied the description of a stochastic period-2 limit cycle in the context of a WR reaction system [13], and the collapse of mean-field dynamics in the non-equilibrium reaction-diffusion system of the WR model [25].

We also proved the positive entropy of two Poincaré maps within a WR system [16], and studied entropy generation in a WR model during both oscillatory and steady-state conditions [24]. We proposed an adaptive parameter control method to control the chaotic behavior of a WR chemical reaction [21], and used the synchronization of two minimal WR systems to improve mixing in chemical reactions [8]. We also proposed and studied a fractal form of the WR model [7], and proposed a modified WR model to account for the finite size of the reactor and the resulting molecular crowding [26]. We explored the chemical network of the WR reaction-diffusion system using Brownian dynamics simulations [7], and studied the existence and stability of equilibria for the proposed modified WR model [29].

We identified six equilibria for the WR system and briefly discussed their stability in both local and global contexts [19]. We developed a sliding mode controller for the fractal model and how

it can be used to synchronize two systems [6]. We also prevented Hopf bifurcation control in the WR system, with the aim of generating or enhancing bifurcation at a specific site using a state feedback method [28]. In doing so, we were able to provide a comprehensive overview of the WR model and its applications in the field of chemical reactions.

1.3 Existence and Global Stability of Equilibria

In this section we classify the system's parameter regions according to the conditions for the presence of equilibria, identify regions that contain all six equilibria, and regions that lack one or more equilibria, analyze these equilibria and their results, and organize the results into a comprehensive table.

The system's equilibria satisfy

$$\begin{cases} a_1 x - k_{-1} x^2 - xy - xz = 0, \\ xy - a_5 y = 0, \\ a_4 z - xz - k_{-5} z^2 = 0. \end{cases} \tag{1.10}$$

First, consider the second equation of the system

$$y(x - a_5) = 0.$$

This leads to either $y = 0$ or $x = a_5$. Substituting $y = 0$ into the first equations of the system (1.10), we obtain

$$a_1 x - k_1 x^2 - xz = 0,$$

this can be rewritten as

$$x(a_1 - k_1 x - z) = 0,$$

this leads to

$$x = 0, \tag{1.11}$$

or

$$z = a_1 - k_1 x. \tag{1.12}$$

Next, substitute the equation (1.11) into the third equation of the system (1.10), we find

$$a_4 z - k_5 z^2 = 0,$$

rewriting it, we get

$$z(a_4 - k_5 z) = 0.$$

Thus, we obtain

$$z = 0, \tag{1.13}$$

or

$$z = \frac{a_4}{k_5}. \tag{1.14}$$

By replacing z from (1.12) into the third equation of system (1.10), we obtain

$$(a_1 - k_1 x)(a_4 - x - k_5(a_1 - k_1 x)) = 0$$

Solving for x, we get

$$x = \frac{a_1}{k_1}, \tag{1.15}$$

or

$$x = \frac{a_1 k_5 - a_4}{k_1 k_5 - 1}. \tag{1.16}$$

Replacing x from (1.15) into (1.12) yields

$$z = a_1 - k_1 \left(\frac{a_1}{k_1} \right),$$

simplifying, we get

$$z = 0.$$

Similarly, substituting (1.16) into (1.12), we find

$$\begin{aligned} z &= a_1 - k_1 \left(\frac{a_1 k_5 - a_4}{k_1 k_5 - 1} \right), \\ &= \frac{a_1 k_1 k_5 - a_1 - a_1 k_1 k_5 + a_4 k_1}{k_1 k_5 - 1}, \\ &= \frac{a_4 k_1 - a_1}{k_1 k_5 - 1}. \end{aligned}$$

Now, we proceed by substituting $x = a_5$ into the first equation of the system

$$a_1 a_5 - k_1 a_5^2 - a_5 y - a_5 z = 0.$$

Rearranging, we express y as

$$y = a_1 - k_1 a_5 - z. \tag{1.17}$$

Similarly, substituting $x = a_5$ into the third equation of (1.10) yields

$$z(a_4 - a_5 - k_5 z) = 0,$$

which leads to

$$z = 0, \tag{1.18}$$

or

$$z = \frac{a_4 - a_5}{k_5}. \tag{1.19}$$

Replacing (1.18) in (1.17), we get

$$y = a_1 - k_1 a_5.$$

Likewise, substituting (1.19) into (1.17) results in

$$y = a_1 - k_1 a_5 + \frac{a_5 - a_4}{k_5}.$$

This yields six solutions. To simplify the solutions, we assume $k_1 \neq \frac{1}{k_5}$, leading to the six equilibria

$$\begin{aligned}
E_1 &= \left(a_5, \frac{a_5 - a_4}{k_5} + a_1 - a_5 k_1, \frac{a_4 - a_5}{k_5}\right). \\
E_2 &= (a_5, a_1 - a_5 k_1, 0). \\
E_3 &= \left(\frac{a_1 k_5 - a_4}{k_1 k_5 - 1}, 0, \frac{a_4 k_1 - a_1}{k_1 k_5 - 1}\right). \\
E_4 &= \left(0, 0, \frac{a_4}{k_5}\right). \\
E_5 &= \left(\frac{a_1}{k_1}, 0, 0\right). \\
E_6 &= (0, 0, 0). \qquad (1.20)
\end{aligned}$$

Since the dependent triplet (x, y, z) represents the concentrations of the intermediate substances, we may impose $x, y, z \geq 0$. It is important to note these equilibria do not necessarily have to co-exist. It is easy to see that equilibria E_4, $E_5$and E_6 exist for all strictly positive parameter values. The equilibrium E_1 exists if the following conditions are satisfied

$$y = \frac{a_5 - a_4}{k_5} + a_1 - a_5 k_1 \geq 0, \qquad (1.21)$$

and

$$z = \frac{a_4 - a_5}{k_5} \geq 0. \qquad (1.22)$$

From $y \geq 0$, we get

$$\frac{a_5 - a_4}{k_5} \geq a_5 k_1 - a_1,$$

multiplying by k_5 gives

$$a_5 - a_4 \geq k_5 \left(a_5 k_1 - a_1,\right)$$

expanding the terms

$$a_5 - a_4 \geq a_5 k_1 k_5 - a_1 k_5,$$

this implies

$$a_5 \left(1 - k_1 k_5\right) \geq a_1 k_5 - a_4 \qquad (1.23)$$

For $z \geq 0$, we find

$$a_4 \geq a_5. \tag{1.24}$$

Combining these inequalities, we obtain the condition

$$\text{(C1)} \quad \begin{cases} (a_1 k_5 - a_4) \geq a_5 \left(k_1 k_5 - 1\right), \\ \text{and} \\ a_4 \geq a_5. \end{cases} \tag{1.25}$$

The equilibrium E_2 exists if

$$y = a_1 - a_5 k_1 \geq 0.$$

This simplifies directly to the condition

$$\text{(C2)} \quad \begin{cases} a_1 - a_5 k_1 \geq 0. \end{cases}$$

The equilibrium E_3 exists if the following conditions hold

$$x = \frac{a_1 k_5 - a_4}{k_1 k_5 - 1} \geq 0, \tag{1.26}$$

and

$$z = \frac{a_4 k_1 - a_1}{k_1 k_5 - 1} \geq 0. \tag{1.27}$$

From $x \geq 0$, we have

$$\begin{cases} (a_1 k_5 - a_4) \geq 0, & \text{if } k_1 k_5 > 1, \\ \text{or} \\ (a_1 k_5 - a_4) \leq 0, & \text{if } k_1 k_5 < 1. \end{cases}$$

For $z \geq 0$, we get

$$\begin{cases} (a_4 k_1 - a_1) \geq 0, & \text{if } k_1 k_5 > 1, \\ \text{or} \\ (a_4 k_1 - a_1) \leq 0, & \text{if } k_1 k_5 < 1. \end{cases}$$

Combining the conditions, we obtain

$$\text{(C3)} \quad \begin{cases} a_1 k_5 \geq a_4 \quad \text{and} \quad a_4 k_1 \geq a_1, & \text{if } k_1 k_5 > 1, \\ \text{or} \\ a_1 k_5 \leq a_4 \quad \text{and} \quad a_4 k_1 \leq a_1, & \text{if } k_1 k_5 < 1. \end{cases} \tag{1.28}$$

To establish the non-existence of equilibria, we negate the conditions for their existence. This allows us to determine the parameter regions where specific equilibria cannot occur. Below, we outline the conditions for the non-existence of the equilibria E_1, E_2, and E_3, respectively

$$(\mathrm{NC1}) \quad \begin{cases} (a_1k_5 - a_4) < a_5\,(k_1k_5 - 1)\,, \\ \text{or} \\ a_4 < a_5, \end{cases}$$

$$(\mathrm{NC2}) \quad \begin{cases} a_1 - a_5k_1 < 0, \end{cases} \tag{1.29}$$

$$(\mathrm{NC3}) \quad \begin{cases} a_1k_5 < a_4 \quad \text{or} \quad a_4k_1 < a_1, \quad \text{if } k_1k_5 > 1, \\ \text{and} \\ a_1k_5 > a_4 \quad \text{or} \quad a_4k_1 > a_1, \quad \text{if } k_1k_5 < 1. \end{cases} \tag{1.30}$$

Before diving into the detailed analysis, we start by classifying the parameter regions of the system based on the existence and non-existence conditions for the equilibria. This classification will allow us to identify regions where all six equilibria coexist, as well as regions where one or more equilibria may be absent. Understanding these parameter regions is essential for identifying where the system's equilibria can coexist and where specific equilibria might fail to exist, ultimately contributing to a deeper understanding of the system's dynamics.

Existence of All Six Equilibria

1. In this region, the existence of all six equilibrium points requires the simultaneous satisfaction of the simplified conditions derived from $(C1)$, $(C2)$, and $(C3)$. The conditions are as follows

   - Condition $(C1)$ requires that

   $$a_5\,(k_1k_5 - 1) \leq (a_1k_5 - a_4)\,, \tag{1.31}$$

   and

   $$a_5 \leq a_4. \tag{1.32}$$

   - Condition $(C2)$ is given by

   $$a_5k_1 \leq a_1. \tag{1.33}$$

- Condition $(C3)$ introduces two cases depending on the value of k_1k_5
  - If $k_1k_5 > 1$, the inequalities become

$$a_4 \leq a_1k_5, \tag{1.34}$$

    and

$$a_1 \leq a_4k_1. \tag{1.35}$$

  - If $k_1k_5 < 1$, the inequalities are

$$a_1k_5 \leq a_4, \tag{1.36}$$

    and

$$a_4k_1 \leq a_1. \tag{1.37}$$

The inequalities are now grouped based on the two cases of k_1k_5 for further simplification.

- **Case 1:** $k_1k_5 < 1$
  - From (1.31) and (1.36), we obtain

$$a_5\left(k_1k_5 - 1\right) \leq a_1k_5 - a_4 \leq 0. \tag{1.38}$$

  - Using (1.32), (1.33), and (1.37), we derive

$$a_5k_1 \leq a_4k_1 \leq a_1.$$

$$\textbf{Case (1.1)} \quad \begin{cases} a_5\left(k_1k_5 - 1\right) \leq a_1k_5 - a_4 \leq 0 \\ a_5k_1 \leq a_4k_1 \leq a_1 \end{cases} \tag{1.39}$$

- **Case 2:** $k_1k_5 > 1$
  - From (1.31), we obtain

$$0 < a_5\left(k_1k_5 - 1\right) \leq \left(a_1k_5 - a_4\right). \tag{1.40}$$

  - Combining (1.33) and (1.35), the conditions simplify to

$$a_5k_1 \leq a_1 \leq a_4k_1. \tag{1.41}$$

    This leads to the satisfaction of all conditions required for the existence of the fixed points

$$\textbf{Case (1.2)} \quad \begin{cases} 0 < a_5\left(k_1k_5 - 1\right) \leq \left(a_1k_5 - a_4\right) \\ a_5k_1 \leq a_1 \leq a_4k_1 \end{cases}$$

Existence of Five Equilibria

1. **All but** E_1: In this region, the fulfillment of conditions $(NC1), (C2)$, and $(C3)$ leads to two distinct cases

   (a) • **Case 1:** $k_1k_5 < 1$. This case is divided into the following two subcases

   i. Consider the conditions

$$\begin{cases} a_1k_5 - a_4 < a_5(k_1k_5 - 1), \\ a_5k_1 \leq a_1, \\ a_1k_5 \leq a_4, \\ a_4k_1 \leq a_1, \end{cases}$$

   by simplifying, we obtain

$$\textbf{Case(2.1)} \quad \begin{cases} a_1k_5 - a_4 < a_5(k_1k_5 - 1) < 0 \\ a_5k_1 \leq a_1 \\ a_4k_1 \leq a_1 \end{cases} \tag{1.42}$$

   ii. Similarly, starting with

$$\begin{cases} a_4 < a_5, \\ a_5k_1 \leq a_1, \\ a_1k_5 \leq a_4, \\ a_4k_1 \leq a_1, \end{cases}$$

   leads to the reduced form

$$\textbf{Case (2.2)} \quad \begin{cases} a_4 < a_5, \\ a_5k_1k_5 \leq a_1k_5 \leq a_4. \end{cases} \tag{1.43}$$

   • **Case 2:** $k_1k_5 > 1$. This case also consists of two subcases

   i. If the following conditions hold

$$\begin{cases} a_1k_5 - a_4 < a_5(k_1k_5 - 1), \\ a_5k_1 \leq a_1, \\ a_4 \leq a_1k_5, \\ a_1 \leq a_4k_1. \end{cases}$$

Which reduces to

$$\textbf{Case (2.3)} \quad \begin{cases} 0 \leq a_1k_5 - a_4 < a_5(k_1k_5 - 1), \\ a_5k_1 \leq a_1 \leq a_4k_1. \end{cases} \tag{1.44}$$

ii. If the following conditions hold

$$\begin{cases} a_4 < a_5, \\ a_5k_1 \leq a_1, \\ a_4 \leq a_1k_5, \\ a_1 \leq a_4k_1. \end{cases}$$

Rewriting, we get

$$\begin{cases} a_4 < a_5, \\ a_5k_1 \leq a_1 \leq a_4k_1, \\ a_4 \leq a_1k_5. \end{cases} \tag{1.45}$$

From the second condition in (1.45), we obtain

$$a_5k_1 \leq a_4k_1,$$

which leads to

$$a_5 \leq a_4,$$

contradicting the first condition in (1.45). Hence, this subcase is not valid.

(b) **All but** E_2: In this region, satisfying conditions $(\text{NC2}), (C1)$, and $(C3)$ leads to two main cases

- **Case 1:** $k_1k_5 < 1$. The conditions to be satisfied are given by

$$\begin{cases} a_1 < a_5k_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_5 \leq a_4, \\ a_1k_5 \leq a_4, \\ a_4k_1 \leq a_1, \end{cases}$$

which simplify to

$$\begin{cases} a_4k_1 \leq a_1 < a_5k_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4) \leq 0, \\ a_5 \leq a_4. \end{cases} \tag{1.46}$$

From the third condition in (1.46), it follows that

$$a_4 < a_5.$$

This contradicts the second condition in (1.46), making this case impossible.

- **Case 2:** $k_1k_5 > 1$. In this case, the condition takes the following form

$$\begin{cases} a_1 < a_5k_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_5 \leq a_4, \\ a_4 \leq a_1k_5, \\ a_1 \leq a_4k_1. \end{cases}$$

Then, simplifying these conditions, we obtain

$$\textbf{Case (3.1)} \quad \begin{cases} a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_1 < a_5k_1 \leq a_4k_1. \end{cases} \tag{1.47}$$

(c) **All but** E_3: In this region, the existence conditions are examined under (NC3), $(C1)$, and $(C2)$, leading to the following cases

- **Case 1:** $k_1k_5 < 1$. This scenario includes the following two subcases

  i. The conditions can be arranged as follows

$$\begin{cases} a_4 < a_1k_5, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_5 \leq a_4, \\ a_5k_1 \leq a_1. \end{cases}$$

Upon simplification, these reduce to

$$\textbf{Case (4.1)} \quad \begin{cases} a_5 \leq a_4 < a_1k_5. \end{cases} \tag{1.48}$$

  ii. Similarly, reorganizing the conditions in this subcase yields

$$\begin{cases} a_1 < a_4k_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_5 \leq a_4, \\ a_5k_1 \leq a_1. \end{cases}$$

After simplification, we obtain

$$\textbf{Case (4.2)} \quad \begin{cases} a_5 k_1 \leq a_1 < a_4 k_1, \\ a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4). \end{cases} \tag{1.49}$$

- **Case 2:** $k_1 k_5 > 1$. This case also includes two subcases

  i. The following conditions define the first subcase

$$\begin{cases} a_1 k_5 < a_4, \\ a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4), \\ a_5 \leq a_4, \\ a_5 k_1 \leq a_1. \end{cases} \tag{1.50}$$

  From the second condition in (1.50), we observe that

$$0 < a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4),$$

  which leads to

$$0 < (a_1 k_5 - a_4).$$

  Thus, we obtain

$$a_4 < a_1 k_5,$$

  which contradicts the first condition in (1.50). Hence, this case is not possible.

  ii. In the second subcase, the conditions are structured as

$$\begin{cases} a_4 k_1 < a_1, \\ a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4), \\ a_5 \leq a_4, \\ a_5 k_1 \leq a_1. \end{cases}$$

  Simplifying leads to

$$\begin{cases} a_5 k_1 \leq a_4 k_1 < a_1, \\ a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4). \end{cases}$$

Existence of four Equilibria

1. **All but** E_1, E_2: In this case, the existence of all equilibria except E_1 and E_2 is ensured by (NC1), (NC2), and $(C3)$, resulting in the following classification

(a)

- **Case 1:** If $k_1k_5 < 1$, This gives rise to two distinct subcases

  i. Under the constraints

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1) < 0, \\ a_1 < a_5k_1 \\ a_1k_5 \leq a_4 \\ a_4k_1 \leq a_1 \end{cases}$$

  they simplify to

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1) < 0, \\ a_4k_1 \leq a_1 < a_5k_1. \end{cases}$$

  ii. If the conditions

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5k_1, \\ a_1k_5 \leq a_4, \\ a_4k_1 \leq a_1. \end{cases}$$

  hold, they reduce to

$$\begin{cases} a_1k_1k_5 \leq a_4k_1 \leq a_1 < a_5k_1. \end{cases}$$

- **Case 2:** If $k_1k_5 > 1$, then two subcases arise.

  i. When the conditions

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_4 \leq a_1k_5, \\ a_1 \leq a_4k_1, \end{cases}$$

  are satisfied, they simplify to

$$\begin{cases} 0 \leq (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_1 \leq a_4k_1. \end{cases}$$

ii. If the conditions

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5 k_1, \\ a_4 \leq a_1 k_5, \\ a_1 \leq a_4 k_1, \end{cases}$$

hold, they simplify to

$$\begin{cases} a_1 \leq a_4 k_1 < a_5 k_1, \\ a_4 \leq a_1 k_5. \end{cases}$$

(b) **All but** E_1, E_3: The conditions (NC1), (NC2), and (C3) lead to the following classifications

- **Case 1:** $k_1 k_5 < 1$.

  i. If the following conditions hold

$$\begin{cases} (a_1 k_5 - a_4) < a_5(k_1 k_5 - 1), \\ a_4 < a_1 k_5, \\ a_5 k_1 \leq a_1, \end{cases} \tag{1.51}$$

  then from the first condition in (1.51), we obtain

$$a_1 k_5 - a_4 < a_5(k_1 k_5 - 1) < 0.$$

  This implies

$$a_1 k_5 - a_4 < 0,$$

  which is equivalent to

$$a_1 k_5 < a_4.$$

  However, this contradicts the second condition in (1.51), leading to a contradiction.

  ii. Another scenario arises as follows

$$\begin{cases} a_4 < a_5, \\ a_4 < a_1 k_5, \\ a_5 k_1 \leq a_1, \end{cases}$$

  which transforms into

$$\mathbf{Case(5.1)} \qquad \begin{cases} a_4 k_1 < a_5 k_1 \leq a_1, \\ a_4 < a_1 k_5. \end{cases} \tag{1.52}$$

iii. A further case occurs when

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_4k_1, \\ a_5k_1 \leq a_1, \end{cases}$$

Rearranging the conditions, we obtain

$$\textbf{Case}(\mathbf{5.2}) \qquad \begin{cases} a_1k_5 - a_4 < a_5(k_1k_5 - 1), \\ a_5k_1 \leq a_1 < a_4k_1. \end{cases} \tag{1.53}$$

iv. Lastly, if

$$\begin{cases} a_4 < a_5, \\ a_1 < a_4k_1, \\ a_5k_1 \leq a_1, \end{cases}$$

reformulating the conditions, we get

$$\begin{cases} a_4 < a_5, \\ a_5k_1 \leq a_1 < a_4k_1, \end{cases} \tag{1.54}$$

From the second condition in (1.54), it follows that

$$a_5 \leq a_4,$$

which contradicts the first condition in (1.54), making this case infeasible.

- **Case 2:** $k_1k_5 > 1$. This scenario results in the following subcases

i. One possible case occurs when

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1k_5 < a_4, \\ a_5k_1 \leq a_1. \end{cases}$$

Rearranging the conditions, we obtain

$$\textbf{Case}(\mathbf{5.3}) \qquad \begin{cases} a_5 < a_5k_1k_5 \leq a_1k_5 < a_4. \end{cases} \tag{1.55}$$

ii. Another possible case arises if

$$\begin{cases} a_4 < a_5, \\ a_1k_5 < a_4, \\ a_5k_1 \leq a_1, \end{cases}$$

which is equivalent to

$$\begin{cases} a_4 < a_5, \\ a_5 < a_5 k_1 k_5 \leq a_1 k_5 < a_4. \end{cases}$$

From the second condition, we derive

$$a_5 < a_4,$$

this contradicts the first condition $(a_4 < a_5)$, making this case impossible.

iii. An alternative case occurs when

$$\begin{cases} (a_1 k_5 - a_4) < a_5(k_1 k_5 - 1), \\ a_1 < a_5 k_1, \\ a_5 k_1 \leq a_1, \end{cases}$$

iv. The final case arises when

$$\begin{cases} a_4 < a_5, \\ a_4 k_1 < a_1, \\ a_5 k_1 \leq a_1. \end{cases}$$

By reordering the constraints, we arrive at

$$\textbf{Case}(\mathbf{5.4}) \begin{cases} a_4 k_1 < a_5 k_1 \leq a_1. \end{cases} \tag{1.56}$$

(c) **All but** E_2, E_3: In this scenario, the conditions $(\text{NC2}), (\text{NC3})$, and (C1) must be satisfied, leading to the following two main cases

- **Case 1:** $k_1 k_5 < 1$. Under this condition, we distinguish two subcases

  i. If

$$\begin{cases} a_1 < a_5 k_1, \\ a_4 < a_1 k_5, \\ a_5(k_1 k_5 - 1) \leq a_1 k_5 - a_4 < 0, \\ a_5 \leq a_4, \end{cases} \tag{1.57}$$

  Then, by combining the second and last conditions in (1.57), the conditions take the following form

$$\begin{cases} a_1 < a_5 k_1, \\ a_5 \leq a_4 < a_1 k_5, \\ a_5(k_1 k_5 - 1) \leq a_1 k_5 - a_4 < 0, \end{cases} \tag{1.58}$$

From the second condition of (1.58), we deduce that

$$a_5 < a_1 k_5,$$

this is equivalent to

$$a_5 k_1 < a_1 k_1 k_5,$$

which implies

$$a_5 k_1 < a_1 k_1 k_5 < a_1,$$

leading to

$$a_5 k_1 < a_1.$$

This contradicts the first condition in (1.58), making this case invalid.

ii. If instead

$$\begin{cases} a_1 < a_5 k_1, \\ a_1 > a_4 k_1, \\ a_5(k_1 k_5 - 1) \leq (a_1 k_5 - a_4) < 0, \\ a_5 \leq a_4, \end{cases} \tag{1.59}$$

then from the first, second and last condition of (1.59), we derive

$$a_4 k_1 < a_1 < a_5 k_1.$$

leading to

$$a_4 k_1 < a_5 k_1.$$

This contradicts the first condition in (1.58), making this case invalid.

- **Case 2:** $k_1 k_5 > 1$. Similarly, we analyze two subcases

i.

$$\begin{cases} a_1 < a_5 k_1, \\ a_1 k_5 < a_4, \\ a_5(k_1 k_5 - 1) \leq a_1 k_5 - a_4, \\ a_5 \leq a_4, \end{cases} \tag{1.60}$$

It follows directly from the third condition of (1.60) that

$$0 < a_5(k_1k_5 - 1) \leq a_1k_5 - a_4.$$

This implies

$$0 < a_1k_5 - a_4,$$

which can be rewritten as

$$a_4 < a_1k_5.$$

This contradicts the condition $(a_1k_5 < a_4)$ in (1.60), making the case invalid.

ii.

$$\begin{cases} a_1 < a_5k_1, \\ a_4k_1 < a_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4) < 0, \\ a_5 \leq a_4. \end{cases}$$

Rearranging the constraints, we obtain

$$\begin{cases} a_4k_1 < a_1 < a_5k_1, \\ a_5(k_1k_5 - 1) \leq (a_1k_5 - a_4), \\ a_5 \leq a_4. \end{cases} \tag{1.61}$$

It is easy to see that a contradiction arises between the first and last conditions in (1.61).making this case invalid.

Existence of three Equilibria

1. (a) **All but** E_1, E_2, E_3: For this case, the conditions $(\text{NC1}), (\text{NC2})$, and (NC3) must hold, ensuring the feasibility of the remaining equilibria.

   - **Case 1:** $k_1k_5 < 1$.

     i. The following conditions must be satisfied

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_4 < a_1k_5. \end{cases}$$

From the first inequality, we deduce that

$$a_1k_5 - a_4 < a_5(k_1k_5 - 1) < 0,$$

which directly implies

$$a_1k_5 < a_4.$$

This contradicts the third condition, making this case impossible.

ii. An alternative case is given by the conditions

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5k_1, \\ a_4 < a_1k_5, \end{cases}$$

by combining the inequalities, we deduce

$$\textbf{Case } (\mathbf{6.1}) \quad \Big\{ a_4 < a_1k_5 < a_5k_1k_5. \tag{1.62}$$

Another possible case arises under the conditions

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_1 < a_4k_1. \end{cases}$$

Reformulating the constraints, we derive

$$\textbf{Case } (\mathbf{6.2}) : \begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < k_1 \min(a_4, a_5). \end{cases} \tag{1.63}$$

1. (a) • i. Considering the conditions

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5k_1, \\ a_1 < a_4k_1, \end{cases}$$

by rewriting these constraints, we obtain

$$\textbf{Case } (\mathbf{6.3}) : \Big\{ a_1 < a_4k_1 < a_5k_1. \tag{1.64}$$

- **Case 2:** $k_1k_5 > 1$. This scenario results in the following subcases

i. One possible case occurs when

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_1k_5 < a_4. \end{cases}$$

Rearranging the conditions, we obtain

$$\textbf{Case } (\mathbf{6.4}) : \begin{cases} a_1 < a_5k_1, \\ a_1k_5 < a_4. \end{cases} \tag{1.65}$$

ii. Another possible case arises if

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5k_1, \\ a_1k_5 < a_4. \end{cases}$$

which is equivalent to

$$\textbf{Case } (\mathbf{6.5}) : \Big\{ a_1k_5 < a_4 < a_5. \tag{1.66}$$

iii. An alternative case occurs when

$$\begin{cases} (a_1k_5 - a_4) < a_5(k_1k_5 - 1), \\ a_1 < a_5k_1, \\ a_4k_1 < a_1. \end{cases}$$

iv.

$$\begin{cases} a_4 < a_5, \\ a_1 < a_5k_1, \\ a_4k_1 < a_1. \end{cases}$$

Equivalently, the conditions simplify to

$$\textbf{Case } (\mathbf{6.6}) \Big\{ a_4k_1 < a_1 < a_5k_1. \tag{1.67}$$

Chapter 2

Local and Global Stability of Equilibria

2.1 The stability nature of equilibria

In this section we analyze the behavior of the system near equilibrium point the goal is to determine whether these point are stable or unstable This classification will allow us to identify regions where all six equilibria coexist.
The conditions of each region

$$\begin{cases} (\mathrm{C2})\ a_5\left(k_1k_5-1\right) \leq \left(a_1k_5-a_4\right), \\ (\mathrm{C3})\ a_5 \leq a_4, \\ (\mathrm{C4})\ a_5k_1 \leq a_1, \\ (\mathrm{C5})\ a_4k_1 \leq a_1, \\ (\mathrm{C6})\ a_1k_5 \leq a_4. \end{cases} \tag{2.1}$$

Note that in the case $k_1k_5 > 1$, conditions (C4) and (C5) change. We have not considered the case $k_1k_5 > 1$ in this paper as its findings are similar and can be obtained by a trivial extension of the work presented herein. In order to investigate the stability of the equilibria, we consider the Jacobian matrix corresponding to (1.8), which takes the form In order to investigate the local stability of the equilibria, we consider the Jacobian matrix corresponding to (1.10), which takes the form(2.2)

First calclulation the gacobian matrix,
sucg that :

$$\begin{cases} f_1 = x(a_1\ f - k_1x - y - z) \\ f_2 = y(x - a_5) \\ f_3 = z(a_4 - x - k_5z) \end{cases}$$

The Jacobian matrix is given by

$$J = \begin{pmatrix} \frac{df_1}{dx} & \frac{df_1}{dy} & \frac{df_1}{dz} \\ \frac{df_2}{dx} & \frac{df_2}{dy} & \frac{df_2}{dz} \\ \frac{df_3}{dx} & \frac{df_3}{dy} & \frac{df_3}{dz} \end{pmatrix}$$

By differentiating f_1, f_2, f_3 with respect to x, y, z we get

$$J(x,y,z) = \begin{pmatrix} a_1 - 2k_1x - y - z & -x & -x \\ y & x - a_5 & 0 \\ -z & 0 & a_4 - x - 2k_5z \end{pmatrix}. \tag{2.2}$$

Before we present the main findings of this study, we intend to simplify the presentation by defining the following notations:

$$\begin{aligned} \beta_1 &= \left(a_1^2k_5^2 - 2a_1a_4k_5^2 - 2a_1a_4k_5 + a_4^2k_5^2 + 2a_4^2k_5 + a_4^2\right) = \left(\left(a_1 - a_4\right)k_5 - a_4\right)^2, \\ \beta_2 &= \left(2a_1^2k_5^2 - 2a_1a_4k_5^2 + 2a_1a_4k_5 - 4a_4^2\right), \\ \beta_3 &= \left(a_1^2k_5^2 - 4a_1^2k_5 + 4a_4a_1\right), \\ \beta_4 &= \frac{\left(a_4k_1 + a_1k_5 - \left(a_1 + a_4\right)k_1k_5\right)}{\left(k_1k_5 - 1\right)}, \\ \beta_5 &= \frac{\left(a_1k_5 - a_4\right)\left(a_4k_1 - a_1\right)}{\left(k_1k_5 - 1\right)}. \end{aligned} \tag{2.3}$$

1. **stability of Equilibria $\mathbf{E}_1$**

Substituting the first equilibrium from (1.10) into the Jacobian yields we get:

$$\begin{aligned} J\left(E_1\right) &= \begin{pmatrix} a_1 - 2a_5k_1 - \frac{a_5-a_4}{k_5} - a_1 + a_5k_1 + \frac{a_5-a_4}{k_5} & -a_5 & -a_5 \\ \frac{a_5-a_4}{k_5} + a_1 - a_5k_1 & a_5 - a_5 & 0 \\ -\frac{a_4-a_5}{k_5} & 0 & a_4 - a_5 - 2k_5\frac{a_4-a_5}{k_5} \end{pmatrix} \\ J\left(E_1\right) &= \begin{pmatrix} -a_5k_1 & -a_5 & -a_5 \\ \frac{1}{k_5}(a_5 - a_4 - a_5k_1k_5 + a_1k_5) & 0 & 0 \\ -(-\frac{a_5-a_4}{k_5}) & 0 & a_5 - a_4 \end{pmatrix} \\ J\left(E_1\right) &= \begin{pmatrix} -a_5k_1 & -a_5 & -a_5 \\ \frac{1}{k_5}\left(a_1k_5 - a_4 - a_5\left(k_1k_5 - 1\right)\right) & 0 & 0 \\ \frac{a_5-a_4}{k_5} & 0 & a_5 - a_4 \end{pmatrix}. \end{aligned}$$

$$J\left(E_1\right) = \begin{pmatrix} -a_5k_1 & -a_5 & -a_5 \\ \frac{1}{k_5}\left(a_1k_5 - a_4 - a_5\left(k_1k_5 - 1\right)\right) & 0 & 0 \\ \frac{a_5-a_4}{k_5} & 0 & a_5 - a_4 \end{pmatrix} \tag{2.4}$$

This leads to

$$\begin{aligned}
\det\left(J\left(E_1\right)\right) &= \frac{1}{k_5}\left(a_4^2a_5 - 2a_4a_5^2 + a_5^3 + a_1a_5^2k_5 - a_5^3k_1k_5 - a_1a_4a_5k_5 + a_4a_5^2k_1k_5\right),\\
\det\left(J\left(E_1\right)\right) &= \frac{a_5}{k_5}\left(a_4^2 - 2a_4a_5 + a_5^2 + a_1a_5k_5 - a_5^2k_1k_5 - a_1a_4k_5 + a_4a_5k_1k_5\right),\\
\det\left(J\left(E_1\right)\right) &= \frac{a_5}{k_5}\left(a_4^2 - a_1a_4k_5 + a_4a_5k_1k_5 - a_4a_5 - a_4a_5 + a_1a_5k_5 - a_5^2k_1k_5 + a_5^2\right),\\
\det\left(J\left(E_1\right)\right) &= \frac{a_5}{k_5}\left(a_4\left(a_4 - a_1k_5 + a_5\left(k_1k_5 - 1\right)\right) - a_5\left(a_4 - a_1k_5 + a_5\left(k_1k_5 - 1\right)\right)\right),\\
\det\left(J\left(E_1\right)\right) &= \frac{1}{k_5}\left(a_5\left(a_4 - a_5\right)\left(a_4 - a_1k_5 + a_5\left(k_1k_5 - 1\right)\right)\right).\\
\operatorname{tr}\left(J\left(E_1\right)\right) &= a_5 - a_4 - a_5k_1.
\end{aligned}$$

1. Using condition (C2) of (2.1), we obtain $\det\left(J\left(E_1\right)\right) < 0$. Similarly, it follows from (C3) that $\operatorname{tr}(J\left(E_1\right)) < 0$. Note that in case If $a_4 = a_5$, then $\det\left(J\left(E_1\right)\right) = 0$, which implies that one or more eigenvalues are equal to zero. In such case, bifurcation analysis is required to determine the stability of E_1. In 3-dimensional space, this is not sufficient to establish the stability of the equilibrium. Following the notation in Lemma 1.1, we may write

$$J^{[2]}\left(E_1\right) = \begin{pmatrix} -a_5k_1 & 0 & a_5 \\ 0 & -a_5k_1 + a_5 - a_4 & a_5 \\ -\frac{a_5-a_4}{k_5} & \frac{1}{k_5}\left(a_1k_5 - a_4 - a_5\left(k_1k_5 - 1\right)\right) & a_5 - a_4 \end{pmatrix}. \tag{2.5}$$

We have

$$\det\left(J^{[2]}\left(E_1\right)\right) = \frac{a_5}{k_5}\left(\left(-a_4a_5k_5\right)k_1^2 + k_5\left(a_1a_5 - \left(a_4 - a_5\right)^2\right)k_1 + \left(a_4 - a_5\right)^2\right),$$

$$\begin{aligned}
\det\left(J^{[2]}\left(E_1\right)\right) &= -k_5a_4^2k_1 + a_4^2 - k_5a_4a_5k_1^2 + 2k_5a_4a_5k_1 - 2a_4a_5 - k_5a_5^2k_1 + a_5^2 + a_1k_5a_5k_1,\\
\det\left(J^{[2]}\left(E_1\right)\right) &= -\left(a_4^2 + a_5^2\right)\left(k_1k_5 - 1\right) + a_1a_5k_1k_5 - a_4a_5k_1^2k_5 + 2a_5^2k_1^2k_5 - 2a_4a_5,\\
\det\left(J^{[2]}\left(E_1\right)\right) &= -\left(a_4^2 + a_5^2\right)\left(k_1k_5 - 1\right) + a_5k_1k_5\left(a_1 - a_4k_1\right) + 2a_5^2k_1\left(k_1k_5 - 1\right),\\
\det\left(J^{[2]}\left(E_1\right)\right) &= +2a_4a_5\left(k_1k_5 - 1\right) - \left(a_4^2 + a_5^2\right)\left(k_1k_5 - 1\right) + a_5k_1k_5\left(a_1 - a_4k_1\right),\\
\det\left(J^{[2]}\left(E_1\right)\right) &= \left((a_4 - a_5)^2 + (a_4 - 2a_5)a_5k_1\right)(k_1k_5 - 1) - a_5k_1(a_4 - a_1k_5).
\end{aligned}$$

Treating this as a quadratic polynomial in k_1, the discriminant is given by

$$\Delta = k_5^2\left(a_1a_5 - \left(a_4 - a_5\right)^2\right)^2 + 4a_4a_5k_5\left(a_4 - a_5\right)^2 > 0.$$

This means that $\det\left(J^{[2]}\left(E_1\right)\right)$ has two positive roots

Since the coefficient of k_1^2 is negative, the determinant is positive between $k_1^{(1)}$ and $k_1^{(2)}$ and negative elsewhere. We need to examine the sign of the term $a_1a_5 - (a_4 - a_5)^2 > 0$ in accordance with Lemma 1.1.

- If $a_1a_5 - (a_4 - a_5)^2 > 0$, it follows directly that $k_1^{(2)} > 0$. As for $k_1^{(1)}$, we can multiply it by its conjugate to obtain

$$k_1^{(1)} = -4a_4a_5k_5\left(a_4 - a_5\right)^2 < 0.$$

  Hence, if $k_1 > k_1^{(2)}$, then $\det\left(J^{[2]}\left(E_1\right)\right) < 0$. This guanrantees the stability of the equilibrium. However, E_1 may either be a node or a focus node depending on the real or complex nature of the eigenvalues.
  Alternatively, if $k_1 < k_1^{(2)}$, then $\det\left(J^{[2]}\left(E_1\right)\right) > 0$ and E_1 is unstable. Consequently, $\lambda_1 < 0$ whereas $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) > 0$ with $\mathrm{Im}\left(\lambda_2\right), \mathrm{Im}\left(\lambda_3\right) \geq 0$. Therefore, E_1 may either be a saddle or saddle focus.
  Finally, if $k_1 = k_1^{(2)}$, then $\det\left(J^{[2]}\left(E_1\right)\right) = 0$. It follows that if $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then $\lambda_1 < 0$ and either $\lambda_2 = -\lambda_1 > 0$ or $\lambda_3 = -\lambda_1 > 0$ leading to E_1 being an unstable saddle. On the other hand, if $\lambda_2, \lambda_3 \in \mathbb{C}$, then $\mathrm{Re}\left(\lambda_2\right) = \mathrm{Re}\left(\lambda_3\right) = 0$, in which case the stability of the equilibrium requires further investigation.

- In case $a_1a_5 - (a_4 - a_5)^2 \leq 0$, using simple multiplication by the conjugate we obtain the same results as the first case.

  equilibrium E_1 is a stable for system (1.10) if

- $k_1k_5 < 1$ and $(a_4 - a_5)^2 + (a_4 - 2a_5)\, a_5k_1\, (k_1k_5 - 1) > a_5k_1\, (a_4 - a_1k_5)$

$$((a_4 - a_5)^2 + (a_4 - 2a_5)a_5k_1)(k_1k_5 - 1) - a_5k_1(a_4 - a_1k_5)$$

2. **Stability of Equilibria** E_2

   Substituting E_2(1.20) into the Jacobian matrix of (2.2) yields

$$J\left(E_2\right) = \begin{pmatrix} a_1 - 2a_5k_1 - a_1 + a_5k_1 & -a_5 & -a_5 \\ a_1 - a_5k_1 & a_5 - a_5 & 0 \\ 0 & 0 & a_4 - a_5 \end{pmatrix}$$

$$J(E_2) = \begin{pmatrix} -a_5k_1 & -a_5 & -a_5 \\ a_1 - a_5k_1 & 0 & 0 \\ 0 & 0 & a_4 - a_5 \end{pmatrix} \tag{2.6}$$

which has eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\sqrt{a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5} - \frac{1}{2}a_5k_1, \\ \lambda_2 &= -\frac{1}{2}\sqrt{a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5} - \frac{1}{2}a_5k_1, \\ \lambda_3 &= a_4 - a_5 \geq 0. \end{aligned} \tag{2.7}$$

Note that in case $\lambda_3 = 0$, then further analysis is required to determine the nature of the equilibrium.
The nature of λ_1 and λ_2 depends on the term $a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5$.

- **Case 1:** If $a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5 < 0$. Therefore, λ_1 and λ_2 are complex roots with

$$\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 = -\frac{1}{2}a_5k_1 < 0.$$

- **Case 2:** If $a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5 = 0$ and the eigenvalues become

$$\lambda_1 = \lambda_2 = -\frac{1}{2}a_5k_1 < 0.$$

  equilibrium E_2 is a stable for system (1.10) if

$$a_4 < a_5.$$

$$E_2\ stable: \begin{cases} a_4 > a_5 : \text{ns} \\ a_4 = a_5 : bur \\ a_4 < a_5 : stable \end{cases}$$

3. **Stability of Equilibria** E_3 :

- Substituting the third equilibrium into the Jacobian yields:

$$J(E_3) = \begin{pmatrix} +k_1\frac{a_4-a_1k_5}{k_1k_5-1} & -\frac{a_1k_5-a_4}{k_1k_5-1} & -\frac{a_1k_5-a_4}{k_1k_5-1} \\ 0 & \frac{a_1k_5-a_4}{k_1k_5-1}-a_5 & 0 \\ -\frac{a_4k_1-a_1}{k_1k_5-1} & 0 & k_5\frac{a_1-a_4k_1}{k_1k_5-1} \end{pmatrix}$$

$$J(E_3) = \begin{pmatrix} +k_1\frac{a_4-a_1k_5}{k_1k_5-1} & -\frac{a_1k_5-a_4}{k_1k_5-1} & -\frac{a_1k_5-a_4}{k_1k_5-1} \\ 0 & \frac{a_1k_5-a_4}{k_1k_5-1}-a_5 & 0 \\ -\frac{a_4k_1-a_1}{k_1k_5-1} & 0 & k_5\frac{a_1-a_4k_1}{k_1k_5-1} \end{pmatrix} \tag{2.8}$$

- The eigenvalues are given by:

$$\begin{aligned} \lambda_1 &= \frac{1}{2(k_1k_5-1)}\left(\sqrt{\beta_1k_1^2+\beta_2k_1+\beta_3}+(a_4-a_1k_5-a_4k_5)k_1+a_1k_5\right), \\ \lambda_2 &= -\frac{1}{2(k_1k_5-1)}\left(\sqrt{\beta_1k_1^2+\beta_2k_1+\beta_3}-(a_4-a_1k_5-a_4k_5)k_1+a_1k_5\right), \\ \lambda_3 &= -\frac{1}{k_1k_5-1}(a_4-a_1k_5-a_5+a_5k_1k_5) = -\frac{a_4-a_1k_5}{k_1k_5-1}-a_5 \le 0. \end{aligned} \tag{2.9}$$

determinant

$$\det J(E_3) = -\left(\lambda^2-\beta_4\lambda-\beta_5\right)\left(\lambda+a_5-\frac{a_4-a_1k_5}{1-k_1k_5}\right).$$

$$(\lambda^2+\lambda\beta_4-\beta_5)\left(\lambda+a_5-\frac{a_4-a_1k_5}{1-k_1k_5}\right)=0,$$

$$\lambda^2+\beta_4\lambda-\beta_5=0,$$

$$\begin{aligned} \beta_4 &= b = \frac{(a_1k_5-a_4)k_1+(a_4k_1-a_1)k_5}{(k_1k_5-1)}. \\ \beta_5 &= c = -\frac{(a_1k_5-a_4)(a_4k_1-a_1)}{(k_1k_5-1)}. \end{aligned}$$

The nature of the remaining two eigenvalues depends on the quadratic polynomial $\lambda^2+\lambda\beta_4-\beta_5$:

- **Case 1:** if $k_1k_5-1>0$ when $a_1k_5-a_4\le 0$ and $a_4k_1-a_1\le 0$ it follows from (C5),(C6) of (2.1) that b^2-4ac Hence, the polynomial $\lambda^2+\lambda\beta_4-\beta_5$
  - First, assuming $b^2-4ac\ge 0$ and given that, it follows that $\lambda_1<0$. As for λ_2, multiplying it by its conjugate we obtain

$$\lambda_1 = \frac{-b-\sqrt{b^2-4c}}{2}<0.$$

$$\lambda_2 = \frac{-b+\sqrt{b^2-4c}}{2} = \frac{\left(-b+\sqrt{b^2-4c}\right)\left(b+\sqrt{b^2-4c}\right)}{2\left(b+\sqrt{b^2-4c}\right)} = \frac{-4c}{2\left(b+\sqrt{b^2-4c}\right)}<0.$$

– Second, let $b^2 - 4ac \leq 0$

$$\operatorname{Re}\lambda_{1,2} = \frac{-b}{2} < 0,$$

which is clearly negative. It is easy to see that in this case, $\lambda_2 > 0$.

- **Case 2:** if $k_1k_5 - 1 > 0$ when $a_1k_5 - a_4 \geq 0$ and $a_4k_1 - a_1 \geq 0$ that $b^2 - 4ac$ Hence, the polynomial $\lambda^2 + \lambda\beta_4 - \beta_5$

$$sng(\lambda_1) = -sng(\lambda_2).$$

Subject to conditions (2.1), then stability equilibrium E_3

$$E_3 \quad : \quad \begin{cases} k_1k_5 < 1 : \frac{a_1k_5 - a_4}{1 - k_1k_5} + a_5 > 0 \ stable \\ k_1k_5 > 1 : \text{ ns} \end{cases}$$

$$E_3 \quad : \quad \begin{cases} k_1k_5 < 1 : \frac{a_1k_5 - a_4}{1 - k_1k_5} + a_5 > 0 \ stable \\ k_1k_5 > 1 : \text{ ns} \end{cases}$$

$$E_3 \quad : \quad \begin{cases} k_1k_5 < 1 : a_1k_5 - a_4 + a_5(1 - k_1k_5) > 0 \ stable \\ k_1k_5 > 1 : \text{ ns} \end{cases}$$

$$E_3 : \begin{cases} k_1k_5 < 1 : a_1k_5 - a_4 > a_5(k_1k_5 - 1) \ stable \\ k_1k_5 > 1 : \text{ ns} \end{cases}$$

4. **Stability of Equilibria** E_4

Into the Jacobian matrix yields

$$J(E_4) = \begin{pmatrix} \frac{a_1k_5 - a_4}{k_5} & 0 & 0 \\ 0 & -a_5 & 0 \\ -\frac{a_4}{k_5} & 0 & -a_4 \end{pmatrix}, \tag{2.10}$$

which has the eigenvalues

$$\begin{aligned} \lambda_1 &= -a_4, \\ \lambda_2 &= -a_5, \\ \lambda_3 &= -\frac{1}{k_5}(a_4 - a_1k_5). \end{aligned} \tag{2.11}$$

Clearly, subject to (C5) in (2.1), as long as $\lambda_3 \neq 0$, these eigenvalues are all negative implying equilibrium E_4 is stable

Subject to conditions (2.1), equilibrium E_4 is a stable for system (1.10) if $a_4 > a_1k_5$

$$E_4 stable : \{a_4 > a_1k_5$$

5. **Stability of Equilibria** E_5

Sub stituting E_5into the Jacobian matrix yields:

$$J(E_5) = \begin{pmatrix} -a_1 & -\frac{a_1}{k_1} & -\frac{a_1}{k_1} \\ 0 & \frac{a_1 - a_5 k_1}{k_1} & 0 \\ 0 & 0 & \frac{a_4 k_1 - a_1}{k_1} \end{pmatrix}, \tag{2.12}$$

which has the eigenvalues

$$\begin{aligned} \lambda_1 &= \frac{1}{k_1}(a_1 - a_5 k_1), \\ \lambda_2 &= -a_1, \\ \lambda_3 &= -\frac{1}{k_1}(a_1 - a_4 k_1). \end{aligned} \tag{2.13}$$

It follows from (C4) and (C5) in (2.1), respectively, that as long as $\lambda_3 \neq 0$, $\lambda_1 > 0$ and $\lambda_2, \lambda_3 < 0$ $E_5$is stable if $\quad a_4 k_1 < a_1 < a_5 k_1$

$$E_5 \quad a_4 k_1 < a_1 < a_5 k_1$$

6. **Stability of Equilibria** E_6

Similarly, substituting E_6 **into the Jacobian matrix yields**

$$J(E_6) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_5 & 0 \\ 0 & 0 & a_4 \end{pmatrix}, \tag{2.14}$$

The eigenvalues are given by

$$\begin{aligned} \lambda_1 &= a_1, \\ \lambda_2 &= -a_5, \\ \lambda_3 &= a_4. \end{aligned} \tag{2.15}$$

which clearly has two positive and one negative eigenvalues it means it is not stable.

The remaining five equilibria E_1 through E_5 are stable subject to the conditions

$$\text{(C4)} \left\{ \begin{array}{c} k_1 k_5 < 1 \; and \; ((a_4 - a_5)^2 + (a_4 - 2a_5)a_5 k_1)(k_1 k_5 - 1) > a_5 k_1 (a_4 - a_1 k_5) \\ k_1 k_5 > 1 \end{array} \right\} \tag{2.16}$$

$$\text{(C5)} \qquad \{a_4 < a_5\} \tag{2.17}$$

$$\text{(C6)} \qquad \left\{ \begin{array}{ll} k_1 k_5 < 1 : a_1 k_5 - a_4 > a_5 \left(k_1 k_5 - 1\right) & stable \\ k_1 k_5 > 1 \qquad \text{ns} & \end{array} \right\} \tag{2.18}$$

$$\text{(C7)} \qquad \{a_4 > a_1 k_5\} \tag{2.19}$$

$$\text{(C8)} \quad \{a_4 k_1 < a_1 < a_5 k_1\} \tag{2.20}$$

2.1.1 Regions and the stability nature of equilibria

In this region, the stability of all six equilibrium points requires the simultaneous satisfaction of the simplied conditions derived from (C4), (C5),(C6),(C7),(C8)

1. **Region (1): All six equilibria**

- Case (1.1) (1.39)
  (a) Subject to (C4) (2.16) So E_1 is stable because the stability condition $(C4)$ (2.16) is satisfied in this condition.
  (b) E_2 is instble because the stability condition$(C5)$(2.17) is not satisfied in this condition.
  (c) E_3 is stble because the stability condition$(C6)$ (2.18) is satisfied in this condition.
  (d) E_4 is stble because the stability condition $(C7)$ (2.19) is satisfied in this condition.
  (e) E_5 is instble because the stability condition $(C8)$ (2.20) is not satisfied in this condition.
- Case (1.2):(**??**)
  (a) Subject to (C4) (2.16) So E_1 is stable because the stability condition $(C4)$ (2.16) is satisfied in this condition.
  (b) E_2 is instble because the stability condition$(C\ 5)$(2.17) is not satisfied in this condition.
  (c) E_3 is instble because the stability condition$(C\ 6)$ (2.18) is not satisfied in this condition.
  (d) E_4 is instble because the stability condition $(C\ 7)$ (2.19) is not satisfied in this condition.
  (e) E_5 is instble because the stability condition $(C\ 8)$ (2.20) is not satisfied in this condition .

Continuing in the same way,we get the following table1

sta stands for stable.

insta stands for instable.

Table 2.1: Regions of equilibrium coexistence and the stability nature of equilibria.

Region of Existence	Cases	Stability of Equilibria				
		E_1	E_2	E_3	E_4	E_5
Region (1): All six equilibria	Case (1.1):	Subject to (C4)	insta	sta	sta	insta
	Case (1.2):	sta	insta	insta	insta	insta
Region (2): All but E_1	Case (2.1):	insta	insta	insta	sta	insta
	Case (2.2):	insta	sta	sta	sta	insta
	Case (2.3):	insta	insta	sta	insta	insta
Region (3): All but E_2	Case (3.1):	sta	insta	insta	insta	insta
		insta	insta	insta	insta	insta
Region (4): All but E_3	Case (4.1):	sta	insta	insta	insta	insta
	Case (4.2):	sta	insta	insta	insta	insta
Region (5): All but E_1, E_3	Case (5.1):	insta	sta	insta	insta	insta
	Case (5.2):	insta	insta	insta	sta	insta
	Case (5.3):	insta	insta	insta	insta	insta
	Case (5.4):	insta	insta	insta	sta	insta
Region (6): Only E_4, E_5, E_6	Case (6.1):	insta	insta	insta	insta	sta
	Case (6.2):	insta	insta	insta	sta	insta
	Case (6.3):	insta	insta	insta	sta	insta
	Case (6.4):	insta	insta	insta	sta	insta
	Case (6.5):	insta	insta	insta	insta	sta
	Case (6.6):	insta	insta	insta	sta	insta

2.2 Global Asymptotic Stability

In this section We will analyze the global asymptotic stability of the system equilibria and study the cases where a single equilibrium is locally stable using the direct Lyapunov method, identifying sufficient conditions for stability and evaluating them within the framework of predefined regions.

First, let us consider equilibrium point

Equilibrium point E_1

$$E_1 = (x^*, y^*, z^*) = \left(a_5, \frac{a_5 - a_4}{k_5} + a_1 - a_5 k_1, \frac{a_4 - a_5}{k_5} \right). \tag{2.21}$$

with a plugin in Systam (1.8) $\frac{a_5-a_4}{k_5}, a_5k_1, -\frac{a_5-a_4}{k_5}, -a_5k_1 \text{in} \frac{dx}{dt}$
and by adding $a_5, -a_5 \text{in} \frac{dz}{dt}$.

$$\begin{cases} \frac{dx}{dt} = -\left(a_1 - k_1 x - y - z + \frac{a_5-a_4}{k_5} - \frac{a_5-a_4}{k_5} - a_5k_1 + a_5k_1 \right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = (a_4 - x - k_5 z - a_5 + a_5)\, z. \end{cases} \tag{2.22}$$

Simplifying the system,we find:

$$\begin{cases} \frac{dx}{dt} = -\left((a_1 + \frac{a_5-a_4}{k_5} - a_5k_1) - k_1 x - y - z + \frac{a_4-a_5}{k_5} + (a_5)k_1 \right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = -\left(-k_5 \frac{(a_4-a_5)}{k_5} + k_5 z + x - a_5 \right) z. \end{cases} \tag{2.23}$$

Compensate $a_5 = x^*$, $y^* = a_1 + \frac{a_5-a_4}{k_5} - a_5k_1$, $\frac{(a_4-a_5)}{k_5} = z^*$.

$$\begin{cases} \frac{dx}{dt} = -\left((y^*) - k_1 x - y - z + z^* + (x^*)k_1 \right) x, \\ \frac{dy}{dt} = (x - x^*)\, y, \\ \frac{dz}{dt} = -\left(-k_5 z^* + k_5 z + x - x^* \right) z. \end{cases} \tag{2.24}$$

can be rewritten in the form

$$\begin{cases} \frac{dx}{dt} = -\left((y - y^*) + k_{-1}\,(x - x^*) + (z - z^*) \right) x, \\ \frac{dy}{dt} = (x - x^*)\, y, \\ \frac{dz}{dt} = -\left(k_5\,(z - z^*) + (x - x^*) \right) z. \end{cases} \tag{2.25}$$

We propose the candidate Lyapunov function defined by

$$V = \left(x - x^* - \ln \frac{x}{x^*} \right) + \left(y - y^* - \ln \frac{y}{y^*} \right) + \left(z - z^* - \ln \frac{z}{z^*} \right). \tag{2.26}$$

Differentiating (2.26) with respect to time yields

$$\frac{dV}{dt} = \frac{dx}{dt} - \frac{dx}{dt}\left(\frac{x^*}{x}\right) + \frac{dy}{dt} - \frac{dy}{dt}\left(\frac{y^*}{y}\right) + \frac{dz}{dt} - \frac{dz}{dt}\left(\frac{z^*}{z}\right), \tag{2.27}$$

simplifying the statement

$$\frac{dV}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt}, \tag{2.28}$$

by compensating the system (1.8),we find :

$$\frac{dV}{dt} = -k_1\left(x - x^*\right)^2 - k_5\left(z - z^*\right)^2 - 2\left(x - x^*\right)\left(z - z^*\right). \tag{2.29}$$

We concider the quadratic to be a polynomial.
Negative polynomial in the case of a delta that is strictly smaller then zero
we put $X = x - x^*$, $Z = z - z^*$
By substituting X , Z in (2.29)
such that

$$\frac{dV}{dt} = -k_1X^2 - 2ZX - k_5Z^2, \tag{2.30}$$

then
we put a=$-k_1$, b= $2Z$, c=$-k_{-5}Z^2$
we calculate the delta $\Delta = b^2 - 4ac = 4Z^2(1 - K_1K_5)$

- if $\Delta < 0$ then $K_1K_5 > 1$

It is easy to see that the resulting quadratic form is negative subject to $k_1k_5 > 1$. Recall from Table 1 that E_1 is uniquely stable across four regions:case (1.1),case (1.2),case (3.1), case(4.1) and case(4.2). It is evident that $k_1k_5 > 1$ is satisfied over (1.2), (3.1), and (4.2). The global asymptotic stability of E_1 over region (4.1) could not be established by means of the Lyapunov function (2.26).
Next, we see that
Equilibrium point E_2

$$E_2 = (x^*, y^*, z^*) = (a_5, a_1 - a_5k_1, 0)$$

Is uniquely stable in the local sense across regions (5.1) and (5.3)
By adding the $+a_5k_1$,$-a_5k_1$in$\frac{dx}{dt}$ and$-a_5$, $+a_5$ in System (1.8) can be rewritten as

$$\begin{cases} \frac{dx}{dt} = (-k_1 x + a_5 k_1 - a_5 k_1 + a_1 - y - z)x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = (a_4 - a_5 + a_5 - x - k_5 z)\, z. \end{cases} \tag{2.31}$$

they simplify to

$$\begin{cases} \frac{dx}{dt} = -\left(k_1 (x - a_5) + (y - a_1 + a_5 k_1) + z\right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = (a_4 - a_5 - (x - a_5) - k_5 z)\, z. \end{cases} \tag{2.32}$$

to compensate $a_5 = x^*$ and $a_1 - a_5 k_1 = y^*$

$$\begin{cases} \frac{dx}{dt} = -\left(k_1 (x - x^*) + (y - y^*) + z\right) x, \\ \frac{dy}{dt} = (x - x^*)\, y, \\ \frac{dz}{dt} = (a_4 - a_5 - (x - x^*) - k_5 z)\, z. \end{cases} \tag{2.33}$$

The proposed candidate Lyapunov function is

$$V = \left(x - x^* - \ln \frac{x}{x^*}\right) + \left(y - y^* - \ln \frac{y}{y^*}\right) + z. \tag{2.34}$$

Its differentiation with respect to time yields

$$\frac{dV}{dt} = \left(\frac{x - x^*}{x}\right) \frac{dx}{dt} + \left(\frac{y - y^*}{y}\right) \frac{dy}{dt} + \left(\frac{z - z^*}{z}\right) \frac{dz}{dt} \tag{2.35}$$

by compensating the system (2.33),we find :

$$\frac{dV}{dt} = -k_1 (x - x^*)^2 - k_5 z^2 - 2 (x - x^*)\, z + (a_4 - a_5)\, z. \tag{2.36}$$

We concider the quadratic to be a polynomial

Negative polynomial in the case of a delta that is strictly smaller then zero

we put $X = x - x^*$

By substituting X , in (**??**)

such that

$$\frac{dV}{dt} = -k_1 X^2 - 2Xz - k_5 z^2 + (a_4 - a_5)\, z. \tag{2.37}$$

then

a=$-k_1$, b= $2z$, c=$-k_5 Z^2 + (a_4 - a_5)\, z\, k_1$.

we calculate the delta : $\Delta = b^2 - 4ac \quad = 4Z^2(1 - K_1 K_5) + (a_4 - a_5)\, k_1 z.$

- **if** $\Delta < 0$ **subject to** $1 - K_1K_5 < 0$ and $a_4 \leq a_5$

  $K_1K_5 > 1$ and $a_4 \leq a_5$

The negativity of the derivative is guaranteed subject to $k_1k_5 > 1$ and $a_4 \leq a_5$. Hence, E_2 is globally asymptotically stable
over region (5.3).

Let us now consider the third equilibrium

$$E_3 = (x^*, y^*, z^*) = \left(\frac{a_1k_5 - a_4}{k_1k_5 - 1}, 0, \frac{a_4k_1 - a_1}{k_1k_5 - 1}\right), \tag{2.38}$$

which according to Table2.1 , is uniquely locally stable over the single region (2.3).
System (1.8) can be rewritten in the form

$$\begin{cases} \frac{dx}{dt} = \left(-k_1x - y - z + k_1\frac{a_1k_5-a_4}{k_1k_5-1} + \frac{a_4k_1-a_1}{k_1k_5-1}\right)x, \\ \frac{dy}{dt} = \left(x - a_5 - \frac{a_1k_5-a_4}{k_1k_5-1} + \frac{a_1k_5-a_4}{k_1k_5-1}\right)y, \\ \frac{dz}{dt} = -\left(x + k_5z - \frac{a_1k_5-a_4}{k_1k_5-1} - k_5\frac{a_4k_1-a_1}{k_1k_5-1}\right)z. \end{cases} \tag{2.39}$$

Simplifying the system,we obtien:

$$\begin{cases} \frac{dx}{dt} = -\left(k_1(x - \frac{a_1k_5-a_4}{k_1k_5-1}) + y + (z - \frac{a_4k_1-a_1}{k_1k_5-1})\right)x, \\ \frac{dy}{dt} = \left((x - \frac{a_1k_5-a_4}{k_1k_5-1}) - (a_5 - \frac{a_1k_5-a_4}{k_1k_5-1})\right)y, \\ \frac{dz}{dt} = -\left((x - \frac{a_1k_5-a_4}{k_1k_5-1}) + k_5\left(z - \frac{a_4k_1-a_1}{k_1k_5-1}\right)\right)z. \end{cases} \tag{2.40}$$

to compensate $\frac{a_1k_5-a_4}{k_1k_5-1} = x^*$, $\frac{a_4k_1-a_1}{k_1k_5-1} = z^*$.

$$\begin{cases} \frac{dx}{dt} = -\left(k_1\left(x - x^*\right) + y + \left(z - z^*\right)\right)x, \\ \frac{dy}{dt} = \left(\left(x - x^*\right) - \left(a_5 - \frac{a_1k_5-a_4}{k_1k_5-1}\right)\right)y, \\ \frac{dz}{dt} = -\left(\left(x - x^*\right) + k_5\left(z - z^*\right)\right)z. \end{cases} \tag{2.41}$$

The proposed Lyapunov function is given by:

$$V = \left(x - x^* - \ln\frac{x}{x^*}\right) + y + \left(z - z^* - \ln\frac{z}{z^*}\right). \tag{2.42}$$

Differentiating with respect to time leads to

$$\frac{dV}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt} \tag{2.43}$$

By compensating the system (2.41),we find :

$$\frac{dV}{dt} = -k_1\left(x - x^*\right)^2 - k_5\left(z - z^*\right)^2 - 2\left(x - x^*\right)\left(z - z^*\right) - \left(a_5 - \frac{a_1k_5 - a_4}{k_1k_5 - 1}\right)y. \tag{2.44}$$

We concider the quadratic to be a polynomial
Negative polynomial in the case of a delta that is strictly smaller then zero
we put
$X = x - x^*$, $Z = z - z^*$
By substituting X ,Z in the phrase (2.44)
such that

$$\frac{dV}{dt} = -k_1 X^2 - 2ZX - k_5 Z^2 - \left(a_5 - \frac{a_1 k_5 - a_4}{k_1 k_5 - 1}\right) y \tag{2.45}$$

then
we put a=$-k_1$, b= $2Z$, c=$-k_{-5} Z^2 - \left(a_5 - \frac{a_1 k_5 - a_4}{k_1 k_5 - 1}\right) y$
we calculate Δ
$\Delta = b^2 - 4ac = 4Z^2(1 - K_1 K_5) - \left(a_5 - \frac{a_1 k_5 - a_4}{k_1 k_5 - 1}\right) y$
if $\Delta < 0$ **, subject to the sufficient conditions** $1 - K_1 K_5 < 0$ and $\left(a_5 - \frac{a_1 k_5 - a_4}{k_1 k_5 - 1}\right) < 0$

$$K_1 K_5 > 1 \quad \text{and} \quad a_1 k_5 - a_4 > a_5 (k_1 k_5 - 1)$$

The derivative in (2.44) is negative subject to the sufficient conditions $k_1 k_5 > 1$ and (C6). Since both conditions are satisfied in region (2.3), it is evident that E_3 is globally asymptotically stable across the region.

Next, we consider the equilibrium point E_4

$$E_4 = (x^*, y^*, z^*) = \left(0, 0, \frac{a_4}{k_5}\right), \tag{2.46}$$

which is uniquely locally stable across seven regions: (2.1), (5.2), (5.4), (6.2), (6.3), (6.4), and (6.6).
System (1.8) can be reformulated as

$$\begin{cases} \frac{dx}{dt} = \left(-k_1 x - y - z + \frac{a_4}{k_5} - \frac{a_4}{k_5} + a_1 \frac{k_5}{k_5}\right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = \left(-x - k_5 z + k_5 \frac{a_4}{k_5}\right) z. \end{cases} \tag{2.47}$$

Simplifying the system,we find

$$\begin{cases} \frac{dx}{dt} = -\left(k_1 x + y + \left(z - \frac{a_4}{k_5}\right) + \frac{1}{k_5}(a_4 - a_1 k_5)\right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = -\left(x + k_5 \left(z - \frac{a_4}{k_5}\right)\right) z. \end{cases} \tag{2.48}$$

to compensate $\frac{a_4}{k_5} = z^*$

$$\begin{cases} \frac{dx}{dt} = -\left(k_1 x + y + (z - z^*) + \frac{1}{k_5}(a_4 - a_1 k_5)\right) x, \\ \frac{dy}{dt} = (x - a_5)\, y, \\ \frac{dz}{dt} = -\left(x + k_5 (z - z^*)\right) z. \end{cases} \tag{2.49}$$

We propose the candidate Lyapunov function

$$V = x + y + \left(z - z^* - \ln \frac{z}{z^*}\right), \tag{2.50}$$

Whose derivative with respect to time is

$$\frac{dV}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt} \tag{2.51}$$

By compensating the system (2.49),we find :

$$\frac{dV}{dt} = -k_1 x^2 - k_{-5}(z - z^*)^2 - 2x(z - z^*) - \frac{1}{k_5}(a_4 - a_1 k_5)\, x - a_5 y. \tag{2.52}$$

We concider the quadratic to be a polynomial
Negative polynomial in the case of a delta that is strictly smaller then zero
we put :
$Z = z - z^*$
By substituting Z in the phrase (2.52)
such that

$$\frac{dV}{dt} = -k_1 x^2 - 2xZ - k_5 Z^2 - \frac{1}{k_5}(a_4 - a_1 k_5)\, x - a_5 y. \tag{2.53}$$

then,we put:
a=$-k_1$, b= $2Z$, c=$-k_5 Z^2 - \frac{1}{k_5}(a_4 - a_1 k_5)\, x - a_5 y.$
we calculate Δ
$\Delta = b^2 - 4ac$
$= 4Z^2(1 - K_1 K_5) - \frac{1}{k_5}(a_4 - a_1 k_5)\, x - a_5 y.$

- **if** $\Delta < 0$, $1 - K_1 K_5 < 0$ and $\frac{1}{k_5}(a_4 - a_1 k_5) > 0$

$K_1 K_5 > 1$ and $a_4 > a_1 k_5$

The derivative in (2.52) is negative subject to the sufficient conditions $k_1 k_5 > 1$ and (C7). These two conditions are clearly satisfied over regions (5.4), (6.4), and (6.6). Hence, the equilibrium point E_4 is globally asymptotically stable across these regions.
Finally,

We consider the fifth equilibrium point

given by

$$E_5 = (x^*, y^*, z^*) = \left(\frac{a_1}{k_1}, 0, 0\right), \tag{2.54}$$

which is uniquely locally stable across seven regions: (6.1) and (6.5).

System (1.8) can be reformulated as in addition $-\frac{a_1}{k_1}, +\frac{a_1}{k_1}$ in$\frac{dz}{dt}$and in$\frac{dy}{dt}$:

$$\begin{cases} \frac{dx}{dt} = \left(-k_1 x + k_1 \frac{a_1}{k_1} - y - z\right) x, \\ \frac{dy}{dt} = \left(x - \frac{a_1}{k_1} - \frac{1}{k_1} k_1 a_5 + \frac{a_1}{k_1}\right) y, \\ \frac{dz}{dt} = \left(\frac{a_1}{k_1} - x - k_5 z - \frac{1}{k_1} a_1 + a_4\right) z. \end{cases} \tag{2.55}$$

Simplifying the system,we find:

$$\begin{cases} \frac{dx}{dt} = -\left(k_1\left(x - \frac{a_1}{k_1}\right) + y + z\right) x, \\ \frac{dy}{dt} = \left(\left(x - \frac{a_1}{k_1}\right) - \frac{1}{k_{-1}}\left(k_1 a_5 - a_1\right)\right) y, \\ \frac{dz}{dt} = \left(\left(\frac{a_1}{k_1} - x\right) - k_5 z - \frac{1}{k_{-1}}\left(a_1 - k_1 a_4\right)\right) z. \end{cases} \tag{2.56}$$

compensate $\frac{a_4}{k_5} = x^*$.

$$\begin{cases} \frac{dx}{dt} = -\left(k_1\left(x - x^*\right) + y + z\right) x, \\ \frac{dy}{dt} = \left(\left(x - x^*\right) - \frac{1}{k_1}\left(k_1 a_5 - a_1\right)\right) y, \\ \frac{dz}{dt} = \left(\left(x^* - x\right) - k_5 z - \frac{1}{k_1}\left(a_1 - k_1 a_4\right)\right) z. \end{cases} \tag{2.57}$$

The proposed Lyapunov function is of the form:

$$V = \left(x - x^* - \ln\frac{x}{x^*}\right) + y + z. \tag{2.58}$$

Differentiating(2.58) with respect to time produces

$$\frac{dV}{dt} = \left(\frac{x - x^*}{x}\right)\frac{dx}{dt} + \left(\frac{y - y^*}{y}\right)\frac{dy}{dt} + \left(\frac{z - z^*}{z}\right)\frac{dz}{dt} \tag{2.59}$$

By compensating the system (2.57),we find :

$$\frac{dV}{dt} = -k_1\left(x - x^*\right)^2 - k_5 z^2 - \frac{1}{k_1}\left(a_1 - k_1 a_4\right) z - \frac{1}{k_1}\left(k_1 a_5 - a_1\right) y. \tag{2.60}$$

We concider the quadratic to be a polynomial

Negative polynomial in the case of a delta that is strictly smaller then zero

we put :

$X = x - x^*$

By substituting X in the phrase (2.60)
such that

$$\frac{dV}{dt} = -k_1 x^2 - k_5 Z^2 - \frac{1}{k_1}\left(a_1 - k_1 a_4\right) z - \frac{1}{k_1}\left(k_1 a_5 - a_1\right) y. \tag{2.61}$$

then
we put: $a = -k_1$, $b = 0$, $c = -k_5 Z^2 - \frac{1}{k_{-1}}\left(a_1 - k_1 a_4\right) z - \frac{1}{k_{-1}}\left(k_1 a_5 - a_1\right) y$
and we calculte delta :
$\Delta = b^2 - 4ac$
$\Delta = -K_1 K_5 Z^2 - \frac{1}{k_1}\left(a_1 - k_1 a_4\right) z - \frac{1}{k_{-1}}\left(k_1 a_5 - a_1\right) y$

- if $\Delta < 0$, $K_1 K_5 > 1$ and $\frac{1}{k_{-1}}\left(a_1 - k_1 a_4\right) > 0$ and $\frac{1}{k_1}\left(k_1 a_5 - a_1\right) > 0$

$K_1 K_5 > 1$ and $a_4 k_1 < a_1 < a_5 k_1$

The derivative in (2.60) is clearly negative subject to the sufficient conditions $k_1 k_5 > 1$ and $(C8)$.these conditions are satisfied in region (6.5). Therefore, E_5 is globally asymptotically stable across region (6.5).

2.3 Global Existence and Positivity of Solutions

Throughout this section we use the following notation: $\mathbb{R}^3_+ := \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\}$.Since the functions f_1, f_2 and f_3 which appeared in (1.8), are continuously differentiable functions. Then, they are locally lipschitz. Thus, we can get the following primary result (see [26]) about the local existence of system (1.8):

Proposition 2.1 *If $k_1, k_5, a_1, a_4, a_5 \in \mathbb{R}_+$, there exists a unique solution of system (1.8) which is defined for $t \in [0, T_{max})$,where $T_{max} < +\infty$ or $T_{max} = +\infty$ is the maximal time-interval of existence.*

In order to show the positivity of the solutions for system (1.8), we need to define the following concept.

Proposition 2.2 *The nonnegative octant $\mathbb{R}^3_+$ is an invariant set with respect to (1.8), for $t \in [0, T_{max})$.*

Proof. Since for $(x, y, z) \in \mathbb{R}^3_+$, we have

$$\begin{cases} f_1(0, y, z) = 0, \\ f_2(x, 0, z) = 0, \\ f_3(x, y, 0) = 0, \end{cases}$$

then the function $(f_1, f_2, f_3)^T$ is essentially nonnegative. By direct application of [15, Proposition 2.1.], we obtain the desired result. ■

Remark 2.1 *If it is assumed that the initial data of system (1.8) is positive, i.e. $x_0, y_0, z_0 > 0$, then its solution remains positive.*

Theorem 2.1 *Let the assumptions of Proposition 2.1 hold. Then, there exists a unique nonnegative global solution of system (1.8), i.e. $T_{max} = +\infty$. Thus, the solution is defined for $t \in [0, +\infty)$.*

Proof.
Let $t \in (0, T]$ with $T \in (0, T_{\max})$. Due to the nonnegativity of x, y and z, we conclude from the first equation of (1.8) we have

$$\begin{aligned} \frac{dx}{dt} &= (a_1 - k_{-1}x - y - z)\, x. \\ &= a_1 x - k_1 x^2 - yx - zx. \\ &\leq a_1 x - k_1 x^2. \end{aligned}$$

we note that x is bounded from above by the solution $\tilde{x}$ of

$$\begin{cases} \frac{d\tilde{x}}{dt} = a_1\tilde{x} - k_1\tilde{x}^2 \text{ in } (0,T), \\ \tilde{x}(0) = x_0. \end{cases} \tag{2.62}$$

We multiply the first equation of (2.62) by $\tilde{x}^{-2}$, which results in the Bernoulli differential equation

$$\tilde{x}^{-2}\frac{d\tilde{x}}{dt} = a_1\tilde{x}^{-1} - k_1.$$

Next, we perform the substitution $u = \tilde{x}^{-1}$, so $\tilde{x} = u^{-1}$, and $\frac{d\tilde{x}}{dt} = -u^{-2}\frac{du}{dt}$. Substituting into the equation

$$-\frac{du}{dt} = a_1u - k_1,$$

which simplifies

$$\frac{du}{dt} + a_1u = k_1.$$

This is a first-order linear ODE, and we solve it using the integrating factor $\mu(t) = e^{a_1t}$. Multiplying both sides by $\mu(t)$ gives

$$e^{a_1t}\frac{du}{dt} + a_1e^{a_1t}u = k_1e^{a_1t},$$

which simplifies

$$\frac{d}{dt}\left(e^{a_1t}u\right) = k_1e^{a_1t}.$$

Integrating both sides

$$e^{a_1t}u = \frac{k_1}{a_1}e^{a_1t} + C.$$

Dividing through by e^{a_1t} gives

$$u = \frac{k_1}{a_1} + Ce^{-a_1t}.$$

Substituting back $u = \tilde{x}^{-1}$, we get

$$\tilde{x}^{-1} = \frac{k_1}{a_1} + Ce^{-a_1t}.$$

Taking the reciprocal of both sides

$$\tilde{x}(t) = \frac{1}{\frac{k_1}{a_1} + Ce^{-a_1t}}. \tag{2.63}$$

Using the initial condition $\tilde{x}(0) = x_0$, we have

$$x_0 = \frac{1}{\frac{k_1}{a_1} + C}. \tag{2.64}$$

Solving for C gives

$$\frac{1}{x_0} = \frac{k_1}{a_1} + C,$$

which implies

$$C = \frac{1}{x_0} - \frac{k_1}{a_1}. \tag{2.65}$$

Substituting C back into the solution, we get the final solution

$$\tilde{x}(t) = \frac{1}{\frac{k_1}{a_1} + \left(\frac{1}{x_0} - \frac{k_1}{a_1}\right) e^{-a_1 t}}.$$

Now, let's find the limits of the solution as $t \to 0$ and $t \to +\infty$.
As $t \to 0$, the solution approaches $\tilde{x}(0) = x_0$.
As $t \to +\infty$, $\tilde{x}(t) \to \frac{a_1}{k_1}$.
Also, from the third equation (1.8), we conclude that z is bounded from above by the solution $\tilde{z}$ of

$$\begin{cases} \frac{d\tilde{z}}{dt} = a_4\tilde{z} - k_5\tilde{z}^2 \ \text{ in } (0,T), \\ \tilde{z}(0) = z_0. \end{cases} \tag{2.66}$$

We note that the initial-value problems (2.62) and (2.66) are Bernoulli differential equations. Hence, according to the explicit solutions and the comparison principle, we deduce that

$$x(t) \leq \mathcal{C}_1 := \max\left(x_0, \frac{a_1}{k_1}\right) \ \text{ and } \ z(t) \leq \mathcal{C}_2 := \max\left(z_0, \frac{a_4}{k_5}\right) \ \text{ in } [0, T_{max}). \tag{2.67}$$

Let us now consider $\chi(t) := x(t) + y(t) + z(t)$. From (1.8), we have

$$\begin{aligned} &= \frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt}, \\ &= a_1 x - k_1 x^2 - a_5 y + a_4 z - 2xz - k_5 z^2, \\ &\leq a_1 x - a_5 y + a_4 z, \\ &= -(a_1 x + a_5 y + a_4 z) + 2a_1 x + 2a_4 z \\ &\leq \mathcal{C} - \sigma\chi. \end{aligned}$$

■

Chapter 3

HOPF-BIFURCATIONS and Reaction Diffusion Systems

3.1 Existence of HOPF-BIFURCATIONS

Evaluating the existence of a Hopf-bifurcation for a certain equilibrium point of our Willamowski-Rössler system (1.8) requires the existence of two complex Jacobian eigenvalues in addition to the characteristic polynomial satisfying conditions (H_3) and$(H_4)$of Lemma 1.2.
We conseder the chemical reaction system(1.8).
This yields six solutions $E_1, E_2, E_3, E_3, E_5, E_6$ To simplify the solutions where $k_1 \neq \frac{1}{k_5}$.We study the hopf bifirecation of each solution:
the hopf bifirecation of Equilibrium Point E_1,

Proposition 3.1 *System (1.8) undergoes a Hopf bifurcation about equilibrium E_1 within parametric regions (1.1) and (4.1) as k_5 varies in a small neighborhood of*

$$k_5^* = \frac{1}{k_1}(a_4 - a_5)\frac{a_4 - a_5 + 2a_5k_1}{(a_4 - a_5)^2 + a_5k_1(a_4 - a_5) + a_5(a_1 - a_5k_1)}. \tag{3.1}$$

iff

$$a_4 \neq a_5. \tag{3.2}$$

and

$$a_1 - a_5k_1 > 0. \tag{3.3}$$

Proof. Evaluating the Jacobian matrix at E_1 yields

$$J(E_1) = \begin{pmatrix} -a_5k_1 & -a_5 & -a_5 \\ \frac{1}{k_5}(a_1k_5 - a_4 - a_5(k_1k_5 - 1)) & 0 & 0 \\ \frac{a_5 - a_4}{k_5} & 0 & a_5 - a_4 \end{pmatrix}, \tag{3.4}$$

with the characteristic polynomial

$$\lambda^3 + a_2(k_5)\lambda^2 + a_1(k_5)\lambda + a_0(k_5) = 0, \tag{3.5}$$

where

$$\begin{cases} a_2(k_5) = a_4 - a_5 + a_5k_1, \\ a_1(k_5) = \frac{a_5}{k_5}(2a_5 - 2a_4 + a_1k_5 + a_4k_1k_5 - 2a_5k_1k_5), \\ a_0(k_5) = -\frac{a_5}{k_5}(a_4 - a_5)(a_4 - a_5 - a_1k_5 + a_5k_1k_5). \end{cases} \tag{3.6}$$

We calculate $D_2(k_5)$ we find:

$$\begin{aligned} D_2(k_5) &= \det\begin{pmatrix} a_1(\alpha^*) & a_0(\alpha^*) \\ a_3(\alpha^*) & a_2(\alpha^*) \end{pmatrix}, \\ &= \left(a_4^2a_5k_1 + a_4a_5^2k_1^2 - 2a_4a_5^2k_1 - 2a_5^3k_1^2 + a_5^3k_1 + a_1a_5^2k_1\right)k_5, \\ &= -2a_4a_5^2 + a_4^2a_5 - 2a_5^3k_1 + a_5^3 + 2a_4a_5^2k_1. \end{aligned}$$

we put $D_2(k_5) = 0$

$(a_4^2a_5k_1 + a_4a_5^2k_1^2 - 2a_4a_5^2k_1 - 2a_5^3k_1^2 + a_5^3k_1 + a_1a_5^2k_1)k_5 = 2a_4a_5^2 - a_4^2a_5 + 2a_5^3k_1 - a_5^3 - 2a_4a_5^2k_1$

$$\begin{aligned} k_5^* &= \frac{-(2a_4a_5^2 - a_4^2a_5 + 2a_5^3k_1 - a_5^3 - 2a_4a_5^2k_1)}{(a_5^3k_1 - 2a_5^3k_1^2 + a_4a_5^2k_1^2 + a_1a_5^2k_1 - 2a_4a_5^2k_1 + a_4^2a_5k_1)}, \\ k_5^* &= \frac{a_5(a_4 - a_5)(a_4 - a_5 + 2a_5k_1)}{a_5k_1(-2a_5^2k_1 + a_4^2 + a_5^2 + a_1a_5 - 2a_4a_5 + a_4a_5k_1)}, \\ k_5^* &= \frac{1}{k_1}(a_4 - a_5)\frac{a_4 - a_5 + 2a_5k_1}{a_4^2 - 2a_5^2k_1 + a_5^2 + a_1a_5 - 2a_4a_5 + a_4a_5k_1}, \\ k_5^* &= \frac{1}{k_1}(a_4 - a_5)\frac{a_4 - a_5 + 2a_5k_1}{a_4^2 - 2a_4a_5 + a_5^2 - 2a_5^2k_1 + a_1a_5 + a_4a_5k_1}, \\ k_5^* &= \frac{1}{k_1}(a_4 - a_5)\frac{a_4 - a_5 + 2a_5k_1}{(a_4 + a_5)^2 - 2a_5^2k_1 + a_1a_5 + a_4a_5k_1}, \\ k_5^* &= \frac{1}{k_1}(a_4 - a_5)\frac{a_4 - a_5 + 2a_5k_1}{(a_4 + a_5)^2 + a_5(a_1 - a_5k_1) + a_5k_1(a_4 - a_5)}. \end{aligned}$$

which exists and is positive subject to $a_4 \neq a_5$ and $a_1 - a_5k_1 > 0$.

Solving (1.6) for $\alpha = k_5$ yields k_5^* as defined in (3.1), which exists and is positive subject to $a_4 \neq a_5$. Condition (H4) leads to

$$\frac{\partial}{\partial k_5}D_2(k_5^*) = a_5k_1^2\frac{\left((a_4 - a_5)^2 + a_5k_1(a_4 - a_5) + a_5(a_1 - a_5k_1)\right)^2}{(a_4 - a_5)(a_4 - a_5 + 2a_5k_1)} \neq 0, \tag{3.7}$$

for $\frac{\partial}{\partial k_5}D_2(k_5^*) \neq 0$:

$$\left(-2a_5^2k_1+(a_4-a_5)^2+a_1a_5+a_4a_5k_1\right)\neq 0$$
$$(a_4-a_5)^2+a_1a_5=a_5k_1\left(2a_5-a_4\right)$$
$$k_1\neq\tfrac{(a_4-a_5)^2+a_1a_5}{a_5(2a_5-a_4)}$$
$$\frac{\partial}{\partial k_5}D_2\left(k_5^*\right) = a_5k_1\left(a_4^2-2a_5^2k_1+(a_4-a_5)^2+a_4a_5k_1\right)\neq 0$$
$$\text{if } 2a_5 > a_4 \text{ and } k_1\neq\frac{(a_4-a_5)^2+a_1a_5}{a_5\left(2a_5-a_4\right)}$$

which is satisfied subject to (3.2) and (3.3). Evaluating $a_0\left(k_5^*\right)$ yields

$$a_0\left(k_5^*\right) = -\frac{a_5}{k_5^*}\left(a_4-a_5\right)\left(a_4-a_5-a_1\left(k_5^*\right)+a_5k_1\left(k_5^*\right)\right)$$
$$a_0\left(k_5^*\right) = a_5\frac{a_4-a_5}{a_4-a_5+2a_5k_1}\left(a_4^2k_1+a_5a_4k_1^2+3a_5a_4k_1-a_1a_4-a_1a_5k_1+a_1a_5\right)$$
$$a_0\left(k_5^*\right) = \frac{1}{\left(a_4-a_5\right)\left(a_4-a_5+2a_5k_1\right)}\left(a_4^2k_1^2+a_5a_4k_1^3+3a_5a_4k_1^2-a_1a_4k_1-a_1a_5k_1^2+a_1a_5k_1\right)$$
$$a_0\left(k_5^*\right) = a_5\frac{a_4-a_5}{a_4-a_5+2a_5k_1}\left(a_4^2k_1+a_5a_4k_1^2+3a_5a_4k_1-a_1a_4-a_1a_5k_1+a_1a_5\right)$$
$$a_0\left(k_5^*\right)=\frac{a_5}{a_4-a_5+2a_5k_1}\left(a_4-a_5\right)\left(-a_4^2k_1-a_5a_4k_1^2-3a_5a_4k_1+a_1a_4+a_1a_5k_1-a_1a_5\right). \quad (3.8)$$

$a_0\left(k_5^*\right)>0$ if

$$\begin{cases} -\left(a_4\left(a_4k_1-a_1\right)+a_1a_5+7a_5a_4k_1\right)+a_5k_1\left(4a_4+a_1-a_4k_1\right)>0 \\ a_1\left(a_4-a_5\right)>a_4k_1\left(a_4+7a_5\right) \\ \text{if } 2a_5>a_4 \text{ and } k_1\neq\frac{(a_4-a_5)^2+a_1a_5}{a_5(2a_5-a_4)} \end{cases}$$

From the existence conditions of E_1, it follows that $a_0\left(k_5^*\right)>0$ is equivalent to (3.2) and (3.3). Similarly, with

$$a_1\left(k_5^*\right) = \frac{a_5}{k_5^*}\left(2a_5-2a_4+a_1\left(k_5^*\right)+a_4k_1\left(k_5^*\right)-2a_5k_1\left(k_5^*\right)\right)$$
$$a_1\left(k_5^*\right) = -\frac{1}{a_4-a_5+2a_5k_1}\left(k_1a_4^2a_5+7k_1a_4a_5^2-a_1a_4a_5+a_1a_5^2\right)$$
$$a_1\left(k_5^*\right) = -\frac{1}{a_4-a_5+2a_5k_1}a_5\left(k_1a_4^2+7k_1a_4a_5-a_1a_4+a_1a_5\right)$$
$$a_1\left(k_5^*\right) = \frac{a_5}{a_4-a_5+2a_5k_1}\left(-a_4^2k_1+a_1a_4-a_1a_5-7a_4a_5k_1\right)>0$$

$a_1\left(k_5^*\right)>0$ if

$$\begin{cases} -\left(a_4\left(a_4k_1-a_1\right)+a_1a_5+7a_5a_4k_1\right)+a_5k_1\left(4a_4+a_1-a_4k_1\right)>0 \\ a_1\left(a_4-a_5\right)-a_4^2k_1-7a_4a_5k_1>0 \\ \text{if } 2a_5>a_4 \text{ and } k_1\neq\frac{(a_4-a_5)^2+a_1a_5}{a_5(2a_5-a_4)} \end{cases}$$

we find:

$$a_1\left(k_5^*\right) = a_5\left(a_4 - a_5\right)\frac{a_1 - a_4k_1}{a_4 - a_5 + 2a_5k_1}, \tag{3.9}$$

it follows that $a_1\left(k_5^*\right) > 0$ is equivalent to (3.2) and (3.3). In order to identify regions where a Hopf bifurcation exists, the two conditions need to to be evaluated across all stability regions of E_1 according to Table **??**:

- **Cases (1.1) and (4.1)**: Since $a_1 - a_4k_1 > 0$, conditions (3.2) and (3.3) leading to a Hopf-bifurcation about equilibrium E_1.
- **Case (1.2) and (3.1)**: Given that $a_1 - a_4k_1 < 0$, condition (3.3) is not fulfilled and thus no Hopf bifurcation exists for system (1.8).
- **Case (4.2)**: From the existence conditions of this region, we have $k_5k_1 > 1$. Substituting for k_5^* yields

  $$\left(a_4 - a_5\right)\frac{a_4 - a_5 + 2a_5k_1}{\left(a_4 - a_5\right)^2 - 2a_5^2k_1 + a_1a_5 + a_4a_5k_1} > 1. \tag{3.10}$$

  This can be reformulated as

  $$\left(a_4 - a_5\right)\left(a_4 - a_5 + 2a_5k_1\right) - \left(a_4 - a_5\right)^2 - \left(-2a_5^2k_1 + a_1a_5 + a_4a_5k_1\right) = -a_5\left(a_1 - a_4k_1\right) > 0, \tag{3.11}$$

  which contradicts with the existence condition $a_1 - a_4k_1 > 0$ of E_1. Hence, no Hopf bifurcation exists for system (1.8) in this region.

■

the hopf bifirecation of Equilibrium Points E_2

Evaluating the Jacobian at E_2 and taking the eigenvalues yields

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\sqrt{a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5} - \frac{1}{2}a_5k_1, \\
\lambda_2 &= -\frac{1}{2}\sqrt{a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5} - \frac{1}{2}a_5k_1, \\
\lambda_3 &= a_4 - a_5.
\end{aligned} \tag{3.12}$$

The third eigenvalue is clearly real. For λ_1 and λ_2 to be complex, the term $a_5^2k_1^2 + 4a_5^2k_1 - 4a_1a_5$ must be negative, which implies that $\frac{1}{2}a_5k_1 \neq 0$. Hence, equilibrium E_2 does not exhibit a Hop bifurcation. The remaining two equilibria require some attention.

the hopf bifirecation of Equilibrium Points E_3

As for equilibrium E_3, the Jacobian is evaluated as

$$J\left(E_3\right) = \frac{1}{k_1k_5 - 1}\begin{pmatrix} \left(a_4 - a_1k_5\right)k_1 & \left(a_4 - a_1k_5\right) & a_4 - a_1k_5 \\ 0 & a_5 - a_4 + \left(a_1 - a_5k_1\right)k_5 & 0 \\ \left(a_1 - a_4k_1\right) & 0 & \left(a_1 - a_4k_1\right)k_5 \end{pmatrix}. \tag{3.13}$$

with the characteristic polynomial
For $n = 3$

$$P(\lambda) = a_3(\alpha)\lambda^3 + a_2(\alpha)\lambda^2 + a_1(\alpha)\lambda - a_0(\alpha).$$

we put $P(\lambda) = 0$

$$\lambda^3 + a_2(k_5)\lambda^2 + a_1(k_5)\lambda + a_0(k_5) = 0. \tag{3.14}$$

wher

$$\begin{cases} a_0(k_5) = \begin{array}{c}(a_1 - a_4k_1)(a_4 - a_1k_5)(a_5 - a_4 + k_1(a_4 - a_1k_5) + k_5(a_1 - a_5k_1)) \\ -k_1(a_1 - a_4k_1)(a_4 - a_1k_5)^2 - k_1k_5(a_1 - a_4k_1)(a_4 - a_1k_5)(a_5 - a_4 + k_5(a_1 - a_5k_1))\end{array} \\ a_1(k_5) = \begin{array}{c}k_1(a_4 - a_1k_5)(a_5 - a_4 + k_5(a_1 - a_5k_1)) - (a_1 - a_4k_1)(a_4 - a_1k_5) \\ +k_5(a_1 - a_4k_1)(a_5 - a_4 + k_1(a_4 - a_1k_5) + k_5(a_1 - a_5k_1))\end{array} \\ a_2(k_5) = a_4 - a_5 - k_1(a_4 - a_1k_5) - k_5(a_1 - a_4k_1) - k_5(a_1 - a_5k_1). \end{cases}, \tag{3.15}$$

using conditions (H_3) and(H_4) of Lemma 1.2
We calculate $D_2(k_5)$

$$\begin{aligned} D_2(k_5) &= \det\begin{pmatrix} a_1(k_5) & a_0(k_5) \\ a_3(k_5) & a_2(k_5) \end{pmatrix}, \\ D_2(k_5) &= \begin{pmatrix} a_1(k_5) & a_0(k_5) \\ 1 & a_2(k_5) \end{pmatrix}, \\ D_2(k_5) &= a_1(k_5)a_2(k_5) - a_0(k_5), \\ D_2(k_5) &= (A)(a_4k_1 + a_1k_5 - a_1k_1k_5 - a_4k_1k_5). \end{aligned}$$

Before we present the main findings of this study, we intend to simplify the presentation by defining the following notations:

$$\begin{aligned} A = {} & -a_1^2k_5 - a_4^2 - a_5^2 - 2a_1^2k_5^2 + a_1a_4 + 2a_4a_5 - a_5^2k_1^2k_5^2 - a_4^2k_1k_5 + 2a_5^2k_1k_5 + 2a_1^2k_1k_5^2 \\ & +3a_1a_4k_5 - a_4a_5k_1 - 3a_1a_5k_5 - a_1a_4k_1^2k_5^2 - a_1a_5k_1^2k_5^2 - a_4a_5k_1^2k_5^2 - 2a_1a_4k_1k_5 \\ & -a_4a_5k_1k_5 + a_1a_4k_1k_5^2 + 3a_1a_5k_1k_5^2 + a_4a_5k_1^2k_5 + a_1a_5k_1k_5 + a_4^2k_1^2k_5 \end{aligned}$$

By putting $D_2(k_5) = 0$ we find:

$$(A)(a_4k_1 + a_1k_5 - a_1k_1k_5 - a_4k_1k_5) = 0$$

$$\begin{cases} (a_4k_1 + a_1k_5 - a_1k_1k_5 - a_4k_1k_5) = 0 \\ or \\ A = 0 \end{cases}$$

to find k_5^* :

$$\begin{aligned} (a_4k_1 + a_1k_5 - a_1k_1k_5 - a_4k_1k_5) &= 0 \\ (a_1k_1 + a_4k_1 - a_1)\, k_5 &= a_4k_1 \end{aligned}$$

$$k_5^* = \frac{a_4}{a_1k_1 + (a_4k_1 - a_1)}k_1,$$

To calculate $a_0\,(k_5^*)$,we substitite k_5^* in $a_0\,(k_5)$:

$$\begin{aligned} a_0\,(k_5^*) &= a_1k_1\,(a_1 - a_4k_1)\,(a_1 - a_5k_1)\left(\frac{a_4}{a_1k_1 + (a_4k_1 - a_1)}k_1\right)^3 \\ &+ \left(\begin{array}{c} a_1\,(a_1 - a_4k_1)\,(a_1k_1 - a_1 + a_5k_1) - k_1\,(a_1 - a_4k_1)\,(a_4\,(a_1 - a_5k_1) + a_1\,(a_4 - a_5)) \\ -a_1^2k_1\,(a_1 - a_4k_1) \end{array}\right) \end{aligned}$$

We simplify:

$$a_0\,(k_5^*) = \left(\frac{a_4}{a_1k_1 + (a_4k_1 - a_1)}k_1\right) + \left(a_4\,(a_1 - a_4k_1)\,(a_5 - a_4 + a_4k_1) - a_4^2k_1\,(a_1 - a_4k_1)\right),$$

$$a_0\,(k_5^*) = a_4\,(k_1 - 1)\,(a_1 - a_4k_1)^4\,\frac{a_4 - a_5 + a_5k_1}{(a_1 - a_1k_1 - a_4k_1)^3}\,,$$

$a_0\,(k_5^*) > 0$ if $(k_1 - 1)\,\frac{a_4 - a_5 + a_5k_1}{(a_1 - a_1k_1 - a_4k_1)} > 0$ end $(k_1 - 1)\,(a_1 - a_4k_1) > 0$

To calculate $a_1\,(k_5^*)$,we substitite k_5^* in $a_1\,(k_5)$

$$a_1\,(k_5^*) = a_4\,(k_1 - 1)\,\frac{(a_1 - a_4k_1)^3}{(a_1k_1 + a_4k_1 - a_1)^2}\,,$$

$a_1\,(k_5^*) > 0$ if $(k_1 - 1)\,(a_1 - a_4k_1) > 0$

$$\begin{cases} (k_1 - 1)\,(a_1 - a_4k_1) > 0 \\ (k_1 - 1)\,\frac{a_4 - a_5 + a_5k_1}{(a_1 - a_4k_1 - a_1k_1)} > 0 \end{cases} \tag{3.16}$$

To calculate $a_2\,(k_5^*)$,we substitite k_5^* in $a_2\,(k_5)$

$$\begin{aligned} a_2\left(k_5^*\right) &= \left(a_1k_1 - 2a_1 + a_4k_1 + a_5k_1\right)\frac{a_4}{\left(a_1+a_4\right)k_1 - a_1}k_1 + \left(a_4 - a_5 - a_4k_1\right) \\ &= \left(a_4 - a_5 + a_5k_1\right)\frac{\left(a_1 - a_4k_1\right)}{\left(a_1 - a_4k_1\right) - a_1k_1} \end{aligned}$$

Applying the second condition (H_4) ,we find:

$$\frac{\partial}{\partial k_5}D_2(k_5^*) = \left(a_1 - a_4k_1\right)^2\frac{a_4\left(k_1-1\right)\left(a_1 - a_4k_1\right) + \left(a_4 - a_5 + a_5k_1\right)^2}{a_1k_1 + \left(a_4k_1 - a_1\right)} \neq 0.$$

Evaluating conditions for hopf bifurcations around equilibrium point E_3 in regions (1.1) to (3.1) ,we will evaluate the conditions specified in the Lemma 1.2 to determine the existence of Hopf bifurcations around equilibrium E_3 in the specified regions.

Objective

To determine whether the conditions are met for the existence of Hopf bifurcations in these regions:

- **case**(1.1)**and**$(2.1),(2.3)$: Since $a_1 - a_4k_1 > 0$ and $a_4 \geq a_5$,conditions (3.16)that no Hopf bifurcations occur around equilibrium point E_3

$$\begin{cases} \left(k_1 - 1\right) > 0 \\ \frac{a_4 + a_5(k_1-1)}{(a_1 - a_4k_1 - a_1k_1)} > 0 \end{cases}$$

$$\begin{cases} \left(k_1 - 1\right) > 0 \\ \left(a_1 - a_1k_1 - a_4k_1\right) > 0 \end{cases}$$

  we find

$$\begin{cases} \left(k_1 - 1\right) > 0 \\ a_1\left(1 - k_1\right) > a_4k_1 \end{cases}$$

- **case (1.2) , case(2.3) , case(3.1)**: Since $a_4 \geq a_5$ and $a_1 - a_4k_1 \leq 0$, conditions (3.16)that no Hopf bifurcations occur around equilibrium point E_3

$$k_5^*k_1 = \frac{a_4}{a_1k_1 + \left(a_4k_1 - a_1\right)}k_1^2 > 1,$$

we calculate

$$a_4k_1^2 > a_1k_1 + \left(a_4k_1 - a_1\right),$$

we find

$$(k_1 - 1)(a_1 - a_4 k_1) < 0,$$

Our analysis led to the conclusion that no Hopf bifurcations occur around equilibrium point E_3. This finding is anticipated, given that E_3 remains stable within these regions, precluding any transition to instability.

the hopf bifirecation of Equilibrium Points $E_4, E_5 \; and \;\; E_6$

E_4 the eigenvalues in (2.11), E_5 in (2.13) and $E_6$of (2.15) the Jacobian eigenvalues are purely real and thus no Hopf bifur

can exist.

3.2 Numerical Simulations

This section presents numerical results that support the main results of the study, where the WR system exhibits a wide range of stability regions. We will focus on specific representative cases to highlight key aspects of the stability of the system. We present four examples, with the first two focusing on the stability of equilibria E_1 and E_3 in regions (1.1) and (4.1), respectively, in order to provide a deeper understanding of the stability of the system under different conditions.

3.2.1 Example 1: System Dynamics in Case (1.1)

In this example, we examine the stability of equilibria E_1 and E_3 in region (1.1). From earlier analysis, we know that in this region, equilibrium E_1 is locally stable subject to condition (C4), whereas equilibrium E_3 is always locally stable.
Figure 1 shows the time evolution of states and state-space portraits of the WR system with parameter values

$$(a_1, a_4, a_5, k_1, k_5) = (89.03, 52.88, 43.55, 0.74, 0.53)\,.$$

These parameter values satisfy the conditions of region (1.1) as well as condition (C4). As long as the initial states lie within the attraction field of E_1, the solution should converge towards it. We set the initial conditions to

$$(x_1, x_2, x_3) = (12.5, 7.2, 9.85)\,.$$

represented by the blue dot. Equilibrium E_1 is represented by the red asterisk, and the system states after 10 seconds are represented by the black circles. Clearly, the solution converges towards equilibrium E_1.
shows the time evolution of states and state-space portraits for

$$(a_1, a_4, a_5, k_1, k_5) = (39.75, 19.58, 10.41, 1.41, 0.45)\,.$$

These values do not satisfy condition (C4), implying that E_1 is locally unstable. We chose a n initial state sufficiently close to E_1, defined by

$$(x_1, x_2, x_3) = E_1 + (1, 2, 5) \times 10^{-2}.$$

As shown in Figure 3.2, the trajectory moves away from E_1 and converges towards E_4, which is the dominant equilibrium in the region.

Figure 3.3 shows the time evolution of states and state-space portraits for

$$(a_1, a_4, a_5, k_1, k_5) = (31.73, 48.91, 39.76, 0.23, 1.24)\,.$$

Again, these values satisfy the conditions of region (1.1) where E_3 should be locally stable. Simulations showed that the attraction region of E_3 is extremely small. The initial conditions were selected as

$$(x_1, x_2, x_3) = E_3 + (1, 2, 5) \times 10^{-12}.$$

It is clear that the solution converges towards E_3. When the distance between the initial condition and E_3 was selected in an order higher than 10^{-11}, the solution diverged away from E_3 towards E_4, which is the dominant equilibrium.
In order to evaluate the bifucation analysis, let us choose the same parameter values used in the literature, i.e.

$$(a_1, a_4, a_5, k_1) = (30, 16.5, 10, 0.415)\,,$$

which belong to region (1.1). Considering k_5 as the critical bifurcation parameter, formula (3.1) yields $k_5^* = 0.70732$. Clearly, conditions (3.2) and (3.3) are satisfied implying the existence of a bifurcation around E_1 in the neiborhood of k_5^*. This is supported by the numerical bifurcation plot depicted in Figure 3.4, which shows that as k_5 enters the neiborhood of k_5^* the system enters into a chaotic state.

3.2.2 Example 2: System Dynamics in Case (4.1)

We will now consider the stability of equilibrium E_3 in region (2.3). The theoretical analysis presented earlier showed that E_3 is globally stable in this region. For this investigation, we chose parameter values

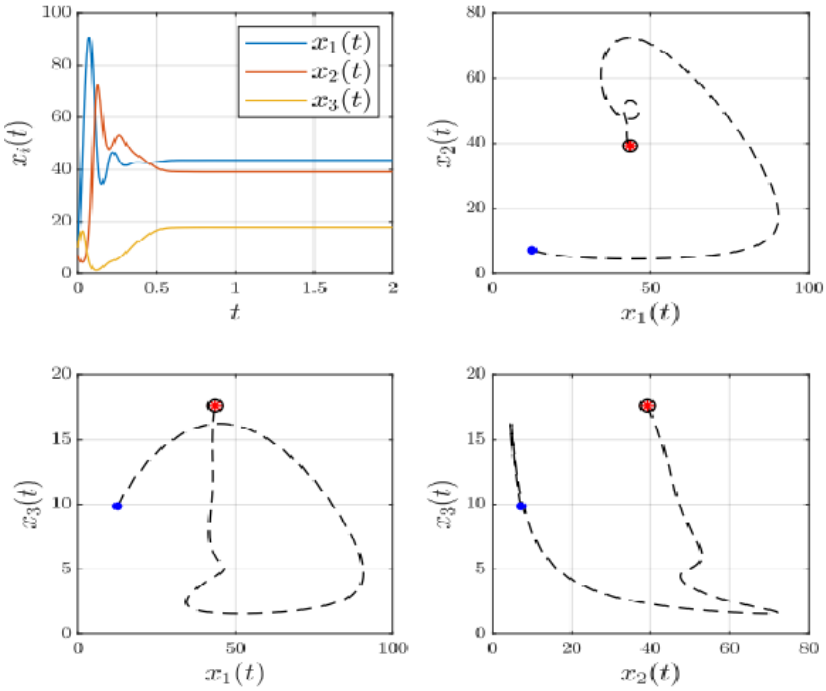
$$(a_1, a_4, a_5, k_1, k_5) = (25.05, 61.98, 7.08, 2.89, 6.77)\,,$$

and initial states

$$(x_1, x_2, x_3) = (12.5, 7.2, 9.85)\,.$$

This parameter set satisfies the conditions of region (2.3). Figure **??** illustrates the states of the WR system. It is observed that the states converge towards E_3. Despite multiple changes in the initial states during the simulations, the trajectory consistently moved towards E_3, affirming its stability in region (2.3).

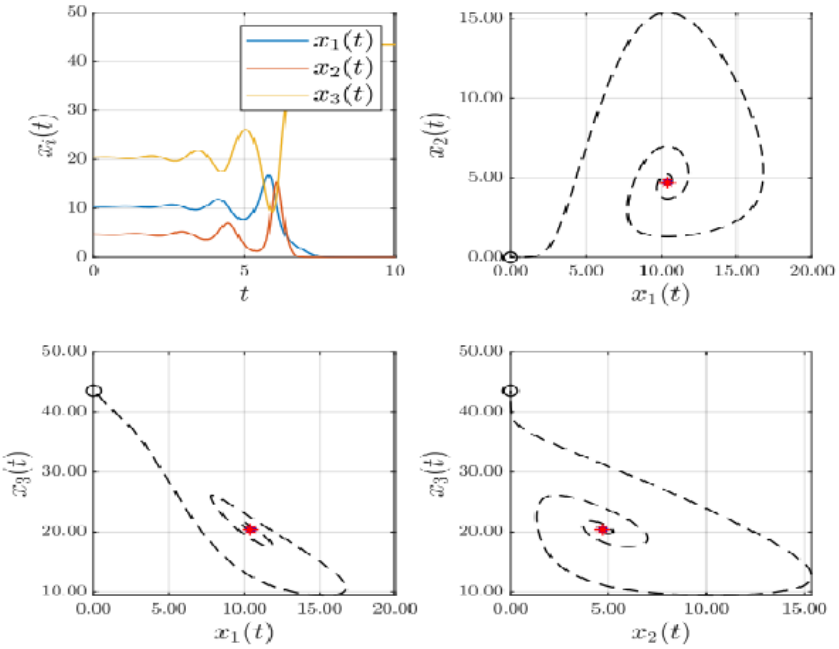
1



1.*png*

Figure 3.1: The time evolution of states and state-space portraits of the WR system subject to $(a_1, a_4, a_5, k_1, k_5) = (89.03, 52.88, 43.55, 0.74, 0.53)$ and $(x_1, x_2, x_3) = (12.5, 7.2, 9.85)$

2



2.*png*

Figure 3.2: The time evolution of states and state-space portraits of the WR system subject to $(a_1, a_4, a_5, k_1, k_5) = (39.75, 19.58, 10.41, 1.41, 0.45)$ and $(x_1, x_2, x_3) = E_1 + (1, 2, 5) \times 10^{-2}$.

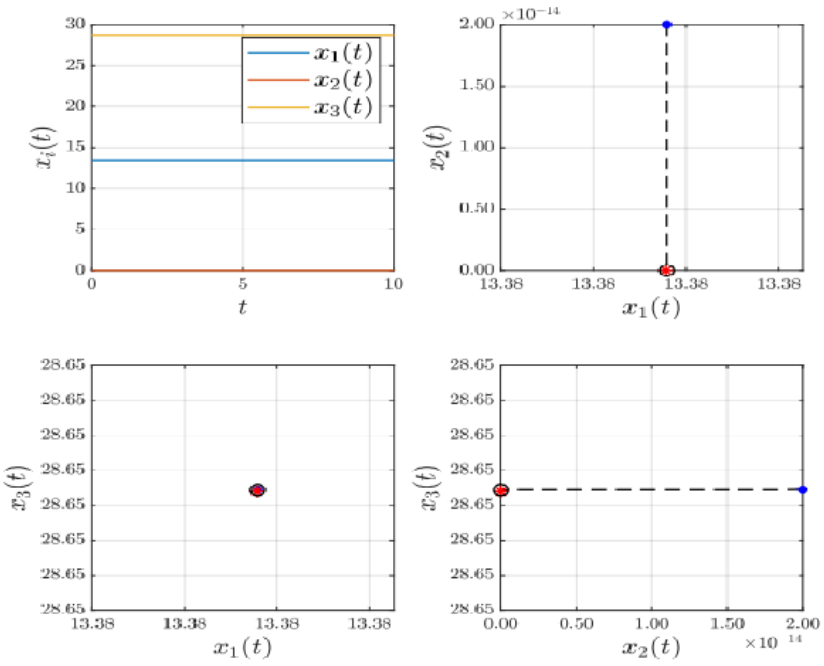


Figure 3.3: The time evolution of states and state-space portraits of the WR system subject to $(a_1, a_4, a_5, k_1, k_5) = (31.73, 48.91, 39.76, 0.23, 1.24)$ and $(x_1, x_2, x_3) = E_3 + (1, 2, 5) \times 10^{-12}$

3.3 Reaction Diffusion Systems

In this section, we will focus on the study of reaction diffusion, where we give a general introduction to these systems and define them in a three-dimensional case and also discuss global asymptotic stability in the partial differential equations (PDE) that represent these systems.

3.3.1 Reaction Diffusion Systems

Reaction–diffusion systems are mathematical models which correspond to several physical phenomena, the most common of which is the change in space and time of the concentration of one or more chemical substances, local chemical reactions in which the substances are transformed into each other, and diusion which causes the substances to spread out over a surface in space. Reaction–diffusion systems are naturally applied in chemistry.However, the system can also describe dynamical processes of non-chemical nature. Many examples are found in biology, geology, physics (neutron diffusion theory) and ecology.Mathematically, reaction–diffusion systems take the form of semilinear parabolic partial differential equations. They can be represented by the general form:

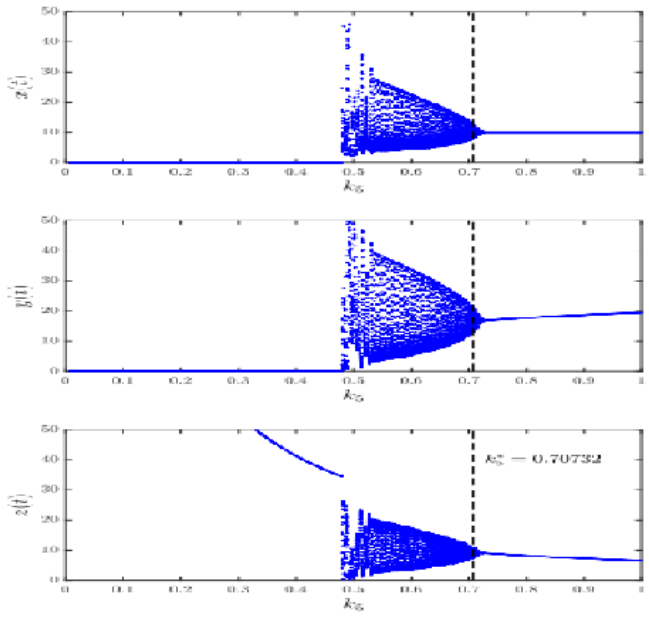


Figure 3.4: Numerical bifurcation diagram of system (1.8) for $(a_1, a_4, a_5, k_1) = (30, 16.5, 10, 0.415)$ with k_5 as the critical bifurcation parameter.

$$\partial_t U - D\Delta U = H(U\ ), \tag{3.17}$$

where $U(x;t)$ denotes the unknown vector function, D is a diagonal matrix of diffusion coeffcients, and H describes the reaction-diffusion mechanics of the system. The solutions of reaction–diffusion equations display a wide range of behaviours, including the formation of travelling waves and wave like phenomena as well as other self organized patterns like stripes, hexagons or more intricate structure like dissipative solitons. Each function, for which a reaction diffusion differential equation holds, represents in fact a concentration variable. [1]

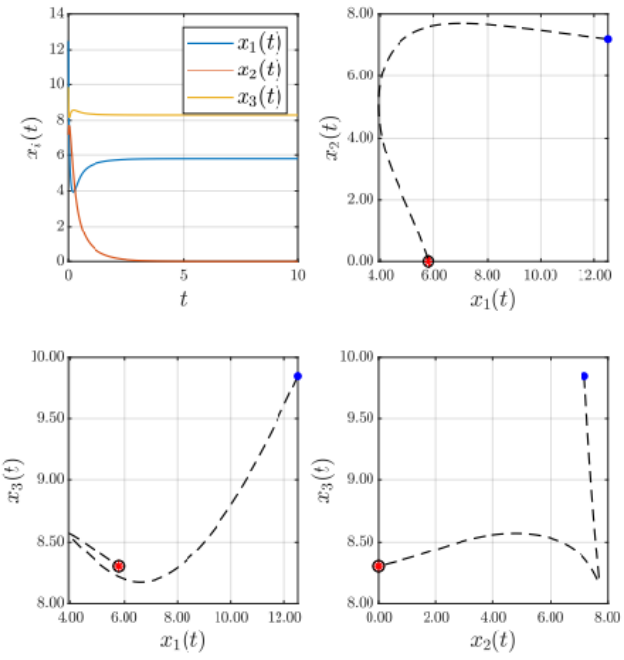
If $U = (u; v; w)$; the system (3.17) can be reduced to

$$\begin{cases} \frac{du}{dt} - d_1\Delta u = f_1(u, v, w), \\ \frac{dv}{dt} - d_2\Delta v = f_2(u, v, w), \\ \frac{dw}{dt} - d_3\Delta w = f_3(x, y, z), \end{cases} \tag{3.18}$$

where$u; v; w$ are the chemical species and $d_1; d_2; d_3$ are the specific diffusion coeffecients.

$$\begin{cases} \frac{du}{dt} - d_1\Delta u = (a_1 - k_1u - v - w)u, \\ \frac{dv}{dt} - d_2\Delta v = (u - a_5)v, \\ \frac{dw}{dt} - d_3\Delta w = (a_4 - u - k_5w)w, \end{cases} \tag{3.19}$$

5



5.png

Figure 3.5: The time evolution of states and state-space portraits of the WR system subject to $(a_1, a_4, a_5, k_1, k_5) = (25.05, 61.98, 7.08, 2.89, 6.77)$ and $(x_1, x_2, x_3) = (12.5, 7.2, 9.85)$

3.4 Global asymptotic stability

In this section, we study the global asymptotic stability of the system(3.19) . The aim is to show that every solution of the system with a positive initial value that is different from the equilibrium point will converge to the equilibrium.

Proposition 3.2 *Let*

$$V\ (t\ ) = \int_{\Omega} [(u - u^* - \ln \frac{u^*}{u}) + (v - v^* - \ln \frac{v^*}{v}) + (w - w^* - \ln \frac{w^*}{w})]. \tag{3.20}$$

Then,$V\ (t)$ is nonnegative and is strictly minimized at the unique equilibrium (u*, v*, w*).Hence, (u*, v*, w*) is globally asymptotically stable.

Proof. We have

$$\frac{dV\ (t\ )}{dt} = \int_{\Omega} \left[(1 - \frac{u^*}{u}) \frac{du}{dt} + (1 - \frac{v^*}{v}) \frac{dv}{dt} + (1 - \frac{w^*}{w}) \frac{dw}{dt} \right] dx. \tag{3.21}$$

Substituting (3.19) here yields

$$\frac{dV\ (t\ )}{dt} = \int_{\Omega} (1 - \frac{u^*}{u}) \left[d_1 \Delta u + (a_1 - k_1 u - v - w) u \right] dx$$

$$+\int_\Omega (1-\tfrac{v^*}{v})\left[d_2\Delta v+(u-a_5)v\right]dx$$
$$+\int_\Omega (1-\tfrac{w^*}{w})\left[d_3\Delta w+(a_4-u-k_5w)w\right]dx$$

Let us split the derivative into two parts

$$\frac{dV\ (t\ )}{dt}=I+J. \tag{3.22}$$

where

$$J=\int_\Omega (1-\frac{u^*}{u})\left[(a_1-k_1u-v-w)u\right]dx+\int_\Omega (1-\frac{v^*}{v})\left[(u-a_5)v\right]dx+\int_\Omega (1-\frac{w^*}{w})\left[(a_4-u-k_5w)w\right]dx. \tag{3.23}$$

and

$$I=I_1+I_2+I_3. \tag{3.24}$$

With

$$I=\int_\Omega (1-\frac{u^*}{u})d_1\Delta udx+\int_\Omega (1-\frac{v^*}{v})d_2\Delta vdx+\int_\Omega (1-\frac{w^*}{w})d_3\Delta wdx. \tag{3.25}$$

$$I_1=\int_\Omega (1-\frac{u^*}{u})d_1\Delta udx.$$
$$I_2=\int_\Omega (1-\frac{v^*}{v})d_2\Delta vdx.$$
$$I_3=\int_\Omega (1-\frac{w^*}{w})d_3\Delta wdx.$$

Using Green's formula and using the Neumann boundary conditions , we obtain

$$\begin{aligned} I_1 &= \int_\Omega (1-\frac{u^*}{u})d_1\Delta udx, \\ I_1 &= -d_1\int_\Omega \nabla(1-\frac{u^*}{u})\nabla udx, \qquad (3.26)\\ I_1 &= -d_1\int_\Omega \frac{u^*}{u^2}\left|\nabla u\right|^2dx, \qquad (3.27)\end{aligned}$$

Similarly, we can write

$$\begin{aligned} I_2 &= \int_\Omega (1-\frac{v^*}{v})d_2\Delta vdx, \\ I_2 &= -d_2\int_\Omega \nabla(1-\frac{v^*}{v})\nabla vdx, \qquad (3.28)\\ I_2 &= -d_2\int_\Omega \frac{v^*}{v^2}\left|\nabla v\right|^2dx, \qquad (3.29)\end{aligned}$$

and

$$\begin{aligned} I_3 &= \int_\Omega (1-\frac{w^*}{w}) d_3 \Delta w dx, \\ I_3 &= -d_3 \int_\Omega \nabla(1-\frac{w^*}{w}) \nabla w dx, \qquad (3.30) \\ I_3 &= -d_3 \int_\Omega \frac{w^*}{w^2} |\nabla w|^2 dx, \qquad (3.31) \end{aligned}$$

Substituting the three terms in yields

$$I = -\int_\Omega \left[d_1 \frac{u^*}{u^2} |\nabla u|^2 + d_2 \frac{v^*}{v^2} |\nabla v|^2 + d_3 \frac{w^*}{w^2} |\nabla w|^2 \right] dx. \qquad (3.32)$$

Hence, it is clear to see that

$$I \leq 0. \qquad (3.33)$$

The second part of the derivative is J as defined in (3.24), ,these are the same results found in the chapter global asymptotic stability and there fore $dt \leq 0$. Furthermore, we can have the equality $\frac{dV}{dt} \leq 0$ if and only if $(u, v, w) = (u^*, v^*, w^*)$. Therefore, by Lyapunov's direct method, we can assert that (u^*, v^*, w^*) is globally asymptotically stable. This concludes the proof.[3]

■

Conclusion

In this study, we conducted an in-depth analysis of the WR dynamical system,which represents an important nonlinear chemical model for understanding complex phenomena such as oscillation and chaos in chemical reactions. Through numerical simulations, our study began by analyzing the system's critical points, identifying six equilibria and analyzing their local stability under specific conditions that depend on the system's parameters. We then moved on to study the global stability of some of these equilibria using Lyapunov's theorem, enabling us to identify regions where the system exhibits global stability. We also addressed the Hopf bifurcation phenomenon, mathematically proven to exist around the equilibrium point E1 in certain regions of parametric space, confirming the system's ability to transition from stability to periodic oscillation or chaotic behavior. We also studied the propagation of the reaction in three dimensions and the stability of approximate solutions to the partial differential equations representing the system.

Finally, a comprehensive numerical simulation confirmed the validity of the theoretical results, providing a comprehensive framework for understanding the dynamics of chaotic chemical reactions. The results obtained in this study not only contribute to deepening the theoretical understanding of the WR model, but also open the way for future practical applications in the fields of chemical engineering and the study of complex models in nature.

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