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Reproducing Kernel Hilbert Space Method For Solving Fractional Integro-Differential Systems

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بسم الله الرحمن الرحيم

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لم تكن الرحلة قصيرة و لا الطريق محفوفا بالتسهيلات ، لكنني فعلتها ، فالحمدلله الذي يسر البدايات و بلغنا النهايات بفضله و كرمه و الشكر لله على القوة حين ضعفت و على الصبر حين تعبت لك الحمد حتى ترضى و لك الحمد اذا رضيت يا واسع الفضل .

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List of Abbreviations

DEs	Differential Equations
FDEs	Fractional Differential Equations
IDEs	Integero Differential Equations
Eq./Eqs.	Equation/ Equations.
CFD	Caputo Fractional Derivative
RKHS	Reproducing Kernel Hilbert Space
IVP	Initial Value Problem
BVP	Boundary Value Problem
IC's	Initial Conditions
BC's	Boundary Conditions

List of Symbols

C	Set of Complex Numbers
R	Set of Real Numbers
K	Set of Real or Complex Numbers
$\mathrm{L}^p[a,b]$	Lebesgue Space
$\mathrm{W}_p^m[a,b]$	Sobolev Space
Lu(x)	Linear Bounded Operator
u(x)	Exact Solution
$\mathrm{u}_n(x)$	Approximate Solution

Abstract

This dissertation develops an efficient numerical method for solving fractional integro-differential systems involving the Caputo fractional derivative. The proposed approach employs the **Reproducing Kernel Hilbert Space (RKHS)** method to construct approximate solutions expressed as uniformly convergent series, ensuring both high accuracy and rapid convergence. Numerical examples demonstrate the effectiveness and precision of the RKHS method in addressing complex fractional systems. The results reveal the significant impact of the fractional derivative order on the qualitative behavior of the solutions, underscoring the importance of fractional calculus in modeling memory-dependent phenomena. This study establishes the RKHS method as a robust and promising tool for the numerical analysis of fractional systems. Future work may extend this framework to incorporate other types of fractional derivatives, higher-dimensional problems, and systems with more intricate initial and boundary conditions.

Résumé

Cette thèse développe une méthode numérique efficace pour résoudre des systèmes intégralo-différentiels fractionnaires faisant intervenir la dérivée fractionnaire de Caputo.Lapproche proposée utilise la méthode de lespace de Hilbert à noyau reproducteur (RKHS) pour construire des solutions approchées sous forme de séries uniformément convergentes, garantissant à la fois une grande précision et une convergence rapide. Des exemples numériques illustrent lefficacité et la précision de la méthode RKHS dans le traitement de systèmes fractionnaires complexes. Les résultats mettent en évidence linfluence significative de lordre de la dérivée fractionnaire sur le comportement qualitatif des solutions, soulignant ainsi limportance du calcul fractionnaire dans la modélisation de phénomènes à mémoire. Cette étude établit la méthode RKHS comme un outil robuste et prometteur pour lanalyse numérique des systèmes fractionnaires. Les travaux futurs pourraient étendre ce cadre à dautres types de dérivées fractionnaires, à des problèmes en dimensions supérieures, ainsi quà des systèmes avec des conditions initiales et aux limites plus complexes.

الملخص

تطور هذه المذكرة طريقة عددية فعالة لحل الانظمة التكاملية التفاضلية الكسرية التي تتضمن مشتقة كابوتو الكسرية لبناء حلول تقريبية تستخدم الطريقة المقترحة طريقة فضاء هلبرت ذي النواة المستنسخة كسلاسل متقاربة بشكل منتظم مما يضمن دقة عالية و سريعة في التقارب . تظهر الامثلة العددية فعالية و دقة في معالجة الانظمة الكسرية المعقدة . تكشف النتائج عن التاثير الكبير لرتبة المشتقة الكسرية على طريقة السلوك النوعي للحلول مما يبرز اهمية الحسابات الكسرية في نمذجة الظواهر التي تعتمد على الذاكرة .تمثل اداة قوية و واعدة للتحليل العددي لانظمة الكسرية .تمكن هذه الدراسة من توسيع العمل المستقبلي ليشمل انواع اخرى من المشتقات الكسرية و مسائل ذات ابعاد اعلى و انظمة ذات شروط اولية و حدودية اكثر تعقيدا.

Keywords: Fractional integro-differential systems; Caputo fractional derivative; Reproducing Kernel Hilbert Space method; Numerical analysis.

Introduction

Systems of fractional differential equations (FDEs) have gained significant importance and popularity over the past few decades due to their demonstrated applications across numerous, seemingly diverse, fields of science and engineering. This growing interest stems from the theory of fractional calculus, which deals with differentiation and integration of non-integer (non-zero) order and has attracted attention not only among mathematicians but also physicists and engineers for its wide-ranging relevance. Fractional derivatives allow for more realistic modeling of physical phenomena by accounting not only for the current state at an instant in time but also for the entire history of the system. For example, fractional derivatives have been successfully applied to model nonlinear earthquake oscillations and fluid-dynamic traffic flow, overcoming deficiencies inherent in classical continuum assumptions. Since most nonlinear fractional equations do not admit exact analytic solutions, approximation and numerical techniques are necessary to analyze these complex systems (Miller and Ross, 1993; Momani, 2005; Momani and Al-Khaled, 2005; Momani and Shawagfeh, 2006; Momani and Odibat, 2007; Podlubny, 1999).

Fractional integro-differential equations (FIDEs) have gained substantial attention due to their critical role in accurately modeling a wide range of complex physical phenomena, such as heat conduction, radiative transfer, equilibrium processes, viscous elasticity, and fracture mechanics (Arqub, 2017; Atangana Baleanu, 2016; Bushnaq, Maayah, Momani, Alsaedi, 2014; Caputo Fabrizio, 2015; Carpinteri Mainardi, 2014). However, obtaining exact solutions for most FIDE systems remains challenging or impossible. To address this, various efficient approximate and numerical techniques have been developed, includ-

ing the decomposition method, homotopy analysis, iterative refinement approaches, and Chebyshev spectral methods. These methods have significantly advanced the capacity to analyze and simulate fractional models, as extensively discussed in the literature (Arqub, 2017; Atangana Baleanu, 2016; Bushnaq et al., 2014; Caputo Fabrizio, 2015; Carpinteri Mainardi, 2014).

The theory of reproducing kernels, introduced in the early 20th century for solving boundary value problems of harmonic and biharmonic functions, has become a versatile tool in numerical analysis, machine learning, and engineering . The Reproducing Kernel Hilbert Space (RKHS) method offers accurate numerical solutions for various integral and differential operators, as detailed in works by Cui and Lin (2008), Al-Smadi (2011), and others.

Caputos fractional derivative notably enhanced this framework by enabling realistic modeling of dynamical systems with memory effects. His contributions have improved RKHS-based numerical methods, providing higher accuracy and stability.

RKHS methods are advantageous because they achieve high accuracy with minimal computational effort, avoid discretization errors, and are broadly applicable across mathematical, physical, and engineering problems.

In this thesis, we extend the application of the cloned kernel Hilbert space method to provide a symbolic approximate solution for a class of FIDE systems. The method relies on presenting analytical and approximate solutions as chains in appropriate spaces, ensuring the accuracy and efficiency of the proposed solutions.

The proposed solution method is based on the construction of an orthogonal basis from the cloned kernel functions, where the asymptotic basis is constructed to generate solu-

tions and represent them as sequences. An asymptotic analysis of the method is presented to ensure its accuracy and efficiency. In addition, numerical examples are presented to demonstrate the effectiveness and applicability of the approach. Analytical and approximate solutions are presented as sequences in the appropriate spaces.

This thesis is organized into four chapters.

- **Chapter 1** introduces foundational concepts in fractional calculus and functional analysis essential for understanding the subsequent work.
- **Chapter 2** presents the RKHS and the related function spaces used to construct the kernel functions. It also includes solution representations and key theoretical results based on RKHS theory.
- **Chapter 3** investigates systems of FIDES of order $0 < \alpha \leq 2$, which are instrumental in modeling processes with memory and non-local effectscommon in many physical and engineering applications.
- **Chapter 4** concludes the thesis with final remarks and potential directions for future research.

CHAPTER 1

Background Materials and Preliminaries.

1.1 Basic Principles of Functional Analysis.

In this chapter, basic definitions and results in functional analysis will be presented and some basic properties of fractional calculus

Definition 1.1.1. *On a vector space X , a norm is a function $\|.\| : X \to [0, \infty)$ that satisfies:*

(i) $\|x\| = 0$ *if and only if* $x = 0$

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$

(iii) $\|x + y\| \leq \|x\| + \|y\|$

for every x,$y \in X$ and $\alpha \in \mathbb{F}$ where $\mathbb{F}$ is a field (complex or real)

Definition 1.1.2. *A vector space with a norm defined on it is called a normed space X.*

Definition 1.1.3. **(i)** *If the sequence (x_n) in the normed space X contains X , then it is convergent such that:*

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

Then we will write $x_n \to x$ and call x a limit of (x_n).

(ii) *If for all $\varepsilon > 0$ there is an N such that*

$$\|x_m - x_n\| < \varepsilon \quad \text{for all } m, n > N.$$

The sequence (x_n) in a normed space X is a Cauchy sequence

(iii) *If all Cauchy sequence (x_n) converges in X , then the moduli space X is complete. A complete normed space is called a Banach space.*

Definition 1.1.4. *The vector space X is called the inner product space if $\langle .,. \rangle : X \times X \to \mathbb{F}$, where for each $x, y, z \in X$ and $\alpha \in \mathbb{F}$ we have:*

(i) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(ii) $\langle \alpha x, z \rangle = \alpha \langle x, z \rangle$

(iii) $\langle x, z \rangle = \overline{\langle z, x \rangle}$

(iv) $\langle x, x \rangle \geq 0$ *and* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$

An inner product on X defines a norm on X that is given by $\|x\| = \sqrt{\langle x, x \rangle}$.

A complet inner product space is called a hilbert space

Definition 1.1.5. *A pre-Hilbert space "inner product space"* H *is a vector space with an inner product defined on it.*

Definition 1.1.6. ***(Cauchy-Schwartz inequality).*** *Let* H *be a pre-Hilbert space. Then*

$$\forall x, y \in \mathrm{H}, \ \ |\langle x, y \rangle| \leq \|x\| \|y\|.$$

The inequality is equal if x and y are linearly dependent.

Definition 1.1.7. *Suppose X and Y are two vector spaces on the same field $\mathbb{F}$ and let $\mathcal{D}(T)$ be a subspace of X. A map $T : \mathcal{D}(T) \subset X \to Y$ so that*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

is said to be a linear operator. If $\mathcal{D}(T) = X$, then, and only then, we write $T : X \to Y$.

Definition 1.1.8. *A linear operator $T : X \to Y$ between two normed spaces X and Y is*

said to be bounded if there is a real number $c \to 0$ *such that for all* $x \in \mathcal{D}(T)$

$$\|Tx\| \leq c\,\|x\|$$

Definition 1.1.9. *Assume that* $T : \mathcal{D}(T) \to Y$ *an operator, which is not necessarily linear, where* $\mathcal{D}(T) \subset X$ *and* X, Y *are normed spaces the operator* T *is continuous at an* $x_0 \in \mathcal{D}(T)$ *if for all* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $\|Tx - Tx_0\| < \varepsilon$ *such that* $\|Tx - Tx_0\| < \varepsilon$ *for every* $x \in \mathcal{D}(T)$ *satisfying* $\|x - x_0\| < \delta$.

T *is continuous if* T *is continuous at all* $x \in \mathcal{D}(T)$

Theorem 1.1.10. *Let* $T : \mathcal{D}(T) \to Y$ *is a linear operator, in which* $\mathcal{D}(T) \subset X$ *and* X, Y *are two normed spaces. Then:*T *will be continuous if and only if* T *is bounded.*

Definition 1.1.11. *A linear operator whose domain in vector space* Y *and whose range in the scalar field* $\mathbb{F}$ *of* y *is a linear function* f *and hence* $f : Y \longrightarrow \mathbb{F}$.

A bounded linear functional f *is a bounded linear operator. Hence there exists a real number* $c > 0$ *so that for every* $y \in \mathcal{D}(f)$, $|f(y)| \leq c\,\|y\|$.

Theorem 1.1.12. ***(Riesz's Theorem)*** *Each bounded linear functional* g *on a Hilbert space* H *can be represented in terms of the inner product, namely,* $g(x) = \langle x, z\rangle$ *where* z *depends on* g *is uniquely determined by* g *and has norm* $\|z\| = \|g\|$.

Definition 1.1.13. *Suppose that* $T : H_1 \longrightarrow H_2$ *is a bounded linear operator,in which* H_1 *and* H_2 *are Hilbert spaces. hence the Hilbert-adjoint operator* T^* *of* T *is the operator*

$$T^* : H_2 \longrightarrow H_1$$

so that for every $x \in H_1$ *and* $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^* y\rangle$$

suppose $T : H \longrightarrow H$ *then* T *is self-adjoint if* $T = T^*$.

Definition 1.1.14. *Let* Ω *be a domain in* $\mathbb{R}^n$, *with* $1 \leq p < \infty$. *We mark by* $L^p(\Omega)$ *the class of all measurable functions* u, *defined on* Ω, *with* $\int_\Omega |u(x)|^p \, dx < \infty$.
$L^p(\Omega)$ *is Banach spaces with respect to the norms* $\|u\| := \left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}}$.

Definition 1.1.15. *A function* $v : [a, b] \longrightarrow \mathbb{R}$ *is said to be absolutely continuous (Abs. C), if for every positive* ε, *there exists a positive* δ *such that for any finite set of disjoint intervals* $(x_1, y_1), (x_2, y_2), ..., (x_k, y_k) (\subset [a, b])$ *with* $\sum_{j=1}^{k} |y_j - x_j| < \delta$, *then*

$$\sum_{j=1}^{k} |v(y_j) - v(x_j)| < \varepsilon$$

Theorem 1.1.16. ***(Fundamental theorem of Lebesgue integral calculus)*** *suppose that* $v : [c, d] \longrightarrow \mathbb{R}$ *is a function. Then* v *is strictly continuous if and only if there is a function* $u \in L^1[c, d]$ *so that* $v(x) = v(c) + \int_c^x u(t) \, dt$, $\forall x \in [c, d]$.

1.2 Fundamentals of Fractional Calculus

Many researchers gave definitions that fit the concept of a non-integer order integral or derivative. The most famous of these definitions are the Riemann-Liouville and Grünwald Letnikov definitions (Oldham, 1974; Podlubny, 1999). Understanding of definitions and use of fractional calculus needs knowledge of special functions.

1.2.1 Special Functions

1. **The Gamma Function**

Euler's Gamma function is one of the common functions utilized in the fractional order differential equations. The definition of the Gamma function $\Gamma(x)$ can be defined as follows it is written in the form

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-1} dt, \quad x \in \mathbb{R}^+ \tag{1.2.1}$$

one can show that the gamma function satisfies the functional equation using integration by parts:

$$\Gamma(x+1) = x\Gamma(x), \ x \in \mathbb{R}^+$$

Also, we note that $\Gamma(1) = \int_0^\infty e^{-t} dt = 1$.

The general property of the Gamma function can be obtained

$$\Gamma(x+1) = x\Gamma(x) = x(x-1)! = x!, \ x \in \mathbb{N}$$

The Gamma function has also a representation by limit :

$$\Gamma(x) = \lim_{n\to\infty} \frac{n! n^x}{x(x+1)(x+2)...(x+n)} \tag{1.2.2}$$

The proof of the Equation(1.2.2) can be found in (Podlubny, 1999)

Some useful relationships of the Gamma function are

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

Thus, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

2. **The Beta Function**

The Beta function is crucial for calculating the fractional derivatives of the power function. It is characterized by the two-parameter integral

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \quad x,y \in \mathbb{R}^+ \tag{1.2.3}$$

It is also possible to define the beta function using the gamma function:

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},\ x,y \in \mathbb{R}^+$$

The symmetric property of the beta function is illustrated as follows $B(x,y) = B(y,x)$.

1.2.2 Fractional Integration and Differentiation

1.2.2.1 Riemann-Liouville Integrals

Definition 1.2.1. ***(Diethelm, 2010)*** *Suppose $\alpha \in \mathbb{R}^+$. The operator J_a^α, defined on $L^1[a,b]$ by*

$$J_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\,dt \tag{1.2.4}$$

for $a \leq x \leq b$, is called the Riemman-Liouville fractional integral operator of order α.

For $\alpha = 0$, we put $J_\alpha^0 = I$, the identity operator.

One of the most important properties of integral operator is also correct in fractional integral as follows:

Theorem 1.2.2. *Let $\alpha, \beta \geq 0$ and $f \in L^1[a,b]$. Then,*

$$J_a^\alpha J_a^\beta f = J_a^{\alpha+\beta} f$$

holds almost anywhere on $[a,b]$. If in addition $f \in C[a,b]$or $\alpha + \beta \geq 1$, then the above identity holds everywhere on $[a,b]$.

Corollary 1.2.3. *Let $\alpha, \beta \geq 0$ and $f \in L^1[a,b]$. Then,*

$$J_a^\alpha J_a^\beta f = J_a^\beta J_a^\alpha f$$

Proof : look (Diethelm, K. 2010). □

Example 1.2.4. *let $f(x) = (x-a)^c$ for some $c > -1$ and $\alpha > 0$. Then,*

$$J_a^\alpha f(x) = \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)} (x-a)^{\alpha+c} \tag{1.2.5}$$

Proof : The derivation is straightforward:

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (t-a)^c (x-t)^{\alpha-1} dt \qquad \text{let } t = a + s(x-a) \\ &= \frac{1}{\Gamma(\alpha)} (x-a)^{\alpha+c} \int_0^1 s^c (1-s)^{\alpha-1} ds \\ &= \frac{\Gamma(c+1)}{\Gamma(\alpha+c+1)} (x-a)^{\alpha+c} \end{aligned}$$

□

Definition 1.2.5. ***(Diethelm, 2010)*** *Let* $\alpha \in \mathbb{R}^+$ *and* $m-1 < \alpha < m$. *The operator* D_a^α, *defined by the formula :*

$$D_a^\alpha f = D^m J_a^{m-\alpha} f = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^m \int_a^x (x-t)^{m-\alpha-1} f(t)\, dt \tag{1.2.6}$$

in order to $a \leq x \leq b$, *is called the Riemman-Liouville fractional differential operator of order* α

For $\alpha = 0$, *we set* $D_a^0 = I$, *the identity operator.*

As a first consequence of this definition we note, that if $\alpha \in \mathbb{N}$ *the operator* D_a^α *in Definition* (1.2.5) *coincides with the classical differential operator* D^α

Lemma 1.2.6. *Let* $\alpha \in \mathbb{R}^+$ *and let* $n \in \mathbb{N}$ *so that* $n > \alpha$. *Then,*

$$D_a^\alpha = D^n J_a^{n-\alpha}.$$

Proof : The assumption on n means that $n > [\alpha]$. and so on,

$$D^n J_a^{n-\alpha} = D^{[\alpha]} D^{n-[\alpha]} J_a^{n-[\alpha]} J_a^{[\alpha]-\alpha} = D^{[\alpha]} J_a^{[\alpha]-\alpha} = D_a^\alpha$$

□

Example 1.2.7. *Let* $f(x) = (x-a)^c$ *for some* $c > -1$ *and* $\alpha > 0$. *Then,*

$$D_a^\alpha f(x) = \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} \tag{1.2.7}$$

Proof : Let $n \in \mathbb{N}$ so that $n = [\alpha]$. Thus,

$$\begin{aligned} D_a^\alpha f(x) &= D_a^\alpha (x-a)^c \\ &= D^n J_a^{n-\alpha} (x-a)^c \\ &= D^n \left(\frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} (x-a)^{n-\alpha+c} \right) \\ &= \frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} D^n (x-a)^{n-\alpha+c} \\ &= \frac{\Gamma(c+1)}{\Gamma(n-\alpha+c+1)} \frac{\Gamma(n-\alpha+c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} \\ &= \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)} (x-a)^{c-\alpha} \end{aligned}$$

The property $D_a^\alpha D_a^\beta f = D_a^{\alpha+\beta} f = D_a^\beta D_a^\alpha f$ is not satisfied in both equalities as seen in the following example □

Example 1.2.8. a. *Let* $f(x) = x^{-\frac{1}{2}}$ *and* $\alpha = \beta = \frac{1}{2}$. *Then,*

$$\begin{aligned} D_0^\alpha f(x) &= D_0^\beta f(x) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma\left(-\frac{1}{2}+1-\frac{1}{2}\right)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} \\ &= \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(0)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} = 0 \end{aligned}$$

and then also $D_0^\alpha D_0^\beta f(x) = 0$,

$D_0^{\alpha+\beta} f(x) = D^1 f(x) = \frac{-x^{-\frac{3}{2}}}{2}$,

b. *Let* $f(x) = x^{\frac{1}{2}}$ *and* $\alpha = \frac{1}{2}$ *and* $\beta = \frac{3}{2}$. *Then,*

$$\begin{aligned} D_0^\alpha f(x) &= \frac{\Gamma\left(\frac{1}{2}+1\right)}{\Gamma\left(\frac{1}{2}+1-\frac{1}{2}\right)} (x-0)^{-\frac{1}{2}-\frac{1}{2}} \\ &= \frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma(1)} = \frac{\sqrt{\pi}}{2} \end{aligned}$$

and $D_0^{\beta} f(x) = 0$ *this This means* $D_0^{\alpha} D_0^{\beta} f(x) = 0$

however $D_0^{\beta} D_0^{\alpha} f(x) = \frac{-x^{-\frac{3}{2}}}{2} = D_0^{\alpha+\beta} f(x)$

We now mention some relationships between the Riemann-Liouville integral and the differential operators :

Theorem 1.2.9. *Let* $\alpha \geq 0$. *Then for each* $f \in L^1[a,b]$,

$$D_a^{\alpha} J_a^{\alpha} f = f \tag{1.2.8}$$

Just about everywhere.

Proof : The case $\alpha = 0$ is intuitive.

For $\alpha > 0$,let $n = [\alpha]$. Then,

$$D_a^{\alpha} J_a^{\alpha} f(x) = D^n J_a^{n-\alpha} J_a^{\alpha} f(x) = D^n J_a^n f(x) = f(x).$$

□

We have shown that D_a^{α} is the left inverse of J_a^{α}. Of course, we cannot claim that it is the right inverse. More precisely, we have the following theorem.

Theorem 1.2.10. *Let* $\alpha \geq 0$ *and* $n-1 < \alpha \leq n$. *Suppose that* f *is so that* $J_a^{n-\alpha} f \in A^n[a,b]$. *Then,*

$$J_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \lim_{z \to a^+} D^{n-k-1} J_a^{n-\alpha} f(z) \tag{1.2.9}$$

In particular, for $0 < \alpha < 1$ *we have*

$$J_a^{\alpha} D_a^{\alpha} f(x) = f(x) - \frac{(x-a)^{\alpha-1}}{\Gamma(\alpha)} \lim_{z \to a^+} J_a^{1-\alpha} f(z) \tag{1.2.10}$$

Proof : look (Diethelm, K. 2010). □

1.2.2.2 Caputo Fractional Derivative

In 1967, a new definition of the rational derivative was published by M. Caputo named the fractional derivative where he showed its relation to the fractional derivative Riemann Liouville .

Definition 1.2.11. *Suppose* $\alpha \in \mathbb{R}^+$ *and* $n-1 < \alpha \leq n$. *The operator* D_{*a}^{α}, *characterized by :*

$$D_{*a}^{\alpha} f(x) = J_a^{n-\alpha} D^n f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(t)\, dt \tag{1.2.11}$$

for $a \leq x \leq b$, *is named the Caputo differential operator of order* α

Example 1.2.12. *We got* $D^{\alpha} C = 0$, *for the Caputo derivative, in which* C *is a constant, and*

$$D^{\alpha} x^c = \begin{cases} 0, & c \leq \alpha - 1, \\ \frac{\Gamma(c+1)}{\Gamma(c-\alpha+1)} x^{c-\alpha}, & c > \alpha - 1. \end{cases}$$

Similar to integer order differentiation, Caputo fractional differentiation is a linear operation:

$$D^{\alpha}(\gamma f(x)+\mu g(x))=\gamma D^{\alpha}f(x)+\mu D^{\alpha}g(x).$$

Example 1.2.13. *Let $\alpha \geq 0$ and $n-1<\alpha \leq n$ and $f(x)=(x-a)^{c}$ to some $c \geq 0$. .Further,*

$$D_{*a}^{\alpha}f(x)=\begin{cases} 0 & \text{if } c \in \{0,1,2,...,n-1\} \\ \frac{\Gamma(c+1)}{\Gamma(c+1-\alpha)}(x-a)^{c-\alpha} & \text{if } c \in \mathbb{N} \text{ and } c \geq n \\ & \text{or } c \notin \mathbb{N} \text{ and } c > n-1 \end{cases}$$

In particular, we observe that the two operators have distinct kernels and that their domains are different when comparing this result with Example (1.2.7), where the Riemann Liouville differential operator was applied to the function f . The Caputo derivative is thus a left inverse of the Riemann-Liouville integral when taking into account the interplay between Riemann-Liouville integrals and Caputo differential operators:

Theorem 1.2.14. *If f is continuous and $\alpha>0$, then*

$$D_{*a}^{\alpha}J_{a}^{\alpha}f=f$$

In addition, the Caputo derivative also is not the right inverse of the Riemann-Liouville integral

Theorem 1.2.15. *Let $n-1<\alpha \leq n$ and $\alpha \geq 0$ and, $f \in A^{n}[a,b]$. Then,*

$$J_{a}^{\alpha}D_{*a}^{\alpha}f(x)=f(x)-\sum_{k=0}^{n-1}\frac{D^{k}f(a)}{k!}(x-a)^{k}. \tag{1.2.12}$$

Proof : look (Diethelm, K. 2010). □

we want in this thesis to solve the fractional differential equation (FDE) of the form:

$$D_{*a}^{\alpha}u\left(x\right)=f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right),\quad a\leq x\leq b,\quad m-1<\alpha\leq m \tag{1.2.13}$$

subject to conditions

$$\begin{aligned} u^{(i)}\left(a\right) &= c_i,\quad i=0,1,2,...,r-1, \\ u^{(i)}\left(b\right) &= d_i,\quad i=r,r+1,...,m-1. \end{aligned} \tag{1.2.14}$$

where $\alpha, c_i, 0\leq i\leq r-1$ and d_i, $r\leq i\leq m-1$ are real constants, $u\left(x\right)$ is an unknown function that needs to be found, and $f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right.$ is either a linear or nonlinear function depending on the problem being solved. D_{*a}^{α} represents the fractional Caputo derivative of order . We now obtain

$$u\left(x\right)-\sum_{k=0}^{m-1}u^{(k)}\left(a\right)\frac{\left(x-a\right)^{k}}{k!}=F\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right),$$

by applying the operator J_a^{α} a to both sides of the equation (1.2.13) and then using (1.2.12). $F\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right)=J_a^{\alpha}\left(f\left(x,u\left(x\right),u^{'}\left(x\right),...,u^{(m-1)}\left(x\right)\right)\right)$.

Theorem 1.2.16. *Let $\alpha\geq 0$ and $n=\lceil\alpha\rceil$. Suppose that f is so that both D_a^{α} and ${}^{C}D_a^{\alpha}$ exist. hence,*

$$({}^{C}D_a^{\alpha}f)(t)=D_a^{\alpha}f(t)-\sum_{j=0}^{n-1}\frac{(t-a)^{j-\alpha}}{\Gamma(j+1-\alpha)}f^{(j)}(a).$$

Specifically, if $0<\alpha<1$, then

$$({}^{C}D_a^{\alpha} f)(t) = D_a^{\alpha} f(t) - \frac{(t-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a).$$

CHAPTER 2

Reproducing Kernel Hilbert Space

2.1 Concept of Reproducing Kernel Hilbert Space

Definition 2.1.1. *Let E be an arbitrary set and K be a kernel on X, that is,*

$K : E \times E \to \mathbb{C}$ The kernel K is called Hermitian if for any finite set of points $\{x_1, ..., x_n\} \subseteq E$ and any complex numbers $\alpha_1, ..., \alpha_n$ we have $\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j K(x_i, x_j) \in \mathbb{R}$, and K is called positive definite if $\sum_{i,j=1}^{n} \overline{\alpha_i}\alpha_j K(x_i, x_j) \geq 0$.

Definition 2.1.2. ***(Aronszajn, 1950)*** *Let H be a Hilbert space of functions $g : E \longrightarrow \mathbb{F}$ on a nonempty set E. A function $K : E \times E \longrightarrow \mathbb{C}$ is a reproducing kernel of H if the following are satisfied:*

1. *$K(., x) \in H$ for all $x \in E$.*
2. *$\langle g, K(., x)\rangle = g(x)$ for all $g \in H$ and for all $x \in E$.*

The last condition is called "The reproducing property",

The first condition can be written as: for all $x \in X$, $k_x(y) = k(x, y)$ as a function of y belongs to H, $y \in E$. So applying the reproducing property to the function k_x at y, we get:

$$k_x(y) = \langle k_x, k_y \rangle, \quad \text{for } x, y \in E.$$

Consequently, for all $x \in E$ we obtain $\|k_x\|^2 = \langle k_x, k_x \rangle = k\langle x, x \rangle$.

Definition 2.1.3. *A Hilbert space H of functions on a set E is called a reproducing kernel Hilbert space (RKHS) if there exists a reproducing kernel k of H.*

The Hilbert space with reproducing kernel k is denoted by H_k. Moreover, $\langle ., . \rangle_{H_k}$, $\|.\|_{H_k}$ denote the inner product and the norm, respectively.

Theorem 2.1.4. ***(Aronszajn, 1950)*** *If a Hilbert space H of functions on a set E recognizes a reproducing kernel, then the reproducing kernel $k(x,y)$is uniquely determined by the Hilbert space H.*

Proof : Let $k(x,y)$ be a reproducing kernel of H. Suppose that there exists another kernel $R(x,y)$of H. Then, for all $x \in E$, applying the reproducing property for k and R we get:

$$\begin{aligned} \|k_x - R_x\|^2 &= \langle k_x - R_x, k_x - R_x\rangle = \langle k_x - R_x, k_x\rangle - \langle k_x - R_x, R_x\rangle \\ &= (k_x - R_x)(x) - (k_x - R_x)(x) = 0 \end{aligned}$$

Hence $k_x = R_x$, that is, $k_x(y) = R_x(y)$, for all $y \in E$. This means that $k(x,y) = R(x,y)$ for all $x, y \in E$ □

Theorem 2.1.5. *For a Hilbert space H of functions on E, there exists a reproducing kernel K for H if and only if for every x of E, the Dirac functional $\delta_x : g \to g(x)$ is a bounded linear functional on H.*

Theorem 2.1.6. *The reproducing kernel $k(x,y)$ of a reproducing kernel Hilbert space H is a positive definite kernel.*

Proof : we have

$$0 \leq \left\| \sum_{i=1}^{n} \alpha_i k_{x_i} \right\|^2 = \left\langle \sum_{i=1}^{n} \alpha_i k_{x_i}, \sum_{i=1}^{n} \alpha_i k_{x_i} \right\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} \langle k_{x_i}, k_{x_j}\rangle = \sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} k\left(_{x_i,x_j}\right)$$

$\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_i \overline{\alpha_j} k\left(_{x_i,x_j}\right) \geq 0$ □

Remark 2.1.7. *Let H be a RKHS and its kernel $k(x,y)$ on X, then, for every $x, y \in X$, we have*

(i) $|k(x,y)|^2 \leq k(x,x)\,k(y,y)$

(ii) *Let* $x_0 \in E$*.hence the following are equivalent:*

(a) $k(x_0,x_0)=0$.

(b) $k(x_0,y)=0$ *for every* $y \in E$.

(c) $f(x_0)=0$ *for every* $g \in H$

Proof : **(i)** By Schwartz Inequality in H we have

$$\begin{aligned} |k(x,y)|^2 &= |\langle k_x, k_y\rangle|^2 \leq (\|k_x\|\,\|k_y\|)^2 = \|k_x\|^2\,\|k_y\|^2 \\ &= \langle k_x, k_x\rangle\,\langle k_y, k_y\rangle = k(x,x)\,k(y,y) \end{aligned}$$

(ii) From (i) it follows that $|k(x_0,y)|^2 \leq k(x_0,x_0)\,k(y,y) = 0$.then $k(x_0,x_0)=0$ is equivalent with $k(x_0,y)=0$ for each $y \in X$ if and only if $f(x_0)=0$ for all $f \in H$

□

Theorem 2.1.8. *There exists a uniquely determined Hilbert space* H_k *of functions on* X*, For any positive definite kernel* $k(x,y)$ *on* X*, admitting the reproducing kernel* $k(x,y)$.

Proof : look (Aronszajn, 1950). □

Theorem 2.1.9. *Each sequence of functions* $f(x_n)_{n\geq 1}$ *which converges strongly to a function* g *in* $H_k(X)$*, also converges in the pointwise sense, i.e,* $\lim_{n\to\infty} f_n(x) = f(x)$*, for every point* $x \in E$*. Moreover, this convergence is uniform on every subset of* E *on which* $x \longmapsto k(x,x)$ *is bounded.*

Proof : For $x \in E$, utilizing the reproducing property and Schwartz Inequality,

$$\begin{aligned} |f(x) - f_n(x)| &= |\langle f, k_x\rangle - \langle f_n, k_x\rangle| \\ &= |\langle f - f_n, k_x\rangle| \\ &\leq \|f - f_n\| . \|k_x\| \\ &= \|f - f_n\| . k(x,x)^{1/2} \end{aligned}$$

Thus, for any point $x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$.

Moreover it is clear from the above inequality that this convergence is uniform on every subset of E on which $x \longmapsto k(x,x)$ is bounded □

Definition 2.1.10. *As follows we have defined the function space* $W_2^m[a,b]$*:*

$$W_2^m[a,b] = \left\{u : u^{(i)} \text{ is Abs.C, } j = 1,2,...,m-1, u^{(m)} \in L^2[a,b]\right\} \tag{2.1.1}$$

The norm and the inner product in the function space $W_2^m[a,b]$ *are defined as follows successively, for any functions* $u_1(x), u_2(x) \in W_2^m[a,b]$

$$\langle u_1, v_1\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u_1^{(i)}(a)\, u_2^{(i)}(a) + \int_a^b u_1^{(m)}(x)\, u_2^{(m)}(x)\, dx \tag{2.1.2}$$

$$\|u\|_{W_2^m[a,b]} = \sqrt{\langle u_1, u_1\rangle_{W_2^m[a,b]}} \tag{2.1.3}$$

Theorem 2.1.11. *The function space* $W_2^m[a,b]$ *is a Hilbert Space.*

Proof : Look (Zhang, 2012). □

Theorem 2.1.12. *The function space* $W_2^m[a,b]$ *is a reproducing kernel space. Its meaning, for every fixed* $x \in [a,b]$ *and any* $u(y) \in W_2^m[a,b]$*, there exists* $k_x(y) \in W_2^m[a,b]$*,*

$y \in [a,b]$ such that $\langle u(y), k_x(y)\rangle = u(x)$, and $k_x(y)$ is called the reproducing kernel function of space $W_2^m[a,b]$.

Proof : Look (Aronszajn, 1950). □

2.2 Reproducing Kernel Function

In this section, we will look at The reproducing kernel function and find the formulas for its algebraic expressions in the space $W_2^m[a,b]$. A piecewise polynomial of degree $2m-1$ can be used to represent these expressions. The reproducing kernel function has a unique representation. Further, we will give some corollaries and remarks related to these kernel functions. At the end of this section, several examples of such kernel functions are given in $W_2^1[a,b]$.

Now, let's find out the expression form of the reproducing kernel function $k_x(y)$ in the space $W_2^m[a,b]$

let $k_x(y)$ is the reproducing kernel function of the space $W_2^m[a,b]$, then for every fixed $x \in [a,b]$ and any $u(y) \in W_2^m[a,b]$, $y \in [a,b]$ we have $\langle u(y), k_x(y)\rangle = u(x)$. Based on (2.1.2) and (2.1.3), we have

$$\langle u(y), k_x(y)\rangle_{W_2^m[a,b]} = \sum_{i=0}^{m-1} u^{(i)}(a)\, k_x^{(i)}(a) + \int_a^b u^{(m)}(y)\, k_x^{(m)}(y)\, dy. \tag{2.2.1}$$

Applying the integration by parts for the second scheme of the right-hand of (2.2.1), we obtain

$$\int_a^b u^{(m)}(y)\, k_x^{(m)}(y)\, dy = \sum_{i=0}^{m-1} (-1)^i\, u^{(m-i-1)}(y)\, k_x^{(m+i)}(y)\,|_{y=a}^{b} + \int_a^b (-1)^m\, u(y)\, k_x^{(2m)}(y)$$

Suppose $j = m - i - 1$, the first term of the right-hand side of the above formula can be rewritten as

$$\sum_{i=0}^{m-1} (-1)^i u^{(m-i-1)}(y) k_x^{(m+i)}(y) |_{y=a}^{b} = \sum_{j=0}^{m-1} (-1)^{m-j-1} u^{(j)}(y) k_x^{(2m-j-1)}(y) |_{y=a}^{b}$$

Equation (2.2.1) was simplified to become

$$\begin{aligned} \langle u(y), k_x(y) \rangle_{W_2^m[a,b]} &= \sum_{i=0}^{m-1} u^{(i)}(a) \left(k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) \right) \\ &\quad + \sum_{i=0}^{m-1} (-1)^{m-i-1} u^{(i)}(b) k_x^{(2m-i-1)}(b) + \int_a^b (-1)^m u(y) k_x^{(2m)}(y) dy \end{aligned}$$

As $k_x(y), u(y) \in W_2^m[a,b]$, we have that

$$k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0,\ k_x^{(2m-i-1)}(b) = 0,\ i = 0, 1, ..., m-1.$$

then $\langle u(y), k_x(y) \rangle_{W_2^m[a,b]} = \int_a^b u(y) ((-1)^m k_x^{(2m)}(y)) dy$

Now, for all $x \in [a,b]$, if $k_x(y)$ realizes $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$, where δ is dirace delta function, so $\langle u(y), k_x(y) \rangle_{W_2^m[a,b]} = \int_a^b u(y) \delta(x-y) dy = u(y)$ clearly, $k_x(y)$ is the reproducing kernel of the space $W_2^m[a,b]$.

Hence, $k_x(y)$ is the solution of the following generalized differential equation:

$$\begin{cases} (-1)^m k_x^{(2m)}(y) = \delta(x-y) \\ k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, i = 0, 1, ..., m-1 \\ k_x^{(2m-i-1)}(b) = 0, i = 0, m-1. \end{cases} \tag{2.2.2}$$

Whereas $x \neq y$

$$(-1)^m k_x^{2m}(y) = 0 \tag{2.2.3}$$

According to the boundary conditions (BCs):

$$k_x^{(i)}(a) - (-1)^{m-i-1} k_x^{(2m-i-1)}(a) = 0, k_x^{(2m-i-1)}(b) = 0, i = 0, 1, ..., m-1 \tag{2.2.4}$$

The characteristic equation of Equation (2.2.3) is $\lambda^{2m} = 0$, and their characteristic values are $\lambda = 0$ with $2m$ multiple roots. So, the general solution of Equation (2.2.3) is expressed by:

$$k_x(y) = \begin{cases} \sum_{i=0}^{2m-1} p_i(x) y^i, \ y \leq x; \\ \\ \sum_{i=0}^{2m-1} q_i(x) y^i, \ y > x. \end{cases} \tag{2.2.5}$$

On the other side, since $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$,we have:

$$k_x^{(i)}(x+0) = k_x^{(i)}(x-0), i = 0, 1, ..., 2m-2 \tag{2.2.6}$$

Integrating $(-1)^m k_x^{(2m)}(y) = \delta(x-y)$ from $x-\varepsilon$ to $x+\varepsilon$ with respect to y and let $\varepsilon \to 0$, we have the jump degree of $k_x^{(2m-1)}(y)$ at $y = x$ given by

$$(-1)^m (k_x^{(2m-1)}(x+0) - k_x^{(2m-1)}(x-0)) = 1. \tag{2.2.7}$$

The equations (2.2.6) and (2.2.7) represent $2m$ essential conditions for solving the coefficients $p_i(x)$ and $q_i(x)$,where $i = 0, 1, ..., 2m-2$, in equation (2.2.6). Additionally, equation (2.2.4) provided $2m$ boundary conditions (BCs), meaning that we have a total of $2m$ equations. It is easy to see that these $4m$ equations are linear with respect to

the variables $p_i(x)$ and $q_i(x)$. The unknown coefficients $p_i(x)$ and $q_i(x)$ in equation (2.2.5)can be computed using the Mathematica 8.0 package.

The following corollary gives some important properties of the reproducing kernel $k_x(y)$

Corollary 2.2.1. *The reproducing kernel $k_x(y)$ is symmetric, unique and $k_x(y) \geq 0$, for every fixed $x \in [a,b]$.*

Proof : By the reproducing property, we have,

$k_x(y) = \langle k_x(.), k_y(.)\rangle = \langle k_y(.), k_x(.)\rangle = k_y(x)$

Now, let $k_x(y)$ and $R_x(y)$ be all the reproducing kernel of the space $W_2^m[a,b]$, then

$k_x(y) = \langle k_x(.), R_y(.)\rangle = \langle R_y(.), k_x(.)\rangle = R_y(x)$.

By the symmetry of $R_x(y)$, we have the unique representation of $k_x(y)$. For the last condition, we note that

$k_x(x) = \langle k_x(.), k_x(.)\rangle = \|k_x(.)\|^2 \geq 0$ □

Now we present some expressions of reproducing kernel function in the space $W_2^1[a,b]$ with respect to different norms by using the approaches proposed in this section,

Example 2.2.2. *Consider the space*

$W_2^1[a,b] = \{u : [a,b] \longrightarrow \mathbb{R} : u(x)$ *is Abs.C and* $u'(x) \in L^2[a,b]\}$. *The inner product and the norm in the space $W_2^1[a,b]$ are given by,*

$\langle u_1, u_2\rangle_{W_2^1} = u_1(a)\,u_2(a) + \int_a^b u_1'(y)\,u_2'(y)\,dy$

and

$\|u_1\|_{W_2^1} = \sqrt{\langle u_1, u_1\rangle}$, *where* $u_1(x), u_2(x) \in W_2^1[a,b]$.

To find the reproducing kernel function $k_x(y)$, *we apply integration by parts to see that*

$$\langle u_1, k_x\rangle_{W_2^1} = u_1(a)\,k_x(a) + u_1(y)\,k_x'(y)\,|_{y=a}^{b} - \int_a^b u_1(y)\,k_x''(y)\,dy$$

Since $u_1(y), k_x(y) \in W_2^1[a,b]$, *we have* $k_x(a) - k_x'(a) = 0$ *and* $k_x'(b) = 0$. *Thus, we need to solve the BVP* $-k_x''(y) = \delta(x-y)$ *subject to* $k_x(a) - k_x'(a) = 0$ *and* $k_x'(b) = 0$.

The characteristic equation is $\lambda^2 = 0$, *and the characteristic value is* $\lambda = 0$ *with 2 multiple roots. So,*

$$k_x(y) = \begin{cases} p_1(x) + p_2(x)\,y, & y \le x, \\ \\ q_1(x) + q_2(x)\,y, & y > x, \end{cases}$$

Also, by using equation (2.2.6) *and* (2.2.7) *we have* $k_x(x+0) = k_x(x-0)$ *and* $k_x'(x+0) - k_x(x-0) = -1$,*hence the unknown coefficients* $p_i(x)$ *and* $q_i(x), i = 1, 2$,*can be obtained by solve the following equations*

$$k_x(a) - k_x'(a) = 0$$

$$k_x'(b) = 0$$

$$k_x(x+0) = k_x(x-0)$$

$$k_x'(x+0) - k_x'(x-0) = -1$$

Applying the First Cds : For $y \le x$:

$$k_x(a) = p_1 + p_2 a, \quad k_x'(a) = p_2$$

Thus,

$$(p_1 + p_2 a) - p_2 = 0$$

which simplifies to:

$$p_1 = p_2(1 - a)$$

Applying the Second Cds : For $y > x$:

$$k'_x(b) = q_2$$

Since $k'_x(b) = 0, weobtain : q_2 = 0$

Applying the Third Cds :

$$p_1 + p_2 x = q_1 + q_2 x$$

Substituting $q_2 = 0 : q_1 = p_1 + p_2 x$

Applying the Fourth Cds :

$$k'_x(x + 0) - k'_x(x - 0) = -1$$

$$q_2 - p_2 = -1$$

Since $q_2 = 0, weget : -p_2 = -1 \Rightarrow p_2 = 1$

So,

$p_1 = p_2(1 - a) = (1 - a), \quad p_2 = 1$

$q_1 = p_1 + p_2 x = (1 - a) + x, \quad q_2 = 0$

Thus, the reproducing kernel function is:

$$k_x(y) = \begin{cases} 1 - a + y, & y \leq x \\ \\ 1 - a + x, & y > x \end{cases}$$

Remark 2.2.3. *Yao (2008) proved that the space* $W_2^1[a,b]$ *in Example* (**??**)*is a complete reproducing kernel space and its reproducing kernel function is*

$$k_x(y) = \begin{cases} 1 + y, & y \leq x, \\ \\ \\ 1 + x, & y > x, \end{cases}$$

Example 2.2.4. *Consider the space* $W_2^1[a,b] = \{u : u(x)$ *is Abs.C and* $u'(x) \in L^2[a,b]$ *and* $u(a) = u(b) = 0\}$. *The inner product and the norm in the space* $W_2^1[a,b]$ *are given, respectively, by*

$$\langle u_1, u_2 \rangle_{W_2^1} = \int_a^b u_1'(y)\, u_2'(y)\, dy \text{ and } \|u\| = \sqrt{\langle u, u \rangle}, \text{ where } u(x), u(x) \in W_2^1[a,b].$$

Similarly, as in Ex.(**??**) *we have*

$$R_x(y) = \begin{cases} p_1(x) + p_2(x)\, y, & y \leq x, \\ \\ \\ q_1(x) + q_2(x)\, y, & y > x, \end{cases}$$

the unknown coefficients $p_i(x)$ *and* $q_i(x), i = 1, 2$,*can be obtained by solve the following equations*

1. $R_x(a) = 0$

2. $R_x(b) = 0$

3. $R_x(x+0) = R_x(x-0)$

4. $R_x'(x+0) - R_x'(x-0) = -1$

So the reproducing kernel function is given by

$$R_x(y) = \begin{cases} \frac{(b-x)(a-y)}{a-b}, & y \leq x, \\ \\ \frac{(a-x)(b-y)}{a-b}, & y > x, \end{cases}$$

Remark 2.2.5. *Paulsen (2009) proved that the space $W_2^1[a,b]$ in Example (3.2.2) is a complete reproducing kernel space and its reproducing kernel function is*

$$R_x(y) = \begin{cases} (1-x)\,y, & y \leq x, \\ \\ (1-y)\,x, & y > x, \end{cases}$$

Example 2.2.6. *Consider the space*

$W_2^3[0,1] = \{u : [0,1] \longrightarrow \mathbb{R} : u(x)$ *is Abs.C and* $u'(x) \in L^2[0,1]$, $u''(x) \in L^2[0,1]$ *and* $u(0) = u(1) = 0\}$.

The inner product in the space $W_2^3[0,1]$ are given by,

$$\langle u(x), R_y(x)\rangle\, \omega_3^2 = \sum_{i=0}^{2} u^i(0) R_y^{(i)}(0) + \int_0^1 u^{(3)}(x) R_y^{(3)}\, dx \tag{2.2.8}$$

the Reproducing Kernel Functions is :

$$R_y(x) = \begin{cases} \sum_{i=1}^{6} C_i(y)x^{i-1}, & x < y \\ \sum_{i=1}^{6} d_i(y)x^{i-1}, & x > y \end{cases} \tag{2.2.9}$$

the unknown coefficients $C_i(y) and d_i(y), i = 1, 2, 3, 4, 5, 6 can be obtained by$:

$$d^k R_y^+(y) = d^k R_y^-(y), \quad k = 0, 1, 2, 3, 4 \tag{2.2.10}$$

$$d^5 R_y^+(y) - d^5 R_y^-(y) = -1 \tag{2.2.11}$$

1.Continuity Condition

$$R_y^+(y) = R_y^-(y) \tag{2.2.12}$$

2.Derivatives

$$\sum_{i=1}^{6} C_i(y)y^{i-1} = \sum_{i=1}^{6} d_i(y)y^{i-1}$$

$$\sum_{i=1}^{6}(i-1)C_i(y)y^{i-2} = \sum_{i=1}^{6}(i-1)d_i(y)y^{i-2}$$

$$\sum_{i=1}^{6}(i-1)(i-2)C_i(y)y^{i-3} = \sum_{i=1}^{6}(i-1)(i-2)d_i(y)y^{i-3}$$

$$\sum_{i=1}^{6}(i-1)(i-2)(i-3)C_i(y)y^{i-4} = \sum_{i=1}^{6}(i-1)(i-2)(i-3)d_i(y)y^{i-4}$$

$$\sum_{i=1}^{6}(i-1)(i-2)(i-3)(i-4)C_i(y)y^{i-5} = \sum_{i=1}^{6}(i-1)(i-2)(i-3)(i-4)d_i(y)y^{i-5}$$

$$\sum_{i=1}^{6}(i-1)(i-2)(i-3)(i-4)(i-5)C_i(y)y^{i-6} = \sum_{i=1}^{6}(i-1)(i-2)(i-3)(i-4)(i-5)d_i(y)y^{i-6}$$

Thus, the reproducing kernel function is:

$$R_y(x) = \frac{1}{11520} \cdot 96y^3(10x^2 - 5xy + y^2) + \frac{8}{13x(1-y)}\Big(120x^4 - 4x^4y^3 + x^4y^4 - 5(x^3 - 24)y^4$$

$$+6(x^4 - 5(x^3 - 24)y + 10x(3+x)y^2) + 2y(15x(-120 - 40x + x^3) + 3x^4y + 10(x^3 - 24)y^2 + 5x(3+x)y^3)$$

$$-2 - 15x^4 - 3(600 - 25x^3 + x^4)y + 15(x^3 - 24)y^2 + 20x(3+x)y^3$$

$$-480x^2y^2(x + y + |y - x|) + 60xy(x + y + |y - x|)^3 - 3(x + y + |y - x|)^5\Big)$$

2.3 Description of Reproducing Kernel Method

This part forms an iterative method to determine and calculate the solution of the generalized m^{th}-order BVP equation.

Consider the following general m^{th}-order BVP of the following form :

$$u^{(m)}(x) + a_1(x)\, u^{(m-1)} + ... + a_{m-1}(x)\, u'(x) = F(x, u(x)),\ a \le x \le b, \tag{2.3.1}$$

subject to the BC's

$$\begin{aligned} u^{(i)}(a) &= c_i,\ i = 0, 1, 2, ..., r-1 \\ u^{(i)}(b) &= d_i,\ i = r, r+1, ..., m-1. \end{aligned} \tag{2.3.2}$$

here $a_i(x)$, $i = 1, 2, ..., m-1$, are continuous real-valued functions, $c_i, 0 \le i \le r-1$ and $d_i, r \le i \le m-1$ are real constants $u(x)$, is unknown function to be determined, $u^{(m)}(x)$ denotes the m^{th} derivative of $u(x)$, and $F(x, u(x))$ is a linear or nonlinear according to the problem discussed.

To solve the BVP (2.3.1) and (2.3.2) with RKHS method, We firstly build the reproducing kernel space $W_2^{m+1}[a,b]$ in which every function respects homogeneous BCs of (2.3.2) and later, we use the space $W_2^1[a,b]$. We can derive the inner product and the norm in the space$W_2^{m+1}[a,b]$ from Equations (2.1.2) and (2.1.3), respectively.

Let $k_x(y)$ and $R_x(y)$ be the reproducing kernel functions of the space$W_2^{m+1}[a,b]$ and $W_2^1[a,b]$, respectively. Pick a differential operator $L: W_2^{m+1}[a,b] \longrightarrow W_2^1[a,b]$ such that $Lu(x) = u^{(m)}(x) + a_1(x)u^{(m-1)}(x) + ... + a_{m-1}(x)u'(x)$. We can also justify that L is bounded operator by using the following lemma;

Lemma 2.3.1. *if* $u(x) \in W_2^{m+1}[a,b]$, *then* $|u(x)| \leq M_0 \|u(x)\|_{W_2^{m+1}}$.

Moreover, $\left|u^{(i)}(x)\right| \leq M_i \|u(x)\|_{W_2^{m+1}}$, *where* M_i *are constants,* $i = 1, 2, ..., m$.

Proof : By reproducing property of $k_x(y)$ and Schwartz inequality, also since $k_x^{(i)}(y)$, $i = 0, 1, 2, ..., m$ is uniformly bounded about x and y, we get

$$|u(x)| = \left|\langle u(y), k_x(y)\rangle_{W_2^{m+1}}\right| \leq \|k_x(y)\|_{W_2^{m+1}} \|u(y)\|_{W_2^{m+1}} \leq M_0 \|u(x)\|_{W_2^{m+1}}.$$

From the representation of $k_x(y)$, we can obtain

$$\left|u^{(i)}(x)\right| = \left|\left\langle u(y), k_x^{(i)}(y)\right\rangle_{W_2^{m+1}}\right| \leq \left\|k_x^{(i)}(y)\right\|_{W_2^{m+1}} \|u(y)\|_{W_2^{m+1}} \leq M_i \|u(x)\|_{W_2^{m+1}}.$$

□

Hence, after homogenization of the BC's (2.3.2), the BVP (2.3.1) and (2.3.2) can be converted into the equivalent form as follows:

$$Lu(x) = F(x, u(x)) \quad , a \leq x \leq b; \tag{2.3.3}$$

$$u^{(i)}(a)=0, i=0,1,2,...,r-1; u^{(i)}(b)=0, i=r,r+1,...,m-1. \tag{2.3.4}$$

where $u(x)\in W_2^{m+1}[a,b]$ and $F(x,u)\in W_2^1[a,b]$.

Now, we construct an orthogonal function system of the space $W_2^{m+1}[a,b]$. For a countable dense set $\{x_i\}_{i=1}^{\infty}$ of $[a,b]$, let $\varphi_i(x)=R_{x_i}(x)$, where $R_x(y)$ is the reproducing kernel of $W_2^1[a,b]$. So, from the properties of $R_x(y)$ for every ,$u(x)\in W_2^1[a,b]$, it follows that $\langle u(x),\varphi_i(x)\rangle_{W_2^1}=\langle u(x),R_{x_i}(x)\rangle_{W_2^1}=u(x_i)$. Additionally, let $\psi_i(x)=L^*\varphi_i(x)$ where ,L^* is the adjoint operator of L.

Obviously, $\psi_i(x)\in W_2^{m+1}[a,b]$. In terms of the properties of $k_x(y)$, we have

$$\langle u(x),\psi_i(x)\rangle_{W_2^{m+1}}=\langle u(x),L^*\varphi_i(x)\rangle_{W_2^{m+1}}=\langle Lu(x),\varphi_i(x)\rangle_{W_2^1}=Lu(x_i), \text{ where } i=1,2,...$$

Lemma 2.3.2. *$\psi_i(x)$ can be written as $\psi_i(x)=L_yk_x(y)\mid_{y=x_i}$. The partial symbol y in the operator L denoted that the operator L holds the function of y.*

Proof : It is clear from the above hypothesis that

$$\begin{aligned}\psi_i(x) &= L^*\varphi_i(x)=\langle L^*\varphi_i(y),k_x(y)\rangle_{W_2^{m+1}}\\ &= \langle\varphi_i(y),Lk_x(y)\rangle_{W_2^1}=L_yk_x\mid_{y=x_i}\end{aligned}$$

□

Theorem 2.3.3. *Assume that the inverse operator L^{-1}in Equation (2.3.3) exist. And, so on $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete function system of the space $W_2^{m+1}[a,b]$,if $\{x_i\}_{i=1}^{\infty}$ is dense in $[a,b]$.*

Proof : For every fixed $u(x) \in W_2^{m+1}[a,b]$,let $\langle u(x), \psi_i(x)\rangle = 0, i = 1, 2, ...,$ that is

$$\langle u(x), \psi_i(x)\rangle_{W_2^{m+1}} = \langle u(x), L^*\varphi_i(y)\rangle_{W_2^{m+1}} = \langle Lu(x), \varphi_i(x)\rangle_{W_2^1} = Lu(x_i) = 0.$$

Note that $\{x_i\}_{i=1}^{\infty}$is dense in $[a,b]$, So , $Lu(x) = 0$. It follows that $u(x) = 0$ from the existence of L^{-1} and the continuity of $u(x)$ □

Using the Gram-Schmidt orthogonalization method of $\{\psi_i(x)\}_{i=1}^{\infty}$, we will now create an orthonormal function $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of the space $W_2^{m+1}[a,b]$ as follows:

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \ i = 1, 2, ... \tag{2.3.5}$$

where β_{ik} are orthogonalization coefficients and is given by

$$\beta_{11} = \frac{1}{\| \psi_1 \|}, \quad \beta_{ii} = \frac{1}{\sqrt{\| \psi_i \|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad \beta_{ij} = \frac{-\sum_{k=j}^{i-1} c_{ik}\beta_{kj}}{\sqrt{\| \psi_i \|^2 - \sum_{k=1}^{i-1} c_{ik}^2}}, \quad j < i, \tag{2.3.6}$$

where $c_{ik} = \langle \psi_i, \psi_k \rangle_{W_2^{m+1}}$.

Theorem 2.3.4. *For every $u(x) \in W_2^{m+1}[a,b]$, the series $\sum_{k=1}^{i} \langle u(x), \overline{\psi}_i(x)\rangle \overline{\psi}_i(x)$ is convergent in the terme of the norm of $W_2^{m+1}[a,b]$. Alternatively, if $\{x_i\}_{i=1}^{\infty}$is dense in $[a,b]$ then the solution of the BVP (2.3.1) and (2.3.2)is unique and given by*

$$u(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}F(x_k, u(x_k))\overline{\psi}_i(x), \tag{2.3.7}$$

Proof : Applying Theorem (2.3.3), it is easy to see that $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ is the complete orthonormal basis of the space $W_2^{m+1}[a,b]$. Therefore, $u(x)$ can be expanded into a Fourier series about the orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$as follows $u(x) = \sum_{i=1}^{\infty} \langle u(x), \overline{\psi}_i(x)\rangle \psi_i(x)$

Further, the space $W_2^{m+1}[a,b]$ is Hilbert space, so the series $\sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle\overline{\psi}_i(x)$ is convergent in the sense of the norm of $W_2^{m+1}[a,b]$. As long as $\langle v(x),\varphi_i(x)\rangle = v(x_i)$ for all $v(x)\in W_2^1[a,b]$, we have:

$$\begin{aligned}
u(x) &= \sum_{i=1}^{\infty}\left\langle u(x),\overline{\psi}_i(x)\right\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\left\langle u(x),\sum_{k=1}^{i}\beta_{ik}\psi_k(x)\right\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),\psi_k(x)\right\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle u(x),L^*\varphi_k(x)\right\rangle_{W_2^{m+1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle Lu(x),\varphi_k(x)\right\rangle_{W_2^{1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}\left\langle F(x,u(x)),\varphi_k(x)\right\rangle_{W_2^{1}}\overline{\psi}_i(x)\\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k))\overline{\psi}_i(x)
\end{aligned}$$

We symbolize the n-term approximate solution to $u(x)$ by □

$$u_n(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}F(x_k,u(x_k))\overline{\psi}_i(x), \tag{2.3.8}$$

Theorem 2.3.5. *For each $u(x)\in W_2^{m+1}[a,b]$, $u_n^{(i)}(x)$ are uniformly convergent to $u^{(i)}(x)$,*

$i=0,1,...,m$.

Proof : By using Lemma (2.3.1), for each $x \in [a,b]$, We obtain

$$\begin{aligned}
\left|u_n^{(i)}(x) - u^{(i)}(x)\right| &= \left|\left\langle u_n^{(i)}(x) - u^{(i)}(x), k_x(x)\right\rangle_{W_2^{m+1}}\right| = \left|\left\langle u_n(x) - u(x), k_x^{(i)}(x)\right\rangle_{W_2^{m+1}}\right| \\
&\leq \left\|k_x^{(i)}(x)\right\|_{W_2^{m+1}} \|u_n(x) - u(x)\|_{W_2^{m+1}} \\
&\leq M_i \|u_n(x) - u(x)\|_{W_2^{m+1}} \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}$$

Hence, the approximate solution $u_n(x)$ and $u_n^{(i)}(x)$ converge uniformly to $u(x)$ and its derivative $u^{(i)}(x)$, respectively □

CHAPTER 3

System of Integro-differential Equations of Fractional Derivative

In this chapter, we will apply the RKHS method to approximately solve the system of fractional Volterra integro-differential equations (FVIDS) of first and second order with initial conditions. The fractional derivative used is of the Caputo type. The approximate solution to the studied system is provided in the Hilbert space $H_2^2[a,b]$ in the form of a series that converges uniformly to the analytic solution.

3.1 Solving FVIDS System of order $0 < \alpha \leq 1$

Consider the following system of first-order FVIDS:

$$ {}^{C}D^{\alpha_r}u_r(x) = G_r(t, u_1(x), \ldots, u_n(x)) + \int_a^x K_r(x, y, u_1(y), \ldots, u_n(y))\, dy \tag{3.1.1} $$

Subject to the initial conditions:

$$ u_r(a) = c_r \tag{3.1.2} $$

where $r = 1, \ldots, n$, $a \leq t \leq b$, and ${}^{C}D^{\alpha_r}$ is the Caputo fractional derivative of order α_r, where $0 \leq \alpha_r \leq 1$, and $u_1, u_2, \ldots, u_n(y)$ are the unknown functions to be determined.

Definition 3.1.1. The function space $H_2^2[a,b]$ is defined as follows:

$$ H_2^2[a,b] = \left\{ u(y) \mid y \text{ is absolutely continuous, } u' \in L^2[a,b], u(y) = 0 \right\} $$

It is associated with the inner product:

$$\langle u_1, u_2 \rangle_{H_2^2} = u_1(a)u_2(a) + u_1'(a)u_2'(a) + \int_a^b u_1''(y)u_2''(y)\, dy$$

and the norm:

$$\|u\|_{H_2^2} = \sqrt{\langle u, u \rangle_{H_2^2}}$$

where $u_1, u_2 \in H_2^2[a, b]$.

Theorem 3.1.2. The reproducing kernel $P_2(x, y)$ attached to the space $H_2^2[a, b]$ is stated as follows:

$$P_2(x, y) = \begin{cases} \sum_{i=0}^{4} a_i(y)x^i & \text{for } x < y, \\ \sum_{i=0}^{4} b_i(y)x^i & \text{for } y \leq x. \end{cases} \tag{3.1.3}$$

Proof: Let $P_2(x, y)$ be the reproducing kernel of $H_2^2[a, b]$. For every $u(y) \in H_2^2[a, b]$, we have:

$$u(y) = \langle u(\tau), P_2(\tau, y) \rangle = u(a)P_2(a, y) + u'(a)P_2'(a, y) + \int_a^b u''(\tau)P_2''(\tau, y)\, d\tau \tag{3.1.4}$$

By integrating by parts twice, we obtain:

$$
\begin{aligned}
u(y) &= \langle u(\tau), P_2(\tau, y)\rangle \\
&= u'(a)\left(P_2'(a,y) - P_2''(a,y)\right) + u'(b)P_2''(b,y) - u(b)P_2^{(3)}(b,y) + \int_a^b u(x)P_2^{(4)}(x,y)\,dx
\end{aligned}
\tag{3.1.1}
$$

Now, consider the following conditions:

$$
\begin{cases}
P_2'(a,y) - P_2''(a,y) \quad = 0, \\
P_2''(b,y) = 0, \\
P_2'''(b,y) = 0, \\
P_2(a,y) = 0.
\end{cases}
\tag{3.1.6}
$$

Due to the reproducing property, from equation (3.1.5), we obtain:

$$
-P_2^{(4)}(x,y) = \delta(x-y) \tag{3.1.7}
$$

For $x \neq y$, the equation becomes:

$$
-P_2^{(4)}(x,y) = 0 \tag{3.1.8}
$$

Thus, the general solution of equation (3.1.8) is:

$$P_2(x,y) = \begin{cases} \sum_{i=1}^{4} c_i(y)x^{i-1} & \text{for } x \le y, \\ \sum_{i=1}^{4} d_i(y)x^{i-1} & \text{for } y < x. \end{cases}$$

From equation (3.1.7), we can deduce that:

$$\begin{cases} P_2(x, y+0) = P_2(x, y-0), \\ P_2'(x, y+0) = P_2'(x, y-0), \\ P_2''(x, y+0) = P_2''(x, y-0), \\ P_2^{(3)}(x, y+0) = P_2^{(3)}(x, y-0) = 1. \end{cases} \tag{3.1.9}$$

Now, from equations (3.1.6) and (3.1.9), we can determine the unknown coefficients $c_i(y)$, $d_i(y)$, for $i = 1, \ldots, 4$. The expression for $P_2(x, y)$ becomes:

$$P_2(x,y) = -\frac{1}{6} \begin{cases} (a-x)(2a^2 - x^2 + 3y(2+x) - a(6+3y+x)), & x \le y, \\ (a-y)(2a^2 - y^2 + 3x(2+y) - a(6+3x+y)), & y \le x. \end{cases}$$

In order to apply the RKHS technique to obtain analytical and numerical solutions to the system (3.1.1)-(3.1.2), we need to transform it into an equivalent system with homogeneous initial conditions. To do this, we set:

$$u_r(x) = \nu_r(y) - c_r, \quad r = 1, 2, \ldots, n.$$

Thus, system (3.1.1)-(3.1.2) becomes:

$$ {}^{C}D^{\alpha_r}u_r(y) = G_r(y, u_1(y), \ldots, u_n(y)) + \int_a^y K_r(y, \tau, u_1(\tau), \ldots, u_n(\tau))\, d\tau \qquad (3.1.10) $$

Subject to the initial conditions:

$$ u_r(a) = 0, \quad r = 1, 2, \ldots, n. \qquad (3.1.11) $$

As a first step in the RKHS algorithm, we define the operators $S_r : H_2^2[a,b] \to H_1^2[a,b]$, for $r = 1, 2, \ldots, n$, by $S_r(u_r(y)) = {}^{C}D^{\alpha_r}(u_r(y))$.

If $u_r(y) \in H_2^2[a,b]$, then $S_r(u_r(y)) \in H_1^2[a,b]$. Therefore, S_r are linear bounded operators over $[a,b]$.

Next, we construct an orthogonal basis for $H_2^2[a,b]$ by setting:

$$ \theta_i^r(y) = S_r^* P_2(y, y_i), $$

where $\{y_i\}_{i=0}^{\infty}$ is a countable dense subset of $[a,b]$, and S_r^* are the adjoint operators of S_r, for $r = 1, 2, \ldots, n$.

Using the Gram-Schmidt orthonormalization process, we can derive the orthonormal basis $\{\theta_i^r(y)\}_{i=1}^{\infty}$ for $r = 1, 2, \ldots, n$ by:

$$ \theta_i^r(y) = \sum_{j=1}^{X_r} \gamma_{r,ij} \theta_j^r(y), $$

where $\gamma_{r,ij}$ are the orthogonalization coefficients.

Using the properties of the reproducing kernel, we can conclude the following result.

Theorem 3.1.3. Let $\{y_i\}_{i=1}^{\infty}$ be a countable dense subset of $[a,b]$ and assume that S_r^{-1} exists for all $r = 1, 2, \ldots, n$. Then $\{\theta_i^r(y)\}_{i=1}^{\infty}$ forms a complete system of functions in $H_2^2[a,b]$.

Proof: For each $r = 1, 2, \ldots, n$, let $u_r(y) \in H_2^2[a,b]$ such that:

$$\langle u_r(y), \theta_i^r(y) \rangle = 0.$$

This gives:

$$\langle u_r(y), \theta_i^r(y) \rangle = \langle u_r(y), S_r^* P_2(y, y_i) \rangle = \langle S_r u_r(y), P_2(y, y_i) \rangle = S_r u_r(y_i).$$

Since $\{y_i\}_{i=1}^{\infty}$ is dense in $[a,b]$, we obtain:

$$S_r u_r(y) = 0.$$

Furthermore, because S_r^{-1} exists, we conclude that:

$$u_r(y) = 0.$$

Thus, $\{\theta_i^r(y)\}_{i=1}^{\infty}$ is a complete system in $H_2^2[a,b]$. □

Theorem 3.1.4. Let $\{y_i\}_{i=1}^{\infty}$ be a countable dense subset of $[a,b]$. If the solution of system (3.1.10)(3.1.11) is unique, then the exact solution is given by:

$$u_r(y) = \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k)\, \theta_i^r(y), \tag{3.1.12}$$

where $\gamma_{r,ik}$ are the orthogonalization coefficients.

Proof: Since $\{\theta_i^r(y)\}_{i=1}^{\infty}$ forms a complete orthonormal basis of $H_2^2[a,b]$, we have:

$$u_r(y) = \sum_{i=1}^{\infty} \langle u_r(y), \theta_i^r(y) \rangle \theta_i^r(y)$$

$$= \sum_{i=1}^{\infty} \left\langle u_r(y), \sum_{k=1}^{i} \gamma_{r,ik} \theta_k^r(y) \right\rangle \theta_i^r(y)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik} \langle u_r(y), \theta_i^r(y) \rangle \theta_i^r(y)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik} \langle S_r u_r(y), P_2(y, y_k) \rangle \theta_i^r(y)$$

$$= \sum_{i=1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k)\, \theta_i^r(y).$$

An approximate solution $u_{rN}(y)$ can be obtained by truncating the series in (3.1.12):

$$u_{rN}(y) = \sum_{i=1}^{N} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k)\, \theta_i^r(y). \tag{3.1.13}$$

Theorem 3.1.5. The approximate solution $u_{rN}(y)$ given in (3.1.13) and its derivative $u'_{rN}(y)$ converge uniformly to $u_r^{(j)}(y)$, where $j = 0, 1$.

Proof: By using the reproducing property of $P_2(x, y)$ and the CauchySchwarz inequality, we have:

$$|u_{rN}(y) - u_r(y)| = \left| \langle u_{rN}(y) - u_r(y), P_2(x, y) \rangle_{H_2^2} \right|$$

$$\leq \|P_2(x, y)\|_{H_2^2} \, \|u_{rN}(y) - u_r(y)\|_{H_2^2} \leq M_0 \, \|u_{rN}(y) - u_r(y)\|_{H_2^2},$$

where M_0 is a positive constant.

Similarly, for the derivative:

$$|u'_{rN}(y) - u'_r(y)| = \left|\langle u'_{rN}(y) - u'_r(y), P_2(x,y)\rangle_{H_2^2}\right|$$

$$\leq \|P_2(x,y)\|_{H_2^2} \|u'_{rN}(y) - u'_r(y)\|_{H_2^2} \leq M_1 \|u'_{rN}(y) - u'_r(y)\|_{H_2^2},$$

where M_1 is a positive constant.

Theorem 3.1.6. Let e_{rN} denote the error between the exact solution $u_r(y)$ and the approximate solution $u_{rN}(y)$. Then the sequence $\{e_{rN}\}_{N=1}^{\infty}$ is monotonically decreasing in $H_2^2[a,b]$.

Proof: We have:

$$\|e_{rN}\|^2 = \|u_r(y) - u_{rN}(y)\|^2 = \left\| \sum_{i=N+1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k)\, \theta_i^r(y) \right\|^2 = \sum_{i=N+1}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k).$$

Similarly:

$$\|e_{r,N-1}\|^2 = \left\| \sum_{i=N}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k)\, \theta_i^r(y) \right\|^2$$

$$= \sum_{i=N}^{\infty} \sum_{k=1}^{i} \gamma_{r,ik}\, S_r u_r(y_k).$$

It is clear that:

$$\|e_{rN}\| \leq \|e_{r,N-1}\|.$$

Thus, the error sequence $\{e_{rN}\}$ is decreasing.

3.2 Numerical Examples

In this section, we investigate the influence of the Caputo fractional derivative on a system of Volterra FIDEs. To illustrate the behavior, efficiency, and applicability of the RKHS method in solving such systems, several examples are provided.

In implementing this method, we select the grid points as $x_l = \frac{l}{N}$, for $l = 1, 2, \ldots, N$, and perform the computations using the *Mathematica* software package.

Example 3.2.1. *Let us consider the following linear system of Volterra FIDEs:*

$$\begin{aligned} {}^{C}D^{\alpha_1}u_1(x) &= 1 + x + x^2 - u_1(x) - \int_0^x (u_1(y) + u_2(y))dy, \\ {}^{C}D^{\alpha_2}u_2(x) &= -1 - x + u_2(x) - \int_0^x (u_1(x) + u_2(x))dx, \\ u_1(0) = 1, &\quad u_2(0) = -1. \end{aligned} \tag{3.2.1}$$

The RKHS method was applied to the system given in (3.2.1), using $N = 35$ points and a regularly spaced grid. Numerical computations were performed for the system with fractional orders α_1 and α_2 varying within the interval $(0, 1]$.

The numerical results for different values of the fractional order α are presented in Tables 3.1 and 3.2, while Figure 3.1 illustrates the approximate behavior of the solutions $u_1(x)$ and $u_2(x)$ along with the absolute error when $\alpha = 1$. Additionally, A comparative curve illustrating the behavior of the approximate solutions for Example (3.2.1) at different fractional orders α, with $N = 35$, is presented in Figure 3.2.

The numerical and graphical results are summarized as follows:

- ***Table 3.1:*** *Presents a detailed comparison between the exact and approximate solutions, including the calculation of the absolute error for the system in Example*

(3.2.1) *at* $\alpha = 1$ *and* $N = 35$.

- ***Table 3.2:*** *Shows the numerical results of the approximate solutions obtained by the RKHS method for* $N = 35$*, highlighting the behavior of the solutions for different fractional orders.*

- ***Figure 3.1:*** *Displays plots of the exact and approximate solutions as well as the absolute error at* $\alpha = 1$ *and* $N = 35$.

- ***Figure 3.2:*** *Shows a comparative plot of the approximate solutions for different fractional orders* α *with* $N = 35$.

Table 3.1: Numerical outcomes for $u_1(x)$ and $u_2(x)$, with $\alpha = 1$, $N = 35$ for Example (3.2.1).

x_i	$u_1(x)$			$u_2(x)$		
	Exact	Approx	Abs. Error	Exact	Approx	Abs. Error
0.0	1.00000	1.00000	0.00000	-1.00000	-1.00000	0.00000
0.1	1.20517	1.20517	1.2e-5	-1.00517	-1.00518	1.1e-5
0.2	1.42140	1.42141	9.8e-6	-1.02140	-1.02141	1.2e-5
0.3	1.64986	1.64987	1.4e-5	-1.04986	-1.04987	1.3e-5
0.4	1.89182	1.89183	2.0e-5	-1.09182	-1.09183	1.4e-5
0.5	2.14872	2.14873	2.7e-5	-1.14872	-1.14873	1.5e-5
0.6	2.42212	2.42214	3.1e-5	-1.22212	-1.22213	1.5e-5
0.7	2.71375	2.71377	3.3e-5	-1.31375	-1.31376	1.6e-5
0.8	3.02554	3.02556	3.6e-5	-1.42554	-1.42555	1.7e-5
0.9	3.35960	3.35963	3.7e-5	-1.55960	-1.55961	1.7e-5
1.0	3.71828	3.71831	3.8e-5	-1.71828	-1.71829	1.8e-5

Example 3.2.2. *Let us consider the following nonlinear system of Volterra FIDEs:*

$$\begin{aligned} {}^{C}D^{\alpha_1+1}u_1(x) &= 1 - \tfrac{x^2}{3} - \tfrac{1}{2}u_1'^2(x) + \tfrac{1}{2}\int_0^x (u_1^2(y) + u_2^2(y))dy, \\ {}^{C}D^{\alpha_2+1}u_2(x) &= -1 + x^2 - xu_1(x) + \tfrac{1}{4}\int_0^x (u_1^2(y) - u_2^2(y))dy, \\ u_1(0) = 1, &\quad u_1'(0) = 2, \\ u_2(0) = -1, &\quad u_2'(0) = 0. \end{aligned} \tag{3.2.2}$$

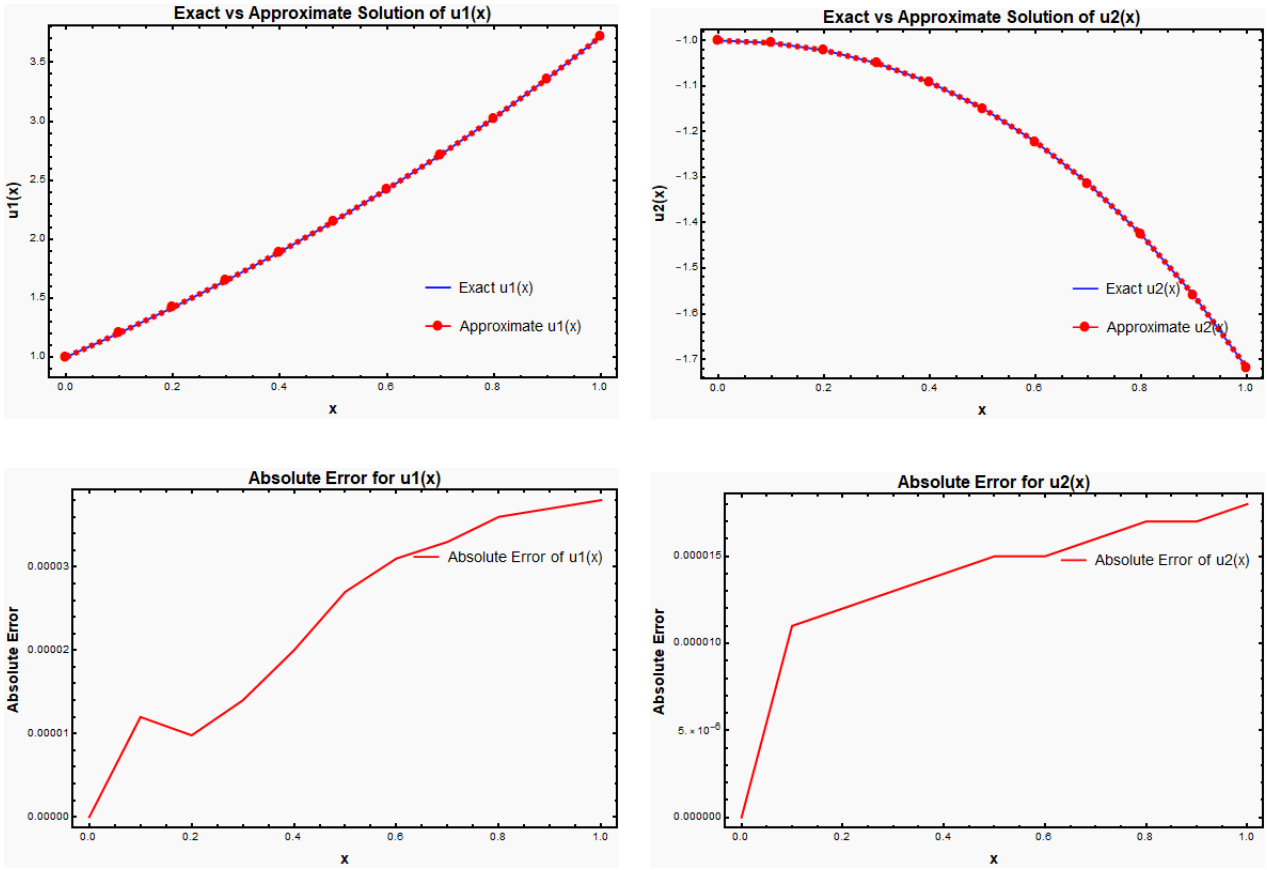


Figure 3.1: Plots of Exact and Approximate Solutions and Their Absolute Errors for Example (3.2.1) at $\alpha = 1$ with $N = 35$ for Example (3.2.1).

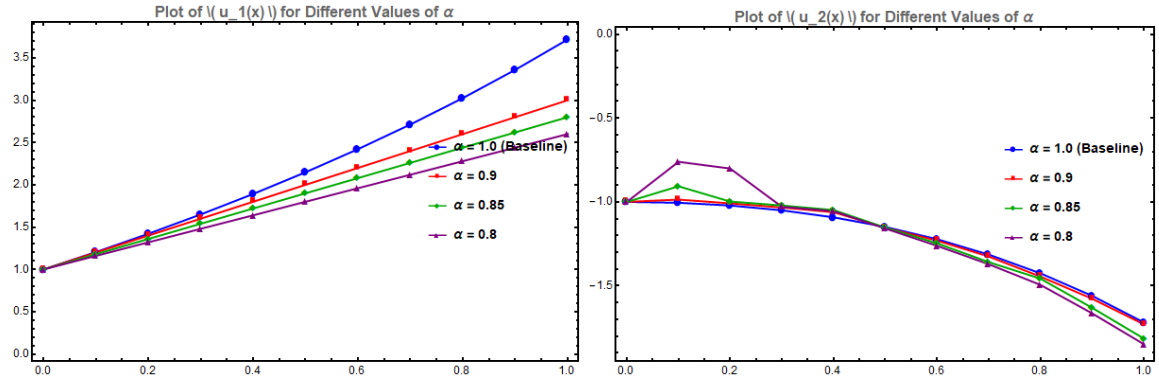


Figure 3.2: Comparative Study of Approximate Solutions $u_1(x)$ and $u_2(x)$ with Varying α at $N = 35$ for Example (3.2.1)

Applying the RKHS method to the system (3.2.2) *where* $N = 35$ *and* $\alpha_1, \alpha_2 \in (0, 1]$. *Then, the exact and approximate solutions with different values of* α *are listed in Tables 3 and 4. Moreover, Figure* (3.4) *shows how the solution curves changes with changing the values of the fractional parameter* α. *Also, it presents the absolute error of approximation at* $\alpha = 1$.

Table 3.2: Numerical outcomes for $u_1(x)$ and $u_2(x)$ for Example (3.2.1) using RKHS method when $N = 35$.

x_l	$u_1(x)$				$u_2(x)$			
	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.85$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.85$	$\alpha = 0.8$
0.0	1.00000	1.00000	1.00000	1.00000	-1.00000	-1.00000	-1.00000	-1.00000
0.1	1.20490	1.34725	1.42220	1.48600	-1.00912	-0.98599	-0.90641	-0.76075
0.2	1.42120	1.55100	1.62410	1.69750	-1.02457	-1.00870	-0.99771	-0.80106
0.3	1.65050	1.76450	1.83230	1.91200	-1.05372	-1.03290	-1.02119	-1.02908
0.4	1.89500	1.99000	2.05000	2.13800	-1.09826	-1.06104	-1.04839	-1.05491
0.5	2.15200	2.22800	2.28000	2.37700	-1.15882	-1.15371	-1.15310	-1.15843
0.6	2.42300	2.48100	2.52500	2.63000	-1.23636	-1.23036	-1.24668	-1.26177
0.7	2.70900	2.75000	2.78800	2.89800	-1.33153	-1.32552	-1.35859	-1.37136
0.8	3.01100	3.03500	3.06700	3.18200	-1.44462	-1.44482	-1.45734	-1.49439
0.9	3.33200	3.33800	3.36900	3.48500	-1.57649	-1.57707	-1.63114	-1.66431
1.0	3.72100	3.75700	3.77600	3.80100	-1.72854	-1.72930	-1.81439	-1.84799

Table 3.3: Numerical outcomes for $u_1(x)$ and $u_2(x)$, with $\alpha = 1$, $N = 35$ for Example (3.2.2).

x_i	$u_1(x)$			$u_2(x)$		
	Exact	Approx	Abs. Error	Exact	Approx	Abs. Error
0.0	1.00000	1.00000	0.00000	-1.00000	-1.00000	0.00000
0.1	1.10517	1.10517	4.1×10^{-6}	-1.10517	-1.10517	1.4×10^{-6}
0.2	1.22140	1.22141	1.1×10^{-5}	-1.22140	-1.22141	4.2×10^{-6}
0.3	1.34986	1.34988	2.1×10^{-5}	-1.34986	-1.34987	8.2×10^{-6}
0.4	1.49182	1.49186	3.5×10^{-5}	-1.49182	-1.49184	1.4×10^{-5}
0.5	1.64872	1.64878	5.6×10^{-5}	-1.64872	-1.64874	2.1×10^{-5}
0.6	1.82212	1.82220	8.4×10^{-5}	-1.82212	-1.82215	3.0×10^{-5}
0.7	2.01375	2.01388	1.2×10^{-4}	-2.01375	-2.01379	4.1×10^{-5}
0.8	2.22554	2.22572	1.8×10^{-4}	-2.22554	-2.22560	5.5×10^{-5}
0.9	2.45960	2.45985	2.5×10^{-4}	-2.45960	-2.45968	7.2×10^{-5}
1.0	2.71828	2.71863	3.5×10^{-4}	-2.71828	-2.71838	9.3×10^{-5}

The results demonstrate the capability of the RKHS method to provide accurate approximate solutions for the studied system of Volterra FIDEs, with a small absolute error that decreases as the number of discretization points increases, reflecting the efficiency and reliability of the method. The absolute error serves as a key indicator of the quality of the numerical solutions, showing a progressive decline as N increases. The behavior of the approximate solutions is significantly influenced by changes in the fractional order

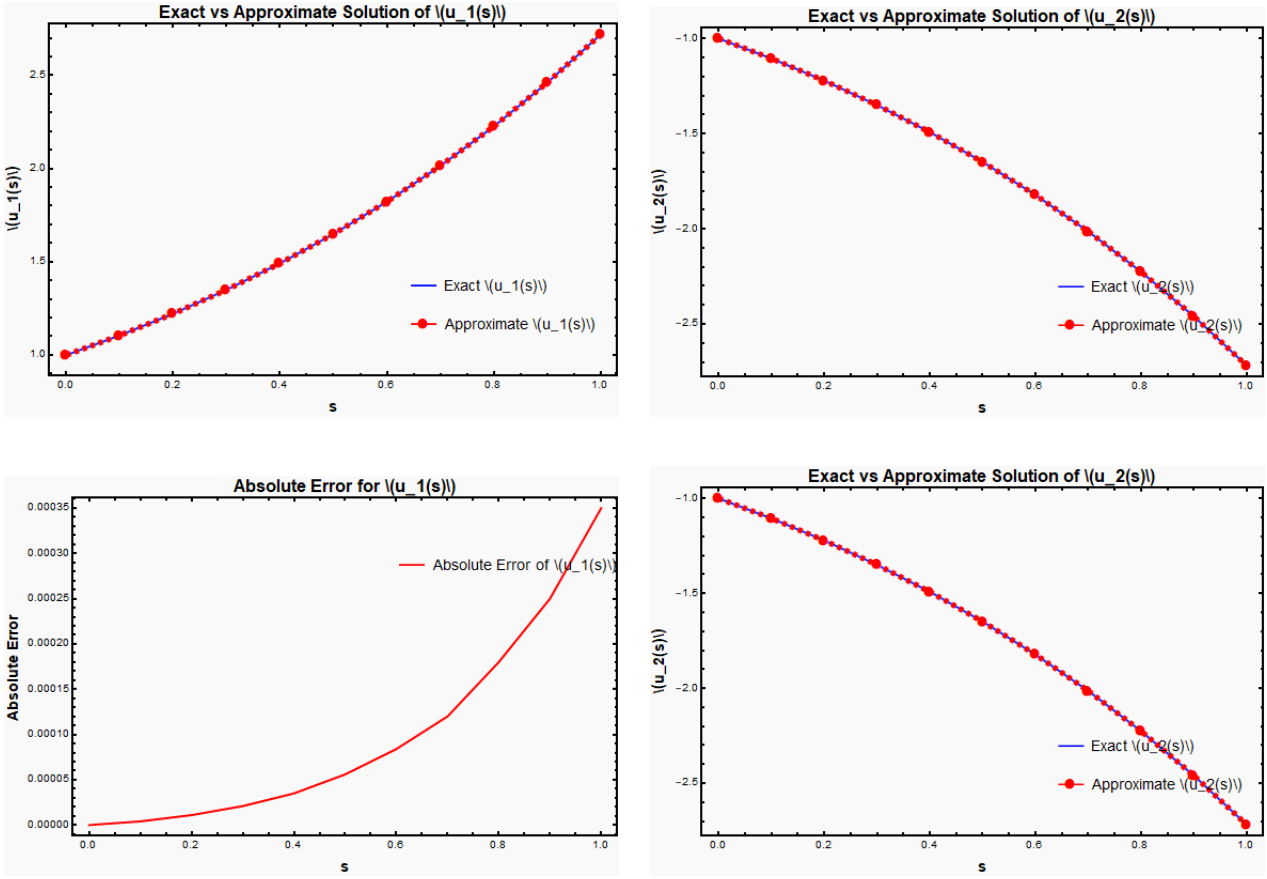


Figure 3.3: Plots of Exact and Approximate Solutions and Their Absolute Errors for Example (3.2.2) at $\alpha = 1$ with $N = 35$.

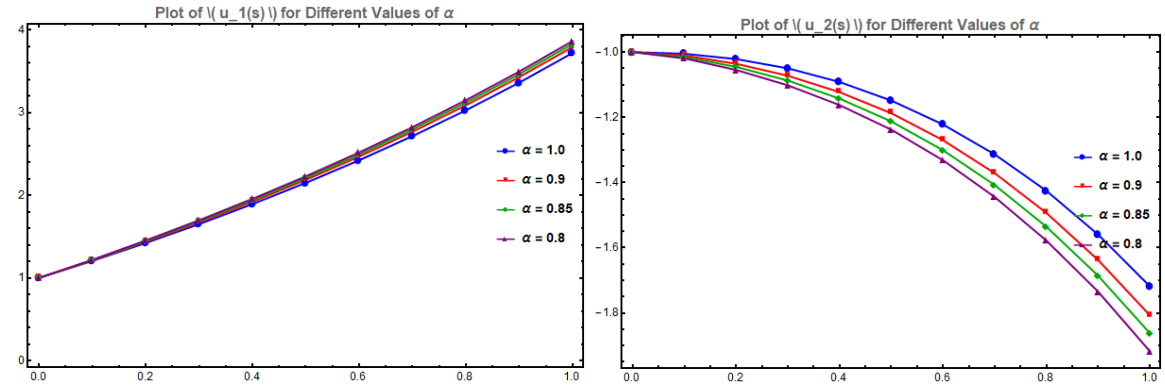


Figure 3.4: Comparative Study of Approximate Solutions $u_1(x)$ and $u_2(x)$ with Varying α at $N = 35$ for Example (3.2.2)

α, underscoring the importance of analyzing the impact of these parameters on the system dynamics. These findings encourage the application of the RKHS method to more complex models and pave the way for its extension to higher-dimensional problems and applications in various fields such as engineering and physics.

Table 3.4: Numerical outcomes for $u_1(x)$ and $u_2(x)$ for Example (3.2.2) using RKHS when $N = 35$.

s_l	$u_1(x)$				$u_2(x)$			
	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.85$	$\alpha = 0.8$	$\alpha = 1$	$\alpha = 0.9$	$\alpha = 0.85$	$\alpha = 0.8$
0.0	1.00000	1.00000	1.00000	1.00000	-1.00000	-1.00000	-1.00000	-1.00000
0.1	1.20517	1.21086	1.21498	1.21889	-1.00517	-1.01077	-1.01511	-1.01945
0.2	1.42140	1.43488	1.44366	1.45209	-1.02140	-1.03589	-1.04532	-1.05479
0.3	1.64986	1.67093	1.68417	1.69678	-1.04986	-1.07280	-1.08736	-1.10198
0.4	1.89182	1.92072	1.93824	1.95476	-1.09182	-1.12295	-1.14273	-1.16261
0.5	2.14872	2.18543	2.20707	2.22725	-1.14872	-1.18765	-1.21277	-1.23805
0.6	2.42212	2.46701	2.49235	2.51602	-1.22212	-1.27036	-1.30102	-1.33186
0.7	2.71375	2.76703	2.79558	2.82212	-1.31375	-1.37156	-1.40784	-1.44444
0.8	3.02554	3.08647	3.11841	3.14772	-1.42554	-1.49328	-1.53576	-1.57836
0.9	3.35960	3.42667	3.46160	3.49338	-1.55960	-1.63776	-1.68593	-1.73476
1.0	3.71828	3.79032	3.82782	3.86178	-1.71828	-1.80732	-1.86205	-1.91743

CHAPTER 4

Conclusions

Conclusions

In this masters thesis, an efficient numerical framework was developed to solve FIDE systems involving the Caputo fractional derivative using the RKHS method. This approach allows constructing approximate solutions as uniformly convergent series to the exact solution, ensuring high accuracy and fast convergence. The numerical results obtained from various applied examples demonstrated the effectiveness and high precision of the method in addressing fractional systems. Furthermore, the impact of the fractional derivative order on the solution behavior was highlighted, emphasizing the importance of fractional models in representing memory-dependent phenomena. This study confirms that the RKHS method is a powerful and promising mathematical tool for solving complex fractional systems. Additionally, the method can be extended to other types of fractional derivatives (such as Atangana or Prabhakar), higher-dimensional systems, and problems with more complex initial and boundary conditions.

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