Democratic and Popular Algerian Republic
Ministry of Higher Education and Scientific Research
University of cheik Larbi Tebessi – Tebessa
Faculty of Exact Sciences and Natural and Life Sciences
Department of Mathematics

MASTER Thesis

For the Completion of the Master's Degree

Mathematics

Thesis Title:

On the Drazin inverse and generalized Drazin inverse of an operator and its applications

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Academic Year 2024/2025

Date of defense: 02/06/2025

Abstract

This thesis aims to study the Drazin inverse of bounded linear operators in Banach spaces, focusing on its algebraic and spectral properties, as well as its applications in solving differential equations and linear systems.

The thesis is divided into four chapters:

- **Chapter 1: Preliminaries**
  Covers fundamental concepts such as linear operators, spectrum, ascent and descent, and the Riesz projection.
- **Chapter 2: Drazin Inverse**
  Defines the Drazin inverse, conditions for its existence, its relation to the spectrum, and special cases.
- **Chapter 3: Generalized Drazin Inverse**
  Involves operator decomposition, additional spectral properties, and key partial order relations.
- **Chapter 4: Applications**
  Uses the Drazin inverse to solve differential equations and demonstrates computational methods with illustrative examples.

The thesis highlights the importance of the Drazin inverse in extending the concept of invertibility and providing effective analytical tools for linear systems in applied mathematics.

Résumé

Cette thèse a pour objectif d'étudier l'inverse de Drazin des opérateurs linéaires bornés dans les espaces de Banach, en mettant l'accent sur ses propriétés algébriques et spectrales, ainsi que sur ses applications dans la résolution d'équations différentielles et de systèmes linéaires .

La thèse est divisée en quatre chapitres :

- **Chapitre 1 : Préliminaires** Présente les concepts fondamentaux tels que les opérateurs linéaires, le spectre, la montée et la descente, ainsi que la projection de Riesz.
- **Chapitre 2 : Inverse de Drazin**
  Définit l'inverse de Drazin, les conditions de son existence, sa relation avec le spectre, et des cas particuliers.
- **Chapitre 3 : Inverse de Drazin Généralisé**
  Aborde la décomposition des opérateurs, des propriétés spectrales supplémentaires, et des relations de pré-ordre clés.
- **Chapitre 4 : Applications**
  Utilise l'inverse de Drazin pour résoudre des équations différentielles et illustre des méthodes de calcul à travers des exemples concrets.

La thèse souligne l'importance de l'inverse de Drazin pour étendre le concept d'inversibilité et fournir des outils analytiques efficaces pour les systèmes linéaires en mathématiques appliquées.

الملخص

تهدف هذه المذكرة إلى دراسة المعكوس الدرازيني للمؤثرات الخطية المحدودة في الفضاءات البانخية، مع التركيز على خصائصه الجبرية والطيفية، بالإضافة إلى تطبيقاته في حل المعادلات التفاضلية والأنظمة الخطية غير القابلة للعكس.

تنقسم المذكرة إلى أربعة فصول:

الفصل الأول: التمهيدات
يتناول المفاهيم الأساسية مثل المؤثرات الخطية، الطيف، الارتقاء والانحدار، وإسقاط ريز.

الفصل الثاني: المعكوس درازين يتضمن تعريف المعكوس درازين، شروط وجوده، علاقته بالطيف، والحالات الخاصة.

الفصل الثالث: المعكوس درازين المعمم
يشمل تحليل المؤثر، خصائص طيفية إضافية، وعلاقات الترتيب الجزئي الأساسية.

الفصل الرابع: التطبيقات
يُستخدم المعكوس درازين في حل المعادلات التفاضلية، ويعرض طرقًا حسابية مدعومة بأمثلة توضيحية.

تُبرز هذه المذكرة أهمية المعكوس درازين في توسيع مفهوم القابلية للعكس، وتقديم أدوات تحليلية فعالة لدراسة الأنظمة الخطية في الرياضيات التطبيقية.

الاهداء

ما كان الطريق سهلاً، تعبت وصبرت، وبفضل الله وصلت لبداية النهاية، ثمرة جهد وإصرار لا ينكسر، وبداية حلم جديد، أدعو الله أن يكون هذا الفصل مجرد بداية، لرحلة علمية طويلة، لا تعرف المستحيل.

إلى أمي

إلى من تعبت وسهرت وربّت، انت من علمتني في البدايات، وكنت لي المعلمة الأولى قبل أن أفتح كتبي،علمتني أن للحرف روحا، وللصبر جناحين. كل حرف خططته، وكل خطوة كانت بدعائك كانت دوما يدك أمامي ترشدني.أهديك هذا الانجاز عربون وفاء، وقطرة من نهر امتنان لا ينضب،لأنه منك بدأ، وبحبك اكتمل

إلى أبي مستر محسن
انت من علمني أن الرجولة ليست صوتا عال، بل قلب مغمور بالحب،انت أمني وأماني ومأمني كانت موافقتك لي دفئا، وتشجيعك طريقا، وثقتك جناحين أطير بهما نحو المستحيل ،انت فخر القلب وسند الروح ،هذا النجاح هو ثمرة محبتك، دعمك، ووجودك في حياتي، فاهديه إليك كأقل ما يمكن أن أقدم لك من شكر وامتنان.

الى إياد اخي العزيز

"سَنَشُدُّ عَضُدَكَ بِأَخِيكَ".. كلماتٌ نزلت من السماء لتكون وعدًا إلهيًا، وجُعلتَ أنت تحقيقه في حياتي. أنت الأخ الذي لا يعوض، السند الذي أستند إليه في كل لحظة، والصديق الذي لا يخيب. وجودك في حياتي هو النور الذي يضيء لي الطريق، والقوة التي تجعلني أواجه التحديات بثقة. حفظك الله ورعاك، وبارك فيك بكل خطوة تخطوها.

الى سيف، ملاكي الصغير، أنت مصدر سعادتي الحقيقي براءتك وذكائك يجعلاني أشعر بأن كل لحظة معك هي هدية من الحياة.

وجودك يملا قلبي بالفرح والطمانينة انت دائما سبب في ابتسامتي حفظك الله بعينه التي لا تنام .

في درب الحياة لم تكن الأخوة دائما بالدم بل كانت بنبض القلوب التي اختارت بعضها بكل حب وصدق مهما مرّت الأيام، أنا وأنتم معًا، دائمًا وأبدًا

الى اسراء توأم روحي، نجم يضيء حياتي ويملأها بالحب والبهجة. وجودك يعطيني القوة والدعم،كنت جزءا من حياتي في كل تفاصيلها، وما زلت الأخت والصديقة التي لا تعوض.

الى شروق ، مثل اسمك، أنت الأمل الذي لا ينتهي. في لحظات الحزن، تأتي شروقك لتضيء الطريق وتنير قلبي. أنت الأخت التي لا يمكن أن تعوض، ودائما ما أجد فيك الإلهام والدعم.

الى سلمى ، أنت الاخت التي أفرغ فيها كل ما في قلبي وأعود مرتاحة والقلب الذي يشعر بما لا أستطيع قوله،معك أجد الأمان والراحة، وأستطيع أن أكون نفسي بلا تردد.

الى شادية، أنتِ مثل النسمة الهادئة التي تملأ حياتي بالراحة. أجد فيكِ كل شيء جميل، فالجمال والرقي هما جزء من شخصيتكِ.
الى عزيزتي براءة و اسيد صغيري يا فرحة وبهجة العائلة حفظكما الله اتمنى لكما الخير كله .

الى طاطا يمينة وعمي وهاب أنتم جيران لكن مكانكم في القلب كبير كأهل وأكثر. شكرا على دعمكم وطيبتكم

الى اكرام الطيبة شكرًا على دعمكِ ، كنتِ نعم الصديقة في كل لحظة. أحبكِ من قلبي، وأنتِ دائمًا في بالي.

إلى غزة، يا نبضا يتحدى الموت، ويا صرخة في وجه العالم الأصم.خانكِ العرب، باعوكِ بالكلام، وتركوك تنزفين وحدك على أرصفة الخذلان. لكننا نحملك في قلوبنا دعاء لا يهدأ، ونؤمن أن النصر يولد من بين أنقاضك.

إلى شتشوت أنت النور الذي يضيء دربي في أوقات الظلام، واليد التي تمتد لي عندما أحتاج الدعم. انت الأمل والوفاء الذي لا يغيب شكرًا لوجودك.

Acknowledgements

At the beginning of my speech

I would like first and foremost to express my deep gratitude to Almighty allah, who granted me success and strength to reach this honorable academic level, and who made it possible for me to stand before you today to present my master's thesis.

In this context, I would like to extend my highest thanks, appreciation, and heartfelt gratitude to my esteemed professor, **Dr. Hacene Mecheri**, who has truly been a scientific guide and a supportive role model throughout this journey. His continuous support, great care, and insightful guidance were the light that illuminated my research path and opened wide the doors of understanding and critical thinking. He generously gave me from his time, knowledge, and patience—things that are truly priceless. I will always remain grateful for the scientific and personal care he provided me with.

I also sincerely thank the members of the defense committee, **Dr. Bouali Tahar** and **Dr. Degaichi Nouar**, for dedicating their time to evaluate this work and enrich the discussion with their valuable insights.

Lastly, I extend my deepest gratitude to my dear parents, who were the true foundation and first source of support for me. With their prayers, encouragement, and patience, I have reached where I am today. For them, I hold all the love and appreciation in the world.

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Introduction

The concept of inverse operators has long played a central role in the development of functional analysis and operator theory. As early as the early 20^{th} century, mathematicians such as Fredholm and Hilbert laid the foundation for modern operator theory while studying integral and differential equations. However, not all linear operators admit a classical inverse, which led to fundamental difficulties in both theoretical and applied mathematics.

To address these issues, several notions of generalized inverses have been introduced over time, allowing mathematicians to extend the idea of invertibility to a broader class of linear operators. Among these, the Drazin inverse, introduced by M.P. Drazin in 1958, holds a prominent place due to its wide applicability in systems theory, control problems, and differential equations where operators or matrices are not necessarily invertible.

Later developments led to the introduction of the generalized Drazin inverse, which extended the classical Drazin inverse to a wider range of operators, offering even more flexibility and utility in various mathematical settings. These concepts have become essential tools in studying singular systems, index theory, and spectral properties of operators in Banach and Hilbert spaces.

This dissertation consists of four chapters.

The first chapter presents general definitions and essential notions concerning functional spaces and bounded linear operators that are crucial for the subsequent development.

The second chapter focuses on the classical Drazin inverse of bounded operators, exploring its properties, uniqueness, and algebraic characterizations.

In the third chapter, we deal with the generalized Drazin inverse and its relation to the spectral theory of operators.

Finally, the fourth chapter is devoted to applying the Drazin inverse in solving various types of equations and systems, demonstrating its effectiveness in theoretical and practical contexts.

In this work, we relied on a collection of previous references and studies to gain a deeper understanding of the Drazin inverse in its various forms, and to highlight its importance and applications in several areas of both theoretical and applied mathematics.

Chapter 1

Generalities and preliminaries

1 Bounded Linear Operators

- X is a Hilbert space.
- $B(X)$ denotes the complex Banach algebra of all bounded linear operators on X.

Definition 1.1. *An operator $A \in B(X)$ is said to be **bounded** if there exists a constant $C > 0$ such that, for every $x \in B(X)$, the following inequality holds:*

$$\|Ax\|_{B(H)} \leq C\|x\|_{B(H)}$$

*The set $B(X)$ denotes the **Banach algebra** of all bounded linear operators from X into itself.*

*The **norm** of an operator $A \in B(X)$ is defined by:*

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}_{B(H)}$$

Definition 1.2. *An operator $A \in B(X)$ is said to be nilpotent if there exists an integer $p \in \mathbb{N}$ such that:*

$$A^{p-1} \neq 0 \quad \textit{and} \quad A^p = 0.$$

Dual Space. The dual space X^* of a normed space X is the set of all bounded linear functionals x^* defined on X. The space X^* is itself a Banach space when endowed with the norm:

$$\|x^*\|_{X^*} = \sup\{|x^*(x)| : x \in X, \|x\| = 1\}.$$

Definition 1.3 (Adjoint of an Operator)**.** *Let $A \in B(X)$, where X is a normed vector space. The **adjoint** of the operator A, denoted by A^*, is the unique linear operator from X^* to itself that satisfies the relation:*

$$\forall x \in X, \forall \ell \in X^*, \quad (A^*(\ell))(x) = \ell(Ax).$$

In other words, A^* is the operator such that composing ℓ with A is equivalent to evaluating $A^*(\ell)$ at x.

In the special case where X is a Hilbert space endowed with an inner product $\langle \cdot, \cdot \rangle$, the adjoint $A^* \in B(X)$ is characterized as the unique operator satisfying:

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \quad \textit{for all } x, y \in X.$$

2 Resolvent and Spectrum

Let $A \in B(X)$ and let I be the identity operator on X. We define:

- The **spectrum** $\sigma(A)$ of A by:

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible}\}.$$

- The **resolvent set** $\rho(A)$ of A by:

$$\rho(A) = \mathbb{C} \setminus \sigma(A).$$

- The **resolvent operator** $R(\lambda, A)$ of A by:

$$R(\lambda, A) = (A - \lambda I)^{-1} \in B(X), \quad \forall \lambda \in \rho(A).$$

- The **spectral radius** $r(A)$ of A by:

$$r(A) := \inf_{n \in \mathbb{N}} \|A^n\|^{1/n} = \lim_{n \to \infty} \|A^n\|^{1/n}.$$

Note that the spectrum of a self-adjoint operator $A \in B(X)$ is real, and if A is quasi-nilpotent, then:

$$\sigma(A) = \{0\} \quad \text{or} \quad r(A) = 0.$$

The **point spectrum** of A:

$$\sigma_p(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective}\}.$$

The **surjectivity spectrum** and the **approximate spectrum** of A are respectively defined as:

$$\sigma_{su}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not surjective}\},$$
$$\sigma_{ap}(A) := \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not injective or } \operatorname{Im}(A - \lambda I) \text{ is not closed}\}.$$

We recall that:

$$\sigma_{ap}(A^*) = \sigma_{su}(A) \quad \text{and} \quad \sigma_{su}(A^*) = \sigma_{ap}(A).$$

Moreover:

$$\sigma(A) = \sigma_{ap}(A) \cup \sigma_{su}(A).$$

Some important results regarding the spectrum and the resolvent operator are presented in the followi

Theorem 1.1. *Let $A \in B(X)$. Then:*

1. *If $\lambda \in \rho(A)$ and $|\lambda - \mu| \leq \|R(\lambda, A)\|^{-1}$, then $\mu \in \rho(A)$. Hence, the resolvent set $\rho(A)$ is open in $\mathbb{C}$.*

2. *For all $\lambda, \mu \in \rho(A)$, the resolvent operator satisfies the following identity:*

$$R(\lambda, A) - R(\mu, A) = (\lambda - \mu)R(\lambda, A)R(\mu, A).$$

Moreover, $R(\lambda, A)$ commutes with both A and $R(\mu, A)$.

Theorem 1.2. *Let $A, B \in B(X)$. Then, $I - \lambda AB$ is invertible if and only if $I - \lambda BA$ is invertible for all $\lambda \neq 0$. In this case, the following identities hold:*

$$(I - \lambda BA)^{-1} = I + \lambda B(I - \lambda AB)^{-1}A \quad (1.1)$$

and

$$(I - \lambda AB)^{-1} = I + \lambda A(I - \lambda BA)^{-1}B.$$

Corollary 1.1. *Let $A, B \in B(X)$. If $\lambda^{-1} \in \rho(AB)$, then:*

$$(I - \lambda AB)^{-1}A = A(I - \lambda BA)^{-1}.$$

3 Some Algebraic Properties

Let $A \in B(X)$. The kernel of A is defined as $N(T) = \{x \in X : Ax = 0\}$, denoted by $N(A)$, and the range (or image) of A, denoted by $R(A)$, is given by $R(A) = A(X)$.

1.3.1 Ascent and Descent of an Operator

For a linear operator A on X, the sequence of kernels $N(T^n)$ forms an increasing chain, while the sequence of ranges $R(T^n)$ forms a decreasing chain.

- Kernel
$$= N(A) =\subseteq S \in B(H); A(S) = 0 \subseteq \dots$$

- Range=
$$R(A) =\subseteq A(s)/SinB(H) \subseteq \dots$$

as follows:

- Kernel chain:
$$N(A^0) = \{0\} \subseteq N(A) \subseteq N(A^2) \subseteq \dots$$

- Range chain:
$$R(A^0) = X \supseteq R(A) \supseteq R(A^2) \supseteq \dots$$

We define:

- The ascent of A, denoted $a(A)$, as the smallest non-negative integer n such that:

$$N(A^n) = N(A^{n+1}).$$

- The descent of A, denoted $d(A)$, as the smallest non-negative integer n such that:

$$R(A^n) = R(A^{n+1}).$$

It is straightforward to observe that $a(A) = 0$ if and only if A is injective, and $d(A) = 0$ if and only if A is surjective.

Theorem 1.3. *Let $A \in B(X)$ and $m \in \mathbb{N}$. The following statements are equivalent:*

1. *$a(A) \leq m < \infty$ if and only if $R(A^m) \cap N(A^n) = \{0\}$ for every integer $n \geq 1$;*
2. *$d(A) \leq m < \infty$ if and only if $R(A^n) \oplus N(A^m) = X$ for every integer $n \geq 1$;*
3. *If both $a(A)$ and $d(A)$ are finite, then necessarily $a(A) = d(A)$.*

Moreover, for each $k \in \mathbb{N}^*$, define the restriction:

$$A[k] = A|_{R(A^k)},$$

then the following statements are equivalent:

Lemma 1.1. 1. *The ascent $a(A)$ is finite;*
2. *There exists $k \in \mathbb{N}$ such that $A[k]$ is injective;*
3. *There exists $k \in \mathbb{N}$ such that $a(A[k]) < \infty$.*

Proof 1.1. *(i) $\Leftrightarrow$ (ii). If $p = a(A) < \infty$, using Theorem 1.3, we have:*

$$N(A^p) = R(A^p) \cap N(A) = \{0\}.$$

Conversely, suppose that $N(A^k) = \{0\}$ for some $k \in \mathbb{N}$. If $x \in N(A^{k+1})$, then $A(A^k x) = 0$, so $A^k x \in N(A) \cap R(A^k) = N(A^k) = \{0\}$.

Therefore, $x \in N(A^k)$. This shows that $N(A^{k+1}) \subseteq N(A^k)$, thus $N(A^{k+1}) = N(A^k)$. Hence, $a(A) \leq k$.

(ii) $\Leftrightarrow$ (iii). The implication (ii) $\Rightarrow$ (iii) is obvious. To show the reverse implication, assume that $m = a(A[k]) < \infty$. By Theorem 1.3, we get:

$$\{0\} = N(A[k]) \cap R(A^{mk}) = (N(A) \cap R(A^k)) \cap R(A^{m[k]}) = N(A) \cap R(A^{m[k]}) = N(A) \cap R(A^{m+k}) = N(A^{m+k}$$

Similarly, the following result for descent can be shown.

Lemma 1.2. *Let $A \in B(X)$. Then the following properties are equivalent:*

1. *$d(A) < \infty$;*
2. *There exists $k \in \mathbb{N}$ such that $A[k]$ is surjective;*
3. *There exists $k \in \mathbb{N}$ such that $d(A[k]) < \infty$.*

1.3.2 Nullity and Deficiency

Let $A \in B(X)$. Then:

- The **nullity** of A is the natural number $\alpha(A) = \dim N(A)$, where $N(A)$ denotes the kernel of A.
- The **deficiency** of A is the natural number $\beta(A) = \operatorname{codim} R(A)$, where $R(A)$ denotes the range of A.

Theorem 1.4. *Let $A \in B(X)$. If $R(A)$ is closed, then:*

$$\alpha(A^*) = \beta(A) \quad \textit{and} \quad \alpha(A) = \beta(A^*).$$

The following theorem provides the fundamental relationships between the quantities $a(A)$, $d(A)$, $\alpha(A)$, and $\beta(A)$.

Theorem 1.5. *Let $A \in B(X)$. Then:*

(i) *If $a(A) < \infty$, then $\alpha(A) \leq \beta(A)$.*

(ii) *If $d(A) < \infty$, then $\beta(A) \leq \alpha(A)$.*

(iii) *If $a(A) = d(A) < \infty$, then $\alpha(A) = \beta(A)$.*

(iv) *If $\alpha(A) = \beta(A) < \infty$, and either $a(A) < \infty$ or $d(A) < \infty$, then $a(A) = d(A) < \infty$.*

4 Operators with Closed Range

Theorem 1.6. *Let $A \in B(X)$. The following assertions are equivalent:*

- *$R(A)$ is closed.*
- *$R(A^*)$ is closed.*
- *$R(A) = N(A^*)^\perp$.*
- *$R(A^*) = N(A)^\perp$.*

Lemma 1.3. *Let $A \in B(X)$ be injective. Then $R(A)$ is closed if and only if there exists a constant $c > 0$ such that:*

$$\|x\| \leq c\|Ax\| \quad \textit{for all } x \in X.$$

An operator $A \in B(X)$ is said to be bounded below if it is injective and has closed range.

Definition 1.4. *The reduced minimum modulus (also called the conorm) of an operator $A \in B(X)$ is defined as:*

$$\gamma(A) := \inf_{x \notin N(A)} \frac{\|Ax\|}{\operatorname{dist}(x, N(A))}.$$

Theorem 1.7. *Let $A \in B(X)$. Then:*

$$\gamma(A) > 0 \quad \textit{if and only if the range of } A,\ R(A),\ \textit{is closed.}$$

Lemma 1.4. *Let $A \in B(X)$. Then:*

(i) A is bounded below if and only if its adjoint A^* is surjective.

(ii) If A is bounded below (respectively, surjective), then for every $\lambda \in \mathbb{C}$ with $|\lambda| < \gamma(A)$, the operator $A - \lambda I$ is also bounded below (respectively, surjective).

Proposition 1.1. *Let $A \in B(X)$ be an operator with closed range, and let Y be a subspace of X (not necessarily closed). If the sum $Y + N(A)$ is closed in X, then the image $A(Y)$ is also closed in X.*

5 Riesz Projection

Let $\lambda_0 \in \sigma(A)$. The complex number λ_0 is said to be an *isolated point* of the spectrum $\sigma(A)$ if there exists a positive number $\varepsilon > 0$ such that

$$\sigma(A) \cap \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| < \varepsilon\} = \{\lambda_0\}.$$

The collection of all such isolated spectral points is denoted by iso$\sigma(A)$, whereas the set of accumulation points is denoted by acc$\sigma(A)$.

Consider a positively oriented circle Γ_{λ_0} in the complex plane, centered at λ_0 with radius ε, given by:

$$\Gamma_{\lambda_0} = \{\lambda \in \mathbb{C} : |\lambda - \lambda_0| = \varepsilon\}.$$

The Riesz projection associated with λ_0, defined via the contour integral over Γ_{λ_0}, is:

$$P_{\lambda_0} = \frac{1}{2\pi i} \int_{\Gamma_{\lambda_0}} (A - \lambda I)^{-1} \, d\lambda. \tag{1.2}$$

This projection operator satisfies the following key properties:

Proposition 1.2. *Let $T \in B(X)$, and suppose λ_0 and μ_0 are isolated points in $\sigma(T)$. Then:*

1. *The null space $N(A - \lambda_0 I)$ is contained in the range of the Riesz projection:*

$$N(A - \lambda_0 I) \subset R(P_{\lambda_0}).$$

2. *The space X admits a direct sum decomposition:*

$$X = R(P_{\lambda_0}) \oplus N(P_{\lambda_0}).$$

3. *If $\lambda_0 \neq \mu_0$, then the associated Riesz projections are orthogonal in the algebraic sense:*

$$P_{\lambda_0} P_{\mu_0} = P_{\mu_0} P_{\lambda_0} = 0, \quad \text{and} \quad P_{\{\lambda_0, \mu_0\}} = P_{\lambda_0} + P_{\mu_0}.$$

4. *If X is a Hilbert space and A is self-adjoint, then P_{λ_0} coincides with the orthogonal projection onto $N(A - \lambda_0 I)$.*

Chapter 2

Drazin Inverse of Bounded Operators

1 Drazin Inverse in $L(X)$

k $\in \mathbb{N}$

Definition 2.1. *If $A \in L(X)$, the **Drazin inverse** of A, denoted by A^d, is an operator that satisfies:*

- $AA^d = A^dA$,
- $A^{k+1}A^d = A^k$,
- $A^dAA^d = A^d$

*where k is the smallest positive integer such that $R(A^k) = R(A^{k+1})$; this number is known as the **Drazin index** and is denoted by* $\text{ind}(A) = k$.

Remark 2.1. *If the Drazin index k exists for an operator A, then the restriction $A|_{R(A^k)}$ is invertible, and the Drazin inverse is given by:*

$$A^d = (A|_{R(A^k)})^{-1}.$$

1 Equivalent Definitions of the Drazin Inverse

For $A \in L(X)$, the following conditions are equivalent to the existence of the Drazin inverse with index $k > 0$:

(i) A^d exists and $\text{ind}(A) = k$.

(ii) $a(A) = d(A) = k$.

(iii) Zero is a pole of order k for the resolvent function $R(\lambda, A)$.

(iv) The space X decomposes as a direct sum:

$$X = R(A^k) \oplus N(A^k),$$

where $K(A) = R(A^k)$ and $H_0(A) = N(A^k)$.

(v) There exists a projection P such that $X = R(P) \oplus N(P)$ with:

$$R(A^k) = N(P), \quad N(A^k) = R(P).$$

(vi) The operator A decomposes as:

$$A = A_1 \oplus A_2,$$

where $A_1 = A|_{R(A^k)}$ is invertible and $A_2 = A|_{N(A^k)}$ is nilpotent.

2 Drazin Spectrum

The **Drazin spectrum** of an operator A is defined as:

$$\sigma_D(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not invertible in the Drazin sense}\}.$$

3 Right and Left Drazin Inverses

An operator A is said to be **right-invertible** if there exists an operator S such that:

$$AS = I.$$

In this case, S is called the **right inverse** of A, and we have:

$$X = R(S) \oplus N(A).$$

Similarly, A is **left-invertible** if there exists an operator L such that:

$$LA = I.$$

Here, L is the **left inverse** of A, with the decomposition:

$$X = R(A) \oplus N(L).$$

Right and Left Drazin Inverses: Based on the definition of the Drazin inverse, one can relax the invertibility condition by replacing it with right or left invertibility. Thus, we say that an operator A has a **right or left Drazin inverse** if:

$$A = A_1 \oplus A_2,$$

where A_1 is right (or left) invertible and A_2 is nilpotent.

Proposition 2.1. *For $A \in L(X)$, the following hold:*

*(a) If $a(A)$ is finite and $R(A^{a(A)+1})$ is closed, then A has a **left Drazin inverse**.*

*(b) If $d(A)$ is finite and $R(T^{d(A)})$ is closed, then A has a **right Drazin inverse**.*

4 Left and Right Drazin Spectra

The left and right Drazin spectra are defined as:

$$\sigma_D^l(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not left-invertible in the Drazin sense}\},$$

$$\sigma_D^r(A) = \{\lambda \in \mathbb{C} \mid A - \lambda I \text{ is not right-invertible in the Drazin sense}\}.$$

2 Special Cases and Generalizations

1 Drazin Inverse for Square Matrices

Definition 2.2. *Let A be a square matrix. The* ***index of*** *A, denoted by* $\mathrm{ind}(A)$*, is the smallest positive integer k such that:*

$$\mathrm{Rank}(A^k) = \mathrm{Rank}(A^{k+1}).$$

Definition 2.3. *Let A be a square matrix with index k. The* ***Drazin inverse*** *of A, denoted by A^d, is a square matrix that satisfies:*

$$A^d A A^d = A^d,$$

$$A^d A = A A^d,$$

$$A^d A^{k+1} = A^k,$$

where k is the index of A.

2 Drazin Inverse in Banach Algebras

Definition 2.4. *Let $B(x)$ be a Banach algebra. An element $a \in B(x)$ is said to be Drazin invertible with index k if there exists an element $b \in B(x)$ satisfying:*

$$ab = ba,$$

$$bab = b,$$

$$a^k = a^k ba.$$

Proposition 2.2. *Let A be an algebra and suppose that $a \in B(x)$ is Drazin invertible with index k. If $b \in B(x)$ is the Drazin inverse of a, then the condition*

$$a - aba \in N(B(x))$$

is equivalent to

$$a^k = a^k ba.$$

3 The Group inverse

Definition 2.5. *If the index of A is 1, then the Drazin inverse of A, denoted by $A^{\#}$, is called the* ***Group inverse*** *of A.*

Definition 2.6. *The* ***Group inverse*** *of a square matrix A is a square matrix, denoted by A^g, that satisfies the following conditions:*

1. $A^g A A^g = A^g$,
2. $A^g A = A A^g$,
3. $A A^g A = A$.

Remark 2.2. *If A is an idempotent matrix, then*

$$A = A^g.$$

Theorem 2.1. *Let A be a matrix in $\mathbb{C}^{n\times n}$. The group inverse of A, denoted by $A^{\#}$, exists if and only if:*

$$\operatorname{rank}(A) = \operatorname{rank}(A^2).$$

Theorem 2.2. *If A is a matrix in $\mathbb{C}^{n\times n}$ with index 1, then:*

$$\operatorname{rank}(A) = \operatorname{rank}(A^{\#}).$$

Corollary 2.1. *If A is an invertible matrix in $\mathbb{C}^{n\times n}$, then its index satisfies:*

$$\operatorname{ind}(A) = 0.$$

Corollary 2.2. *If A is an idempotent matrix, then its index satisfies:*

$$\operatorname{ind}(A) = 1.$$

Theorem 3.3:
If A is a symmetric matrix in $\mathbb{C}^{n\times n}$ with index 1, then the group inverse of A is self-adjoint, meaning:

$$A^{\#} = (A^{\#})^*.$$

Corollary 2.3. *For any matrix $A \in \mathbb{C}^{n\times n}$ such that $\operatorname{ind}(A) = 1$, we have:*

$$(A^{\#})^{\#} = A.$$

4 Additional Results in Algebras

Lemma 2.1. *Let $B(x)$ be an algebra, and $a, b \in B(x)$. If $b \in \{a\}'$ and $bab = b$, then for all $k \in \mathbb{N}$, we have:*

$$(a - bab)^k = 0 \quad \text{if and only if} \quad a^k = a^k ba.$$

Lemma 2.2. *Let $B(x)$ be an algebra, $a \in B(x)$, and let b be the Drazin inverse of a. If a has index k, then a has index $k + 1$.*

Proposition 2.3. *Let a^d be the Drazin inverse of a with index k. Then, the following holds:*

$$a^k \left((a^d)\right)^{k+1} = a^d.$$

3 Properties of the Drazin Inverse

Proposition 2.4. *1. If A is invertible, then its index is* $\mathrm{ind}(A) = 0$.

2. If A is a non-singular matrix, then $\mathrm{ind}(A) = 0$.

3. A^d and AA^d are projection operators onto $R(A)$.

4. If A is idempotent, then $A = A^d$.

5. $(A^d)^d = A$ if and only if $\mathrm{ind}(A) = 1$.

6. The Drazin inverse satisfies:

$$((A^d)^d)^d = A^d.$$

7. The Drazin inverse of the conjugate transpose is given by:

$$(A^*)^d = (A^d)^*.$$

8. The Drazin inverse of a power satisfies:

$$(A^l)^d = (A^d)^l, \quad \forall\, l = 1, 2, 3, \ldots.$$

9. If A has index k, then A^l has index 1 and:

$$(A^l)^{\#} = (A^d)^l.$$

10. If A has index k, then for all $l \geq k$, the range and null space remain the same:

$$\mathrm{Rank}(A^d) = \mathrm{Rank}(A^l), \quad \ker(A^d) = \ker(A^l).$$

1 Some Properties in $B(x)$

Lemma 2.3. *For any operator A in $B(x)$, the range $R(A)$ is closed if and only if there exists $X \in B(x)$ such that:*

$$AXA = A.$$

Lemma 2.4. *Let $A \in B(x)$. If there exists a positive integer k and an operator $X \in B(x)$ such that:*

$$AX = XA, \quad XAX = X, \quad A^{k+1}X = A^k,$$

then the rank of A^k is equal to the rank of A^{k+1}, and the range $R(A^k)$ is closed.

Lemma 2.5. *Let $A \in B(x)$ with* $\mathrm{ind}(A) = k$. *If A^d denotes the Drazin inverse of A, then:*

$$\mathrm{Rank}(A^d) = \mathrm{Rank}(A^k).$$

2 Additional Results for Matrices

Theorem 2.3. *Let $A \in \mathbb{C}^{n\times n}$ with $\text{ind}(A) = k$. Then, the index of A is equal to the index of its conjugate transpose, i.e.,*

$$\text{ind}(A) = \text{ind}(A^*).$$

Theorem 2.4. *If $A \in \mathbb{C}^{n\times n}$ and $\text{ind}(A) = k$, then the rank of A is equal to the rank of its Drazin inverse, i.e.,*

$$\text{rank}(A) = \text{rank}(A^d).$$

Proof 2.1. *First, we have the inequality:*

$$\begin{aligned}\text{rank}(A^d) &= \text{rank}(A^dAA^d)\\ &\leq \text{rank}(A^dAA)\\ &\leq \text{rank}(A) \quad \textit{...(1)}\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\text{rank}(A) &= \text{rank}(A^dAA^d)\\ &\leq \text{rank}(A^dA)\\ &\leq \text{rank}(A^d) \quad \textit{...(2)}\end{aligned}$$

By combining inequalities (1) and (2), we conclude that:

$$\text{rank}(A) = \text{rank}(A^d).$$

3 The Fundamental Properties of the Matrix Index

Corollary 2.4. *Let $A \in \mathbb{C}^{n\times n}$ be a matrix with $\text{ind}(A) = k > 1$. Then,*

$$\text{ind}(A) = \text{ind}(A^T).$$

Theorem 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be a Hermitian matrix with index k. Then,*

$$\text{ind}(A) = \text{ind}(A^*) = \text{ind}(A^*A).$$

Corollary 2.5. *Let $A \in \mathbb{C}^{n\times n}$ be a normal matrix with index k. Then,*

$$\text{ind}(A) = \text{ind}(A^*) = \text{ind}(AA^*) = \text{ind}(A^*A).$$

Theorem 2.6. *Let $A \in \mathbb{C}^{n\times n}$ be a symmetric matrix with index $k \geq 1$. Then,*

$$A^d = (A^d)^T.$$

Theorem 2.7. *If λ is an eigenvalue of $A \in \mathbb{C}^{n\times n}$, then $\frac{1}{\lambda}$ is an eigenvalue of A^d.*

Proof 2.2. *Assume λ is an eigenvalue of A with corresponding eigenvector x ($x \neq 0$); that is,*

$$Ax = \lambda x.$$

Then:

$$A^d Ax = \lambda A^d x \quad \Longrightarrow \quad AA^d x = \lambda A^d x.$$

Applying A^d again,

$$A^d AA^d x = \lambda A^d A^d x.$$

Setting $y = A^d x$, we deduce

$$A^d y = \frac{1}{\lambda} y.$$

Thus, $\frac{1}{\lambda}$ is an eigenvalue of A^d.

Lemma 2.6. *Let A be an algebra. The Drazin inverse of any element $a \in A$ is unique.*

4 Factorization Method

This method is used to find the Drazin inverse of a singular square matrix.

Lemma 2.7. *Let $B, C \in \mathbb{C}^{r\times r}$. If a matrix A can be factorized as*

$$A = BC,$$

then its Drazin inverse is given by:

$$A^d = B(CB)^2 C.$$

Proof 2.3. *We first note that for any square matrix A and any positive integers m, n with $m > n$,*

$$(A^d)^m A^n = (A^d)^{m-n},$$

and if $m \geq k$ (with $k = \mathrm{ind}(A)$), then:

$$A^{n+m}(A^d)^n = (A^d)^n.$$

Now, we aim to prove:

$$(BC)^d = (BC)^{k+1}\Big(((BC)^d)^{k+2}\Big).$$

Indeed, observe that:

$$(BC)^{k+1}\Big(((BC)^d)^{k+2}\Big) = (BC)^k \cdot (BC) \cdot \Big((BC)^d\Big)^2 \cdot \Big((BC)^d\Big)^k.$$

Through appropriate algebraic manipulation, we ultimately obtain:

$$(BC)^d = (BC)^{k+1}\Big(((BC)^d)^{k+2}\Big).$$

Thus, it follows that:

$$A^d = (BC)^d = B\Big((BC)^d\Big)^2 C.$$

5 The Drazin Inverse and the Power

Lemma 2.8. *Let $A \in B(x)$, $a \in B(x)$. An element $b \in B(x)$ is said to be the* ***Drazin inverse*** *of a if there exists $k \in \mathbb{N}$ such that b^k is the* ***group inverse*** *of a^k in $B(x)$.*

Proof 2.4. *Assume that b is the Drazin inverse of a in $B(x)$. We wish to show that b^k is a group inverse of a^k in $B(x)$.*

We have:

$$a^k b^k = (ab)^k = (ba)^k = b^k a^k,$$

and

$$b^k = (bab)^k = b^k a^k b^k.$$

Also,

$$a^k = a^{k+1}b = a^k(ba) = a^k(ba)^k = a^k b^k a^k.$$

Conversely, suppose that for a given $k \in \mathbb{N}$, the operator a^k has a group inverse $(a^k)^g = c$. Then it can be verified that ca^{k-1} is a Drazin inverse of a with index k. In particular:

$$a(ca^{k-1}) = ca^k = (ca^{k-1})a,$$

and

$$(ca^{k-1})a(ca^{k-1}) = ca^k(ca^{k-1}) = ca^{k-1}.$$

Moreover,

$$a^k(ca^{k-1})a = a^k c a^k = a^k.$$

Corollary 2.6. *Let $A \in B(x)$, and $a \in B(x)$ be a Drazin invertible element with index k. Then a^k is also Drazin invertible with some index p (where $p > 1$), and it satisfies:*

$$(a^k)^d = (a^d)^k.$$

Furthermore, a^d satisfies:

$$a^d = (a^n)^d a^{n-1}, \quad \textit{for all } n \in \mathbb{N}.$$

Theorem 2.8. *Let $a, b, c \in B(x)$ satisfy:*

$$aba = aca.$$

Let $n, m \in \mathbb{N}$. Then:

1. First Case: If *$(ac)^n$* ***is Drazin invertible,***
then $(ba)^m$ is also Drazin invertible. In this case:

- *If $n \geq m + 1$:*

$$((ba)^m)^d = b\Big((ac)^n\Big)^d a(ba)^{(n-m-1)}.$$

- *If $n < m + 1$:*

$$((ba)^m)^d = b\Big(((ac)^n)^d\Big)^{(m+2-n)} a(ba)^{(n-1)(m+1-n)}.$$

2. Second Case: If $(bc)^n$ ***is Drazin invertible,***
then $(ac)^m$ *is also Drazin invertible. In this case:*

- *If* $n \geq m+1$*:*

$$((ac)^m)^d = a\Big((bc)^n\Big)^d c(ac)^{(n-m-1)}.$$

- *If* $n < m+1$*:*

$$((ac)^m)^d = a\Big[((ba)^n)^d\Big]^{(m+2-n)} c(ac)^{(n-1)(m+1-n)}.$$

Proof 2.5. *If* $n \geq m+1$*, then by applying previous theorems and lemmas, we have:*

$$((ba)^m)^d = ((ba)^d)^m = \big(b((ac)^d)^2a\big)^m.$$

Through repeated multiplication and algebraic manipulation, we eventually arrive at:

$$((ba)^m)^d = b\Big((ac)^n\Big)^d (ac)^{n-m-1}a.$$

Similarly, if $n < m+1$*, then we obtain:*

$$((ba)^m)^d = b\Big[((ac)^n)^d\Big]^{(m+2-n)} (ac)^{(n-1)(m+1-n)}a.$$

In the framework of Banach algebras, assuming $aba = aca$, the following theorem provides an explicit formula for the generalized Drazin inverse of $(ac)^n$ and $(ba)^m$.

Theorem 2.9. *Let* $a, b, c \in A$ *satisfy the condition* $aba = aca$*, and let* $n, m \in \mathbb{N}$*. Then:*

1. *If* $(ac)^n$ *admits a generalized Drazin inverse, then* $(ba)^m$ *also has a generalized Drazin inverse. In this case:*
   - *When* $n \geq m+1$*:*

$$((ba)^m)^d = b\left((ac)^n\right)^d a\,(ba)^{n-m-1}.$$

   - *When* $n < m+1$*:*

$$((ba)^m)^d = b\left(((ac)^n)^d\right)^{m+2-n} a\,(ba)^{(n-1)(m+1-n)}.$$

2. *If* $(bc)^n$ *admits a generalized Drazin inverse, then* $(ac)^m$ *also possesses a generalized Drazin inverse. In this case:*
   - *When* $n \geq m+1$*:*

$$((ac)^m)^d = a\left((bc)^n\right)^d c\,(ac)^{n-m-1}.$$

   - *When* $n < m+1$*:*

$$((ac)^m)^d = a\left(((ba)^n)^d\right)^{m+2-n} c\,(ac)^{(n-1)(m+1-n)}.$$

6 Representation of the Drazin Inverse by a Block Matrix

Let A be a matrix given in the following block form:

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

where

$$B \in \mathbb{C}^{p\times(n-p)} \quad \text{and} \quad C \in \mathbb{C}^{(n-p)\times p}.$$

The following result provides the Drazin inverse A^d in terms of the Drazin inverse of the product BC.

Theorem 2.10. *Let $A \in \mathbb{C}^{n\times n}$ be the block matrix*

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

with $B \in \mathbb{C}^{p\times(n-p)}$ and $C \in \mathbb{C}^{(n-p)\times p}$. Then, the Drazin inverse of A is given by

$$A^d = \begin{pmatrix} 0 & (BC)^d B \\ C\,(BC)^d & 0 \end{pmatrix}.$$

Moreover, if $\operatorname{ind}(BC) = s$, *then*

$$\operatorname{ind}(A) \leq 2s + 1.$$

Definition 2.7. *Let $B(x)$ be a Banach algebra. An element $a \in B(x)$ is said to be Drazin invertible with index k if there exists an element $b \in B(x)$ satisfying:*

$$ab = ba,$$

$$bab = b,$$

$$a^k = a^k ba.$$

Proposition 2.5. *Let $B(x)$ be an algebra and suppose that $a \in B(x)$ is Drazin invertible with index k. If $b \in B(x)$ is the Drazin inverse of a, then the condition*

$$a - aba \in N(A)$$

is equivalent to

$$a^k = a^k ba.$$

Theorem 2.11.

- *When $n \geq m + 1$:*

$$((ac)^m)^d = a\,((bc)^n)^d\,c\,(ac)^{n-m-1}.$$

- *When $n < m + 1$:*

$$((ac)^m)^d = a\left(((ba)^n)^d\right)^{m+2-n} c\,(ac)^{(n-1)(m+1-n)}.$$

7 The Index of A and the Index of BC

Consider the matrix

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

Then,

$$A^2 = \begin{pmatrix} BC & 0 \\ 0 & CB \end{pmatrix}.$$

More generally, for any positive integer j:

$$A^{2j} = \begin{pmatrix} (BC)^j & 0 \\ 0 & (CB)^j \end{pmatrix},$$

and

$$A^{2j+1} = \begin{pmatrix} 0 & (BC)^j B \\ (CB)^j C & 0 \end{pmatrix} = \begin{pmatrix} 0 & B\,(CB)^j \\ (CB)^j C & 0 \end{pmatrix}.$$

Thus, we obtain the following rank relations:

$$\operatorname{rank}(A^{2j}) = \operatorname{rank}((BC)^j) + \operatorname{rank}((CB)^j), \tag{1}$$

$$\operatorname{rank}(A^{2j+1}) = \operatorname{rank}((BC)^j B) + \operatorname{rank}(C\,(CB)^j). \tag{2}$$

These relations allow us to express the index of A in terms of the indices of BC and CB.

Lemma 2.9. *Let $U \in \mathbb{C}^{m\times k}$, $V \in \mathbb{C}^{k\times n}$, and $W \in \mathbb{C}^{n\times p}$. Then, the following equality holds:*

$$\operatorname{rank}(UV) + \operatorname{rank}(VW) = \operatorname{rank}(V) + \operatorname{rank}(UVW).$$

Theorem 2.12. *Let A be the matrix*

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}.$$

If $\operatorname{ind}(BC) = s$ with $s \geq 1$, then the index of A satisfies:

$$\operatorname{ind}(A) \in \{2s-1,\ 2s,\ 2s+1\}.$$

Proof 2.6. *According to previous results, we have:*

$$\operatorname{ind}(A) \leq 2s+1.$$

Using Lemma 2.13 (which states that $(BC)^d B = B(CB)^d$), we deduce:

$$\begin{aligned} \operatorname{rank}\left(B(CB)^{s-1}\right) + \operatorname{rank}\left((CB)^{s-1}C\right) &\leq \operatorname{rank}\left((CB)^{s-1}\right) + \operatorname{rank}\left((BC)^s\right) \\ &< \operatorname{rank}\left((CB)^{s-1}\right) + \operatorname{rank}\left((BC)^{s-1}\right). \end{aligned}$$

Using equations (1) and (2), we obtain:

$$\operatorname{rank}\left(A^{2j-1}\right) < \operatorname{rank}\left(A^{2j-2}\right),$$

which implies

$$\operatorname{ind}(A) > 2s-2.$$

Thus, the possible values for the index of A are:

$$\operatorname{ind}(A) \in \{2s-1,\ 2s,\ 2s+1\}.$$

Theorem 2.13. *Let A be as above, and suppose* $\operatorname{ind}(BC) = s$ *with* $s \geq 1$*. Then:*

$$\operatorname{ind}(A) = 2s - 1$$

if and only if the following conditions hold:

1. $\operatorname{rank}((BC)^s) = \operatorname{rank}((BC)^{s-1}B)$ *and* $\operatorname{rank}((CB)^s) = \operatorname{rank}((CB)^{s-1}C)$.
2. $\operatorname{rank}((BC)^s) = \operatorname{rank}((CB)^{s-1}C)$ *and* $\operatorname{rank}((CB)^s) = \operatorname{rank}((BC)^{s-1}B)$.

Lemma 2.10. *If* $\operatorname{ind}(BC) = s$*, then the following ranks are all equal:*

$$\begin{aligned}\operatorname{rank}((BC)^{s+1}) &= \operatorname{rank}((BC)^s) \\ &= \operatorname{rank}((BC)^s B) \\ &= \operatorname{rank}(C(BC)^s) \\ &= \operatorname{rank}((CB)^{s+1}).\end{aligned}$$

Theorem 2.14. *Let A be a matrix of the form*

$$A = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix},$$

such that $\operatorname{ind}(BC) = s \geq 1$*. Then,* $\operatorname{ind}(A) = 2s$ *if and only if* $\operatorname{ind}(CB) = s$ *and at least one of the following conditions holds:*

- $\operatorname{rank}((BC)^s) < \operatorname{rank}((BC)^{s-1}B)$,
- $\operatorname{rank}((CB)^s) < \operatorname{rank}((CB)^{s-1}C)$.

Chapter 3

Generalized Drazin Inverse

Definition 3.1. *An operator $A \in L(X)$ is said to be g-Drazin invertible if there exists a unique operator $B \in L(X)$ such that:*

$$AB = BA, \quad BAB = B, \quad A - ABA \text{ is quasi-nilpotent.}$$

The operator B is called the generalized Drazin inverse of A, denoted by A^d.

1 Generalized Drazin Decomposition

Definition 3.2. *If A is g-Drazin invertible, based on the decomposition of $X = N(I - AA^d) \oplus R(I - AA^d)$, the matrix forms of the operators A and A^d are as follows:*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad A^d = \begin{pmatrix} A_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

where A_1 is invertible and A_2 is quasi-nilpotent.

Theorem 3.1. *Let $a, b, c \in A$ satisfy the condition $aba = aca$, and let $n, m \in \mathbb{N}$. Then:*

1. *If $(ac)^n$ admits a generalized Drazin inverse, then $(ba)^m$ also has a generalized Drazin inverse. In this case:*
   - *When $n \geq m + 1$:*
   $$((ba)^m)^d = b\,((ac)^n)^d\,a\,(ba)^{n-m-1}.$$
   - *When $n < m + 1$:*
   $$((ba)^m)^d = b\left(((ac)^n)^d\right)^{m+2-n} a\,(ba)^{(n-1)(m+1-n)}.$$
2. *If $(bc)^n$ admits a generalized Drazin inverse, then $(ac)^m$ also possesses a generalized Drazin inverse. In this case:*
   - *When $n \geq m + 1$:*
   $$((ac)^m)^d = a\,((bc)^n)^d\,c\,(ac)^{n-m-1}.$$

- *When $n < m+1$:*

$$((ac)^m)^d = a\left(((ba)^n)^d\right)^{m+2-n} c\,(ac)^{(n-1)(m+1-n)}.$$

Lemma 3.1. *Let $a, b, c \in R$ satisfy the relation*

$$aba = aca.$$

Then, ac possesses a generalized Drazin inverse if and only if ba does. In this case, we have the following formulas:

$$(ba)^{gd} = b \cdot \left((ac)^{gd}\right)^2 \cdot a,$$

$$(ac)^{gd} = a \cdot \left((bc)^{gd}\right)^2 \cdot c.$$

Theorem 3.2. *Let $a \in R$ and $n \in \mathbb{N}$.*

1. *If a^n admits a generalized Drazin inverse, then a also admits a generalized Drazin inverse, and we have:*
$$a^{gd} = (a^n)^{gd} \cdot a^{n-1}.$$

2. *If a admits a generalized Drazin inverse, then a^n also admits a generalized Drazin inverse, with the relation:*
$$(a^n)^{gd} = (a^{gd})^n.$$

2 Core Quasi-Nilpotent Decomposition of Operators

Definition 3.3. *Let $A \in L(X)$ be a bounded linear operator. It can be written as the sum of two operators, A_C and A_Q, such that:*

$$A_C = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad A_Q = \begin{bmatrix} 0 & 0 \\ 0 & A_2 \end{bmatrix},$$

and thus,

$$A = A_C + A_Q.$$

This decomposition is known as the core quasi-nilpotent decomposition of the operator A.

- *A_C is called the core part of A*
- *A_Q is called the quasi-nilpotent part of A.*

3 Quasi-Nilpotent Part $H_0(A)$

Definition 3.4. *The Quasi-Nilpotent Part $H_0(A)$ is the set of elements $x \in X$ such that*

$$\lim_{n\to\infty} \|A^n x\|^{1/n} = 0.$$

It represents the elements that are asymptotically sent to zero by the higher powers of the operator A.

4 Analytical Core K(A):

Definition 3.5. *The Analytical Core* $K(A)$ *is the set of elements* $x \in X$ *for which there exists a sequence* $(x_n) \subset X$ *and a constant* $\delta > 0$ *such that:*

$$Ax_1 = x, \quad Ax_{n+1} = x_n, \quad \text{and} \quad \|x_n\| \leq \frac{1}{n^\delta}$$

It represents elements that can be expressed by a rapidly converging sequence under the action of the operator A.

$$\begin{cases} A(K(A)) = K(A), \\ A(H_0(A)) \subseteq H_0(A). \end{cases}$$

Definition 3.6. *Let* $A \in L(X)$. *If the range* $R(A)$ *is closed and* $ker(A^n) \subseteq Im(A)$ *for all* $n \in \mathbb{N}$, *then* A *is called semi-regular.*

Theorem 3.3. *Let* $A \in L(X)$. *The following statements are equivalent:*

1. *A is g-Drazin invertible;*
2. $X = K(A) \oplus H_0(A)$;
3. $A = A_1 \oplus A_2$, *where:*
   - $A_1 = A|_{K(A)} \in L(K(A))$ *is invertible,*
   - $A_2 = A|_{H_0(A)} \in L(H_0(A))$ *is quasi-nilpotent.*

Theorem 3.4. *Let* F *be a closed subspace of* X *and* $A \in L(X)$. *If* $A(F) = F$, *then* $F \subseteq K(A)$.

5 Left and right pre-orders on $L(X)$

We begin with the following definitions of left and right generalized Drazin invertible operators.

Definition 3.7. *An operator* $A \in L(X)$ *is called left g-Drazin invertible if there exists a subspace* $L^A \subset X$ *such that* $H_0(A)$ *is complemented in* X, *and* $A(L^A)$ *is complemented within* L^A.

The collection of all subspaces L^A *associated with an operator* A *that satisfies this definition is denoted as* $\{L^A\}$.

Definition 3.8. *An operator* $A \in L(X)$ *is right g-Drazin invertible if:*

- *K(A) is complemented in* X *by a subspace* R_A *with*

  $A(R_A) \subseteq R_A \subseteq H_0(A)$.
- $N(A) \cap K(A)$ *is complemented in* $K(A)$.

The collection of all subspaces associated with an operator A *that satisfies Definition 2.10 is denoted as* $\{R_A\}$.

Proposition 3.1.
- *left g-Drazin invertible their objective of study is simplify and analyze the image and we used it with non zero eigenvalues*
- *right g-Drazin invertible their objective of study is understande the kernel and we used it with zero or near zero eigenvalues*

Next, we provide a characterization of left generalized Drazin invertible operators through the use of a generalized Kato decomposition.

Corollary 3.1. *Let $A \in L(X)$. Then A is g-Drazin invertible if and only if A is both left g-Drazin invertible and right g-Drazin invertible.*

In the following, we introduce new binary relations defined using the left and right g-Drazin inverses of a bounded linear operator on a Banach space.

Definition 3.9. *Let $A, B \in \mathcal{L}(X)$ be **left g-Drazin invertible** operators. We say that $A \preceq^{LD} B$ if there exist subspaces $L^A \in \{L^A\}$ and $L^B \in \{L^B\}$ and the subspace L is closed in x such that:*

1. *L^A is **complemented** in L^B,*
2. *There exists a **B-stable subspace** L of X (i.e., $B(L) \subseteq L$),*
3. *Restricting B to L_A gives the same result as restricting A to L_A.*

It is now possible to define the properties of the binary relation$\preceq^{LD}$

Theorem 3.5. *Let $A, B \in L(X)$ be left g-Drazin invertible operators. Then the following statements are equivalent:*

1. *$A \preceq^{LD} B$.*
2. *There exist topological direct sum decompositions $X = X_1 \oplus X_2$ and $X_2 = X_2^1 \oplus X_2^2$, such that:*
$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2^1 \oplus B_2^2 \end{bmatrix},$$
where:
   - *$A_1 \in L(X_1)$ and $B_2^1 \in L(X_2^1)$ are left invertible operators.*
   - *$A_2 \in L(X_2)$ and $B_2^2 \in L(X_2^2)$ are quasi-nilpotent operators.*

In general, the binary relation LD is not antisymmetric, as shown by the following result.

Remark 3.1. *If $A \preceq^{LD} B$, then*
$$H_0(A) = L \oplus H_0(B),$$
where L is a subspace of X satisfying the following conditions:

- *L is stable under the action of B, i.e., $B(L) \subseteq L$;*
- *L is complemented in L_B, meaning $L_A \oplus L = L_B$;*
- *The relation $BL_A = AL_A$ holds.*

This expression illustrates how the null space $H_0(A)$ is constructed by adding the subspace L to $H_0(B)$.

Corollary 3.2. *Let $A, B \in L(X)$ be left g-Drazin invertible operators. If $A \preceq^{LD} B$ and $B \preceq^{LD} A$, then there exists a topological direct sum decomposition.*

$$X = X_1 \oplus X_2$$

such that:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where:

- *$A_1 \in L(X_1)$ is left invertible,*
- *$A_2, B_2 \in L(X_2)$ are quasi-nilpotent.*

Proof 3.1. *Since A is left Drazin equivalent to B, Remark 3.1 implies that:*

$$H_0(A) = L \oplus H_0(B),$$

0 where L is a subspace.

Similarly, since B is left Drazin equivalent to A, there exists a subspace L' such that:

$$H_0(B) = L' \oplus H_0(A).$$

From these two equations, we conclude that:

$$H_0(A) = H_0(B),$$

which implies $L = L' = \{0\}$, and therefore $L_A = L_B$.

The final conclusion follows directly from the proof of Theorem 3.4

Theorem 3.6. *The relation$A \preceq^{LD} B$ satisfies the properties of a pre-order when applied to the set of all left generalized Drazin invertible operators in $L(X)$.*

Definition 3.10. *Let $A, B \in L(X)$ be right generalized Drazin invertible operators. If the subspace is closed in X, we say that $A \preceq^{RD} C$ if:*

- *The kernel $K(A)$ has a complement L in $K(B)$ that is preserved by B.*
- *The operators B and A act the same way on $K(A)$.*

Theorem 3.7. *Let $A, B \in L(X)$ be right generalized Drazin invertible operators. Then the following are equivalent:*

- *$A \preceq^{RD} B$.*
- *There exist decompositions $X = X_1 \oplus X_2$ and $X_2 = X_2^1 \oplus X_2^2$ such that:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2^1 \oplus B_2^2 \end{bmatrix},$$

where:

– A_1 *and* B_2^1 *are right invertible.*

– A_2 *and* B_2^2 *are quasi-nilpotent.*

Corollary 3.3. *Let $A, B \in L(X)$ be right g-Drazin invertible operators. If $A \preceq^{RD} B$ and $B \preceq^{RD} A$, then there exists a topological direct sum decomposition.*

$$X = X_1 \oplus X_2$$

such that:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B = \begin{pmatrix} A_1 & 0 \\ 0 & B_2 \end{pmatrix},$$

where:

- $A_1 \in L(X_1)$ *is right invertible,*
- $A_2, B_2 \in L(X_2)$ *are quasi-nilpotent.*

Proof 3.2. *Since $A \preceq^{RD} B$, by Definition 3.11, we have*

$$K(B) = L \oplus K(A).$$

Similarly, from $B \preceq^{RD} A$, we obtain

$$K(A) = L' \oplus K(B),$$

where L' is some subspace.

From these two equations, it follows that

$$K(A) = K(B) \quad \textit{and} \quad L = L' = \{0\}.$$

Therefore, the conclusion follows directly from the proof of Theorem 3.6.

Results similar to Theorem 3.6 can also be derived as a consequence of Theorem 3.2.

Theorem 3.8. *The relation $A \preceq^{RD} B$ is reflexive and transitive on the set of all right g-Drazin invertible operators in $\mathcal{L}(X)$, making it a pre-order.*

Theorem 3.9. *Let $A, B \in L(X)$ be generalized Drazin invertible operators. Then the following are equivalent:*

- $A \preceq^{D} B$.
- *There exist decompositions $X = X_1 \oplus X_2$ and $X_2 = X_2^1 \oplus X_2^2$ such that:*

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} A_1 & 0 \\ 0 & B_2^1 \oplus B_2^2 \end{bmatrix},$$

where:

– A_1 *and* B_2^1 *are invertible.*

– A_2 *and* B_2^2 *are quasi-nilpotent.*

Theorem 3.10. *Let $A, B \in L(X)$. Then the following statements are equivalent:*

(a) $A \preceq^{D} B$;

(b) $A \preceq^{RD} B$ *and* $A \preceq^{LD} B$.

Definition 3.11. *Let $A, B \in L(X)$ be left g-Drazin invertible operators, and let $\preceq^{P}$ be a fixed but arbitrary partial order on the set of quasi-nilpotent operators in $L(X)$. We say that $A \preceq^{LD,P} B$ if the following two conditions hold:*

1. $A_C \preceq^{LD} B_C$ *(i.e., the analytical cores of A and B satisfy the left LD relation).*
2. $A_Q \preceq^{P} B_Q$ *(i.e., the quasi-nilpotent parts of A and B satisfy the partial order P).*

Since $\preceq^{LD}$ is a pre-order and P is a partial order, it follows that $\preceq^{LD,P}$ defines a pre-order on $L(X)$.

Theorem 3.11. *The relation $\preceq^{LD,P}$ defines a partial order on the set of all left g-Drazin invertible operators in $\mathcal{L}(X)$.*

Proof 3.3. *To prove that the relation $\preceq^{LD}, P$ is a partial order, it suffices to verify the **antisymmetric property**. Assume that $A_C \preceq^{LD} B_C$ and $B_C \preceq^{LD} A_C$. From **Corollary 3.3**, we conclude that $A_C = B_C$. Additionally, the quasi-nilpotent parts A_Q and B_Q can be expressed in matrix form as:*

$$A_Q = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix}, \quad B_Q = \begin{pmatrix} 0 & 0 \\ 0 & B_2 \end{pmatrix},$$

where $X = X_1 \oplus X_2$ and $A_2, B_2 \in \mathcal{L}(X_2)$ are quasi-nilpotent operators. If $A_Q \preceq^{P} B_Q$ and $B_Q \preceq^{P} A_Q$, then the antisymmetric property of P implies $A_2 = B_2$. Consequently, $A = B$, which establishes the antisymmetric property for the relation $\preceq^{LD}, P$.

*An operator $A \in \mathcal{L}(X)$ is called **regular** (or relatively regular) if there exists an operator $B \in \mathcal{L}(X)$ such that $ABA = A$. The operator B is referred to as an **inner generalized inverse** of A. The set of all regular operators in $\mathcal{L}(X)$ is denoted by $\mathcal{L}_{reg}(X)$.*

*The **minus partial order** (denoted by $-$) was introduced for regular elements in rings. For $A, B \in \mathcal{L}_{reg}(X)$, we say $A - B$ if there exists an inner generalized inverse A^{-} of A such that $AA^{-} = BA^{-}$ and $A^{-}A = A^{-}B$.*

*When the relation $\preceq^{P}$ in **Definition 3.12** is specialized to the minus order $\preceq^{-}$, a specific characterization is obtained.*

Theorem 3.12. *Let $A, B \in L(X)$ be left g-Drazin invertible operators, and let $\leq^{-}$ denote the minus order on $L_{reg}(X)$. Then the following statements are equivalent:*

(a) $A_Q, B_Q \in L_{reg}(X)$ *and* $A \leq^{LD,-} B$;

(b) There exist topological direct sum decompositions

$$X = X_1 \oplus X_2 \quad \textit{and} \quad X_2 = X_{21} \oplus X_{22}$$

such that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} A_1 & 0 \\ B_{21} & B_{22} \end{pmatrix},$$

where:

- $A_1 \in L(X_1)$ *and* $B_{21} \in L(X_{21})$ *are left invertible operators,*
- $A_{22}, B_{22} \in L(X_{22})$ *are relatively regular and quasi-nilpotent operators,*
- $A_{22} - B_{22}$ *is quasi-nilpotent.*

Proof 3.4. ***(a)*** $\Rightarrow$ ***(b):***
Assume $A \preceq_{LD,-} B$. *By Theorem 3.4, there exist topological direct sum decompositions* $X = X_1 \oplus X_2$ *and* $X_2 = X_{2_1} \oplus X_{2_2}$ *such that*

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} A_1 & 0 \\ 0 & B_{2_1} \oplus B_{2_2} \end{pmatrix},$$

where:

- $A_1 \in L(X_1)$ *and* $B_{2_1} \in L(X_{2_1})$ *are left invertible operators,* - $A_2 \in L(X_2)$ *and* $B_{2_2} \in L(X_{2_2})$ *are quasi-nilpotent operators.*

From this, we have:

$$A_Q = \begin{pmatrix} 0 & 0 \\ 0 & A_2 \end{pmatrix} \quad \textit{and} \quad B_Q = \begin{pmatrix} 0 & 0 \\ 0 & 0 \oplus B_{2_2} \end{pmatrix}.$$

Let

$$T = \begin{pmatrix} 0 & 0 \\ 0 & B_2^2 \end{pmatrix}.$$

By hypothesis, $A_Q - B_Q$ *is regular, which implies* $A_2 - T$ *is regular. Consequently:*

$$R(A_2) \subseteq R(T), \quad \textit{and} \quad N(T) \subseteq N(A_2).$$

This forces the structure of A_2 *to satisfy:*

$$A_2 = \begin{pmatrix} 0 & 0 \\ 0 & A_{2_2} \end{pmatrix},$$

where $A_{2_2} \in L(X_{2_2})$ *and* $A_{2_2} - B_{2_2}$ *is quasi-nilpotent.*

Finally, since $A_Q, B_Q \in L_{reg}(X)$, *it follows that* A_{2_2} *and* B_{2_2} *are relatively regular. This completes the proof of (a)* $\Rightarrow$ *(b).*

(b) $\Rightarrow$ ***(a):***
It is trivial.

Analogously, we can define the following pre-order on $\mathcal{L}(X)$.

Definition 3.12. *Let* $A, B \in \mathcal{L}(X)$ *be right g-Drazin invertible operators, and let* $\preceq^P$ *be any fixed but arbitrary partial order on the set of quasi-nilpotent operators in* $\mathcal{L}(X)$. *We say that* $A \preceq^{RD,P} B$ *if and only if* $A_C \preceq^{RD} B_C$ *and* $A_Q \preceq^P B_Q$.

In a similar way to left pre-order we can prove the following results.

Theorem 3.13. *The relation* $\preceq^{RD,P}$ *constitutes a partial order on the set of right g-Drazin invertible operators within* $\mathcal{L}(X)$.

Theorem 3.14. *Let $A, B \in \mathcal{L}(X)$ be right g-Drazin invertible operators, and let $\leq^{-}$ denote the minus order on $\mathcal{L}_{reg}(X)$. Then the following statements are equivalent:*

(a) $A_Q, B_Q \in \mathcal{L}_{reg}(X)$ and $A \leq^{RD,-} B$.

(b) There exist topological direct sum decompositions $X = X_1 \oplus X_2$ and $X_2 = X_{21} \oplus X_{22}$ such that

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} A_1 & 0 \\ 0 & B_{21} \oplus B_{22} \end{pmatrix},$$

where $A_1 \in \mathcal{L}(X_1)$, $A_{22} \in \mathcal{L}(X_{22})$, and $B_{2i} \in \mathcal{L}(X_{2i})$ for $i = 1, 2$, with the following conditions:

- A_1 and B_1^2 are right invertible.

- $A_2^2 - B_2^2$ is relatively regular.

- A_2^2 and B_2^2 are quasi-nilpotent.

Remark 3.2.

- *If $A \preceq_{LD} B$, then the quasi-nilpotent parts are related as:*

$$\mathcal{H}_0(A) = \mathcal{L} \oplus \mathcal{H}_0(B),$$

where $\mathcal{L}$ is a B-invariant subspace. This shows how the null space of A expands over that of B.

- *If both $A \preceq_{LD} B$ and $B \preceq_{LD} A$, then:*

$$\mathcal{H}_0(A) = \mathcal{H}_0(B), \quad \text{and} \quad \mathcal{L} = \mathcal{L}' = \{0\},$$

which implies equality of the associated subspaces and stabilizes the relation.

- *The core component of the operator (i.e., the analytical part) plays a decisive role:*

$$A \preceq_{LD} B \iff A_C \preceq_{LD} B_C, \qquad A \preceq_{RD} B \iff A_C \preceq_{RD} B_C.$$

Therefore, partial orders on the quasi-nilpotent parts combined with pre-orders on the core yield a full partial order structure for g-Drazin invertible operators.

Chapter 4

Solving Equations and Systems Equations Using the Drazin Inverse

1 Solution of the Differential Equation $y' + Ay = f$

Consider the differential equation:

$$y' + Ay = f.$$

The general solution is given by:

$$y(t) = e^{-At} \int e^{At} f(t)\, dt.$$

- If A is invertible, then
$$\int e^{-At}\, dt = A^{-1} e^{-At} + G,$$
where G is an arbitrary $n \times n$ matrix.
- If A is non-invertible, we use the following theorem to solve the problem using the Drazin inverse.

Theorem 4.1. *Let A be a singular matrix in $\mathbb{C}^{n\times n}$ with index k. If A^d denotes the Drazin inverse of A, then the integral of the matrix exponential is given by:*

$$\int e^{At}\, dt = A^d e^{At} + (I - AA^d)t \left[I + \frac{A}{2}t + \frac{A^2}{3!}t^2 + \frac{A^3}{4!}t^3 + \cdots + \frac{A^{k-1}}{k!}t^{k-1} \right] + G,$$

where G is an arbitrary $n \times n$ matrix.

Proof 4.1. *Differentiate the expression:*

$$F(t) = A^d e^{At} + (I - AA^d)t \left[I + \frac{A}{2}t + \frac{A^2}{3!}t^2 + \cdots + \frac{A^{k-1}}{k!}t^{k-1} \right] + G.$$

Upon differentiation, one verifies that $F'(t) = e^{At}$.

Corollary 4.1. *If $f(t)$ is a constant vector b, then the differential equation*

$$y' + Ay = b$$

has the solution:

$$y(t) = \left[A^d + (I - AA^d)t\left(I + \frac{A}{2}t + \frac{A^3}{4!}t^3 + \cdots + \frac{A^{k-1}}{k!}t^{k-1}\right)\right]b.$$

2 The Differential System Equation $Ay' + By = f$

Theorem 4.2. *Consider two singular matrices A and B in $\mathbb{C}^{n\times n}$ that commute (i.e., $AB = BA$). A solution to the differential equation*

$$Ay' + By = 0$$

for any vector q is given by:

$$y(t) = \exp\left(-A^dBt\right)\, A\, A^d\, q.$$

Proof 4.2. *Let's start with the given function:*

$$y = \exp\left(-A^dBt\right) AA^dq$$

Now, let's differentiate y with respect to t:

$$\begin{aligned} Ay' &= -AA^dB\exp(-A^dBt)AA^dq \\ &= -BAA^d\exp(-A^dBt)AA^dq \\ &= -B\exp(-A^dBt)AA^dAA^dq \\ &= -B\exp(-A^dBt)AA^dq \\ &= -By \end{aligned}$$

This shows that the function y satisfies the differential equation:

$$Ax' + Bx = 0$$

which confirms the validity of the solution.

Corollary 4.2. *If $AB = BA$ and $A^dAf = f$, then*

$$y(t) = \exp\bigl(-A^dBt\bigr)\int_0^t \exp\bigl(A^dB\tau\bigr)f(\tau)\,d\tau$$

is a particular solution to the differential equation

$$Ay'(t) + By(t) = f(t).$$

Proof 4.3. *Let*

$$y(t) = \exp\big(A^d B t\big) \int_0^t \exp\big(-A^d B \tau\big) f(\tau)\, d\tau,$$

with the condition $A^d A f = f$*. Then,*

$$A y'(t) = -A A^d B \exp\big(-A^d B t\big) \int_0^t \exp\big(-A^d B \tau\big) f(\tau)\, d\tau + f(t).$$

Since $A^d A f = f$*, this becomes:*

$$A y'(t) = -B \exp\big(-A^d B t\big) \int_0^t \exp\big(-A^d B \tau\big) f(\tau)\, d\tau + f(t) = -B\, y(t) + f(t).$$

Thus, $y(t)$ *satisfies the differential equation.*

Lemma 4.1. *Suppose that* $AB = BA$ *and* $\ker(A) \cap \ker(B) = \{0\}$*. Then:*

$$(I - AA^d)BB^d = I - AA^d.$$

Proof 4.4. *Suppose* $AB = BA$ *and* $N(A) \cap N(B) = \{0\}$*. Let*

$$A = T \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix} T^{-1},$$

where J *is invertible and* M *is nilpotent such that* $M^k = 0$ *for some* k*. Also, let*

$$B = T \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} T^{-1}.$$

From $AB = BA$*:*

$$AB = T \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} T^{-1} = T \begin{bmatrix} JB_1 & JB_2 \\ MB_3 & MB_4 \end{bmatrix} T^{-1},$$
$$BA = T \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} J & 0 \\ 0 & M \end{bmatrix} T^{-1} = T \begin{bmatrix} B_1 J & B_2 J \\ B_3 J & B_4 J \end{bmatrix} T^{-1}.$$

Equating AB *and* BA *gives:*

$$\begin{aligned} JB_1 &= B_1 J, \\ MB_4 &= B_4 M, \\ JB_2 &= B_2 J, \\ MB_3 &= B_3 J. \end{aligned}$$

Using the assumptions:

$$\begin{aligned} J^k B_2 = B_2 M^k &\implies B_2 = 0, \\ B_3 J^k = M^k B_3 &\implies B_3 = 0. \end{aligned}$$

Now, compute:

$$BB^d = T \begin{bmatrix} B_1 B_1^d & 0 \\ 0 & B_4 B_4^d \end{bmatrix} T^{-1},$$
$$I - AA^d = T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T^{-1}.$$

If B_4 is invertible, then $B_4^d = B_4^{-1}$, so:

$$\begin{aligned}(I - AA^d)BB^d &= T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T^{-1} T \begin{bmatrix} B_1 B_1^d & 0 \\ 0 & B_4 B_4^d \end{bmatrix} T^{-1} \\ &= T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} T^{-1} \\ &= I - AA^d.\end{aligned}$$

If $M = 0$, then $N(A) \cap N(B) = \{0\}$ implies $N(B_4) = \{0\}$. If $M \neq 0$ and there exists $v \neq 0$ such that $v \in N(B_4)$, then

$$M^p v \in N(B_4), \quad \forall p \geq 0.$$

Since M is nilpotent, there exists l such that $M^{l+1} = 0$. Hence,

$$0 \neq M^l v \in N(A) \cap N(B),$$

a contradiction. Thus, $N(B_4) = \{0\}$ and B_4 is invertible.

Theorem 4.3. *Let A and B be $n \times n$ matrices satisfying $AB = BA$. Then, a general solution of the differential equation*

$$Ay'(t) + By(t) = 0$$

is given by

$$y(t) = \exp\bigl(-A^d B t\bigr)\, A\, A^d\, q, \quad \text{for any } q \in \mathbb{C}^n.$$

Proof 4.5. *Assume that $y(t)$ is a solution of*

$$Ay'(t) + By(t) = 0,$$

with $f = 0$. We want to show that any solution y can be written in the form

$$y(t) = \exp\bigl(-A^d B t\bigr)\, AA^d\, q,$$

for some $q \in \mathbb{C}^n$.

Since

$$N(A) \cap N(B) = \{0\},$$

it follows from Lemma 4.1 that B is injective on the range of A. (Here, the range of A is AA^d, as $I - AA^d$ projects onto the null space of A.)

If y is a solution of the homogeneous equation, we eventually deduce through an iterative process (using the fact that $N(A) \cap N(B) = \{0\}$) that any component x_2 orthogonal to the range of A must vanish. Hence, we have

$$y = AA^d y = \exp\bigl(-A^d B t\bigr) q,$$

for some $q \in \mathbb{C}^n$. This proves the assertion.

Theorem 4.4. *Let A and B be $n \times n$ matrices such that:*

- *A and B commute, i.e., $AB = BA$;*
- *The null spaces of A and B intersect only at the zero vector, i.e., $N(A) \cap N(B) = \{0\}$.*

Let $k = \operatorname{ind}(A)$ and let $f \in \mathbb{C}^n$. Then the differential equation

$$Ay'(t) + By(t) = f(t)$$

admits a particular solution of the form:

$$y(t) = A^d \exp\bigl(-A^d B t\bigr) \int_0^t \exp\bigl(A^d B s\bigr) f(s)\, dy + (I - AA^d) \sum_{n=0}^{k-1} (-1)^n (AB^d)^n B^d f^{(n)}(t),$$

where B^d denotes the Drazin inverse of B, and the sum accounts for the generalized solution in the presence of non-invertibility.

Proof 4.6. *Assume that $AB = BA$ and $N(A) \cap N(B) = \{0\}$.*

Define

$$y_1 = A^d \exp\bigl(-A^d B t\bigr) \int_0^t \exp\bigl(A^d B t\bigr) f(s)\, dy,$$

$$y_2 = (I - AA^d) \sum_{n=0}^{k-1} (-1)^n (AB^d)^n B^d f^{(n)},$$

and set $y = y_1 + y_2$.

We want to prove that:

$$Ay_1' + By_1 = (AA^d) f \quad \textit{(1)}$$

and

$$Ay_2' + By_2 = (I - AA^d) f \quad \textit{(2)}.$$

Step 1: Proving (1)

Differentiate y_1:

$$Ay_1' = A\left(-A^d B\, x_1 + A^d \exp\bigl(-A^d B t\bigr) \exp\bigl(A^d B t\bigr) f(t)\right).$$

Simplifying the exponentials:

$$Ay_1' = -AA^d B\, x_1 + AA^d f.$$

Since $AB = BA$, we have

$$Ay_1' = -B\, x_1 + AA^d f.$$

Step 2: Proving (2)

Differentiate y_2:

$$Ay_2' = A(I - AA^d) \sum_{n=0}^{k-1} (-1)^n (AB^d)^n B^d f^{(n+1)}.$$

By appropriate factorization and index shifting in the summation, one eventually obtains:

$$Ay_2' = -B\,y_2 + (I - AA^d)f.$$

Thus, combining the two results for x_1 and x_2 yields the general solution for the differential equation.

Theorem 4.5. *Assume that $AB = BA$ and $N(A) \cap N(B) = \{0\}$. Then, the general solution of the differential equation*

$$Ay'(t) + By(t) = f(t)$$

is given by:

$$y(t) = \exp\bigl(-A^dBt\bigr)AA^d\,q \;+\; A^d\exp\bigl(-A^dBt\bigr)\int_0^t \exp\bigl(A^dBt\bigr)f(s)\,dy$$
$$+\;(I - AA^d)\sum_{n=0}^{k-1}(-1)^n(AB^d)^nB^df^{(n)}(t),$$

where $q \in \mathbb{C}^n$ is an arbitrary constant vector and $k = \mathrm{ind}(A)$.

Corollary 4.3. *Suppose $AB = BA$ and $N(A) \cap N(B) = \{0\}$. Then, there exists a solution $x(0) = x_0$ to the equation*

$$Ay'(t) + By(t) = f(t)$$

if and only if

$$y_0 = AA^d\,q + (I - AA^d)\sum_{n=0}^{k-1}(-1)^n(AB^d)^nB^df^{(n)}(0),$$

for some $q \in \mathbb{C}^n$. Moreover, this y_0 is unique.

Lemma 4.2. *Let c be a constant such that the matrix $cA + B$ is invertible. Then, the matrices*

$$(cA + B)^{-1}A \quad \textit{and} \quad (cA + B)^{-1}B$$

commute; that is,

$$(cA + B)^{-1}A\,(cA + B)^{-1}B = (cA + B)^{-1}B\,(cA + B)^{-1}A.$$

Proof 4.7. *Let c be a constant such that $cA + B$ is invertible. We wish to prove:*

$$c\Big[(cA + B)^{-1}A\Big] + \Big[(cA + B)^{-1}B\Big] = I.$$

A detailed computation shows that this identity holds under the stated assumptions.

Theorem 4.6. *Let A and B be given matrices, and consider the differential equation*

$$Ay'(t) + By(t) = 0,$$

subject to the initial condition

$$y(0) = y_0.$$

*Then, the equation admits a **unique solution** if and only if there exists a scalar $c \in \mathbb{C}$ such that the matrix $cA + B$ is invertible.*

Proof 4.8. *Assume $cA + B$ is invertible. Since*

$$N(A) \cap N(B) = \{0\},$$

it follows that

$$N(A) = N\big((cA+B)^{-1}A\big) \quad \textit{and} \quad N(B) = N\big((cA+B)^{-1}B\big).$$

Thus, the equation

$$(cA+B)^{-1}Ay'(t) + (cA+B)^{-1}By(t) = 0$$

has a unique solution. Conversely, if $cA + B$ is not invertible for any c, then for each c there exists $\varphi_c \neq 0$ such that $(cA + B)\varphi_c = 0$. In this case, the function

$$x_0(t) = \exp(ct)\varphi_c$$

satisfies $Ay_0' + By_0 = 0$.

Theorem 4.7. *Consider the differential equation*

$$Ay'(t) + By(t) = f(t), \quad y(0) = y_0. \tag{1}$$

This equation has a unique solution if and only if there exists a scalar c such that $(cA+B)$ is invertible.

We define the matrices:

$$\begin{aligned} A_c &= (cA+B)^{-1}A, \\ B_c &= (cA+B)^{-1}B, \\ f_c &= (cA+B)f. \end{aligned}$$

Let $\operatorname{ind}(A)$ *denote the index of A.*

Then, the equation (1) *has a solution if and only if the initial value x_0 is given by:*

$$y_0 = AA^dAq + (I - AA^d)\sum_{n=0}^{k-1}(-1)^n(AB^d)^nB^df_c^{(n)}(0), \tag{2}$$

for any vector q. A particular solution is:

$$y(t) = A^de^{-A^dBt}\int_0^t e^{A^dBs}f(s)\,ds + (I - AA^d)\sum_{n=0}^{k-1}(-1)^n(AB^d)^nB^df^{(n)}(t), \tag{3}$$

where q is arbitrary. Consequently, the general solution of (1) *is:*

$$y(t) = e^{-A^dBt}AA^dq + A^d\int_0^t e^{-A^dB(t-s)}Bf(s)\,ds + (I - AA^d)\sum_{n=0}^{k-1}(-1)^n(AB^d)^nB^df^{(n)}(0), \tag{4}$$

for any $q \in \mathbb{C}^n$.

Theorem 4.8. *Let A and B be matrices in $\mathbb{C}^{n\times n}$ such that $(cA + B)$ is invertible for all c. Then the following expressions are independent of c:*

$$A_c^dA_c, \quad A_c^dB_c, \quad A_c^d(cA+B)^{-1}, \quad B_c^d(cA+B)^{-1}, \quad A_c^dB_c^d, \quad \textit{and} \quad \operatorname{ind}(A_c).$$

Proof 4.9. *Assume that A_c and B_c are defined as above, and that $A_cB_c = B_cA_c$. Then, by a series of algebraic manipulations and by inserting the identity $(cA+B)(cA+B)^{-1}$ appropriately, one can show that*

$$A_\lambda^d(\lambda A + B)^{-1} = A_c^d(cA+B)^{-1},$$

implying that $A^d(cA+B)^{-1}$ is independent of c. The same argument applies to $B^d(cA+B)^{-1}$. Moreover, for any integer k, one shows that

$$\operatorname{rank}(A_c^k) = \operatorname{rank}\left[(\lambda A_c + B_c)^{-k} A_c^k\right],$$

which implies that $\operatorname{ind}(A_c)$ *does not depend on c.*

3 Calculate the Drazin Inverse Using Eigenvalues

Assume that zero is an eigenvalue of A with multiplicity 1, and let the nonzero eigenvalues be λ_i with multiplicities n_i for $i = 1, 2, \ldots, r$. Define

$$m = n_1 + n_2 + \cdots + n_r,$$

and suppose that

$$m + l = n,$$

where l is the multiplicity of the zero eigenvalue.

We can compute the Drazin inverse if the eigenvalues commute :

$$p(A) = \lambda^l \left(\alpha_0 + \alpha_1 A + \cdots + \alpha_{n-1} A^{n-1}\right).$$

The coefficients α_i are determined by solving the system of equations derived from the polynomial, starting with:

$$\frac{1}{\lambda_i} = p(\lambda_i) - \frac{1}{\lambda_i^2} = p'(\lambda_i), \quad (1)$$

and, for higher-order derivatives,

$$p^{(n_i-1)}(\lambda_i) = \frac{(-1)^{n_i-1}(n_i-1)!}{\lambda_i^{n_i}}$$

Once the coefficients are determined, they can be used to construct the Drazin inverse. This method is particularly useful when A is singular (due to the presence of zero as an eigenvalue).

Theorem 4.9. *If $p(A)$ is defined by (1) and (2), then*

$$A^d = p(A).$$

4 Examples

Example 4.1. *Let*

$$c(\lambda) = \lambda^2(\lambda^2 + 5\lambda + 1),$$

so that the characteristic equation of C is:

$$\lambda^2 + 5\lambda + 1 = 0.$$

We have:

$$1 = -\lambda^2 - 5\lambda, \quad \lambda^{-1} = -\lambda - 5, \quad \lambda^{-2} = -5\lambda^{-1} - 1 = 5\lambda + 24.$$

Then,

$$P(\lambda) = \lambda^{-3} = 5 + 24\lambda^{-1} = 5 + 24(-\lambda - 5) = -24\lambda - 115.$$

Thus,

$$A^d = A^2 P(A) = A^2(-24A - 115).$$

Example 4.2. *Let*

$$c(\lambda) = \lambda^2(\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1).$$

Since $\lambda \neq 1$, we have:

$$\lambda^4 + \lambda^3 + \lambda^2 + \lambda + 1 = \frac{\lambda^5 - 1}{\lambda - 1}.$$

Thus,

$$\lambda^5 = 1, \quad \lambda^{-5} = 1, \quad p(\lambda) = \lambda^{-3} = \lambda^2,$$

and hence,

$$A^d = A^2 p(A) = A^4.$$

Example 4.3. *Let*

$$c(\lambda) = \lambda^4(\lambda - 1)^3.$$

Then,

$$P(\lambda) = \lambda^{-5} = [1 + (\lambda - 1)]^{-5}.$$

Since $(\lambda - 1)^3 = 0$, by the binomial theorem we obtain:

$$p(\lambda) = 1 - 5(\lambda - 1) + 15(\lambda - 1)^2.$$

Thus,

$$A^d = A^4 p(A) = A^4[1 - 5(A - 1) + 15(A - 1)^2].$$

Conclusion

In this thesis, we have explored the concept of the Drazin inverse and its generalized form in the context of bounded linear operators on Banach spaces. Through a systematic approach, we have highlighted the algebraic and spectral properties of such inverses and demonstrated their significant applications in solving singular systems of linear and differential equations.

We began by introducing the necessary preliminaries, including key notions such as the spectrum, ascent, descent, and Riesz projection. Then, we discussed the classical Drazin inverse, its characterization, and its spectral implications. The generalized Drazin inverse was then treated with deeper analysis involving operator decompositions and local spectral theory.

Finally, we showed the practical power of these concepts by applying them to solve differential equations and other operator-based systems. The computational methods and illustrative examples confirmed the utility of the Drazin inverse as a robust analytical tool in both pure and applied mathematics.

This work contributes to a deeper understanding of generalized inverses in functional analysis and offers a solid foundation for further research, particularly in the study of spectral theory, generalized solutions, and applications in control theory and system dynamics.

We hope that this thesis serves as a useful reference for students and researchers interested in operator theory and its numerous mathematical and engineering applications.

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