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Analytical and Numerical Analysis of A Generalized Non Standard Volterra Integro-Differential Equations

Presented by:
KHALED Soumia

Jury Members:

BOUZENADA Smail	Prof	Tebessa University	President
NABTI Abderrazak	Prof	Tebessa University	Examiner
LEMITA Samir	M.C.A	Tebessa University	Supervisor

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KHALED Soumia
Echahid Cheikh Larbi Tebessi University
Tebessa, Algeria.

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Dedication

To those whose prayers were the secret of my success and whose support illuminated my path…

*To my **dear parents**, the source of love and generosity, the reason for my existence and achievement… I offer you all my gratitude, for I would not have reached this point without your sacrifices, patience, and unlimited support.*

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ملخص

- تحتل معادلات فريدهولم وفولتيرا التكاملية مكانة بارزة في الرياضيات نظرًا لتطبيقاتها الواسعة في مختلف المجالات العلمية.

- في هذه المذكرة، نركز على شكل محدد من معادلات فولتيرا التكاملية التفاضلية غير الخطية وغير القياسية.

- باستخدام نظرية شودر للنقطة الثابتة، نثبت وجود و وحدانية حل معادلة فولتيرا التكاملية التفاضلية المقدمة في ظل شروط محددة جيدًا.

- نطبق طريقة موجات فيبوناتشي للحصول على حل تقريبي لهذه المعادلة التكاملية التفاضلية، متبوعة بأمثلة عددية توضح النتائج التي تم الحصول عليها.

الكلمات المفتاحية:

- معادلة فولتيرا التكاملية التفاضلية؛ المعادلات الغير قياسية؛ نظرية النقطة الثابتة لشودر؛ التقريب العددي؛ طريقة موجات فيبوناتشي.

Résumé

- Les équations intégrales de Fredholm et de Volterra occupent une place importante dans les mathématiques en raison de leurs nombreuses applications dans divers domaines scientifiques.

- Dans ce mémoire, nous intéressons particulièrement à une forme spécifique de l'équation intégro-différentielle non linéaire, non standard de Volterra.

- Grâce au théorème du point fixe de Schauder, nous établissons l'existence et l'unicité de la solution de cette équation intégro-différentielle de Volterra sous certaines conditions précises.

- Par la suite, nous appliquons la méthode de **Fibonacci Wavelet Collocation**, afin d'obtenir une solution approximative de cette équation intégro-différentielle, accompagnée de quelques exemples concrets illustrant les résultats obtenus.

Mots clés

- Equation intégro-différentielle de Volterra ; Equation non standard ; Théorème du point fixe de Schauder ; Solution approchée ; La méthode de Fibonacci Wavelet Collocation.

Abstract

- Fredholm and Volterra integral equations hold a significant position in mathematics due to their wide-ranging applications across various scientific fields.

- In this thesis, we focus on a specific form of the non linear, non standard, integro-differential Volterra equation.

- Using Schauder fixed point theorem, we establish the existence and uniqueness for the solution of this integro-differential Volterra equation under well-defined conditions.

- We then apply the **Fibonacci Wavelet Collocation** method to obtain an approximate solution to this integro-differential equation, followed by illustrative examples demonstrating the obtained results.

Keywords

- Volterra integro-differential equation ; Non standard equation ; Schauder fixed point theorem ; Numerical approximation ; Fibonacci Wavelet Collocation method.

Introduction

At the beginning of the 20th century, the Italian mathematician **Vito Volterra** constructed a new type of mathematical equation which called the Volterra integral equation [1], where he also laid the theoretical foundations of these equations. These integral equations play a very important role in many scientific fields [2, 3, 4, 5] : Physique, biologic, electronic, mechanic and others. Moreover, many researchers have developed different methods [6, 7, 8] to treat these integral equations.

For this reason, our thesis is dedicated to study a new kind of Volterra integral equation that given by :

$$u(t) = \int_0^t k(t,s)\phi(u(t), u(s), u'(s))ds + g(t), \quad \forall t \in [0,1].$$

This equation describes a new form of the non-standard Volterra equations, where we can see that the integral operator contains inside the unknown function u and its derivative u'.

We will treat the proposed equation from the analytical and numerical side, and in such a way that our study will be easy to understand, we have divided the thesis into three chapters :

Chapter 1 (Notions and preliminaries) : We recall some concepts in the functional analysis and the numerical analysis in order to use them in the following chapters.

Chapter 2 (Analytical study) : We add to our integral equation the necessary hypotheses that allow us to judge that this equation has a unique solution.

Chapter 3 (Numerical study) : We apply the Fibonacci Wavelets Collocation method to find the approximate solution of the proposed equation.

Chapitre 1

Preliminaries

In this part, we recall some fundamental concepts in order to use in the next chapters.

1.1 Banach spaces

1.1.1 Normed space

Definition 1.1.1

We say that the application $\|.\|_V$ defined from the vector space V to $\mathbb{R}^+$, is a norm, if the following three properties are satisfied :

1. $\forall v \in V,\ \|v\|_V = 0 \Leftrightarrow v = 0.$
2. $\forall \lambda \in \mathbb{R}, \forall v \in V,\ \|\lambda v\|_V = |\lambda| \|v\|_V.$
3. $\forall v, w \in V,\ \|v + w\|_V \leq \|v\|_V + \|w\|_V$. *(Triangle inequality).*

Then, $(V, \|.\|_V)$ becomes a normed vector space (n.v.s).

Example 1.1.1

Let $C([a, b], \mathbb{R})$ be the vector space of continuous functions on $[a, b]$ with real values.

We set for all $f \in C([a,b],\mathbb{R})$:

$$\begin{aligned} ||f||_1 &= \int_a^b |f(x)|dx, \\ ||f||_\infty &= \sup_{x\in[a,b]} |f(x)|. \end{aligned}$$

The applications $||.||_1$ et $||.||_\infty$ are considered as norms on the space $C([a,b],\mathbb{R})$.

Definition 1.1.2 *(Cauchy sequence)*

The sequence $(U_n)_{n\in\mathbb{N}} \in V$ is called a Cauchy sequence for the norm $\|.\|_V$ if :

$$\forall\, \varepsilon > 0,\ \exists\, N \geq 0,\ \forall\, n > N, \forall\, p \geq 0,\ \|U_{n+p} - U_n\|_V \leq \varepsilon.$$

Proposition 1.1.1

1. *Every Cauchy sequence is a bounded sequence.*
2. *Every convergent sequence is a Cauchy sequence.*
3. ***Warning !*** *Every Cauchy sequence is not necessarily convergent.*

1.1.2 Banach spaces

Definition 1.1.3 *(Banach space or complete normed vector space)*

We say that the normed vector space $(V, \|.\|_V)$ is complete (or Banach space) if any Cauchy sequence of V is convergent in V according the norm $\|.\|_V$.

Example 1.1.2

We consider $C^0([a,b],\mathbb{R})$ the space of continuous functions on $[a,b]$, we equip this space with the following norm :

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|, \quad \forall f \in C^0([a,b],\mathbb{R}).$$

Then, $\left(C^0([a,b],\mathbb{R}), \|.\|_\infty\right)$ is a Banach space.

Example 1.1.3

We consider $C^1([a,b],\mathbb{R})$ the space of continuous and differentiable functions on $[a,b]$, we equip this space with the following norm :

$$\|f\|_\infty = \sup_{x\in[a,b]} |f(x)| + \sup_{x\in[a,b]} |f'(x)|, \quad \forall f \in C^1([a,b],\mathbb{R}).$$

Then, $\left(C^1([a,b],\mathbb{R}), \|.\|_\infty\right)$ is a Banach space.

1.2 Fixed Point Theorems

Fixed point theorems play an essential role in studying nonlinear equations, for this reason we recall some types of these theorems.

Definition 1.2.1

Let X be a Banach space and T an operator defined from X to itself ($T : X \longrightarrow X$). We say that x is a fixed point of T if and only if :

$$x = T(x).$$

1.2.1 Banach fixed point

Theorem 1.2.1 *[9]*

Let X a Banach space and the operator $T : X \longrightarrow X$. If the operator T is a contraction, then we say that it has a unique fixed point.

1.2.2 Schauder fixed point

Theorem 1.2.2 *[9]*

Let S be a non-empty, closed and convex subset of the Banach space X. If the operator $T : S \longrightarrow S$ verifies :

1. *T is continuous,*

2. *$T(S)$ is relatively compact,*

then, T admits at least one fixed point in S.

1.2.3 Krasnoselskii fixed point

Theorem 1.2.3 *[9]*

Let M be a non-empty, closed and convex subset of the Banach space X. If the two operators A and B verify the following properties :

1. *$\forall x, y \in M,\ Ax + By \in M$,*

2. *A is continuous and compact,*

3. *B is contraction,*

then, there exists at least one fixed point of the operator $A + B$, which means that :

$$\exists\, x \in M, \qquad x = Ax + Bx.$$

Theorem 1.2.4 *(Arzéla-Ascoli)*

Let S be a subset of the Banach space $C([a, b], \mathbb{R})$. We say that S is relatively compact if and only if the following conditions must be satisfied :

1. *The subset S is bounded.*

2. *The subset S is equicontinuous.*

Definition 1.2.2 *(Uniformly equicontinuous)*

Let S be a subset of the Banach space $C([a, b], \mathbb{R})$. We say that S is uniformly equicontinuous if :

$$\forall \epsilon > 0,\ \exists \delta > 0,\ \forall x, y \in [a, b],\ |x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \epsilon,\ \forall\, f_n \in S.$$

1.3 Some used properties

Definition 1.3.1 *(Lipschitz function)*
Let f be a function defined on the interval $[a, b]$ with values in $\mathbb{R}$. We say that f is Lipschitz function if there exists a constant $L \in \mathbb{R}^+$ satisfies :

$$\forall\, x,\, y \in [a, b],\ |f(x) - f(y)| \leq L|x - y|.$$

Remark 1.3.1

1. *The previous function f is said contraction if we can choose $0 < L < 1$.*
2. *The definition of a contraction function in a Banach space X, remains the same previous definition, but we use the distance associated with the norm of this space $\|x - y\|_X$.*

Theorem 1.3.1 *(Mean value theorem)*
For any continuous and differentiable function defined on the interval $[a, b]$ with values in $\mathbb{R}$, $(f : [a, b] \longrightarrow \mathbb{R})$, then there exists a constant c in $]a, b[$ such that :

$$f(b) - f(a) = f'(c)(b - a).$$

Remark 1.3.2
The definition of a Lipschitz function is closely related to the differentiable function. Moreover, according to the mean value theorem, a differentiable function on interval $[a, b]$ is a Lipschitz function, if and only if its derivative is bounded on $[a, b]$, we write :

$$\forall\, x,\, y \in [a, b],\ |f(x) - f(y)| \leq L|x - y|, \quad \text{where} \quad L = \sup_{t \in [a,b]} |f'(t)|.$$

Lemma 1.3.1

Let $f(x), g(x)$ be functions in the Banach space $C^1([a,b],\mathbb{R})$. If there exists $A_1, A_2, B_1, B_2 \in \mathbb{R}^+$ such that $|f(x)| < A_1, |f'(x)| < A_2, |g(x)| < B_1$ and $|g'(x)| < B_2$ for all $x \in [a,b]$, then we have :

$$\forall\, x,\, y \in [a,b],\ |f(x)g(x) - f(y)g(y)| \leq (A_1B_2 + A_2B_1)|x-y|.$$

Proof 1.3.1

It is clear that for all $x,\, y \in [a,b]$:

$$\begin{aligned}
|f(x)g(x) - f(y)g(y)| &= |f(x)g(x) - f(x)g(y) + f(x)g(y) - f(y)g(y)|,\\
&\leq |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)|,\\
&\leq |f(x)||g(x) - g(y)| + |g(y)||f(x) - f(y)|,\\
&\leq A_1|g(x) - g(y)| + B_1|f(x) - f(y)|,\\
&\leq A_1B_2|x-y| + B_1A_2|x-y|,\\
&\leq (A_1B_2 + A_2B_1)|x-y|.
\end{aligned}$$

Lemma 1.3.2 *[10]*

Let $\gamma(t)$ be a continuous positive function on $[a,b]$ verifying :

$$\exists\, L > 0,\ \ \gamma(t) \leq L \int_a^t \gamma(s)ds,$$

then,

$$\gamma(t) = 0,\ \ \forall t \in [a,b].$$

Proof 1.3.2

As the function $\gamma(t)$ is continuous over interval $[a,b]$, then $\exists \eta > 0$ such that $\gamma(t) \leq \eta$, $\forall t \in [a,b]$. So, we get :

$$\gamma(t) \leq L \int_a^t \gamma(s)ds \leq L\eta(t-a).$$

Also,

$$\begin{aligned}\gamma(t) &\leq L\int_a^t \gamma(s)ds,\\ &\leq L\int_a^t L\eta(s-a)ds,\\ &\leq L^2\eta\frac{(t-a)^2}{2}.\end{aligned}$$

By repeating n times, we obtain :

$$\gamma(t) \leq L^n\eta\frac{(t-a)^n}{n!} \underset{n\to\infty}{\longrightarrow} 0.$$

1.4 Orthogonal polynomials

In this section, we address the topic of orthogonal polynomials, and explore some of their basic properties [11].

Definition 1.4.1

We say that the functions $f(x)$ and $g(x)$ are orthogonal with respect to the weight function $w(x)$ on $[a,b]$ if :

$$\langle f,g\rangle = \int_a^b w(x)f(x)g(x)\,dx = 0.$$

Proposition 1.4.1

Let $(P_i)_{i\geq 0}$ be a family of orthogonal polynomials. Then, there is a recurrence relation between P_{n+1}, P_n and P_{n-1} given by :

$$\forall n \in \mathbb{N}^*, \exists a_n, b_n, c_n \in \mathbb{R}, \quad P_{n+1}(x) = (a_n x + b_n)P_n(x) + c_n P_{n-1}(x).$$

1.4.1 Legendre polynomials

Definition 1.4.2

The Legendre polynomials take the following form by the Rodrigués formula :

$$L_n(x) = \frac{1}{n!\,2^n}\frac{d^n}{dx^n}\left((x^2-1)^n\right),$$

or can be given by the following recurrence formula

$$(n+1)L_{n+1}(x) = (2n+1)xL_n(x) - nL_{n-1}(x), \quad \textit{with} \quad L_0(x) = 1, \quad L_1(x) = x, n \geq 1.$$

Example 1.4.1

For $n = 4$ we get the following six polynomials of Legendre :

$$\begin{aligned}
L_0(x) &= 1,\\
L_1(x) &= x,\\
L_2(x) &= \frac{1}{2}(3x^2-1),\\
L_3(x) &= \frac{1}{2}(5x^3-3x),\\
L_4(x) &= \frac{1}{8}(35x^4-30x^2+3),\\
L_5(x) &= \frac{1}{8}(63x^5-70x^3+15x).
\end{aligned}$$

Proposition 1.4.2

Legendre polynomials are orthogonal on $[-1,1]$ and satisfy :

$$\int_{-1}^{1} L_i(x)L_j(x)\,dx = \begin{cases} \dfrac{2}{2i+1}, & i = j,\\ 0, & i \neq j. \end{cases}$$

Corollary 1.4.1

The Legendre polynomials can be defined by a differential equation for any integer n, such that $L_n(x)$ is a solution for the following differential equation :

$$(1-x^2)y''(x) - 2xy'(x) + n(n+1)y(x) = 0.$$

1.4.2 The first kind of Chebyshev polynomials

Definition 1.4.3

The Chebyshev polynomials of the first kind are given by the Rodrigués representation as :

$$U_n(x) = \frac{(-1)^n \sqrt{\pi(1-x^2)}}{2^n \left(n-\frac{1}{2}\right)!} \cdot \frac{d^n}{dx^n} \left(1-x^2\right)^{n-\frac{1}{2}},$$

or by :

$$U_n(x) = \cos\left(n \cos^{-1}(x)\right), \quad \forall x \in [-1,1],$$

also can be defined by the following recurrence formula :

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad with \quad U_0(x) = 1, U_1(x) = x, n \geq 1.$$

Proposition 1.4.3

These polynomials are orthogonal with respect to the weight function $w(x) = 1/\sqrt{1-x^2}$ on the interval $[-1,1]$, and

$$\int_{-1}^{1} w(x)\, U_n(x)\, U_m(x)\, dx = \begin{cases} \pi, & \text{if } n = m = 0, \\ \frac{\pi}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

Corollary 1.4.2

The Chebyshev polynomial of degree n represent a solution for the following differential equation :

$$(1-x^2)y''(x) - xy'(z) + n^2 y(x) = 0, \quad \forall x \in [-1,1].$$

1.4.3 The second kind of Chebyshev polynomials

Definition 1.4.4

The Chebyshev polynomials of the second kind $U_n(x)$ are given by the following for-

mula :

$$U_n(x) = \frac{(-1)^n(n+1)\sqrt{\pi}}{2^{n+1}\left(n+\frac{1}{2}\right)!} \cdot \sqrt{1-x^2} \cdot \frac{d^n}{dx^n}\left(1-x^2\right)^{n+\frac{1}{2}},$$

or can defined by the following recurrence formula :

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), \quad \textit{with} \quad U_0(x) = 1, U_1(x) = 2x, n \geq 1.$$

Example 1.4.2

The first six Chebyshev polynomials of the second kind are :

$$U_0(x) = 1,$$
$$U_1(x) = 2x,$$
$$U_2(x) = 4x^2 - 1,$$
$$U_3(x) = 8x^3 - 4x,$$
$$U_4(x) = 16x^4 - 12x^2 + 1,$$
$$U_5(x) = 32x^5 - 32x^3 + 6x.$$

Proposition 1.4.4

The present polynomials are orthogonal with respect to the weight function $w(x) = \sqrt{1-x^2}$ *on the interval* $[-1, 1]$*. Moreover, we have :*

$$\int_{-1}^{1} w(x)\, U_i(x)\, U_j(x)\, dt = \begin{cases} \frac{\pi}{2}, & \textit{if } i = j, \\ 0, & \textit{if } i \neq j. \end{cases}$$

Corollary 1.4.3

The Chebyshev polynomials of the second kind of the degree n represent a solution for the following differential equation :

$$(1-x^2)y''(x) - 3xy'(x) + n(n+2)y = 0.$$

1.4.4 Hermite polynomials

Definition 1.4.5

The Hermite polynomials can be defined by the Rodrigués formula :

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right),$$

or by the following relation :

$$2xH_{n+1}(x) = H_{n+2}(x) + 2(n+1)H_n(x), \quad \text{with} \quad H_0(x) = 1, H_1(x) = 2x, n \geq 0.$$

Example 1.4.3

The first six Hermite polynomials $H_n(z)$ are :

$$H_0(x) = 1,$$
$$H_1(x) = 2x,$$
$$H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x,$$
$$H_4(x) = 16x^4 - 48x^2 + 12,$$
$$H_5(x) = 32x^5 - 160x^3 + 120x.$$

Proposition 1.4.5

These polynomials are orthogonal on the interval $\mathbb{R}$ with respect to the weight function $w(x) = e^{-x^2}$. Moreover we have :

$$\int_{-1}^{1} w(x)H_n(x)H_m(x)\,dx = \begin{cases} n!\,2^n\sqrt{\pi}, & n = m, \\ 0, & n \neq m. \end{cases}$$

Corollary 1.4.4

The Hermite polynomial $H_n(x)$ is a solution for the following differential equation :

$$y''(x) - 2xy'(x) + 2ny(x) = 0.$$

1.4.5 Laguerre polynomials

Definition 1.4.6

The Rodrigués representation for Laguerre polynomials is :

$$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}\left(x^n e^{-x}\right),$$

also, The Laguerre polynomials satisfy the recurrence formula :

$$(n+2)L_{n+2}(x) = (2n+3-x)L_{n+1}(x) - n+1L_n(x), \quad \textit{with} \quad L_0(x) = 1, L_1(x) = -x+1, n \geq 0.$$

Example 1.4.4

The first six Laguerre polynomials are :

$$\begin{aligned}
L_0(x) &= 1,\\
L_1(x) &= -x + 1,\\
L_2(x) &= \frac{1}{2}x^2 - 4x + 2,\\
L_3(x) &= \frac{1}{6} - x^3 + 9x^2 - 18x + 6,\\
L_4(x) &= \frac{1}{24}x^4 - 16x^3 + 72x^2 - 96x + 24,\\
L_5(x) &= \frac{1}{120} - x^5 + 25x^4 - 200x^3 + 600x^2 - 600x + 120.
\end{aligned}$$

Proposition 1.4.6

The Laguerre polynomials are orthogonal with respect to the weight function $w(x) = e^{-x}$ *on the interval* $[0, +\infty[$*. Moreover, we have :*

$$\int_0^\infty e^{-x} L_n(x) L_m(x)\, dz = \begin{cases} 1, & n = m, \\ 0, & n \neq m, \end{cases}$$

Corollary 1.4.5

The Laguerre polynomial $L_n(x)$ *satisfies the following differential equation :*

$$xy''(x) + (1-x)y'(x) + ny(x) = 0.$$

Remark 1.4.1

We can see that an alternative polynomials employed in engineering and scientific applications can be listed as follows :

Bernoulli polynomials [12], Fibonacci polynomials [13], Genocchi polynomials [14], Hosoya polynomials [15], Lucas polynomials [16], Pell polynomials [17], Touchard polynomials [18]. For more detail see [19].

Chapitre 2

Analytical study of the proposed equation

In this chapter, using the fixed point techniques, we will demonstrate the existence and uniqueness of the solution for the proposed integro-differential equation.

2.1 Existence of the solution

We consider the nonlinear Volterra integro-differential equation given by the following formula :

$$u(t) = \int_0^t k(t,s)\phi(u(t), u(s), u'(s))ds + g(t), \quad \forall t \in [0,1], \tag{2.1}$$

where $k \in C^1([0,1]^2, \mathbb{R})$, $\phi \in C^1(\mathbb{R}^3, \mathbb{R})$, $g \in C^1([0,1])$ and $u(t)$ is the solution that we want to find in the space $C^1([0,1])$. We point out that the unknown solutions $u(t), u(s)$ and $u'(t)$ are nonlinearly founded inside the kernel of equation, which is called a general non-standard integro-differential equation.

For technical reason, we need to derive equation (2.1) with respect to variable t to get an alternate equation that provides additional insight into the solution $u(t)$

as follows :

$$\begin{aligned} u'(t) &= \int_0^t \frac{\partial k}{\partial t}(t,s)\phi(u(t),u(s),u'(s))ds + u'(t)\int_0^t k(t,s)\frac{\partial \phi}{\partial t}(u(t),u(s),u'(s))ds \\ &+ k(t,t)\phi(u(t),u(t),u'(t)) + g'(t). \end{aligned} \tag{2.2}$$

To begin to study the system of equations (2.1)-(2.2), we must impose the following hypothesis :

$$(\mathcal{H}) : \begin{cases} \forall\, t,s \in [0,1],\ \forall\, x,\ y,\ z \in \mathbb{R}, \exists K,\ M,\ G,\ R \in \mathbb{R}^+, \quad \textit{such that}: \\ \bullet\ \max\left\{k(t,s), \frac{\partial k}{\partial t}(t,s)\right\} \leq K, \\ \bullet\ \max\left\{\phi(x,y,z), \frac{\partial \phi}{\partial x}(x,y,z)\right\} \leq M, \\ \bullet\ \max\left\{g(t), g'(t)\right\} \leq G, \\ \bullet\ \left|u'(t)\right| \leq R. \end{cases}$$

Theorem 2.1.1

If the hypothesis $(\mathcal{H})$ *is satisfied, then equation* (2.1) *has at least one solution in the space* $C^1([0,1])$.

Proof 2.1.1

Consider the Banach space $X = C^1([0,1])$ *equipped with the standard norm :*

$$\forall\, u \in X, \|u\| = \sup_{t\in[0,1]} |u(t)| + \sup_{t\in[0,1]} |u'(t)|.$$

We define the operator A *from the space* X *into itself by :*

$$A(u)(t) = \int_0^t k(t,s)\phi(u(t),u(s),u'(s))ds.$$

It's clear that if the operator A *admits at least one fixed point, then equation* (2.1) *admits at least one solution in* X. *For this reason, we will demonstrate that the operator* A *satisfies the conditions of Schauder fixed point theorem [11], according to the*

following steps :

Step 1 : *We must verify that for all $u \in F$, then $A(u) \in F$, where F is a subset of X, given by :*

$$F := \{u \in X,\ \|u\| \leq 3KM + RKM + 2G\}.$$

First, we can see that the subset F is non-empty, closed, and convex. For any $u \in F$ and $t \in [0,1]$, we have :

$$\begin{aligned} |A(u)(t)| &= \left| \int_0^t k(t,s)\phi(u(t),u(s),u'(s))ds + g(t) \right|, \\ &\leq \left| \int_0^t k(t,s)\phi(u(t),u(s),u'(s))ds \right| + |g(t)|, \\ &\leq KM + G. \end{aligned}$$

Similarly, one may find :

$$\begin{aligned} |A'(u)(t)| &\leq \int_0^t \left| \frac{\partial k}{\partial t}(t,s)\phi(u(t),u(s),u'(s)) \right| + |u'(t)| \int_0^t \left| k(t,s)\frac{\partial \phi}{\partial t}(u(t),u(s),u'(s)) \right| ds \\ &+ |k(t,t)\phi(u(t),u(t),u'(t))| + |g'(t)|, \\ &\leq KM + RKM + KM + G. \end{aligned}$$

Therefore, we deduce that $||A(u)|| \leq 3KM + RKM + 2G$, which means $A(u) \in F$.

Step 2 : *We must verify that operator A is compact and continuous.*
*It is clear that A is continuous. To study the compactness of operator A, it remains to prove that the set $A(F)$ is uniformly bounded and equicontinuous. From **Step 1**, we find directly that $A(F)$ is uniformly bounded. Going now to prove that $A(F)$ is equicontinuous.*
For any $u \in F$, for all $t_1, t_2 \in [0,1]$, with $t_1 > t_2$, we have :

$$|A(u)(t_1) - A(u)(t_2)| \leq I_1 + I_2,$$

where,

$$\begin{aligned} I_1 &= \left| \int_0^{t_1} k(t_1,s)\phi(u(t_1),u(s),u'(s))ds - \int_0^{t_2} k(t_2,s)\phi(u(t_2),u(s),u'(s))ds \right|, \\ I_2 &= |g(t_1)-g(t_2)|. \end{aligned}$$

Then,

$$\begin{aligned} I_1 &= \left| \int_0^{t_1} k(t_1,s)\phi(u(t_1),u(s),u'(s))ds - \int_0^{t_2} k(t_2,s)\phi(u(t_2),u(s),u'(s))ds \right|, \\ &\leq \int_0^{t_1} |k(t_1,s)\phi(u(t_1),u(s),u'(s)) - k(t_2,s)\phi(u(t_2),u(s),u'(s))|\, ds \\ &+ \int_{t_1}^{t_2} |k(t_2,s)\phi(u(t_2),u(s),u'(s))|\, ds, \end{aligned}$$

and by using Lemma 1.3.1, we obtain :

$$\begin{aligned} I_1 &\leq (KMR+KM)|t_1-t_2| + (KM)|t_1-t_2|, \\ &= (KMR+2KM)|t_1-t_2|. \end{aligned}$$

Also, the mean value Theorem gives us :

$$I_2 = |g(t_1)-g(t_2)| \leq G|t_1-t_2|.$$

So, we conclude that :

$$|A(u)(t_1)-A(u)(t_2)| \leq (KMR+2KM+G)|t_1-t_2|.$$

In the same way, we can get :

$$|A'(u)(t_1)-A'(u)(t_2)| \leq \widetilde{I}_1+\widetilde{I}_2+\widetilde{I}_3+\widetilde{I}_4+\widetilde{I}_5,$$

where,

$$\begin{aligned}
\widetilde{I}_1 &= |k(t_1,t_1)\phi(u(t_1),u(t_1),u'(t_1)) - k(t_2,t_2)\phi(u(t_2),u(t_2),u'(t_2))|,\\
\widetilde{I}_2 &= \int_0^{t_1}\left|\frac{\partial k}{\partial t}(t_1,s)\phi(u(t_1),u(s),u'(s)) - \frac{\partial k}{\partial t}(t_2,s)\phi(u(t_2),u(s),u'(s))\right| ds,\\
\widetilde{I}_3 &= \int_{t_1}^{t_2}\left|\frac{\partial k}{\partial t}(t_2,s)\phi(u(t_2),u(s),u'(s))\right| ds,\\
\widetilde{I}_4 &= \left|u'(t_1)\int_0^{t_1} k(t_1,s)\frac{\partial \phi}{\partial t}(u(t_1),u(s),u'(s))ds - u'(t_2)\int_0^{t_2} k(t_2,s)\frac{\partial \phi}{\partial t}(u(t_2),u(s),u'(s))ds\right|,\\
\widetilde{I}_5 &= |g'(t_1) - g'(t_2)|.
\end{aligned}$$

By applying Lemma 1.3.1, we obtain :

$$\widetilde{I}_1 \leq (KMR + KM)|t_1 - t_2|.$$

Also, we have :

$$\begin{aligned}
\widetilde{I}_2 &= \int_0^{t_1}\left|\frac{\partial k}{\partial t}(t_1,s)\phi(u(t_1),u(s),u'(s)) - \frac{\partial k}{\partial t}(t_2,s)\phi(u(t_2),u(s),u'(s))\right| ds,\\
&\leq \int_0^{t_1}\left|\frac{\partial k}{\partial t}(t_1,s)\right| |\phi(u(t_1),u(s),u'(s)) - \phi(u(t_2),u(s),u'(s))|\, ds\\
&+ \int_0^{t_1} |\phi(u(t_2),u(s),u'(s))|\left|\frac{\partial k}{\partial t}(t_1,s) - \frac{\partial k}{\partial t}(t_2,s)\right| ds,\\
&\leq KMR|t_1 - t_2| + M\int_0^{t_1}\left|\frac{\partial k}{\partial t}(t_1,s) - \frac{\partial k}{\partial t}(t_2,s)\right| ds,
\end{aligned}$$

on the other hand, the function $\frac{\partial k}{\partial t}(t,s)$ *is continuous over the interval* $[0,1]$*, so it is uniformly continuous, which permits us to write :*

$$\left|\frac{\partial k}{\partial t}(t_1,s) - \frac{\partial k}{\partial t}(t_2,s)\right| \leq \delta_1(|t_1 - t_2|) \to 0 \quad as \quad t_1 \to t_2,$$

so, we obtain :

$$\widetilde{I}_2 \leq KMR|t_1 - t_2| + M\delta_1(|t_1 - t_2|).$$

It is clear that :

$$\widetilde{I}_3 \leq KM|t_1 - t_2|.$$

Now, For the term $\widetilde{I}_4$*, we have :*

$$\begin{aligned}
\widetilde{I}_4 &= \left| u'(t_1)\int_0^{t_1} k(t_1,s)\frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s))ds - u'(t_2)\int_0^{t_2} k(t_2,s)\frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s))ds \right|, \\
&\leq |u'(t_1)| \left| \int_0^{t_1} k(t_1,s)\frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s))ds - \int_0^{t_2} k(t_2,s)\frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s))ds \right| \\
&+ |u'(t_1) - u'(t_2)| \int_0^{t_2} \left| k(t_2,s)\frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s)) \right| ds, \\
&\leq R \left| \int_0^{t_1} k(t_1,s)\frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s))ds - \int_0^{t_2} k(t_2,s)\frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s))ds \right| \\
&+ KM\left|u'(t_1) - u'(t_2)\right|,
\end{aligned}$$

and as we have done before, we find :

$$\begin{aligned}
\widetilde{I}_4 &\leq R\int_0^{t_1} |k(t_1,s)| \left| \frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s)) - \frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s)) \right| ds \\
&+ R\int_0^{t_1} \left| \frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s))ds \right| |k(t_1,s) - k(t_2,s)|\, ds \\
&+ R\int_{t_1}^{t_2} \left| k(t_2,s)\frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s))ds \right|, \\
&+ KM\left|u'(t_1) - u'(t_2)\right|, \\
&\leq RK\int_0^{t_1} \left| \frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s)) - \frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s)) \right| ds \\
&+ RMK|t_1 - t_2| + RMK|t_1 - t_2| + KM\left|u'(t_1) - u'(t_2)\right|,
\end{aligned}$$

also, the functions $\frac{\partial\phi}{\partial t}(u(t),u(s),u'(s))$ *and* $u'(t)$ *are uniformly continuous, so we can write :*

$$\begin{aligned}
\left| \frac{\partial\phi}{\partial t}(u(t_1),u(s),u'(s)) - \frac{\partial\phi}{\partial t}(u(t_2),u(s),u'(s)) \right| &\leq \delta_2(|t_1 - t_2|) \to 0 \quad as \quad t_1 \to t_2, \\
|u'(t_1) - u'(t_2)| &\leq \delta_3(|t_1 - t_2|) \to 0 \quad as \quad t_1 \to t_2,
\end{aligned}$$

so, we get :

$$\widetilde{I}_4 \leq 2KMR|t_1 - t_2| + KR\delta_2(|t_1 - t_2|) + KM\delta_3(|t_1 - t_2|).$$

According to the uniformness continuously of the function $g'(t)$, we have :

$$|g'(t_1) - g'(t_2)| \leq \delta_4(|t_1 - t_2|) \to 0 \quad as \quad t_1 \to t_2,$$

then

$$\widetilde{I}_5 \leq \delta_4(|t_1 - t_2|).$$

Consequently,

$$\begin{aligned} |A'(u)(t_1) - A'(u)(t_2)| &\leq (4KMR + 2KM)|t_1 - t_2| + M\delta_1(|t_1 - t_2|) + KR\delta_2(|t_1 - t_2|) \\ &+ KM\delta_3(|t_1 - t_2|) + \delta_4(|t_1 - t_2|). \end{aligned}$$

Finally, we conclude that :

$$|A(u)(t_1) - A(u)(t_2)| \leq ((2 + R)KM + G)|t_1 - t_2| \to 0 \quad as \quad t_1 \to t_2,$$

$$\begin{aligned} |A'(u)(t_1) - A'(u)(t_2)| &\leq (4KMR + 2KM)|t_1 - t_2| + M\delta_1(|t_1 - t_2|) + KR\delta_2(|t_1 - t_2|) \\ &+ KM\delta_3(|t_1 - t_2|) + \delta_4(|t_1 - t_2|) \to 0 \quad as \quad t_1 \to t_2, \end{aligned}$$

*which confirms us that $A(F)$ is equicontinuous. From the **Steps 1** and **2**, the operator A has at least one fixed point, i.e. the equation* (2.1) *has at least one solution.*

2.2 Uniqueness of the solution

In the previous section, we have demonstrated that equation (2.1) has at least one solution. Now, we will prove that this solution is unique, but we must first add

the new hypothesis :

$$(\widetilde{\mathcal{H}}):\begin{cases}\forall\, x,\, y,\, z,\, x',\, y',\, z' \in \mathbb{R}, \exists A_1,\, A_2,\, A_3,\, B_1,\, B_2,\, B_3 \in \mathbb{R}^+, \quad \textit{such that}:\\ \bullet\ \left|\phi(x,y,z)-\phi(x',y',z')\right| \leq A_1|x-x'|+A_2|y-y'|+A_3|z-z'|,\\ \bullet\ \left|\frac{\partial\phi}{\partial t}(x,y,z)-\frac{\partial\phi}{\partial t}(x',y',z')\right| \leq B_1|x-x'|+B_2|y-y'|+B_3|z-z'|,\\ \bullet\ KA_1<1,\\ \bullet\ KA_3+KM<1.\end{cases}$$

Theorem 2.2.1

If hypotheses $(\mathcal{H})$ *and* $(\widetilde{\mathcal{H}})$ *are satisfied, then the solution of equation* (2.1) *is unique.*

Proof 2.2.1

Let $u(t), v(t)$ *be two different solutions of equation* (2.1). *We define the function* $\gamma(t)$ *by :*

$$\gamma(t)=|u(t)-v(t)|+|u'(t)-v'(t)|, \quad \forall\, t\in[0,1].$$

First, equation (2.1) *gives us :*

$$\begin{aligned}|u(t)-v(t)| &= \left|\int_0^t k(t,s)\phi(u(t),u(s),u'(s))ds-\int_0^t k(t,s)\phi(v(t),v(s),v'(s))ds\right|,\\ &\leq \int_0^t |k(t,s)|\,|\phi(u(t),u(s),u'(s))-\phi(v(t),v(s),v'(s))|\,ds,\end{aligned}$$

and from $(\mathcal{H})$, $(\widetilde{\mathcal{H}})$ *:*

$$\begin{aligned}|u(t)-v(t)| &\leq K\int_0^t |\phi(u(t),u(s),u'(s))-\phi(v(t),v(s),v'(s))|ds,\\ &\leq K\int_0^t \left(A_1|u(t)-v(t)|+A_2|u(s)-v(s)|+A_3|u'(s)-v'(s)|\right)ds,\\ &\leq KA_1|u(t)-v(t)|+K\max\{A_2,A_3\}\int_0^t \gamma(s)ds,\end{aligned}$$

then

$$(1-KA_1)|u(t)-v(t)| \leq K\max\{A_2,A_3\}\int_0^t \gamma(s)ds,$$

and according to $KA_1 < 1$, we obtain :

$$|u(t) - v(t)| \leq \frac{K \max\{A_2, A_3\}}{(1 - KA_1)} \int_0^t \gamma(s)ds. \tag{2.3}$$

In a similar way, we get :

$$|u'(t) - v'(t)| \leq S_1 + S_2 + S_3,$$

where,

$$\begin{aligned}
S_1 &= |k(t,t)||\phi(u(t), u(t), u'(t)) - \phi(v(t), v(t), v'(t))|, \\
S_2 &= \int_0^t \left|\frac{\partial k}{\partial t}(t,s)\right| |\phi(u(t), u(s), u'(s)) - \phi(v(t), v(s), v'(s))|ds, \\
S_3 &= \left|u'(t) \int_0^t k(t,s)\frac{\partial \phi}{\partial t}(u(t), u(s), u'(s))ds - v'(t) \int_0^t k(t,s)\frac{\partial \phi}{\partial t}(v(t), v(s), v'(s))ds\right|.
\end{aligned}$$

It's clear that :

$$\begin{aligned}
S_1 &= |k(t,t)||\phi(u(t), u(t), u'(t)) - \phi(v(t), v(t), v'(t))|, \\
&\leq K(A_1|u(t) - v(t)| + A_2|u(t) - v(t)| + A_3|u'(t) - v'(t)|), \\
&= K(A_1 + A_2)|u(t) - v(t)| + KA_3|u'(t) - v'(t)|.
\end{aligned}$$

Also,

$$\begin{aligned}
S_2 &= \int_0^t \left|\frac{\partial k}{\partial t}(t,s)\right| |\phi(u(t), u(s), u'(s)) - \phi(v(t), v(s), v'(s))|ds, \\
&\leq K \int_0^t |\phi(u(t), u(s), u'(s)) - \phi(v(t), v(s), v'(s))|ds, \\
&\leq K \int_0^t (A_1|u(t) - v(t)| + A_2|u(s) - v(s)| + A_3|u'(s) - v'(s)|)ds, \\
&\leq KA_1|u(t) - v(t)| + K \max\{A_2, A_3\} \int_0^t \gamma(s)ds.
\end{aligned}$$

For the term S_3, we have :

$$\begin{aligned} S_3 &= \left| u'(t)\int_0^t k(t,s)\frac{\partial\phi}{\partial t}(u(t),u(s),u'(s))ds - v'(t)\int_0^t k(t,s)\frac{\partial\phi}{\partial t}(v(t),v(s),v'(s))ds \right|, \\ &\leq |u'(t)|\int_0^t |k(t,s)|\left|\frac{\partial\phi}{\partial t}(u(t),u(s),u'(s)) - \frac{\partial\phi}{\partial t}(v(t),v(s),v'(s))\right| ds \\ &+ |u'(t)-v'(t)|\int_0^t |k(t,s)|\left|\frac{\partial\phi}{\partial t}(v(t),v(s),v'(s))\right| ds, \\ &\leq KM|u'(t)-v'(t)| + RK\int_0^t \left|\frac{\partial\phi}{\partial t}(u(t),u(s),u'(s)) - \frac{\partial\phi}{\partial t}(v(t),v(s),v'(s))\right| ds, \end{aligned}$$

and from hypothesis $(\widetilde{\mathcal{H}})$, we get :

$$\begin{aligned} S_3 &\leq KM|u'(t)-v'(t)| + RK\int_0^t (B_1|u(t)-v(t)| + B_2|u(s)-v(s)| + B_3|u'(s)-v'(s)|)ds, \\ &\leq KM|u'(t)-v'(t)| + RKB_1|u(t)-v(t)| + RK\max\{B_2,B_3\}\int_0^t \gamma(s)ds, \end{aligned}$$

so, we conclude :

$$\begin{aligned} |u'(t)-v'(t)| &\leq S_1+S_2+S_3, \\ &= (KA_3+KM)\,|u'(t)-v'(t)| + (2KA_1+KA_2+RKB_1)\,|u(t)-v(t)| \\ &+ (K\max\{A_2,A_3\} + RK\max\{B_2,B_3\})\int_0^t \gamma(s)ds, \end{aligned}$$

and by using the condition $KA_3+KM<1$ and the previous inequality (2.3)*, we obtain :*

$$\begin{aligned} |u'(t)-v'(t)| &\leq \frac{(2KA_1+KA_2+RKB_1)K\max\{A_2,A_3\}}{(1-KA_3-KM)(1-KA_1)}\int_0^t \gamma(s)ds \\ &+ \frac{(K\max\{A_2,A_3\}+RK\max\{B_2,B_3\})}{(1-KA_3-KM)}\int_0^t \gamma(s)ds, \qquad (2.4) \end{aligned}$$

by combining the inequalities (2.3) *and* (2.4)*, we find :*

$$\exists\, L>0, \quad \textit{such that,} \quad \gamma(t) \leq L\int_0^t \gamma(s)ds,$$

where,

$$\begin{aligned} L &= \frac{K\max\{A_2,A_3\}}{(1-KA_1)} + \frac{K\max\{A_2,A_3\}+RK\max\{B_2,B_3\}}{(1-KA_3-KM)} \\ &+ \frac{(2KA_1+KA_2+RKB_1)K\max\{A_2,A_3\}}{(1-KA_3-KM)(1-KA_1)}. \end{aligned}$$

Thus, by applying Lemma 1.3.2 described above, we get $\gamma(t)=0$*, i.e.* $u(t)=v(t)$ *and* $u'(t)=v'(t)$*. Therefor, the equation* (2.1) *has a unique solution.*

Chapitre 3

Numerical study of the proposed equation

In general, there are no analytical methods for solving integral equations. Therefore, numerical approximation techniques must be used to approximate these equations. In this chapter, we will use the Fibonacci Wavelets Collocation method [20] to determine the approximate solution of our integral equation.

3.1 Fibonacci Wavelet and its properties

3.1.1 General Wavelets' Family

To build a family of functions, the mother wavelets dilate and translate themselves. We coined the term wavelets to describe them. The family's wavelets are as follows :

$$\psi_{p,q}(t) = |p|^{-\frac{1}{2}}\psi\left(\frac{t-q}{p}\right), \quad p,\, q \in \mathbb{R}, \quad p \neq 0.$$

The dilation and translation parameters are indicated by the letters p and q, respectively. Furthermore, q varies continuously.

If we choose $p = p_0^{-k},\ q = np_0^{-k}q_0,\ p_0 > 1,\ q_0 > 1$, and n and k are positive integers,

the discrete wavelet family is introduced as :

$$\psi_{k,n}(t) = |p_0|^{k/2} \psi \left(p_0^k t - n q_0\right),$$

where $\psi_{k,n}(t)$ is the wavelet basis in $L^2(\mathbb{R})$.

3.1.2 Fibonacci Polynomials

First, the Fibonacci polynomials are defined in terms of the recurrence relation for every $t \in \mathbb{R}+$ as follows :

$$P_{m+2}(t) = tP_{m+1}(t) + P_m(t), \quad \forall m \geq 0,$$

with initial conditions $P_0(t) = 1$, $P_1(t) = t$. They can be defined similarly using the following general formula :

$$P_m(t) = \begin{cases} 1, & m = 0, \\ t, & m = 1, \\ tP_{m-1}(t) + P_{m-2}(t), & m > 1. \end{cases}$$

3.1.3 Fibonacci Wavelets

Now, we define the Fibonacci Wavelets $\psi_{n,m}(t) = \psi(k, n, m, t)$ On the interval $[0, 1]$ by :

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k-1}{2}}}{\sqrt{w_m}} P_m \left(2^{k-1} t - \hat{n}\right), & \frac{\hat{n}}{2^{k-1}} \leq t < \frac{\hat{n}+1}{2^{k-1}}, \\ 0, & \text{otherwise}, \end{cases}$$

where,

- $P_m(t)$ are the Fibonacci polynomial of degree $m = 0, 1, ...,$
- k is the level of resolution $k = 1, 2, ...,$
- $\hat{n}$ and n are the translation parameters $\hat{n} = n - 1,\ n = 1, 2, ..., 2^{k-1}$,
- t is the normalized time,
- $\frac{1}{\sqrt{w_m}}$ is a normalization factor, $w_m = \int_0^1 \left(P_m(t)\right)^2 \, dt$.

3.1.4 Function Approximation Using Fibonacci Wavelets

Let $\psi_{i,j}$ be the sequence of Fibonacci Wavelets for $m = 0, 1, ..., M-1$ and $n = 1, 2, ..., 2^{k-1}$. For every fixed n, there is a Fibonacci space spanned by the elements of the sequence $\psi_{i,j}$. So, we fairly accurate the function $u(x)$ under the Fibonacci Wavelet as follows :

$$u(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} C_{n,m}\, \psi_{n,m}(x).$$

By truncating this series, we get :

$$u(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} C_{n,m}\, \psi_{n,m}(x) = C^T \Psi(x), \tag{3.1}$$

where C and $\Psi(x)$ are $2^{k-1}M \times 1$ vectors given as :

$$C^T = [C_{1,0},\, \ldots,\, C_{1,M-1},\, C_{2,0},\, \ldots,\, C_{2,M-1},\, \ldots,\, C_{2^{k-1},0}, \ldots, C_{2^{k-1},M-1}],$$

$$\Psi(x) = [\psi_{1,0}(x),\, \ldots,\, \psi_{1,M-1}(x),\, \psi_{2,0}(x),\, \ldots,\, \psi_{2,M-1}(x),\, \ldots,\, \psi_{2^{k-1},0}(x),\, \ldots,\, \psi_{2^{k-1},M-1}(x)].$$

We point out that then Fibonacci Wavelet Expansion of the function $u(x)$ is uniformly converges to it.

3.1.5 Integration Operational Matrix of Fibonacci Wavelets

Consider the Fibonacci Wavelet basis which tackled for $k = 1$ and $M = 10$:

$$\psi_{1,0}(t) = 1,$$
$$\psi_{1,1}(t) = \sqrt{3}t,$$
$$\psi_{1,2}(t) = \frac{1}{2}\sqrt{\frac{15}{7}}(1 + t^2),$$
$$\psi_{1,3}(t) = \sqrt{\frac{105}{239}}(2t + t^3),$$
$$\psi_{1,4}(t) = 3\sqrt{\frac{35}{1943}}(1 + 3t^2 + t^4),$$
$$\psi_{1,5}(t) = \frac{3}{4}\sqrt{\frac{385}{2582}}(3t + 4t^3 + t^5),$$
$$\psi_{1,6}(t) = 3\sqrt{\frac{5005}{1268209}}(1 + 6t^2 + 5t^4 + t^6),$$
$$\psi_{1,7}(t) = 3\sqrt{\frac{5005}{2827883}}(4t + 10t^3 + 6t^5 + t^7),$$
$$\psi_{1,8}(t) = \frac{3}{2}\sqrt{\frac{85085}{28195421}}(1 + 10t^2 + 15t^4 + 7t^6 + t^8),$$
$$\psi_{1,9}(t) = 3\sqrt{\frac{1616615}{5016284989}}(5t + 20t^3 + 21t^5 + 8t^7 + t^9).$$

Let $\Psi(t) = \Psi_{10}(t)$ be basis given as :

$$\Psi_{10}(t) = [\psi_{1,0}(t),\ \psi_{1,1}(t),\ \psi_{1,2}(t),\ \psi_{1,3}(t),\ \psi_{1,4}(t),\ \psi_{1,5}(t),\ \psi_{1,6}(t),\ \psi_{1,7}(t),\ \psi_{1,8}(t),\ \psi_{1,9}(t)]^T.$$

Now integrate the above first ten basis concerning variable s, limit from 0 to t, then express as a linear combination of Fibonacci Wavelet basis $\Psi_{10}(t)$ as follows :

$$\int_0^t \psi_{1,0}(s)\,ds = \left[0\ \frac{1}{\sqrt{3}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,1}(s)\,ds = \left[-\frac{1}{\sqrt{3}}\ 0\ \sqrt{\frac{7}{5}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,2}(s)\,ds = \left[0\ \frac{\sqrt{5}}{6\sqrt{7}}\ 0\ \frac{\sqrt{239}}{42}\ 0\ 0\ 0\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,3}(s)\,ds = \left[-\frac{\sqrt{105}}{2\sqrt{239}}\ 0\ \sqrt{\frac{7}{2\sqrt{239}}}\ 0\ \frac{\sqrt{1943}}{4\sqrt{717}}\ 0\ 0\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,4}(s)\,ds = \left[0\ 0\ 0\ -\frac{\sqrt{717}}{5\sqrt{1943}}\ 0\ \frac{4\sqrt{2582}}{5\sqrt{21373}}\ 0\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,5}(s)\,ds = \left[\frac{\sqrt{385}}{4\sqrt{2582}}\ 0\ 0\ 0\ -\frac{\sqrt{21373}}{24\sqrt{2582}}\ 0\ \frac{\sqrt{1268209}}{24\sqrt{33566}}\ 0\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,6}(s)\,ds = \left[0\ 0\ 0\ 0\ 0\ \frac{4\sqrt{33566}}{7\sqrt{1268209}}\ 0\frac{\sqrt{28195421}}{4\sqrt{48074011}}\ 0\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,7}(s)\,ds = \left[\frac{3\sqrt{5005}}{4\sqrt{2827883}}\ 0\ 0\ 0\ 0\ 0\ \frac{\sqrt{1268209}}{8\sqrt{2827883}}\ 0\ \frac{\sqrt{28195421}}{4\sqrt{48074011}}\ 0\right]\Psi_{10}(t),$$

$$\int_0^t \psi_{1,8}(s)\,ds = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{\sqrt{48074011}}{18\sqrt{28195421}}\ 0\ \frac{\sqrt{5016284989}}{18\sqrt{535712999}}\right]\Psi_{10}(t).$$

Moreover, the base function $\psi_{1,9}(s)$ is a polynomial of degree 9. So, we must define its integral from 0 to t using the new base $\psi_{1,10}(t)$ by the following formula :

$$\int_0^t \psi_{1,9}(s)\,ds = \left[-3\sqrt{\frac{323323}{25081424945}}\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{\sqrt{535712999}}{5\sqrt{5016284989}}\ 0\right]\Psi_{10}(t)+\frac{\sqrt{11941544471}}{10\sqrt{5016284989}}\psi_{1,10}(t),$$

where,

$$\psi_{1,10}(t) = 3\sqrt{\frac{1616615}{11941544471}}(1+15t^2+35t^4+28t^6+9t^8+t^{10}).$$

Hence, we conclude that :

$$\int_0^t \Psi_{10}(s)\,dt = B_{10\times 10}\Psi_{10}(t) + \overline{\Psi_{10}(t)},$$

where, the matrix $B_{10\times 10}$ is given by :

$$B_{10\times 10} = \begin{bmatrix}
0 & \frac{1}{\sqrt{3}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{3}}{2} & 0 & \sqrt{\frac{7}{5}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{5}}{6\sqrt{7}} & 0 & \frac{\sqrt{239}}{42} & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{\sqrt{105}}{2\sqrt{239}} & 0 & \sqrt{\frac{7}{2\sqrt{239}}} & 0 & \frac{\sqrt{1943}}{4\sqrt{717}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{\sqrt{717}}{5\sqrt{1943}} & 0 & \frac{4\sqrt{2582}}{5\sqrt{21373}} & 0 & 0 & 0 & 0 \\
\frac{\sqrt{385}}{4\sqrt{2582}} & 0 & 0 & 0 & -\frac{\sqrt{21373}}{24\sqrt{2582}} & 0 & \frac{\sqrt{1268209}}{24\sqrt{33566}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{4\sqrt{33566}}{7\sqrt{1268209}} & 0 & \frac{\sqrt{28195421}}{4\sqrt{48074011}} & 0 & 0 \\
\frac{3\sqrt{5005}}{4\sqrt{2827883}} & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{1268209}}{8\sqrt{2827883}} & 0 & \frac{\sqrt{28195421}}{4\sqrt{48074011}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{48074011}}{18\sqrt{28195421}} & 0 & \frac{\sqrt{5016284989}}{18\sqrt{535712999}} \\
-3\sqrt{\frac{323323}{25081424945}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{535712999}}{5\sqrt{5016284989}} & 0
\end{bmatrix}$$

and the vector basis $\overline{\Psi_{10}(t)}$ by :

$$\overline{\Psi_{10}(t)} = \left[0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \frac{\sqrt{11941544471}}{10\sqrt{5016284989}}\psi_{1,10}(t)\right]^T.$$

In a similar manner, we can generate matrices for our handiness by :

$$\int_0^t \Psi(s)\,ds = B\Psi(t) + \overline{\Psi(t)}. \tag{3.2}$$

3.2 Description of Approximate Solution's Method

The primary goal of this part is to approach the solution of the system of equations (2.1)-(2.1), using the Fibonacci Wavelets. We mention that we need to add the initial condition $u(0) = \varrho_0$ and we will use just the second equation of the previous system :

$$\begin{aligned} u'(t) &= \int_0^t \frac{\partial k}{\partial t}(t,s)\phi(u(t),u(s),u'(s))ds + u'(t)\int_0^t k(t,s)\frac{\partial \phi}{\partial t}(u(t),u(s),u'(s))ds \\ &+ k(t,t)\phi(u(t),u(t),u'(t)) + g'(t). \end{aligned}$$

First, we approach the function $u'(t)$ by using (3.1) the approximate formula of Fibonacci Wavelets :

$$u'(t) \approx C^T \Psi(t). \tag{3.3}$$

Then, we integrate (3.3) from 0 to t, when the operational matrix formula (3.2) gives us that :

$$\begin{aligned} \int_0^t u'(s)ds &\approx \int_0^t C^T \Psi(s)ds, \\ u(t) - u(0) &\approx C^T \int_0^t \Psi(s)ds, \\ u(t) &\approx \varrho_0 + C^T \left[B\Psi(t) + \overline{\Psi(t)} \right]. \end{aligned} \tag{3.4}$$

Now, substitute (3.3) and (3.4) in (2.2), we obtain :

$$\begin{aligned} C^T\Psi(t) &= \int_0^t \frac{\partial k}{\partial t}(t,\, s)\phi\left(\varrho_0 + C^T\left[B\Psi(t) + \overline{\Psi(t)}\right],\, \varrho_0 + C^T\left[B\Psi(s) + \overline{\Psi(s)}\right],\, C^T\Psi(s)\right) ds \\ &+ C^T\Psi(t)\int_0^t k(t,s)\frac{\partial \phi}{\partial t}\left(\varrho_0 + C^T\left[B\Psi(t) + \overline{\Psi(t)}\right],\, \varrho_0 + C^T\left[B\Psi(s) + \overline{\Psi(s)}\right],\, C^T\Psi(s)\right) ds \\ &+ k(t,t)\phi\left(\varrho_0 + C^T\left[B\Psi(t) + \overline{\Psi(t)}\right],\, \varrho_0 + C^T\left[B\Psi(t) + \overline{\Psi(t)}\right],\, C^T\Psi(t)\right) + g'(t). \end{aligned}$$

By collocating this equation on the uniform following grids points :

$$t_i = \frac{i+1}{2^{k-1}(M+1)}, \qquad i = 0, 1, 2, \ldots, M-1,$$

we obtain the following nonlinear algebraic system :

$$C^T A = D^T, \tag{3.5}$$

where A is a matrix given by :

$$A = (a_{i,m})_{0 \le i,m \le 2^{k-1}M-1}, \quad a_{i,m} = \psi_{k,m}(t_i),$$

and the vector $D = [d_i]_{0\leq i\leq 2^{k-1}M-1}$ by :

$$\begin{aligned} d_i &= \int_0^{t_i} \frac{\partial k}{\partial t}(t_i,\, s)\phi\left(\varrho_0 + C^T\left[B\Psi(t_i) + \overline{\Psi(t_i)}\right],\, \varrho_0 + C^T\left[B\Psi(s) + \overline{\Psi(s)}\right],\, C^T\Psi(s)\right) ds \\ &+ C^T\Psi(t_i)\int_0^{t_i} k(t_i,s)\frac{\partial \phi}{\partial t}\left(\varrho_0 + C^T\left[B\Psi(t_i) + \overline{\Psi(t_i)}\right],\, \varrho_0 + C^T\left[B\Psi(s) + \overline{\Psi(s)}\right],\, C^T\Psi(s)\right) ds \\ &+ k(t_i,t_i)\phi\left(\varrho_0 + C^T\left[B\Psi(t_i) + \overline{\Psi(t_i)}\right],\, \varrho_0 + C^T\left[B\Psi(t_i) + \overline{\Psi(t_i)}\right],\, C^T\Psi(t_i)\right) + g'(t_i). \end{aligned}$$

We solve the above algebraic system (3.5) using MATLAB Software to determine the values of the coefficients C^T, then by substituting them in formulas (3.3) and (3.4) we get the approximate solution $u(t)$ and its derivative $u'(t)$, respectively.

3.3 Numerical Examples

In this section, we provide two illustrative examples to show the efficacy of the Fibonacci Wavelets Collocation method. First, we take $k = 1$ and varying the parameter $M \in \{1, 2, ..., 9\}$ in order to find an approximate solution $u_{ap}(t)$ with its derivative $u'_{ap}(t)$ of the given examples according to the system (3.5) described above. Moreover, to compare between the exact solution $u_{ex}(t)$ and its derivative $u'_{ex}(t)$ with the approximate solutions we define the error function E as :

$$E = \|u_{ex}(t) - u_{ap}(t)\|_\infty + \|u'_{ex}(t) - u'_{ap}(t)\|_\infty.$$

3.3.1 Example 1

Consider the following equation :

$$u(t) = \frac{1}{2}\int_0^t e^{(s-t)}\cos\left(u(t)(u(s) + u'(s)) - u(t)\right) ds + g(t),$$

where,

$$g(t) = \frac{1}{2} + \frac{3}{2}e^{-t},\ \varrho_0 = 2,\ t \in [0, 1],$$

and the exact solution and its derivative are given by :

$$u(t) = e^{-t} + 1, \quad u'(t) = e^{-t}.$$

M	E	Iteration	CPU time (sec)
2	8.27E-2	9	0.28
3	1.10E-2	10	0.28
5	8.63E-5	11	0.31
7	3.12E-7	12	0.35
9	1.66E-9	20	3.21

TABLE 3.1 – The error function E of example 1.

Interpretation : Table 3.1 shows us that the error function E converges to zero quickly by increasing the values of M, which means that the approximate solution and its derivative close to the exact solution and its derivative (Figures 3.1-3.2), respectively. So, these numerical results confirm the efficiency and validity of the Fibonacci Wavelets Collocation method.

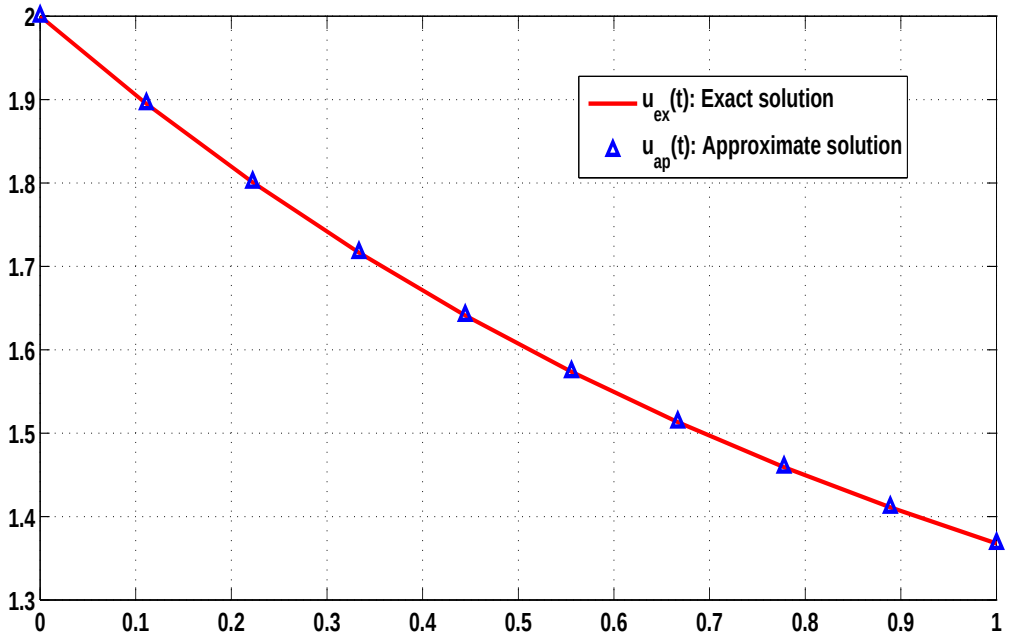


FIGURE 3.1 – Exact and approximate solution of example 1 with $M = 8$.

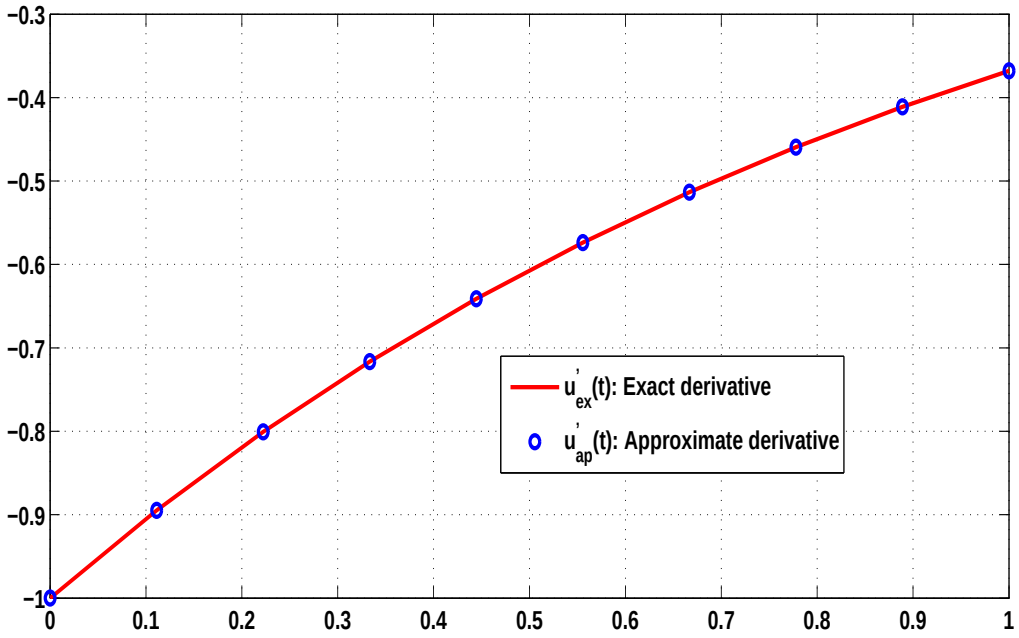


FIGURE 3.2 – Exact and approximate derivative of solution of example 1 with $M = 8$.

3.3.2 Example 2

Consider the second equation :

$$u(t) = \int_0^1 \frac{\sin(s-t)}{u(t)^2(Ru(s)^2 - u'(s) + R)^2 + 1} ds + g(t),$$

where,

$$g(t) = 1 + \frac{\sin(Rt) - \cos(t)\cos(Rt)}{\cos(Rt)}, \ R > 0, \ \varrho_0 = 0, \ t \in [0,1],$$

and the exact solution and its derivative are given by :

$$u(t) = \tan(Rt), \quad u'(t) = \frac{R}{\cos^2(Rt)}.$$

M	$R=0$	$R=\frac{\pi}{8}$	$R=\frac{\pi}{4}$	$R=\frac{\pi}{2}$	$R=\pi$	$R=\frac{3\pi}{2}$
2	3.05E-16	2.18E-2	3.68E-1	4.18E+32	2.33E+1	2.51E+33
3	1.85E-16	1.83E-3	1.11E-1	4.18E+32	2.79E+33	1.39E+32
5	5.55E-16	3.96E-5	1.23E-2	4.18E+32	9.46E+33	1.25E+34
7	2.92E-15	7.06E-7	1.16E-3	4.18E+32	3.15E+34	1.39E+32
9	1.78E-14	8.97E-9	1.00E-4	4.18E+32	1.11E+35	1.39E+32

TABLE 3.2 – The error function E of example 2 by varying the values of R.

Interpretation : First, wa have found in the theoretical analysis of our equation that the boundedness of the derivative of its solution is a necessary condition for the existences and uniqueness study. However, in example 2, we have chosen $u'(t) = \frac{R}{\cos^2(Rt)}$, i.e. $|u'(t)| \leq 3.5R$, then we vary the values of R in order to see its influence.

Table 3.2 shows us that for $R \leq \frac{\pi}{4}$, the error function E vanishes by increasing the values of M. As well as, the graphical representations of the exact solution with its derivative and the approximate solution with its derivative given in Figures 3.3 and 3.4 are nearly identical, respectively, for $R = \frac{\pi}{4}$ and $M = 8$.

Regarding to $R \geq \frac{\pi}{2}$, Table 3.2 and Figure3.5 show that the approximate solution completely diverges for any value of M. So, we conclude that the boundedness of solution's derivative is very important condition to study these types of equations.

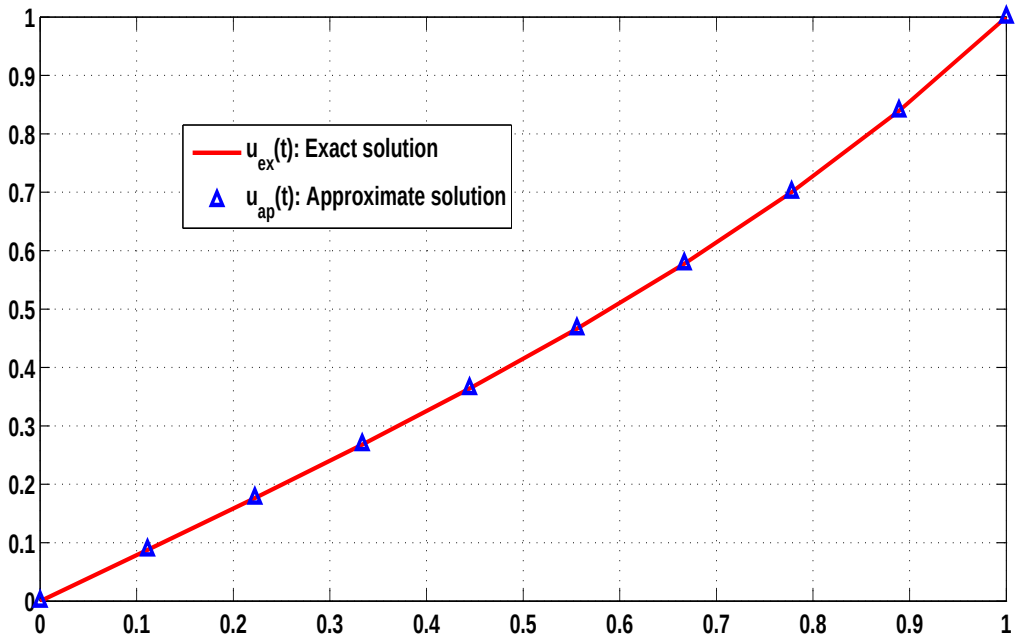


FIGURE 3.3 – Exact and approximate solution of example 2 for $R = \frac{\pi}{4}$ and $M = 8$.

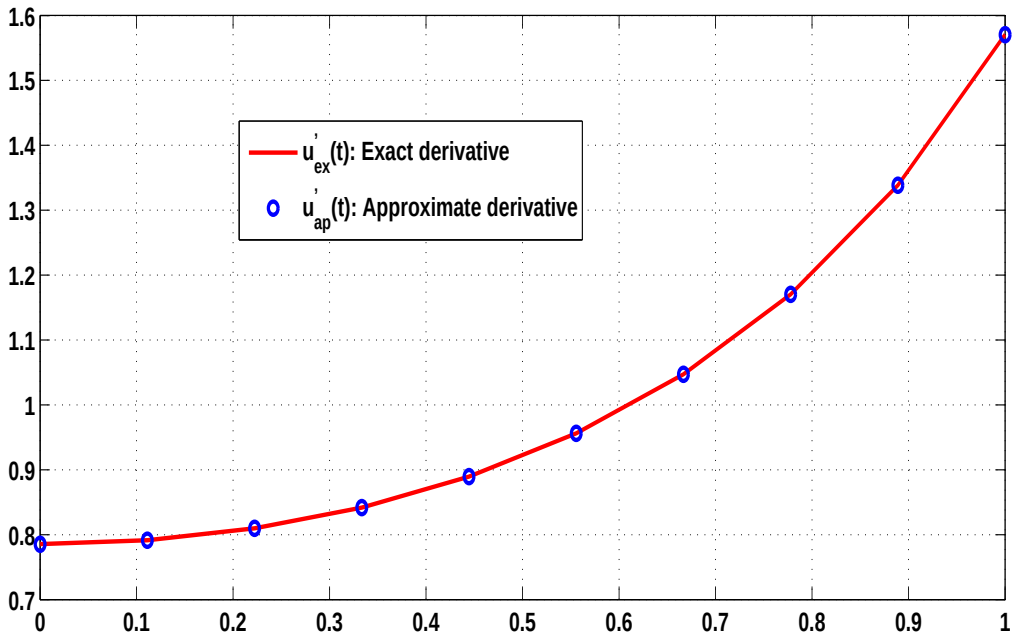


FIGURE 3.4 – Exact and approximate derivative of solution of example 2 for $R = \frac{\pi}{4}$ and $M = 8$.

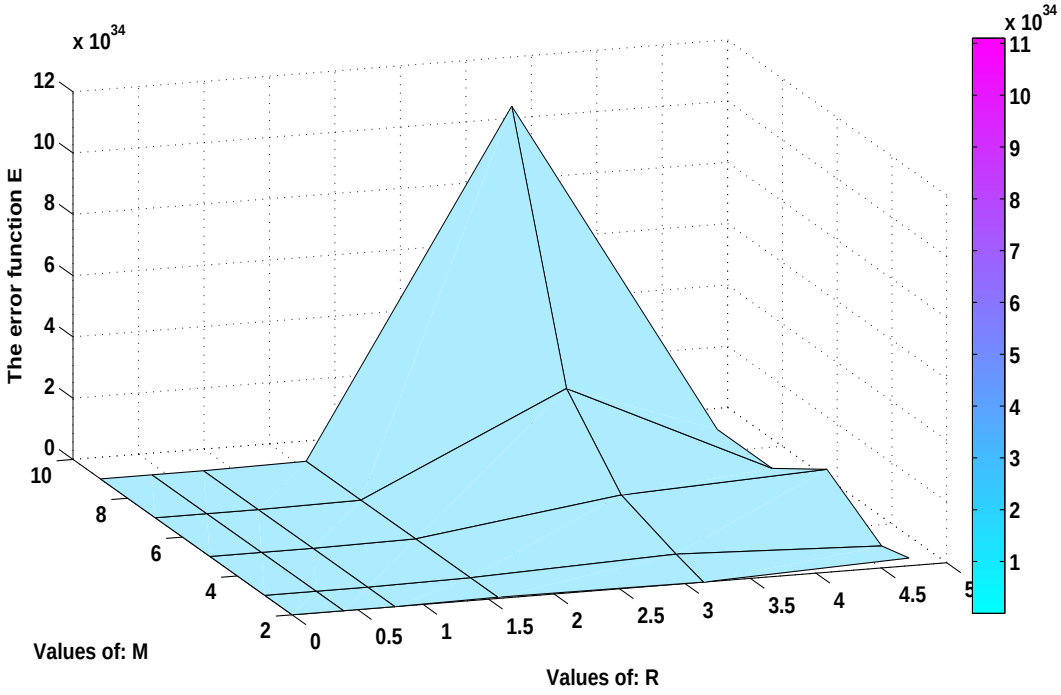


FIGURE 3.5 – The error function E of example 2 by varying of R and M.

Conclusion

- In this thesis, we studied a specific form of the non standard integro-differential Volterra equation.
- Our study started with establishing the existence and uniqueness for the solution of this equation based on the Schauder fixed point theorem.
- Alternatively, Fibonacci Wavelet Collocation method is employed to obtain an approximate solution for this integral equation, where the numerical results show its effectiveness and performance.

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