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Introduction to control theory of systems

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Notations & abbreviations

$\mathbb{R}_+$	Set of positive real numbers.
$\|.\|_H$	A norm in Banach space H.
$(.,.)_H$	A scalar product in Hilbert space H.
$\langle .,. \rangle_{H',H}$	Duality product between H and $H^{'}$.
$\|.\|_H$	A semi-norm in H.
C^2	The class of functions with continuous first and second derivative.
$\frac{\partial y}{\partial \nu} = \nabla y.\nu$	The conormal derivative.
$\Delta = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}$	The laplacien operator.
$\nabla = \left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)^T$	The gradient operator.
div	Divergence.
$A > 0$	A is a positive matrix.
$\mathcal{A}^*$	The adjoint operator of $\mathcal{A}$.
$d\Gamma$	Lebesgue measure on boundary Γ.
χ_ω	Characteristic function of the set ω.
$\mathcal{L}(\mathcal{Y}, \mathcal{Z})$	The space of linear bounded operators from $\mathcal{Y}$ to $\mathcal{Z}$.
$\mathcal{D}(Q)$	The space of functions in C^∞ with a compact support tin Q.
${}^{RL}D_t^\alpha$	Reimann-Liouville fractional derivative of order α with respect to tim
w.r.t.	With respect to.
PDE	Partial differential equation.
s.t.	Such that.
iff	If and only if.
a.e.	Almost every where.

Introduction

Every system can communicate with the outside through two types of variables. On the one hand, the system receives actions from the outside. These actions are exerted through inputs on the system or controllers. On the other hand, the system provides information on its evolution to the outside. This information completes the model and is called outputs (or observations). These outputs are provided through sensors.

We call input (or control) the excitation applied to the system from an external source in order to cause a specific or a desired response. The output is called observation.

When systems depend on time and space variables, the notions of control and observation must be completed by some spatial information.

The systems, as it will be defined, can be seen from the outside as a black box or from the inside if we can define intrinsic variables internal to its operation (state variable).

The modeling or external description, is still called modeling of operation or modeling input output. The external presentation of the systems can be in the form

$$y(t) = f\left(u\left(t\right), t\right);$$

which gives the output of the system at time t due to the control $u\left(t\right)$.

To do this by introducing the notion of state note z, of the system. The state of a system can be defined as a minimal set of variables such that from their knowledge at minutant t_0 and those of the system environment it is possible to predict the behavior of these variables for all future times have $(t \geq t_0)$. These are the variables describing and summarizing the internal history of the system constitutes the so-called system state.

Controllability is a crucial concept in control systems design as it determines whether a system can be controlled effectively. Systems that are controllable are typically easier to manage and can be designed to achieve desired performance characteristics.

In other words, u and y are related through an equation of state.

$$\begin{cases} u \longrightarrow z \text{ with } z' = f\left(t, z, u\right), \\ z \longrightarrow y \text{ with } z' = h\left(t, z\right), \end{cases}$$

Mathematically, what is added to the external representation is the data of the function f and the dynamic model described by

$$z' = f\left(t, z, u\right) \tag{0.1}$$

This is the equation of state, its solution is called (trajectory).

The internal description of the systems supposes a rather fine knowledge of the various components of the system, their behavior and their interactions mathematically, this description can be represented in the standard form of a set of differential equation integro-differential, delays or partial derivatives, etc..

We place ourselves in the case of an internal representation in the case of mathematical localization, this description can be represented in the standard form of a set of differential equations in the case of continuous system in time. In the case of discrete systems, it will be a difference equation (or a system of equations). This is the so-called state vector representation. We designate by Z the state space which will be in m first time identified with $\mathbb{R}^n$. This space contains in the set of admissible values of the state. Thus, for any $t, z(t) \in Z$ is n vectors with n components, each of which is a state variable note

$$z_i(t); i = 1, ..., n$$

For any $t, u(t)$ is m input vector (control) with component $u_i(t); i = 1, \ldots, p$. Analogously, we will define m control space V, with $V \subset \mathbb{R}^p$. An equation of state will take the form of the following general differential system:

$$\begin{cases} z' = f(z(t), u(t)), t \geq t_0 \\ z(t_0) = z_0 \end{cases}$$

Thus the study of the system is reduced to that of the set of solutions of the above equation for an input $u(t)$ assumed to be known.

Let's go back to the autonomous state equation described by :

$$\begin{cases} z'(t) = Az(t) \\ z(0) = z_0 \end{cases} \tag{0.2}$$

Where the state $z, [0, t] \longrightarrow \mathbb{R}^n$ and $A \in L(\mathbb{R}^n)$ represents the system dynamics.

From the precision of autonomous vehicles to the stabilization of power grids and the regulation of biological processes, control theory lies at the heart of engineering and science. At its essence, control theory answers a timeless question: How can we design inputs to steer complex systems toward desired behaviors? While classical methods have long addressed finite-dimensional systems modeled by ordinary differential equations (ODEs), the frontiers of modern science—smart materials, climate resilience, quantum technologies, and biomedical engineering—demand mastery over systems governed by partial differential equations (PDEs).

These infinite-dimensional systems, where states evolve across both space and time, challenge us to extend classical principles into uncharted mathematical terrain.

This course bridges the gap between finite and infinite dimensions, offering a rigorous yet practical journey through controllability, observability, and optimal control. Designed for mathematicians, engineers, and scientists, it equips learners to analyze, control, and optimize systems ranging from robotic networks to turbulent flows.

Why This Course?

The 21st century has ushered in an era of distributed complexity. Consider:

Smart Cities: Controlling traffic flow modeled by hyperbolic PDEs.

Renewable Energy: Stabilizing power grids with stochastic load dynamics.

Medical Therapeutics: Optimizing drug delivery via reaction-diffusion models.

These challenges defy reduction to simple ODEs. Instead, they demand tools for spatially distributed control, boundary actuation, and data-driven synthesis—tools this course delivers through a blend of theory, computation, and real-world relevance.

Course Structure and Innovations

1. Foundations in Finite Dimensions (ODEs)

Controllability: Kalman rank conditions, Gramian-based methods, and the Hautus test.

Observability: Duality principles, observer design, and robust detection.

2. Infinite-Dimensional Challenges (PDEs)

Controllability: Hilbert Uniqueness Method (HUM) for wave equations, Carleman estimates for parabolic systems.

Observability: Geometric Control Condition (GCC), spectral techniques for Schrödinger equations.

Optimal Control: Adjoint methods, Pontryagin's principle in Banach spaces, and PDE-constrained optimization.

Pedagogical Distinctions

Theory Meets Practice: Each chapter pairs rigorous proofs (e.g., HUM's duality) with computational labs (e.g., FEniCS/Python for PDE control).

Interdisciplinary Breadth: Examples span robotics, fluid dynamics, neuroscience, and climate science.

Expanded Exercises: Over 50 problems, from analytical derivations (e.g., Gramian positivity) to coding challenges (e.g., implementing DMD for model reduction).

Visual Learning: Phase portraits, FEM meshes, and energy propagation diagrams.

A Course for the Future

This expanded edition is not merely a textbook—it is a toolkit for innovation. By unifying classical control theory with cutting-edge PDE methods, it prepares students to:

Analyze: Prove exact controllability of a vibrating membrane.

Compute: Design a neural network-based controller for a heat equation.

Whether you aim to stabilize fusion reactors, optimize smart grids, or pioneer quantum technologies, this course transforms theoretical mastery into actionable insight.

This introduction positions the course as a vital bridge between classical theory and modern challenges, emphasizing its relevance to both academic research and industrial innovation. Let me know if you'd like to refine specific themes!

Chapter 1

Controllability and Observability for ODEs

Control begins with understanding what we can influence and observe. In this chapter, we explore the foundational principles of controllability and observability for linear time-invariant (LTI) and time-varying (LTV) systems governed by ordinary differential equations (ODEs). Starting with the celebrated Kalman rank condition, we establish criteria to determine whether a system's state can be steered to any desired target (controllability) or reconstructed from its outputs (observability). Through duality principles and spectral tests like the Hautus criterion, we reveal the hidden symmetry between control and observation.

Consider the following linear controlled system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), t \in (0,T) \\ x(0) = x_0 \end{cases}, \tag{1.1}$$

where
A is a square matrix of $\mathbb{R}^n \times \mathbb{R}^n$,
B is a matrix of $\mathbb{R}^n \times \mathbb{R}^p$,
u is the control function.

1.1 Reachable Set and Controllability

Let's sart by

Definition 1.1 (**Reachable Set**)
For a dynamical system, the reachable set $R(T)$ at time T, starting from time $t = 0$, is the set of all states $x(T)$ that can be reached by applying admissible control inputs $u(t)$ over $t \in [0, T]$.

Lemma 1.1 *The reachable set is a subspace of* $\mathbb{R}^n$.

Proof. Exercise. □

Definition 1.2 **(Controllability)**
A system is controllable if, for any initial state x_0 and target state x_f, there exists a control $u(t)$ that steers the system from x_0 to x_f in finite time T.

For linear systems, controllability is equivalent to the reachable set spanning the entire state space $\mathbb{R}^n$.

Theorem 1.1 ***(Kalman rank condition)***
A linear system (1.1) *is controllable iff*

$$rankK = rank[B\ AB \cdots A^{n-1}B] = n. \tag{1.2}$$

Proof. Constructive proof using Cayley-Hamilton theorem.
Necessity:
Assume the system is controllable but $rankK = k_0 < n$.
Then, there exists a nonzero vector $v \in \mathbb{R}^n$ such that $v^\perp K = 0$.
This implies $v^\perp A^k B = 0$ for all $k \geq k_0 \geq 0$.
Using the solution

$$x(t) = e^{tA}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

, for $x_0 = 0$:

$$v^\perp x(t) = v^\perp e^{tA}x_0 + \int_0^t v^\perp e^{A(t-\tau)}Bu(\tau)d\tau$$

which contradicts controllability (cannot reach states outside the null space of v).

Sufficiency:

If $rankK = n$, the reachable set spans $\mathbb{R}^n$.

By the Cayley-Hamilton theorem, $e^{A(t-\tau)}B$ can be expressed as a linear combination of $B, AB, \ldots, A^{n-1}B$.

Thus, any state $x_f \in \mathbb{R}^n$ can be reached by a suitable control $u(t)$. □

1.1.1 *Examples*

Example 1.1 (**Controllable System**)

Consider the equation

$$y\text{”}\,(t) = u\,(t)$$

.

Write it in Cauchy's form

$$x' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

where $x = \begin{pmatrix} y \\ y' \end{pmatrix}$.

Kalman controllability Matrix:

$$\begin{aligned} K &= [B\ AB] \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$rank(K) = 2$, so the system is controllable.

Example 1.2 (**Uncontrollable System**)

Consider

$$x' = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

Controllability Matrix:

$$K = [B\ AB] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

$rank(K) = 1 < 2$, so the system is uncontrollable.

Theorem 1.2 ***(Hautus Test fâr controllability)***

The (2.1) system is controllable iff

$$rank\left[\begin{array}{cc} A-\lambda I & B \end{array}\right]=n \tag{1.3}$$

for all eigenvalues λ of A, where n is the dimension of the state space.

Proof. A system is controllable if the controllability matrix

$$rank[BAB\cdots A^{n-1}B]=n.$$

Equivalently, there is no nontrivial A-invariant subspace contained in $\ker\left(B^{\perp}\right)$ .Assume the system is controllable. Suppose, for contradiction, that there exists an eigenvalue λ of A such that:

$$rank\left[\begin{array}{cc} A-\lambda I & B \end{array}\right]<n$$

Then, there exists a non-zero left eigenvector $v\neq 0$ such that:

$$v^{\perp}\left(A-\lambda I\right)=0 \text{ and } v^{\perp}B=0$$

This implies:

$$v^{\perp}A=\lambda v^{\perp} and v^{\perp}B=0$$

The eigenvector v spans a 1-dimensional A-invariant subspace orthogonal to B. Thus, $v^{\perp}K=0$, contradicting $rank(K)=n$.

Hence, the Hautus condition holds.

Step 3: (Reverse Direction (Hautus Condition implies Controllability)

Assume the Hautus condition holds. Suppose, for contradiction, the system is not controllable. Then, there exists a nontrivial $A-$invariant subspace $W\subset\mathbb{R}^n$ such that $W\subset\ker\left(B^{\perp}\right)$.

Let λ be an eigenvalue of $A|_W$ (the restriction of A to W) and let $v\neq 0$ be a corresponding left eigenvector in W: $v^{\perp}A=\lambda v^{\perp} and v^{\perp}B=0$.This implies:

$$v^{\perp}\left(A-\lambda I\right)=0 \text{ and } v^{\perp}B=0$$

so $rank(A-\lambda I)<n$, violating the Hautus condition.

Thus, the system must be controllable. □

Example 1.3 Consider the system:

$$A=\begin{pmatrix}1 & 1\\ 0 & 2\end{pmatrix}, B=\begin{pmatrix}1\\ 0\end{pmatrix}$$

Eigenvalues of A: $\lambda_1=1$, $\lambda_2=2$.

Check Hautus condition:

For $\lambda_1=1$:

$rank\left[\begin{array}{cc} A-\lambda_1 I & B \end{array}\right]=2.$

For $\lambda_2=2$:

$rank\left[\begin{array}{cc} A-\lambda_2 I & B \end{array}\right]=1<2$

System is uncontrollable.

1.2 Observability

Let's consider the following observed system

$$\begin{cases} x' = Ax \\ x(0) = x_0 \\ y(t) = Cx(t) \end{cases} \tag{1.4}$$

where
A is a square matrix of $\mathbb{R}^n \times \mathbb{R}^n$,
C is a matrix of $\mathbb{R}^m \times \mathbb{R}^n$,
x_0 is unknown.

Definition 1.3 (Observability)
(1.4) is said to be observable if one could reconstruct $x(0)$ from the knowldge of the output $y(t) = Cx(t)$.

Definition 1.4 In other words,(1.4) is said to be observable if

$$x_0 \to Ce^{tA}x_0$$

is injective.

1.2.1 *Observability Kalman Rank Condition*

Theorem 1.3 *System is observable iff*

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

1.2.2 *Duality between controllability and observability*

The duality between controllability and observability in linear time-invariant (LTI) systems is a foundational result in control theory. It establishes that controllability of a system is equivalent to observability of its dual system, and vice versa. Below is a structured proof of this duality:

The dual system is defined as:

$$\begin{cases} z' = A^{\perp} z + C^{\perp} v \\ w(t) = B^{\perp} z(t) \end{cases}$$

where $z \in \mathbb{R}^n$, $v \in \mathbb{R}^p$, and $w \in \mathbb{R}^m$.
Controllability Matrix :

$$K = [B \; AB \cdots \; A^{n-1}B]$$

Observability Matrix:

$$O = \begin{bmatrix} C^{\perp} \\ C^{\perp}A^{\perp} \\ \vdots \\ C^{\perp}\left(A^{\perp}\right)^{n-1} \end{bmatrix}$$

.

Theorem 1.4 *The pair (A, C) is observable iff $\left(A^{\perp}, C^{\perp}\right)$ is obsevable.*

Proof. The observability matrix of

$$\begin{cases} x' = A^{\perp} x \\ x\,(0) = x_0 \\ y(t) = C^{\perp} x(t) \end{cases}$$

is:

$$O = \begin{bmatrix} C^{\perp} \\ C^{\perp}A^{\perp} \\ \vdots \\ C^{\perp}\left(A^{\perp}\right)^{n-1} \end{bmatrix}$$

look at

$$O^{\perp} = [C \; AC \cdots \; A^{n-1}C]$$

wich give the result. □

1.2.3 *Hautus for obsevability*

The Hautus conditions also reflect this duality

Theorem 1.5 (1.4) *is observable*

$$rank \begin{bmatrix} A^{\perp} - \lambda I \\ C^{\perp} \end{bmatrix} = n$$

Proof. Duality Principle: Observability of (A, C) is equivalent Controllability of $\left(A^{\perp}, C^{\perp}\right)$. □

1.3 Controllability Gramian Condition

Consider the finite-dimensional linear time-invariant (LTI) system (1.1):

$$\begin{cases} x'(t) = Ax(t) + Bu(t), t \in (0, T) \\ x(0) = x_0 \end{cases}.$$

The controllability Gramian W_c over the time interval $(0, T)$ is defined as:

$$Wc(T) = \int_0^T e^{A(T-t)} BB^{\perp} e^{A^{\perp}(T-t)} dt$$

Theorem 1.6 *The system* (2.1) *is controllable on* $(0, T)$ *if and only if* $Wc(T)$ *is positive definite.*

Proof. ($\Rightarrow$) Controllability implies $Wc(T) > 0$ then the system is controllable. By definition, for any x_0,$x_T \in \mathbb{R}^n$, there exists a control $u(t)$ such that:

$$x_T = e^{AT} x_0 + \int_0^T e^{A(T-t)} Bu(t)\, dt$$

For $x = 0$, we can reach any x_T, so the map

$$u \to \int_0^T e^{A(T-t)} Bu(t)\, dt$$

is surjective. The controllability Gramian $Wc(T)$ is the matrix representation of this linear operator. Surjectivity implies $Wc(T)$ is full rank. Since $Wc(T)$ is symmetric and positive semi-definite, full rank implies $Wc(T) > 0$.

($\Leftarrow$) $Wc(T) > 0$ implies controllability:Assume $Wc(T) > 0$. For $x = 0$, define the control:

$$u(t) = B^{\perp} e^{A^{\perp}(T-t)} \left(Wc(T)\right)^{-1} x_T$$

Substitute $u(t)$ into the solution:

$$x(T) = \int_0^T e^{A(T-t)} Bu(t)\, dt = Wc(T) \left(Wc(T)\right)^{-1} x_T = x_T$$

Thus, any state x_T is reachable, so the system is controllable. □

Theorem 1.7 ***(Minimum Energy Control)***

If the system is (1.1) *controllable, the control* $u^*(t)$ *that transfers* $x(0) = x_0$ *to* $x(T) = x_T$ *with minimum energy*

$$J(u) = \int_0^T \|u(t)\|^2 dt$$

is:

$$u^*(t) = B^\perp e^{A^\perp (T-t)} (Wc(T))^{-1} \left(x_T - e^{AT} x_0\right)$$

The minimum energy is:

$$E^* = \left(x_T - e^{AT} x_0\right)^\perp (Wc(T))^{-1} \left(x_T - e^{AT} x_0\right)$$

Proof. The energy functional $J(u) = \int_0^T u^\perp(t)\, u(t)\, dt$ is minimized subject to the constraint:

$$x_T = e^{AT} x_0 + \int_0^T e^{A(T-t)} B u(t)\, dt$$

Using Lagrange multipliers, define:

$$L = J(u) + \lambda^\perp \left(x_T - e^{AT} x_0 + \int_0^T e^{A(T-t)} B u(t)\, dt \right)$$

Taking variations δu, the optimality condition gives:

$$u^*(t) = B^\perp e^{A^\perp (T-t)} \lambda$$

where $\lambda = (Wc(T))^{-1} \left(x_T - e^{AT} x_0\right)$. Substituting λ yields $u^*(t)$. The minimum energy $\lambda^\perp Wc(T)\, \lambda$ follows directly. □

1.4 Linear time variant Systems (LTV)

1.4.1 *Controllability*

Consider the finite-dimensional linear time-variant (LTV) system :

$$\begin{cases} x'(t) = A(t)\, x(t) + B(t)\, u(t), t \in (t_0, t_1) \\ x(0) = x_0 \end{cases} \tag{1.5}$$

where

$$A : (t_0, t_1) \to \mathbb{R}^n \times \mathbb{R}^n$$

the space of square matrix of $\mathbb{R}^n \times \mathbb{R}^n$,

$$B : (t_0, t_1) \to \mathbb{R}^n \times \mathbb{R}^n$$

u is the control function.
$x_0 \in \mathbb{R}^n$

Theorem 1.8 ***(Controllability Gramian Condition)***

The LTV system (1.5) is controllable on (t_0, t_1) if the controllability Gramian

$$Wc(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) B^{\perp}(\tau) \Phi^{\perp}(t_1, \tau) d\tau$$

is positive definite, where $\Phi(t_1, \tau)$ is the state transition matrix.

Proof. Show that $Wc(t_0, t_1)$ is invertible iff the system can transfer any x_0 to $x(t_1) = 0$ (null controllability).

Use the control law

$$u(t) = -B^{\perp}(t) \Phi^{\perp}(t, \tau) W^{-1} c(t_0, t_1) \Phi(t_1, t_0) x_0$$

□

Example 1.4 For $x' = tx + ux'$, compute $Wc(0, T)$.

$$\Phi(T, \tau) = e^{\frac{T^2 - \tau^2}{2}},$$

Since $Wc(0, T) > 0$, the system is controllable.

Theorem 1.9 ***(Silverman-Meadows Controllability)***

For the LTV system (1.5), define the sequence:

$$M_0(t) = B(t),$$
$$M_{k+1}(t) = -A(t) M_k(t) + M_k'(t)$$

The system is controllable on (t_0, t_1) if there exists $\tau \in (t_0, t_1)$ where

$$rank[M_0(\tau) \;\; M_1(\tau) \cdots M_{n-1}(\tau)] = n.$$

Proof. A Construct the Controllability Matrix by

Define the matrix

$$\mathcal{M}(t) = [M_0(t) \;\; M_1(t) \cdots M_{n-1}(t)]$$

Full Rank at τ:

By assumption, $\mathcal{M}(\tau)$ has full rank n. By continuity, there exists an interval $I_\varepsilon = (\tau - \varepsilon, \tau + \varepsilon) \subset [t_0, t_1]$ where $\mathcal{M}(t)$ retains full rank.

Express Solutions via Peano-Baker Series:

The state transition matrix $\Phi(t,\tau)$ for LTV systems satisfies:

$$\Phi(t,\tau) = I + \int_{\tau}^{t} A(s_1)\,ds_1 + \int_{\tau}^{t} A(s_1) \int_{\tau}^{s_1} A(s_2)\,ds_2 ds_1 + ...$$

For small ε, approximate $\Phi(t,\tau) = I + \int_{\tau}^{t} A(s_1)\,ds_1 \approx I + A(\tau)(t-\tau)$

Expand the Controllability Gramian:

The controllability Gramian over I_ε is:

$$W(\tau-\varepsilon, \tau+\varepsilon) = \int_{\tau-\varepsilon}^{\tau+\varepsilon} \Phi(\tau+\varepsilon, s)\, B(s)\, B^{\perp}(s)\, \Phi^{\perp}(\tau+\varepsilon, s)\,ds$$

Using the full rank of $\mathcal{M}(t)$, show W is positive definite.

Spanning the State Space:

Full rank of $\mathcal{M}(\tau)$ implies the control input can generate arbitrary motions in all nn directions of the state space. Use induction on kk to show each $M_k(\tau)$ contributes to spanning $\mathbb{R}^n$.

Conclusion:

Positive definiteness of W guarantees controllability on I_ε, which extends to $[t_0, t_1]$ by piecewise continuity. □

Example

Consider the scalar system

$$x'(t) = a(t)\,x(t) + b(t)\,u(t)$$

We have

$$M_0(t) = b(t)$$
$$M_1(t) = a(t)\,b(t) - b'(t)$$

Controllability requires the existence of τ such that

$$rank\,[M_0(\tau)\, M_1(\tau)] = 1$$

i.e., $b(\tau) \neq 0$.

Theorem 1.10 *(Uniform Controllability)*

An LTV (1.5) system is uniformly controllable on $[t_0, \infty[$ if $\exists \alpha, \beta > 0$ such that

$$\alpha I \leq Wc(t, t+T) \leq \beta I \text{ for all } t \geq t_0$$

In other words controllability is preserved over any interval of length T.

1.4.2 *Observability*

Let's consider the following LTV observed system

$$\begin{cases} x'(t) = A(t)\,x(t) \\ x(0) = x_0 \\ y(t) = C(t)\,x(t) \end{cases} \tag{1.6}$$

where

$$A : (t_0, t_1) \to \mathbb{R}^n \times \mathbb{R}^n$$

the space of square matrix of $\mathbb{R}^n \times \mathbb{R}^n$,

$$C : (t_0, t_1) \to \mathbb{R}^m \times \mathbb{R}^p$$

x_0 is unknown.

Theorem 1.11 ***(Observability Gramian Condition)***

The LTV system (1.5) is observable on (t_0, t_1) if the observability Gramian

$$W_O(t_0, t_1) = \int_{t_0}^{t_1} \Phi(\tau, t_0)\ C^{\perp}(\tau)\, C(\tau)\, \Phi^{\perp}(\tau, t_0)\, d\tau$$

is positive definite.

Proof. Dual to controllability. Use the fact that $W_O(t_0, t_1)$ invertible implies $y(t) \equiv 0 \Rightarrow x(t_0) = 0$. □

Theorem 1.12 ***(Dual Observability for LTV)***

The LTV system (1.6) is observable on (t_0, t_1) if and only if its dual system

$$z'(t) = A^{\perp}(t)\, z(t) + C^{\perp}(t)\, u(t)$$

is controllable.

Proof. Use duality between $W_C(t_0, t_1)$ and $W_O(t_0, t_1)$. □

1.4.3 *Lyapunov Equations (LTI Case)*

Proposition 1.1 ***(Controllability Gramian for LTI)***

For the stable LTI systems, the controllability Gramian Wc satisfies the Lyapunov equation:

$$AWc + WcA^{\perp} + BB^{\perp} = 0$$

If $Wc > 0$, the system is controllable.

Proof. Differentiate $Wc(T) = \int_0^T e^{A(T-t)} BB^{\perp} e^{A^{\perp}(T-t)} dt$ and take $t \to \infty$. □

Theorem 1.13 ***(Kalman Decomposition)***
The following LTI system

$$\begin{cases} x'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) + Du(t) \end{cases} t \in (0, T)$$

is decomposed into four mutually exclusive subsystems:
Controllable and observable (Affected by inputs and influences outputs),
Controllable and unobservable (Affected by inputs but does not influence outputs),
Uncontrollable and observable (Not affected by inputs but influences outputs.),
Uncontrollable and unobservable (Neither affected by inputs nor influences outputs).

Proof. Controllable Subspace :

The span of the controllability matrix i.e. $\mathrm{Im}(K)$

$$\mathcal{K} = [B\ AB \cdots\ A^{n-1}B]$$

Unobservable Subspace :The kernel of the observability matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

$\mathcal{O}^{\perp} = Ker(\mathcal{O})$
$AK \subset K$ follows from $A^k B \in K$.
$A\mathcal{O}^{\perp} \subset \mathcal{O}^{\perp}$ If $z \in \mathcal{O}^{\perp}$, then $CA^k z = 0$ for all $k \geq 0$, implying $CA^k(Az) = CA^{k+1}z = 0$.
Define the four subspaces:
$\mathcal{C}\mathcal{O} = \mathcal{C} \cap \mathcal{O}$ (controllable and observable).
$\mathcal{C}\mathcal{O}^{\perp} = \mathcal{C} \cap \mathcal{O}^{\perp}$(controllable and unobservable)
$\mathcal{C}^{\perp}\mathcal{O}$=$\mathcal{C}^{\perp} \cap \mathcal{O}$ (uncontrollable and observable).
$\mathcal{C}^{\perp}\mathcal{O}^{\perp}$=$\mathcal{C}^{\perp} \cap \mathcal{O}^{\perp}$(neither).
These subspaces are mutually orthogonal and span $\mathbb{R}^n$i.e.

$$\mathbb{R}^n = \mathcal{C}\mathcal{O} \oplus \mathcal{C}\mathcal{O}^{\perp} \oplus \mathcal{C}^{\perp}\mathcal{O} \oplus \mathcal{C}^{\perp}\mathcal{O}^{\perp}$$

Choose bases $\{e_1, ..., e_r\}$for $\mathcal{C}\mathcal{O}$, $\{e_{r+1}, ..., e_{r+s}\}$ for $\mathcal{C}\mathcal{O}^{\perp}$,$\{e_{r+s+1}, ..., e_{r+s+t}\}$ for $\mathcal{C}^{\perp}\mathcal{O}$, and $\{e_{r+s+t+1}, ..., e_n\}$ for $\mathcal{C}^{\perp}\mathcal{O}^{\perp}$. Let $T = [e_1, ..., e_n]$
Under $z = T^{-1}x$, the system becomes:

$$\begin{cases} z'(t) = T^{-1}ATz(t) + T^{-1}Bu(t), \\ y(t) = T^{-1}CTz(t) + Du(t) \end{cases}$$

$T^{-1}AT$ has a block-triangular structure.

$T^{-1}B$ affects only $\mathcal{CO}$ and $\mathcal{CO}^{\perp}$.CT reads only $\mathcal{CO}$ and $\mathcal{CO}^{\perp}$.

Explicitly:

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix},$$

$$T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}$$

Minimal Realization:

The $\mathcal{CO}$ subsystem

$$z_1' = A_{11}z_1 + B_1u \tag{1.1}$$

$$y = C_1z_1 \tag{1.2}$$

is minimal (controllable and observable). □

1.4.4 *Algorithm for Kalman Decomposition*

Find bases for subspaces:

Compute $\mathcal{C}$(controllable subspace).

Compute $\mathcal{O}^{\perp}$ (unobservable subspace).

Define four subspaces:

$\mathcal{CO} = \mathcal{C} \cap \mathcal{O}$ (controllable and observable).

$\mathcal{CO}^{\perp} = \mathcal{C} \cap \mathcal{O}^{\perp}$(controllable and unobservable).

$\mathcal{C}^{\perp}\mathcal{O}$=$\mathcal{C}^{\perp} \cap \mathcal{O}$ (uncontrollable and observable).

$\mathcal{C}^{\perp}\mathcal{O}^{\perp}$=$\mathcal{C}^{\perp} \cap \mathcal{O}^{\perp}$(neither).

Construct transformation matrix T:

Let

$$T = \begin{bmatrix} T_{\mathcal{CO}} & T_{\mathcal{CO}^{\perp}} & T_{\mathcal{C}^{\perp}\mathcal{O}} & T_{\mathcal{C}^{\perp}\mathcal{O}^{\perp}} \end{bmatrix}$$

where each T is a basis for the corresponding subspace.

Transform the system:

Apply $z = T^{-1}x$. The transformed system becomes:

$$\begin{cases} z'(t) = T^{-1}ATz(t) + T^{-1}Bu(t), \\ y(t) = T^{-1}CTz(t) + Du(t) \end{cases}$$

Block structure:

In the new coordinates, the system matrices take the form:

$$T^{-1}AT = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix},$$

$$T^{-1}B = \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} C_1 & 0 & C_3 & 0 \end{bmatrix}$$

Properties of the Decomposition

The control u affects only $\mathcal{CO}$ and $\mathcal{CO}^{\perp}$
Output y depends only on $\mathcal{CO}$ and $\mathcal{CO}^{\perp}$
hidden dynamics:
$\mathcal{CO}^{\perp}$, and $\mathcal{CO}^{\perp}$ do not affect the input-output behavior.

Example 1.5 Consider a system with:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix},$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$$

Controllability:

$$K = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 1 & 1 \\ 1 & 4 & 12 & 32 \\ 2 & 4 & 8 & 16 \end{bmatrix}$$

$Rank(K) = 3$■ Controllable subspace is $\mathbb{R}^3$.

Observability:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 4 & 4 \\ 1 & 3 & 8 & 12 \end{bmatrix}$$

$Rank(O) = 3$ ■ Unobservable subspace is $\mathbb{R}^3$.

Intersections:

$\mathcal{C}\mathcal{O} = \mathcal{C} \cap \mathcal{O} = \mathbb{R}^2$

$\mathcal{C}\mathcal{O}^{\perp} = \mathcal{C} \cap \mathcal{O}^{\perp} = \mathbb{R}$

$\mathcal{C}^{\perp}\mathcal{O} = \mathcal{C}^{\perp} \cap \mathcal{O} = \mathbb{R}$

$\mathcal{C}^{\perp}\mathcal{O}^{\perp} = \mathcal{C}^{\perp} \cap \mathcal{O}^{\perp} = \{0\}$

Theorem 1.14 ***(Hautus-Keymura Observability for Descriptor Systems)***

For the descriptor system

$$\begin{cases} Ex'(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \end{cases}$$

observability holds iff:

$$Rank \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n, \forall \lambda \in \mathbb{C}.$$

where n is the dimension of the state vector.

Proof. Definitions:

A descriptor system is observable if the output $y(t) = 0$ implies $x(t) = 0$ for all solutions of $Ex'(t) = Ax(t)$.

Observability requires no non-trivial solutions where $x(t) \neq 0$ produce $y(t) = 0$.

Necessity :

Assume the system is observable.

Finite Modes ($\lambda \in \mathbb{C}$):

Suppose there exists $\lambda \in \mathbb{C}$ and $v \neq 0$ such that $(A - \lambda E)v = 0$ and $Cv = 0$.

The solution $x(t) = e^{st}v$ satisfies $Ex'(t) = Ax(t)$ and $y(t) = Cx(t)$, contradicting observability. Hence,

$$Rank \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n, \forall \lambda \in \mathbb{C}$$

Infinite Modes (Impulse Observability):

Suppose there exists $v \neq 0$ such that $Ev = 0$ and $Cv = 0$.

This corresponds to an impulsive solution (e.g., $x(t) = \delta(t)\, v$) with $y(t) = Cx(t) = 0$, violating observability. Thus,

$$Rank \begin{bmatrix} E \\ C \end{bmatrix} = n, \forall \lambda \in \mathbb{C}$$

Sufficiency :

Assume the rank conditions hold.

Finite Modes:

If $x(t)$ is a solution with $y(t) = 0$, taking Laplace transforms gives $(A - sE) X(s) = 0$ and $CX(s) = 0$.

The rank condition

$$Rank \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n$$

implies $X(s) = 0$, so $x(t) = 0$.

Infinite Modes:

Using the Weierstrass canonical form, decompose the system into slow (differential) and fast (algebraic) subsystems. The condition

$$Rank \begin{bmatrix} E \\ C \end{bmatrix} = n$$

ensures no non-trivial solutions in the fast subsystem (impulsive modes) can exist without being detected by $y(t)$.

Thus, the system is observable if and only if both rank conditions hold. □

1.5 First chapter exercises

Prove that the set C of reachable states is a subspace space of $\mathbb{R}^n$.

Exercise 1 (**Canonical form of controllability, observability for an ODE**)

Prove that the system described by the ordinary differential equation of order n with constant coefficient

$$\frac{d^n y}{dt^n}(t) + a_n \frac{d^{n-1} y}{dt^{n-1}}(t) + \ldots + a_1 y(t) = u(t), a_i \in \mathbb{R}, i = 1, \ldots, n$$

and expanded by the following output function

$$y = x(t),$$

completed by initial conditions, is controllable and observable, then give a control that realizes the controllability to the desired state $z_d = 0$.

Hint. Write the equation on the standard form of control systems, then apply the Kalman criterion.

Exercise 2 (**Electrical circuit: controllability, observability)**

Consider the electrical circuit in figure 1.

According to Kirchhoff's laws we have

$$\begin{aligned} i_1 &= C_1 v_1, \\ i_2 &= C_2 v_2, \\ i_3 &= C_3 v_3, \\ i_1 + i_2 + i_3 &= 0, \\ v_3 - v_2 &= R_2(i - i_1 - i_3), \\ v_1 - v_3 &= R_1(i - i_1), \end{aligned}$$

with the output

$$V = v_1 - v_2.$$

1) Express the dynamics of this circuit in the standard form

$$\begin{aligned} \dot{x} &= Ax + Bu, \\ y &= Cx, \end{aligned}$$

where $x = (v_1, v_2, v_3)^t, u = i$ and $y = V$. (Find the matrices A, B and C)

2)Assume that

$$\begin{aligned} R_1 &= R_2 = R \text{ and} \\ C_1 &= C_2 = C_3 = C. \end{aligned}$$

Compute the controllability and observability matrix and its ranks, conclude.

Fig.2 Electrical cercuit

Exercise 3 (Controllability and observability of discrete systems)

Exercise 4 Let $A \in \mathcal{M}_n(\mathbb{R}), B \in \mathcal{M}_{n,m}(\mathbb{R})$. Prove that the system

$$\frac{d^2y}{dt^2} = Ay + Bu,$$
$$\frac{dy}{dt}(0) \in \mathbb{R}^n$$

is controllable in $\mathbb{R}^{2n}$ if (A, B) is controllable in $\mathbb{R}^n$.
(**Hint :** Write the equation in standard form then use the identity

$$rang(A + B) = rang(A) + rang(B) - dim(ImA \cap ImB)$$

).

Exercise 5 **(PBH test)**
Let $A = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \\ \beta \end{bmatrix}$ and $\alpha, \beta \in \mathbb{R}$.
1) Determine all values of (A, B) s.t. (A, B) is controllable.

2) For the values of α, β for which (A, B) is uncontrollable, determine the uncontrollable eigenvalues.(Use the PBH test).

Exercise 6 (**Like- diagonal matrix**)

Prove that the pair $(A, B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,m}(\mathbb{R})$ is controllable if and only if the matrix,

$$\left.\begin{array}{cccccccc} B & I & & & & & & \\ -A & B & I & & & & & \\ & -A & B & I & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & -A & B & I & & \\ & & & & -A & B & I & \\ & & & & & -A & B \end{array}\right\} n \text{ blocks}$$

has rank n^2.

State a similar result for observability of a pair (C, A).

Exercise 7 **(Systems similarities)**

In this exercise all matrices are assumed to be of suitable size.

a) Let S be a regular matrix. Prove that (A, B) is controllable iff (SAS^{-1}, SB) is also controllable.

b) Prove that (A, B) is controllable iff $(A + BF, B)$ is also controllable.

c) Let R be a regular matrix. Prove that (A, B) is controllable iff (A, BR) is also controllable.

d)Let S and R be regular matrices. Prove that (A, B) is controllable iff $(S(A + BF)S^{-1}, SBR)$ is also controllable.

Exercise 8 (**According to Lemma 1.2**)

Let $(A, B) \in \mathcal{M}_n(\mathbb{R}) \times \mathcal{M}_{n,p}(\mathbb{R})$ be an uncontrollable pair with $B \neq 0$,

$$0 < rankK = rank \left[\begin{array}{ccccc} B & AB & A^2B & \ldots & A^{n-1}B \end{array} \right] = k < n.$$

From Lemma1.2, (A, B) is similar to a pair of the form

$$\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right).$$

with $(A_{11}, B_1) \in \mathcal{M}_k(\mathbb{R}) \times \mathcal{M}_{k,p}(\mathbb{R})$.

Prove that (A_{11}, B_1) is controllable.

Exercise 9 **(Linear satellite problem in circular orbit)**

The linearized model of a satellite orbiting a circular orbit of radius $r > 0$ and which has an angular velocity $\omega \neq 0$ is described by the equations

$$\frac{dx}{dt} = Ax + Bu = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 2r\omega^2 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-2\omega}{r} & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \begin{bmatrix} u_r \\ u_\theta \end{bmatrix}$$

with

$$y = Cx = \begin{bmatrix} 1\,0\,0\,0 \\ 0\,0\,1\,0 \end{bmatrix} x = \begin{bmatrix} r \\ \theta \end{bmatrix}.$$

This satellite is controlled by two motors: the first provides a radial force u_r and the second an angular force u_θ. (See figure 2)

1) Prove that this system is controllable.

2) Prove that this system is observable.

3) Due to a technical problem, you must shut down one of the two motors. Which one will you choose ?

Fig.3 Linear satellite problem

Exercise 10 (**Controllability of a population dynamic system**)

Consider the system in $\mathbb{R}^5$ the following dynamical system

$$\begin{cases} \frac{dx_1}{dt} = -2\alpha x_1 + \beta x_4, \\ \frac{dx_2}{dt} = \alpha x_1 - (m_1 + \alpha)\, x_2, \\ \frac{dx_3}{dt} = \alpha x_1 - (m_1 + \alpha)\, x_3, \\ \frac{dx_4}{dt} = \alpha x_2 - m_2 x_4, \\ \frac{dx_5}{dt} = \alpha x_3 - m_2 x_5, \end{cases}$$

which represents a population structured into five classes:

x_1 denotes the density of newborns, which give rise to two possible types x_2 and x_3, which in turn give rise to x_4 and x_5. We assume all parameters to be strictly positive.

Let A be the matrix of the previous system. We can bring or remove individuals from a single population, which is equivalent to considering the controlled system

$$\frac{dx}{dt} = Ax + bu\,(t)$$

where b is a column vector with a 1 at row k, $1 \leq k \leq 5$ and zeros elsewhere, and $u\,(t) \in [-1, 1]$.

We ask to find the integer k to make the system controllable (i.e., the population we should target).

Exercise 11 **(Drug absorption)**

A simplified mathematical model can be used to illustrate the progression of drug absorption, circulation, and eventual breakdown within an individual's body. This model is represented by the following equations:

$$\begin{cases} \frac{dx_1}{dt} = -a_1 x_1 + u, \\ \frac{dx_1}{dt} = a_1 x_1 - a_2 x_2, \end{cases}$$

with

$$y = x_2$$

In this model, x_1, x_2, and v denote deviations from initial steady state values for the drug quantity in the digestive system, the drug quantity in the circulatory system, and the rate of drug consumption, respectively. The output (observation) y represents the drug quantity in the bloodstream at a given time t. The parameters a_1 and a_2 are strictly positive constants that reflect the physiological characteristics of the individual being studied.

1) Study the controllability of the above system

2) Study the observability of the above system

Assume that the pair (A, B) is controllable with a state space $\mathbb{R}^n$, $v \in \mathbb{R}^m$.

Prove that the system

$$\begin{cases} \frac{dy}{dt} = Ay + Bv \\ \frac{dv}{dt} = u \end{cases}$$

is controllable with a state space $\mathbb{R}^{n+m}$ by the action of the control u.

Exercise 12 Consider the following system

$$\begin{cases} \frac{dx}{dt} = Ax \\ y = Cx \end{cases}$$

where A and C are block-matrix given by

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 & \cdots & 0 \\ A_{21} & A_{22} & A_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k-11} & A_{k-12} & A_{k-13} & \cdots & A_{k-1k} \\ A_{k1} & A_{k2} & A_{k3} & \cdots & A_{kk} \end{bmatrix},$$

and

$$C = \begin{bmatrix} C_1 \, 0 \, 0 \cdots 0 \end{bmatrix}$$

where A_{ii} are square matrix for all $1 \leq i \leq k$, A_{i-1i} and C_1all having full column rank for all $1 \leq i \leq k$.

1)Show that the system is observable. (Hint: Show that if P and Q have full column rank,

then PQ has full column rank.)

2) What can you say when $A_{i-1i} = 0$ for all $1 \leq i \leq k$?

Exercise 13 **(Discrete Linear Time invariant Control Systems)**

Consider the following discrete linear control system

$$\begin{cases} z_{n+1} = Az_n + Bu_n, n \in \mathbb{N} \\ y_n = Cz_n \\ z_0 \in \mathbb{R} \end{cases}$$

1) Prove that for all $n \in \mathbb{N}$

$$z_n = A^n z_0 + \sum_{k=0}^{n-1} A^{n-1-k} B u_k,$$

and

$$z_n = A^n z_0 + \begin{pmatrix} B \; AB \; ... \; A^{n-1}B \end{pmatrix} \begin{pmatrix} u_{n-1} \\ u_{n-2} \\ \vdots \\ u_0 \end{pmatrix},$$

Exercise 14 **(Tank control problem)**

In this exercise, you will be guided through the essential steps of modeling and simulating a physical system. The tank has two inflows, which are controlled by valves, and one outflow.

The water tank has the cross sectional area F . The mass flow m_{out} exits the tank through the cross sectional area E.

There are two inflows: The mass flow m_u is regulated through a valve, which is controlled by the input signal u; the mass flow m_z is disturbance, which is controlled by the disturbance signal z.

For the control of the water level, we can only use u, not z. The output signal of the sensor, which measures the water level h, is denoted y. The dynamics of the sensor are not negligible.

Fig.5 Tank control problem

1) What are the inputs of the system? What is/are the output(s)?

2) Identify the relevant reservoirs and name their corresponding level variables.

3) Draw the causality diagram of the system. Highlight dynamic subsystems (containing a reservoir) with a shade.

Denote all signals according to the nomenclature introduced in the description above.

Exercise 15 **(One link robotic arm)**

Consider a one link robotic arm described by the following second order ODEs.

$$\begin{cases} J_1 \frac{d^2 q_1}{dt^2} + F_1 \frac{dq_1}{dt} + \frac{K}{N}\left(q_2 - \frac{q_1}{N}\right) = T \\ J_2 \frac{d^2 q_2}{dt^2} + F_2 \frac{dq_2}{dt} + K\left(q_2 - \frac{q_1}{N}\right) + mgd\cos q_2 = 0 \end{cases}$$

For simplicity, assume

$$J_1 = J_2 = F_1 = F_2 = K = N = m = g = d = 1,$$

and take the angle of the arm as the system's output.

1) Write the system in the standard form

Determine the system's state, input, and output.

(ii) Find the system's equilibria corresponding to $T^* = 0$.

(iii) Linearize the system around each of the above equilibrium points.

Put the resulting linear system in the standard LTI form.

(iv) Compute and plot the impulse responses of each of the linearized systems. Do your results match your intuition?

(v) Determine the internal and input-output stability of each linearization.

Does your answers match your intuition?

(vi) Are the linearized systems controllable? Observable?

(vii) Discretize each of the linearized systems using Euler discretization and a sample time of T_s.

Are the discretized systems internally stable? Controllable? Observable? How does your responses depend on T_s?

(viii) Find a control signal that takes each of discretized systems from the initial state of downward rest to the final state of upward rest in tf = 1000Ts.

*Fig.*6 One link robotic arm

Exercise 16 **(Algebraic and Geometric Criteria)**

Given the system $x' = Ax + Bu$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -4 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 17 (a) Compute the controllability matrix and determine if the system is controllable.

(b) Use the Popov-Belevitch-Hautus (PBH) test to verify controllability.

Exercise 18 **(Parametric Controllability)**

Analyze how the parameter α affects the controllability of:

$$x' = \begin{pmatrix} \alpha & 1 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 1 \\ \alpha \end{pmatrix} u$$

Identify values of α that make the system uncontrollable and interpret geometrically.

Exercise 19 (**Controllability Gramian**)

For the system $x' = -x + u$, compute the controllability Gramian G_T over $t \in [0, T]$. Derive the minimum energy input $u(t)$ required to drive $x(0) = 0$ to $x(T) = x_f$.

Exercise 20 **(Kalman Decomposition)**

Decompose the following uncontrollable system into controllable and uncontrollable subsystems:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

Exercise 21 (**Observability Matrix**)

For the system:

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}, C = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

(a) Compute the observability matrix and determine observability.

(b) Design a full-state observer if the system is observable.

Exercise 22 (**Duality Principle**)

Show that the observability of $x' = Ax, y = Cx$ is equivalent to the controllability of $z' = A^{\perp} z + C^{\perp} u$.

Exercise 23 **(Observability Gramian)**

For $x' = -x, y = x$, compute the observability Gramian Wo over $t \in [0, T]$. Use it to reconstruct $x(0)$ from $y(t)$.

Exercise 24 **(Unobservable Subspace)**

Given A=[2102]A=[2012] and C=[11]C=[11], find the unobservable subspace and interpret its physical meaning.

Exercise 25 **(Sampled-Data Systems)**

A continuous system $x' = Ax + Bu$ is sampled with period T. Derive conditions under which the discrete-time system $x_{k+1} = e^{AT}x_k + \left(\int_0^T e^{As}ds\right) Bu_k$ retains controllability.

Exercise 26 **(Pole Placement)**

Prove that a system must be controllable to arbitrarily assign eigenvalues via state feedback $u = -kx$. Illustrate with an example.

Exercise 27 **(Stabilizability vs. Controllability)**

Construct a system that is stabilizable but not controllable. Show how unstable modes become controllable.

Exercise 28 **(Invariance Under Feedback)**

Prove that controllability is preserved under linear state feedback $u = -kx + v$.

Exercise 29 **(PBH Test Equivalence)**

Show that the PBH test (no eigenvector v of A satisfies $v^T B = 0$) is equivalent to the Kalman rank condition.

Exercise 30 **(Duality Formal Proof)**

Rigorously prove the duality between controllability and observability using matrix transpose properties.

Exercise 31 **(Loss of Observability)**

Prove that adding an output to a system cannot reduce its observability.

Exercise 32 **(Controllability of Composite Systems)**

For interconnected systems

$$\begin{cases} x_1' = A_1 x_1 + B_1 u_1 \\ x_2' = A_2 x_2 + B_2 u_2 \end{cases}$$

derive conditions under which the combined system

$$x' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} x + \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} u$$

is controllable.

Exercise 33

Prove that for LTV systems, controllability on (t_0, t_1) implies controllability on $(t_0, t_2)$for $t_2 > t_1$.

Chapter 2

Controllability and Observability for PDEs

When dynamics span both space and time, ordinary differential equations give way to partial differential equations (PDEs) and control theory ascends to new complexity. This chapter tackles the infinite-dimensional realm, where controllability means shaping wave propagation, heat diffusion, or quantum probability densities. Through Hilbert Uniqueness Method (HUM) and Carleman estimates, we conquer the challenges of boundary control and null controllability for wave and heat equations. The Geometric Control Condition (GCC) becomes our compass, ensuring control regions intersect all wave trajectories.

Consider a system governed by the abstract Cauchy problem:

$$\begin{cases} y'(t) = Ay(t) + Bu(t) \\ y(0) = y_0 \in Y \end{cases} \tag{2.1}$$

where:

Y is the state space, an infinite-dimensional Hilbert.

$A : D(A) \subset Y \to Y$ is an unbounded operator generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$.

$B : U \to Y$ is the bounded control operator.

$u(t)$ is the control input in $L^2(0,T;U)$.

2.0.1 *Controllability Definitions:*

Definition 2.1 (**Exact Controllability**):

We say that (2.1) is exactly controllable if for all $y_0, y_T \in Y$ there exists a control $u \in L^2(0,T;U)$ such that

$$y(T) = y_T$$

Definition 2.2 (**Approximate Controllability**)

We say that (2.1) is approximately controllable if for all $y_0, y_T \in Y$ and all $\varepsilon > 0$ there exists a control $u \in L^2(0,T;U)$ such that

$$\|y(T) - y_T\| < \varepsilon$$

Definition 2.3 (**Null Controllability**)

We say that (2.1) is nul controllable if for all $y_0 \in Y$ there exists a control $u \in L^2(0,T;U)$ such that

$$y(T) = 0.$$

Theorem 2.1 *(**Exact Controllability**)*

Let A generate a C_0-semigroup $\{S(t)\}_{t\geq 0}$ on a the Hilbert space Y, and let $B \in L(U,Y)$, where U is the control space.

The system:

$$\begin{cases} y'(t) = Ay(t) + Bu(t) \\ y(0) = y_0 \in Y \end{cases}$$

is exactly controllable in time T>0 if and only if there exists c>0 such that:

$$\int_0^T \|B^* S^*(t) y\|_U \, dt \geq c \|y\|_Y^2 \text{ for all } y \in Y \tag{2.2}$$

Proof. (■) Assume exact controllability. The controllability operator

$$\mathcal{B}_T : L^2(0,T;U) \to Y$$

defined by:

$$\mathcal{B}_T u = \int_0^T S(T-t) B u(s)$$

is surjective.

By the Open Mapping Theorem, $\mathcal{B}_T$ is bounded below. Its adjoint

$$\mathcal{B}_T^* : Y \to L^2(0,T;U)$$

is given by:

$$\left(\mathcal{B}_T^* y\right)(t) = B^* S(T-t) x(t)$$

Surjectivity of $\mathcal{B}_T$ implies $\mathcal{B}_T^*$ is injective and bounded below:

$$\left\|\mathcal{B}_T^* y\right\|_{L^2(0,T;U)} \geq \|y\|_Y$$

(■) Conversely, assume the integral inequality holds. Then $\mathcal{B}_T^*$ is injective and bounded below, so $\mathcal{B}_T$ has dense and closed range, hence is surjective. Thus, the system is exactly controllable. □

Theorem 2.2 ***(Approximate Controllability)***

The system (2.3) *is approximately controllable in time* $T > 0$ *if and only if:*

$$B^* S^*(t) y = 0 \text{ for all } t \in (0,T) \Rightarrow y = 0$$

Proof. Approximate controllability is equivalent to the reachable set being dense in Y. By duality, this holds if the adjoint system:

$$\begin{cases} z'(t) = A^* z(t) \\ w(t) = B^* z(t) \end{cases}$$

is approximately observable. This requires that $B^* S^*(t) y = 0$ for all $t \in (0,T)$ implies $y = 0$. □

Theorem 2.3 ***(Null controllability)***

The system (2.1) *is null controllable in time* $T > 0$ *if and only if there exists a constant* $C > 0$ *such that for all* $\varphi_0 \in D(A^*)$ *the solution* $\varphi(t)$ *to the adjoint system:*

$$\begin{cases} \varphi'(t) = A^* \varphi(t) \\ \varphi(0) = \varphi_0 \end{cases}$$

satisfies the observability inequality:

$$\|\varphi(T)\|_Y \leq C \int_0^T \|B^* \varphi(t)\|_U \, dt$$

Proof. Null controllability implies the observability inequality

Assume the system is null controllable. Then, for every $y_0 \in Y$, there exists $u \in L^2(0,T;U)$ such that:

$$y(T) = S(T)\, y_0 + \int_0^T S(T-t)\, Bu(t)\, dt = 0$$

By the Closed Graph Theorem, the linear operator mapping x_0 to u is bounded. Thus, there exists $C > 0$ such that:

$$\|u\|^2_{L^2(0,T;U)} \leq C \|y_0\|^2_Y$$

Consider the adjoint system

$$\begin{cases} \varphi'(t) = A^* \varphi(t) \\ \varphi(0) = \varphi_0 \end{cases}$$

The solution is

$$\varphi(t) = S^*(T-t)\, \varphi_0$$

. For $y_0 = \varphi_0$, we have:

$$\begin{aligned} \|\varphi_0\|^2 &= (y_0, \varphi_0) = (S(T)\, y_0, \varphi_0) \\ &= -\left(\int_0^T S(T-t)\, Bu(t)\, dt, \varphi_0 \right) \end{aligned}$$

Using duality and the Cauchy-Schwarz inequality:

$$\begin{aligned} \|\varphi_0\|^2 &= \left| \int_0^T (S(T-t)\, Bu(t), \varphi_0)\, dt \right| \\ &\leq \|u\|_{L^2(0,T;U)} \left(\int_0^T \|B^* \varphi(t)\|\, dt \right)^{\frac{1}{2}} \end{aligned}$$

Substituting $\|u\|_{L^2(0,T;U)} \leq \sqrt{C}\, \|\varphi_0\|_Y$

$$\|\varphi_0\|^2 \leq \sqrt{C}\, \|\varphi_0\|_Y \left(\int_0^T \|B^* \varphi(t)\|\, dt \right)^{\frac{1}{2}}$$

Simplifying, we obtain:

$$\|\varphi(T)\|_Y \leq C \int_0^T \|B^* \varphi(t)\|_U\, dt$$

Reverse Direction : Observability inequality implies null controllability

Assume the observability inequality holds. Define the observability Gramian

$$\begin{aligned} G_T &: Y \to Y \\ G_T\varphi_0 &= \int_0^T S(T-t) BB^* S^*(T-t)\varphi_0 dt \end{aligned}$$

The observability inequality ensures G_T is coercive:

$$\begin{aligned} (G_T\varphi_0, \varphi_0) &= \int_0^T \|B^* S^*(T-t)\varphi_0\|_U \, dt \\ &\geq \frac{1}{C} \|\varphi_0\|_Y \end{aligned}$$

Thus, G_T is invertible. For any $\varphi_0 \in Y$, define the control:

$$u(t) = -B^* S^*(T-t) G_T^{-1} S(T) y_0$$

Substituting $u(t)$ into the solution formula:

$$\begin{aligned} y(T) &= S(T) y_0 - \int_0^T S(T-t) BB^* S^*(T-t) G_T^{-1} S(T) y_0 dt \\ &= S(T) y_0 - G_T G_T^{-1} S(T) y_0 = 0 \end{aligned}$$

Hence, the system is null controllable. □

The observability inequality guarantees that the Gramian G_T is invertible, enabling explicit construction of the control $u(t)$.

This theorem bridges controllability and observability via semigroup theory, leveraging duality and coercivity properties in Hilbert spaces.

2.1 Observability

Consider the following ouput system

$$\begin{cases} y'(t) = Ay(t), t \in (0,T) \\ y(0) = y_0 \\ z(t) = Cy(t) \end{cases} \tag{2.3}$$

where:

$A : D(A) \subset Y \to Y$ is an unbounded operator generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$.

$C : Y \to Z$ is the bounded control operator.

Y is the state space infinite-dimensional Hilbert

Z is the observation space

Y is an infinite-dimensional Hilbert.

Definition 2.4 (Exact Observability)

A system (2.3) is exactly observable on a time interval $(0,T)$ if t the output uniquely determines the initial state y_0 (i.e., stability or with bounded sensitivity to noise).

In other words, system (2.3) is exactly observable on a time interval $(0,T)$ if the operator

$$\begin{aligned} &\mathcal{O} \ : \ Y \to L^2(0,T;Z) \\ &y_0 \to CS(t)\, y \end{aligned}$$

is injective with bounded inverse on $\operatorname{Im}\mathcal{O}$.

Definition 2.5 (**Approximate observability**)

The system (2.3) is approximately observable if the observability map $\mathcal{O}$ is injective. Equivalently, if $z(t) = 0$ for all t[0,T]$t \in (0,T)$ implies $y_0 = 0$. This guarantees uniqueness but not necessarily stability in state estimation.

Theorem 2.4 ***(Exact Observability characterization)***

The system (2.3) is exactly observable in time $T > 0$ if and only if there exists $c > 0$ such that:

$$\int_0^T \|CS(t)\, y\|_U \, dt \geq c \|y\|_Y^2 \text{ for all } y \in Y$$

Proof. If the system is exactly observable, the observability operator $\mathcal{O}$ is injective and bounded below. Hence:

$$\|\mathcal{O}y\|_{L^2(0,T;Y)} \geq c \|y\|_Y$$

Assume the inequality holds. Then $\mathcal{O}$is injective with closed range, implying exact observability. □

Theorem 2.5 ***(Approximate Observability)***

The system (2.3) *is approximately observable in time* $T > 0$ *if and only if:*

$$CS(t)\, y = 0 \text{ for all } t \in (0,T) \Rightarrow y = 0$$

Proof. Approximate observability requires that distinct initial states produce distinct outputs. If $CS(t)\, y = 0$ for all $t \in (0,T)$, then $y = 0$, ensuring injectivity of the observability map. □

2.1.1 *Duality between Controllability and observability*

A system (A,C) is observable if its dual system (A^*, C^*) is controllable. For example:

Exact observability corresponds to exact controllability of the dual system.

Approximate observability corresponds to approximate controllability of the dual system.

Infinite-dimensional systems, such as those governed by partial differential equations (PDEs) or delay equations, require a functional-analytic framework. The duality between controllability and observability persists but involves adjoint operators, reflexive spaces, and semigroup theory. Below are key theorems and proofs adapted to infinite dimensions.

Theorem 2.6 *(Duality of Approximate Controllability and Observability)*

The system (A, B) is approximately controllable on $(0.T)$ if and only if its dual system (A^*, B^*) is approximately observable on $(0.T)$.

Proof. (Forward Direction)(A, B)approximately controllable then (A^*, B^*) approximately observable):

Suppose (A, B) is not approximately observable. Then, there exists $x \neq 0$ in X such that

$CS\,(t)\,x = 0$, i.e. for all $t \in (0.T)$.

By duality, ■x0*,T(T-s)Bu■X*,X=■B*T*(T-s)x0*,u■U*,U=0■x0*,T(T-s)Bu■X*,X=■B*T*(T-s)x0*,u■U*,U=0 for all u■Uu■U.

This implies x0*■R(T)x0*■R(T). Since R(T)R(T) is dense, x0*=0x0*=0, a contradiction. Hence, ■d■d is approximately observable.

Converse Direction (■d■d approximately observable ■ ■■■■ approximately controllable):

If (A, B) is not approximately controllable, $R\,(T)$is not dense. By Hahn-Banach, there exists x0*■0x0*■=0 in X*X* such that ■x0*,R(T)■=0■x0*,R(T)■=0.

This implies B*T*(T-s)x0*=0B*T*(T-s)x0*=0 for all s■[0,T]s■[0,T], so ■T*x0*=0■T*x0*=0. This contradicts the approximate observability of ■d■d. □

Theorem 2.7 *(Duality of Exact Controllability and Observability*

Proof. Proof Sketch:

Exact controllability implies the controllability Gramian GT=■0TT(s)BB*T*(s)dsGT=■0TT(s)BB*T*(s)ds is coercive: GT■■IGT■■I.

The dual observability Gramian is GTd=GT*GTd=GT*, which is also coercive. Hence, ■d■d is exactly observable. □

Example: Heat Equation

Consider the controlled heat equation on $\Omega \subset \mathbb{R}^n$:

$$\frac{\partial y}{\partial t} = \Delta y + Bu$$
$$z = Cy$$

Controllability: Approximate controllability holds if the control acts on an open subset of Ω.

Dual Observability: The dual system involves the adjoint heat equation

$$\frac{\partial \phi}{\partial t} = \Delta \phi + Bu$$

with observations $C^*\phi$. Approximate observability requires that $C^*\phi = 0$ unless $\phi = 0$.

2.2 Hilbert uniqueness method (HUM)

The HUM method (Hilbert Uniqueness Method) developed by J.L. Lions in [**?**, Lions 88] allows us to study the controllability of solutions of PDEs. This method consists in proving a uniqueness theorem, through a majorization of the energy (inverse inequality) by means of the method of multipliers, which allows us to construct data spaces for which there is a control that allows to bring the solution of the system to the desired state.

Hilbert Uniqueness Method is a constructive method used to prove controllability of linear time-invariant systems, it guarantees the existence of a control that transfers the system from any initial state to any final state within a given time interval. The method constructs the necessary control by solving a system of linear adjoint equations, which makes it a constructive method.

Let Ω be a bounded open of $\mathbb{R}^n$ ($n = 1, 2, 3$ in applications), of regular boundary $\Gamma = \partial\Omega$. Let $T > 0$ be a given positive time. We define the space-time cylinder $Q = \Omega \times]0, T[$ of lateral boundary $\Sigma = \Gamma \times]0, T[$.

We consider in Q the wave equation with a non-homogeneous boundary condition of Dirichlet type

$$\left|\begin{array}{cc} \frac{\partial^2 y}{\partial t^2} - \Delta y = 0 & \text{in } Q, \\ y = v & \text{on } \Sigma, \\ y(x,0) = y^0(x), \frac{\partial y}{\partial t}(x,0) = y^1(x) & \text{in } \Omega, \end{array}\right.$$

whose state $y = y(x,t)$, $v = v(x,t)$ is the control function that models the action at the boundary Γ on the system (2.4).

The question is: How to stop these vibrations? That is to say: What is the control that brings the vibrations at time T to rest?

In order to explain the dependence of the solution $y = y(x,t)$ of the problem (2.4) on the control v we will use the notation $y(v)$

The studied problem is the following one: Given $T > 0$, can we, for any pair of initial data $\{y^0, y^1\}$, given in a suitable space, find a control v that brings the system back to the equilibrium state $\{0, 0\}$ at time T, i.e. we want the solution $y = y(x, t; v)$ of (), to verify the condition

The evolution system (2.4) describes, for example, the vibrations of an n-dimensional body Ω subjected to the action of a force v on the boundary Γ and starting from an initial state described by the data $\{y^0, y^1\}$.

$$\boxed{y(x,T;v) = y'(x,T;v) = 0 \ \text{ in } \Omega.}$$

It's a null controllability problem.

Remark 2.1 *Because of the finite speed of wave propagation, the system* (2.5) *can only be exactly nul controllable if T is large enough.*

In applications, it is rare to find systems that can be controlled on the whole boundary. For this reason we consider the action of the control only on a part of the boundary.

henceforward, consider a non-empty open part Σ_0 of Σ and we act on the system by the following boundary condition

$$y = \begin{vmatrix} v & \text{on} & \Sigma_0 \\ 0 & \text{on} & \Sigma \backslash \Sigma_0. \end{vmatrix}$$

The formulation of the exact null controllability problem is now: "Given a time $T > 0$ and initial conditions $\{y^0, y^1\}$ given in a suitable space, is there a control v defined on Σ_0 such that if $y = y(v)$ is the solution of (2.4) we have (2.5)?"

Our goal is to study this exact controllability problem. To do so, we introduce the main ideas of the Hilbert uniqueness method (HUM) which we will use to solve the system (2.4) and (2.5).

2.2.1 *Description of HUM method*

The HUM methodology is based on certain uniqueness criteria for the associated homogeneous system and the construction (by completion processes) of Hilbertian spaces adapted to the structure of the system.

The following algorithm describes the fundamental steps of the application of the HUM method to the solution of the exact controllability problem of the system (2.4) and (2.5).

Fist step:

Let $\{\phi^0, \phi^1\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, and consider the following homogeneous wave equation

$$\begin{vmatrix} \phi'' - \Delta\phi = 0 & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(0) = \phi^0, \ \phi'(0) = \phi^1 & \text{in } \Omega. \end{vmatrix}$$

The problem (2.6) admits a unique solution $\phi = \phi(x,t)$ which satisfies the following regularity property:

$$\frac{\partial \phi}{\partial \nu} \in L^2(\Sigma).$$

Second step:
Then, we solve the backward equation

$$\left| \begin{array}{ll} \psi'' - \Delta\psi = 0 & \text{in } Q, \\ \psi = \left\{ \begin{array}{l} \dfrac{\partial \phi}{\partial \nu} \\ 0 \end{array} \right. & \begin{array}{l} \text{on } \Sigma_0, \\ \text{on} \Sigma \backslash \Sigma_0 \end{array} \\ \psi(T) = \psi'(T) = 0 & \text{in } \Omega. \end{array} \right. \tag{2.1}$$

where ν denotes the normal vector outside Ω and "$\dfrac{\partial}{\partial \nu}$" the derivative in that direction i.e.

$$\begin{aligned} \frac{\partial \phi}{\partial \nu} &= \nabla \phi \cdot \nu \\ &= \sum_{k=1}^{n} \frac{\partial \phi}{\partial x_k} \nu_k. \end{aligned}$$

Problem (2.7) is a non-homogeneous backward system, because we have a final condition
$\psi(T) = \psi'(T) = 0$ in Ω and non-homogeneous boundary conditions. This does not change the "well-posedness" of the system, which therefore admits a unique solution ψ.

The operator Λ

We define a linear operator Λ that associates with $\left\{\phi^0, \phi^1\right\}$ by

$$\mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \to \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \tag{2.2}$$

$$\boxed{\Lambda\left\{\phi^0, \phi^1\right\} = \left\{\psi'(0), -\psi(0)\right\}.} \tag{2.3}$$

The operator Λ is well defined because ψ is sufficiently regular.
Third step: Multiplying equation $(2.7)_1$ by $\phi = \phi(x,t)$ and integrating over Q, we obtain

$$\underbrace{\int_Q \psi'' \phi dx dt}_{(I)} - \underbrace{\int_Q \phi \Delta \psi dx dt}_{(II)} = 0.$$

Analysis of the first integral (I)
We have

$$\left(\psi', \phi\right)' = \left(\psi'', \phi\right) + \left(\psi', \phi'\right).$$

By integrating over $(0, T)$, we obtain

$$\left\langle \psi'(T), \phi(T) \right\rangle - \left\langle \psi'(0), \phi(0) \right\rangle = \int_Q \psi'' \phi dx dt + \int_Q \psi' \phi' dx dt$$

with condition (2.7), we find

$$\int_Q \psi'' \phi dx dt = -\left\langle \psi'(0), \phi^0 \right\rangle - \int_Q \psi' \phi' dx dt \tag{2.4}$$

on the other hand, by integration by parts, we have

$$\int_Q \psi' \phi' dx dt = \left(\psi(T), \phi'(T)\right) - \left(\psi(0), \phi'(0)\right) - \int_Q \psi \phi'' dx dt$$

therefore

$$\int_Q \psi' \phi' dx dt = -\left(\psi(0), \phi^1\right) - \int_Q \psi \phi'' dx dt. \tag{2.5}$$

Substituting () into (), we obtain

$$\boxed{\int_Q \psi'' \phi dx dt = -\left\langle \psi'(0), \phi^0 \right\rangle + \left(\psi(0), \phi^1\right) + \int_Q \psi \phi'' dx dt.} \tag{2.11}$$

Analysis of the second integral

According to Green's second identity, we have

$$\boxed{-\int_Q \phi \Delta \psi dx dt = -\int_Q \psi \Delta \phi dx dt + \int_\Sigma \psi \frac{\partial \phi}{\partial \nu} d\Gamma dt.} \tag{2.6}$$

Adding () to(), we obtain

$$\int_Q \left(\psi'' - \Delta\psi\right) \phi dx dt = \int_Q \left(\phi'' - \Delta\phi\right) \psi dx dt - \left\langle \psi'(0), \phi^0 \right\rangle + \left(\psi(0), \phi^1\right) + \int_\Sigma \psi \frac{\partial \phi}{\partial \nu} d\Gamma dt = 0,$$

since $\phi'' - \Delta\phi = 0$ and $\psi'' - \Delta\psi = 0$ in Q, then we find

$$-\left\langle \psi'(0), \phi^0 \right\rangle + \left(\psi(0), \phi^1\right) + \int_\Sigma \psi \frac{\partial \phi}{\partial \nu} d\Gamma dt = 0. \tag{2.7}$$

Remark that $\psi = \dfrac{\partial \phi}{\partial \nu}$ on Σ_0 and $\psi = 0$ on $\Sigma \backslash \Sigma_0$. Then

$$\left\langle \psi'(0), \phi^0 \right\rangle - \left(\psi(0), \phi^1\right) = \int_{\Sigma_0} \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma dt. \tag{2.8}$$

We consider $\left\langle \psi'(0), \phi^0 \right\rangle - \left(\psi(0), \phi^1\right)$ as a scalar product $\left\{\psi'(0), -\psi(0)\right\}$ and $\left\{\phi^0, \phi^1\right\}$.

Thus, from () we have

$$\left\langle \Lambda\left\{\phi^0, \phi^1\right\}, \left\{\phi^0, \phi^1\right\}\right\rangle = \left\langle \psi'(0), \phi^0 \right\rangle - \left(\psi(0), \phi^1\right) = \int_{\Sigma_0} \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Gamma dt. \tag{2.9}$$

We then introduce the semi-norm

$$\left\|\left\{\phi^0, \phi^1\right\}\right\|_F = \left(\int_{\Sigma_0} \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Gamma dt\right)^{\frac{1}{2}} = \left\|\frac{\partial \phi}{\partial \nu}\right\|_{L^2(\Sigma_0)} \quad \forall \left\{\phi^0, \phi^1\right\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega) \tag{2.10}$$

and we assume that in fact $\|.\|_F$ defines a norm in the space $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$.

The fact that $\|.\|_F$ defines a norm is equivalent to the following uniqueness theorem being verified.

Theorem 2.8 [14] ***(Uniqueness theorem)***

If $\phi = \phi(x,t)$ checks () for $\left\{\phi^0, \phi^1\right\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ and the condition

$$\frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \Sigma_0,$$

then

$$\phi = 0 \quad \text{in } Q.$$

Remark 2.2 *This uniqueness criterion is true by the inverse inequality or by Holmgren's theorem (see L. Hörmander* [10] *and J. L. Lions* [14]*)*

If the theorem () verified, then the norm () induces the following scalar product

$$\boxed{\left(\left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\}\right)_F = \int_{\Sigma_0} \frac{\partial \phi}{\partial \nu} \frac{\partial \zeta}{\partial \nu} d\Gamma dt} \tag{2.11}$$

where $\zeta = \zeta(x,t)$ is the solution of the problem () corresponding to the initial data $\left\{\zeta^0, \zeta^1\right\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$.

So from () and (), we deduce that

$$\left\langle \Lambda\left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\}\right\rangle = \left(\left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\}\right)_F \ , \forall \left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega). \tag{2.12}$$

According to the Cauchy-Schwarz inequality, we obtain

$$\left|\left\langle \Lambda\left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\}\right\rangle\right| \leq \left\|\left\{\phi^0, \phi^1\right\}\right\|_F \left\|\left\{\zeta^0, \zeta^1\right\}\right\|_F \ , \forall \left\{\phi^0, \phi^1\right\}, \left\{\zeta^0, \zeta^1\right\} \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega). \tag{2.13}$$

The inequality () shows the continuity of the bilinear form defined by ■ in $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$.

We define by F the completion of this space with respect to the norm (), so we obtain a Hilbert space. The continuous bilinear form $\left\{\left\{\phi^0,\phi^1\right\},\left\{\zeta^0,\zeta^1\right\}\right\} \rightarrow \left\langle \Lambda\left\{\phi^0,\phi^1\right\},\left\{\zeta^0,\zeta^1\right\}\right\rangle$ admits an extension by continuity to the closure F, we present this extension with the same notation. Then, we obtain a continuous bilinear form on the Hilbert space F which is coercive (the coercivity of the bilinear form is equivalent to the inverse inequality which we will state in the sequel), so by the Lax-Milgram lemma, for every $\left\{\eta^0,\eta^1\right\} \in F'$ (dual of F), there exists a unique $\left\{\phi^0,\phi^1\right\} \in F$ such that

$$\left\langle \Lambda\left\{\phi^0,\phi^1\right\},\left\{\zeta^0,\zeta^1\right\}\right\rangle = \left\langle \left\{\eta^0,\eta^1\right\},\left\{\zeta^0,\zeta^1\right\}\right\rangle_{F'\times F} \tag{2.14}$$

therefore, for any $\left\{\zeta^0,\zeta^1\right\} \in F$ and for every $\left\{\eta^0,\eta^1\right\} \in F'$, there exists a unique $\left\{\phi^0,\phi^1\right\} \in F$ which is the solution of the equation $\Lambda\left\{\phi^0,\phi^1\right\} = \left\{\eta^0,\eta^1\right\}$ in F'. From all this, it follows that $\Lambda : F \longrightarrow F'$ is an isomorphism.

Fourth step: Conclusion

We then consider the equation

$$\boxed{\Lambda\left\{\phi^0,\phi^1\right\} = \left\{y^1,-y^0\right\}.} \tag{2.15}$$

Since Λ is an isomorphism of F onto F', equation () admits a unique solution $\left\{\phi^0,\phi^1\right\} \in F$ for any initial data pair $\left\{y^0,y^1\right\}$ such that $\left\{y^1,-y^0\right\} \in F'$.

On the other hand, the operator Λ has been defined by the application () giving

$$\psi'(0) = y^1 \quad \text{and} \quad \psi(0) = y^0,$$

where ψ denotes the solution of problem ().

We choose the control v by

$$\boxed{v = \frac{\partial\phi}{\partial\nu} \quad \text{on } \Sigma_0.}$$

We note that $\psi = \psi(x,t)$ and $y = y(x,t)$ are two solutions of the same non-homogeneous problem. Then, from the uniqueness of the solution of problem () we have

$$y(x,t) = \psi(x,t) \quad \forall\,(x,t) \in Q.$$

From (), we deduce that

$$\boxed{y(x,T) = y'(x,T) = 0 \text{ in } \Omega.}$$

We see that $y = y(v)$ satisfies the condition ().

Finally, we have constructed a control v that gives exact controllability.

The next step is to characterize the spaces F and F' as Sobolev spaces.

First, we will state some preliminary results that will be needed in the application of HUM.

2.3 Some reminders on the existence and uniqueness of solutions of the wave equation

In this section we study some results of existence and regularity of the solutions of problem ().

We consider the homogeneous problem () and the associated energy

$$\boxed{E(t)=\frac{1}{2}\left\{\left\|\phi'(t)\right\|^2_{L^2(\Omega)}+\left\|\nabla\phi(t)\right\|^2_{[L^2(\Omega)]^n}\right\}\quad\forall t\in(0,T).} \tag{2.16}$$

We have the following existence and uniqueness result of the solution of ():

Lemma 2.1 *Let Ω be a bounded domain of $\mathbb{R}^n$ with a Lipchitzian boundary. Then, For initial conditions $\phi^0\in H_0^1(\Omega)$ and $\phi^1\in L^2(\Omega)$ there exists a unique solution $\phi=\phi(x,t)$ of () with*

$$\phi\in C\left(0,T;H_0^1(\Omega)\right)\cap C^1\left(0,T;L^2(\Omega)\right)\cap C^2\left(0,T;H^{-1}(\Omega)\right). \tag{2.17}$$

The solution ϕ of () (which belongs to the class () is called a weak solution of the equation).

In addition, we have the conservation of energy

$$\boxed{E(t)=E(0)=\frac{1}{2}\left\{\left|\phi^1\right|^2+\left|\nabla\phi^0\right|^2\right\}\quad\forall t\in(0,T).} \tag{2.18}$$

Proof. We have the existence and uniqueness result for the solution ϕ of () from [14].

To show the conservation of energy, we multiply equation () by $\phi'(x,t)$ and integrating over Ω we get

$$\begin{aligned}0&=\int_\Omega\left(\phi''-\Delta\phi\right)\phi'dx\\&\overset{(1.3)}{=}\frac{d}{dt}\left[\frac{1}{2}\int_\Omega\left\{\left|\phi'\right|^2+\left|\nabla\phi\right|^2\right\}dx\right]-\int_\Gamma\left(\nabla\phi\cdot\nu\right)\phi'd\Gamma\\&=\frac{dE(t)}{dt}-\int_\Gamma\left(\nabla\phi\cdot\nu\right)\phi'd\Gamma.\end{aligned}$$

According to the boundary conditions in (), we obtain

$$\frac{dE(t)}{dt}=0\qquad\forall t\in(0,T)$$

and consequently we have conservation of energy. □

For the regularity of the solution ϕ, we have the following lemma which will justify the integrations performed in the sequel

Lemma 2.2 [14] *We now assume that* Γ *is of class* C. *Then, if* $\left\{\phi^0, \phi^1\right\} \in H^2(\Omega) \cap H_0^1(\Omega)$ *the solution* ϕ *of* () *checks the following regularity property:*

$$\phi \in C\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(0, T; H_0^1(\Omega)\right) \cap C^2\left(0, T; L^2(\Omega)\right). \tag{2.19}$$

we now consider the inhomogeneous wave equation

$$\begin{cases} \phi'' - \Delta\phi = f & \text{in } Q, \\ \phi = 0 & \text{on } \Sigma, \\ \phi(0) = \phi^0, \phi'(0) = \phi^1 & \text{on } \Omega. \end{cases} \tag{2.20}$$

We have the following result

Lemma 2.3 ***(i)*** *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *with a Lipchitzian boundary. For any* $f \in L^1\left(0, T; L^2(\Omega)\right)$, $\phi^0 \in H_0^1(\Omega)$ *and* $\phi^1 \in L^2(\Omega)$ *there exists a unique solution* ϕ *of* () *with*

$$\phi \in C\left(0, T; H_0^1(\Omega)\right) \cap C^1\left(0, T; L^2(\Omega)\right). \tag{2.21}$$

Moreover, there is a constant $C > 0$ *such that*

$$\|\phi\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} + \left\|\phi'\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C\left\{\left|\nabla\phi^0\right| + \left|\phi^1\right| + \|f\|_{L^1(0,T;L^2(\Omega))}\right\}. \tag{2.22}$$

(ii) *We now assume that* Ω *is of class* C^2. *Then, for all* $f \in L^1\left(0, T; H_0^1(\Omega)\right)$, $\phi^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *and* $\phi^1 \in H_0^1(\Omega)$ *there exists a unique solution* ϕ *such that*

$$\phi \in C\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(0, T; H_0^1(\Omega)\right) \tag{2.23}$$

with the estimate

$$\|\phi\|_{L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)} + \left\|\phi'\right\|_{L^\infty\left(0,T;H_0^1(\Omega)\right)} \leq C\left\{\left\|\phi^0\right\|_{H^2(\Omega)\cap H_0^1(\Omega)} + \left\|\phi^1\right\|_{H_0^1(\Omega)} + \|f\|_{L^1\left(0,T;H_0^1(\Omega)\right)}\right\}. \tag{2.30}$$

Proof. See [14]. □

Now, we establish a fundamental identity that will then allow us to obtain the necessary a priori estimates in the application of HUM.

Lemma 2.4 [14] *Let* $q = (q_k)$ *be a vector field of class* $\left[C^1\left(\overline{\Omega}\right)\right]^n$. *Then for any weak solution* $\phi = \phi(x, t)$ *of equation* (), *that is, for any* $\left\{\phi^0, \phi^1\right\} \in H_0^1(\Omega) \times L^2(\Omega)$

and $f \in L^1\left(0,T;L^2(\Omega)\right)$, *we have*

$$\begin{aligned}\frac{1}{2}\int_{\Sigma} q_k\nu_k\left|\frac{\partial\phi}{\partial\nu}\right|^2 d\Sigma = \left(\phi'(t), q_k\frac{\partial\phi(t)}{\partial x_k}\right)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\left[\left|\phi'\right|^2 - \left|\nabla\phi\right|^2\right]dxdt \\ + \int_Q \frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\frac{\partial\phi}{\partial x_i}dxdt - \int_Q f q_k\frac{\partial\phi}{\partial x_k}dxdt.\end{aligned} \tag{2.24}$$

The convention of repeated indices has been applied here so that, for example

$$q_k\nu_k = \sum_{k=1}^{n} q_k\nu_k.$$

Before proving the identity (), we will state the following lemma which will be necessary in the following

Lemma 2.5 [19] *If* $\phi \in H_0^1(\Omega)\cap H^2(\Omega)$, *then*

$$\frac{\partial\phi}{\partial x_i} = \nu_i\frac{\partial\phi}{\partial\nu} \quad \textit{and } \Gamma.$$

and

$$\left|\nabla\phi\right|^2 = \left(\frac{\partial\phi}{\partial\nu}\right)^2.$$

Proof. See [19]. □

Now we return to the proof of Lemma 2.4.

Proof. We first establish the identity in the case of a strong solution ϕ, i.e., that corresponds to data $\left\{\phi^0,\phi^1\right\} \in \left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)$, $f \in L^1\left(0,T;H_0^1(\Omega)\right)$.

Multiplying equation () by $q_k\dfrac{\partial\phi}{\partial x_k}$ and integrating over Q, it follows that

$$\underbrace{\int_Q \phi'' q_k\frac{\partial\phi}{\partial x_k}dxdt}_{(I)} - \underbrace{\int_Q \Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdt}_{(II)} = \int_Q f q_k\frac{\partial\phi}{\partial x_k}dxdt. \tag{2.25}$$

Analysis of (II)

We have

$$-\int_Q \Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdt = -\int_0^T\int_\Omega \Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdt$$

according to Green's identity, we obtain

$$-\int_\Omega \Delta\phi q_k\frac{\partial\phi}{\partial x_k}dx = -\int_\Gamma \frac{\partial\phi}{\partial\nu}q_k\frac{\partial\phi}{\partial x_k}d\Gamma + \int_\Omega \nabla\phi\cdot\nabla\left(q_k\frac{\partial\phi}{\partial x_k}\right)dx. \tag{2.26}$$

On the other hand, one has

$$\begin{aligned}\nabla\phi\cdot\nabla\left(q_k\frac{\partial\phi}{\partial x_k}\right) &= \frac{\partial\phi}{\partial x_i}q_k\frac{\partial}{\partial x_k}\left(\frac{\partial\phi}{\partial x_i}\right)+\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\\ &= \frac{1}{2}q_k\frac{\partial}{\partial x_k}\left(\frac{\partial\phi}{\partial x_i}\right)^2+\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\\ &= \frac{1}{2}q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2+\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}\end{aligned}$$

therefore

$$(2.33)\Leftrightarrow -\int_\Omega\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dx=-\int_\Gamma\frac{\partial\phi}{\partial\nu}q_k\frac{\partial\phi}{\partial x_k}d\Gamma+\frac{1}{2}\int_\Omega q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2dx+(2.34)$$
$$\int_\Omega\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dx \qquad (2.27)$$

Again by using Green's identity, we obtain

$$\frac{1}{2}\int_\Omega q_k\frac{\partial}{\partial x_k}\left|\nabla\phi\right|^2dx=\frac{1}{2}\int_\Gamma q_k\left|\nabla\phi\right|^2\nu_kd\Gamma-\frac{1}{2}\int_\Omega\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dx \qquad (2.28)$$

Substituting () into (), we find

$$-\int_\Omega\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dx=-\int_\Gamma\frac{\partial\phi}{\partial\nu}q_k\frac{\partial\phi}{\partial x_k}d\Gamma+\frac{1}{2}\int_\Gamma q_k\left|\nabla\phi\right|^2\nu_kd\Gamma-\frac{1}{2}\int_\Omega\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dx+\int_\Omega\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}$$

Moreover, thanks to lemma 2.5, we have

$$\begin{aligned}-\int_\Omega\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dx=&-\int_\Gamma q_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Gamma+\frac{1}{2}\int_\Gamma q_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Gamma-\\ &\frac{1}{2}\int_\Omega\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dx+\int_\Omega\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dx.\end{aligned}$$

Finally, the integration over $(]0,T)$ gives us

$$\boxed{-\int_Q\Delta\phi q_k\frac{\partial\phi}{\partial x_k}dxdt=-\frac{1}{2}\int_\Sigma q_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2d\Gamma dt-\frac{1}{2}\int_Q\frac{\partial q_k}{\partial x_k}\left|\nabla\phi\right|^2dxdt+\int_Q\frac{\partial\phi}{\partial x_i}\frac{\partial q_k}{\partial x_i}\frac{\partial\phi}{\partial x_k}dxdt} \qquad (2.36)$$

Analysis of (I)
We have

$$\int_Q\phi''q_k\frac{\partial\phi}{\partial x_k}dxdt=\int_0^T\int_\Omega\phi''q_k\frac{\partial\phi}{\partial x_k}dxdt,$$

and we know that

$$\int_0^T\frac{\partial}{\partial t}\left(\phi',q_k\frac{\partial\phi}{\partial x_k}\right)dt=\int_Q\phi''q_k\frac{\partial\phi}{\partial x_k}dxdt+\int_Q\phi'q_k\frac{\partial\phi'}{\partial x_k}dxdt,$$

then

$$\int_Q \phi'' q_k \frac{\partial \phi}{\partial x_k} dxdt = \left(\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k} \right) \Big|_0^T - \frac{1}{2} \int_Q q_k \frac{\partial}{\partial x_k} \left| \phi' \right|^2 dxdt.$$

on the other hand, we observe that

$$\int_\Omega \frac{\partial}{\partial x_k} \left(\frac{q_k}{2} \left| \phi' \right|^2 \right) dx = \int_\Gamma \frac{q_k}{2} \left| \phi' \right|^2 \nu_k d\Gamma = 0$$

as a consequence of the Gauss divergence formula and the fact that $\phi = 0$ on Σ. Then it follows that

$$-\frac{1}{2} \int_Q q_k \frac{\partial}{\partial x_k} \left| \phi' \right|^2 dxdt = \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \phi' \right|^2 dxdt. \tag{2.29}$$

Substituting () into (), we obtain

$$\boxed{\int_Q \phi'' q_k \frac{\partial \phi}{\partial x_k} dxdt = \left(\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k} \right) \Big|_0^T + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \phi' \right|^2 dxdt} \tag{2.30}$$

Then, by transferring () and () into ()(2.32), we deduce

$$\left(\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k} \right) \Big|_0^T + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \phi' \right|^2 dxdt - \frac{1}{2} \int_\Sigma q_k \nu_k \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma dt -$$
$$\frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \nabla \phi \right|^2 dxdt$$
$$+ \int_Q \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} dxdt = \int_Q f q_k \frac{\partial \phi}{\partial x_k} dxdt,$$

then

$$\frac{1}{2} \int_\Sigma q_k \nu_k \left| \frac{\partial \phi}{\partial \nu} \right|^2 d\Gamma dt = \left(\phi'(t), q_k \frac{\partial \phi(t)}{\partial x_k} \right) \Big|_0^T + \frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \phi' \right|^2 dxdt -$$
$$\frac{1}{2} \int_Q \frac{\partial q_k}{\partial x_k} \left| \nabla \phi \right|^2 dxdt$$
$$+ \int_Q \frac{\partial \phi}{\partial x_i} \frac{\partial q_k}{\partial x_i} \frac{\partial \phi}{\partial x_k} dxdt - \int_Q f q_k \frac{\partial \phi}{\partial x_k} dxdt$$

hence the identity (2.31).

Now consider the general case of a weak solution $\phi = \phi(x,t)$, i.e., one that corresponds to data $\left\{ \phi^0, \phi^1 \right\} \in H_0^1(\Omega) \times L^2(\Omega)$, $f \in L^1\left(0,T; L^2(\Omega)\right)$.

We approach these data with more regular data $\left\{ \phi_n^0, \phi_n^1 \right\} \in \left(H^2(\Omega) \cap H_0^1(\Omega) \right) \times H_0^1(\Omega)$, $\{f_n\} \in L^1\left(0,T; H_0^1(\Omega)\right)$ such that

$$\begin{array}{l} \left\{ \phi_n^0, \phi_n^1 \right\} \to \left\{ \phi^0, \phi^1 \right\} \text{ in } H_0^1(\Omega) \times L^2(\Omega) \text{ and} \\ f_n \to f \text{ in } L^1\left(0,T; L^2(\Omega)\right), \text{ when } n \to +\infty. \end{array} \tag{2.31}$$

The identity () is thus verified by the strong solutions ϕ_n that correspond to the data $\left\{\phi_n^0, \phi_n^1, f_n\right\}$, and from the estimate () and the properties () we see that

$$\begin{array}{l}\phi_n \to \phi \text{ in } C\left(0,T;H_0^1(\Omega)\right) \text{ and} \\ \phi_n' \to \phi' \text{ in } C\left(0,T;L^2(\Omega)\right), \text{ when } n \to +\infty.\end{array}$$

This allows us to go to the limit when $n \to +\infty$ in the right-hand member of the identity (), to deduce that

$$q_k \nu_k \left|\frac{\partial \phi}{\partial \nu}\right|^2 \in L^1(\Sigma)$$

and that the identity () and also verified in this case. □

2.4 Regularity of weak solutions

2.4.1 *The homogeneous problem*

Theorem 2.9 [14] *Let Ω be a bounded domain of $\mathbb{R}^n$ of boundary Γ of class C^2. Then there exists a constant $C > 0$ such that*

$$\begin{array}{c}\displaystyle\int_\Sigma \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Sigma \leq C(T+1)\left\{\left|\nabla \phi^0\right|^2 + \left|\phi^1\right|^2 + \|f\|_{L^1(0,T;L^2(\Omega))}\right\}, \\ \forall \left\{\phi^0, \phi^1, f\right\} \in H_0^1(\Omega) \times L^2(\Omega) \times L^1\left(0,T;L^2(\Omega)\right)\end{array} \tag{2.32}$$

where $\phi = \phi(x,t)$ denotes the solution of problem ().

Proof. See [14]. □

Remark 2.3 [14] *From () it follows that $\dfrac{\partial \phi}{\partial \nu} \in L^2(\Sigma)$. This is a hidden regularity property of weak solutions of the wave equation.*

2.4.2 *The non-homogeneous problem*

Here we will study the existence and regularity of the solutions of the inhomogeneous problem ().

Consider the following problem:

$$\left|\begin{array}{ll} z'' - \Delta z = 0 & \text{in } Q, \\ z = v & \text{on } \Sigma, \\ z(0) = z^0,\ z'(0) = z^1 \text{ in } \Omega. & \end{array}\right. \tag{2.33}$$

The solution z of the problem () is defined by the transposition method (see J. L. Lions and E. Magenes [18]). More precisely, we have the following definition

Definition 2.6 [16] For any $\left\{z^0, z^1, v\right\} \in L^2(\Omega) \times H^{-1}(\Omega) \times L^2(\Sigma)$, we call an ultra-weak solution of problem (), a function $z \in L^\infty\left(0,T;L^2(\Omega)\right)$ satisfying the

condition

$$\int_Q fzdxdt = \left\langle z^1, \theta\left(0\right)\right\rangle - \left(z^0, \theta'\left(0\right)\right) - \int_\Sigma \frac{\partial\theta}{\partial\nu} v d\Gamma dt.$$

For any $f \in L^1\left(0,T;L^2\left(\Omega\right)\right)$, where $\theta = \theta\left(x,t\right)$ denotes the solution of the problem

$$\left|\begin{array}{ll} \theta'' - \Delta\theta = f & \text{in } Q, \\ \theta = 0 & \text{on } \Sigma, \\ \theta\left(T\right) = \theta'\left(T\right) = 0 & \text{in } \Omega. \end{array}\right. \tag{2.34}$$

Where $<.,.>$ denotes the duality product between $\langle .,. \rangle$ and $H_0^1\left(\Omega\right)$.

Lemma 2.6 [16] *The problem () admits a single ultra weak solution z verifies*

$$\left\|z\right\|_{L^\infty(0,T;L^2(\Omega))} \leq C\left(\left|z^0\right| + \left\|z^1\right\|_{H^{-1}(\Omega)} + \left\|v\right\|_{L^2(\Sigma)}\right). \tag{2.35}$$

Proof. See [19]. □

Theorem 2.10 [16] *For all $z^0 \in L^2\left(\Omega\right)$ and $z^1 \in H^{-1}\left(\Omega\right)$ and for all $v \in L^2\left(\Sigma\right)$ there exists a unique solution z of problem () with*

$$z \in C\left(0,T;L^2\left(\Omega\right)\right) \cap C^1\left(0,T;H^{-1}\left(\Omega\right)\right), \tag{2.36}$$

and

$$\left\|z\right\|_{L^\infty(0,T;L^2(\Omega))} + \left\|z'\right\|_{L^\infty(0,T;H^{-1}(\Omega))} \leq C\left(\left|z^0\right| + \left\|z^1\right\|_{H^{-1}(\Omega)} + \left\|v\right\|_{L^2(\Sigma)}\right). \tag{2.37}$$

Proof. See [16]. □

Now we return to the identification of the spaces F and F' .

First, we notice that with the particular choice $f = 0$ in Theorem () we get the

Corollary 2.1 ***(The direct inequality)*** [14]

Let Ω be a bounded domain of $\mathbb{R}^n$ of boundary Γ of class C^2. Then there exists a constant $C > 0$ such that

$$\int_\Sigma \left|\frac{\partial\phi}{\partial\nu}\right|^2 d\Gamma dt \leq C\left(T+1\right)E\left(0\right) = C\left(T+1\right)\left\|\left\{\phi^0,\phi^1\right\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)}, \tag{2.38}$$

for all ϕ weak solution of ().

Since F is the complement with respect to the norm defined by the right-hand side of the inequality (), then we deduce that $H_0^1\left(\Omega\right) \times L^2\left(\Omega\right) \subset F$. To prove that $F \subset H_0^1\left(\Omega\right) \times L^2\left(\Omega\right)$ one needs to prove that there exists a constant C_1 such that

$$C_1\left\|\left\{\phi^0,\phi^1\right\}\right\|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq \int_\Sigma \left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt. \tag{2.39}$$

If we prove () which combined with the "direct inequality" of Corollary 2.1, will allow us to identify the space F by $F = H_0^1(\Omega) \times L^2(\Omega)$ and its dual $F' = H^{-1}(\Omega) \times L^2(\Omega)$.

The inequality () is called the inverse inequality which we will prove in the next paragraph

2.4.3 *The inverse inequality*

The purpose of this paragraph is to establish a second estimate of the "inverse inequality" which will at the same time lead to a uniqueness result of the type of Theorem 2.1 and a fortiori to obtain additional information about the space of initial data in which the exact controllability takes place.

We first introduce some notations.

Let $x^0 \in \mathbb{R}^n$, we define the vector

$$m(x) = x - x^0,$$

with the components

$$m_k(x) = x_k - x_k^0 \ , \quad 1 \leq k \leq n,$$

and a partition of the boundary Γ as follows

$$\Gamma\left(x^0\right) = \{x \in \Gamma; m(x) \cdot \nu(x) > 0\}$$

with

$$\Gamma_*\left(x^0\right) = \Gamma \backslash \Gamma\left(x^0\right) = \{x \in \Gamma; m(x) \cdot \nu(x) \leq 0\}$$

and

$$\Sigma\left(x^0\right) = \Gamma\left(x^0\right) \times]0, T[\,.$$

$$\Sigma_*\left(x^0\right) = \Sigma \backslash \Sigma\left(x^0\right) = \Gamma_*\left(x^0\right) \times]0, T[\,.$$

In addition, we introduce

$$R\left(x^0\right) = \sup_{x \in \overline{\Omega}} \left\|x - x^0\right\| = \left\|m(x)\right\|_{L^\infty(\Omega)}$$

and we consider

$$T\left(x^0\right) = 2R\left(x^0\right).$$

Theorem 2.11 ***(The inverse inequality)*** [14]

Let Ω be a bounded domain of $\mathbb{R}^n$ of boundary Γ of class C^2. Then, for any $T > T\left(x^0\right)$ and any weak solution ϕ of () the following inequality holds.

$$\left(T - T\left(x^0\right)\right) E\left(0\right) \leq \frac{R\left(x^0\right)}{2} \int_{\Sigma(x^0)} \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Gamma dt. \tag{2.40}$$

Proof. We write the identity stated in lemma (), in the case of homogeneous solutions ($f = 0$) we obtain

$$\frac{1}{2}\int_{\Sigma} q_k \nu_k \left|\frac{\partial \phi}{\partial \nu}\right|^2 d\Sigma = \left(\phi'\left(t\right), q_k \frac{\partial \phi\left(t\right)}{\partial x_k}\right)\Big|_0^T + \frac{1}{2}\int_Q \frac{\partial q_k}{\partial x_k}\left[\left|\phi'\right|^2 - \left|\nabla \phi\right|^2\right] dxdt \tag{2.51}$$

$$\int_Q \frac{\partial q_k}{\partial x_i}\frac{\partial \phi}{\partial x_k}\frac{\partial \phi}{\partial x_i} dxdt. \tag{2.41}$$

with the choice of multipliers

$$q_k\left(x\right) = m_k\left(x\right) = x_k - x_k^0,\ 1 \leq k \leq n,$$

then

$$\frac{\partial q_k}{\partial x_i} = \delta_{ik},\ \sum_{k=1}^n \frac{\partial q_k}{\partial x_k} = n \ \text{ and } \ \frac{\partial q_k}{\partial x_i}\frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_k} = \left|\nabla \phi\right|^2.$$

We put

$$X = \left(\phi'\left(t\right), q_k \frac{\partial \phi\left(t\right)}{\partial x_k}\right)\Big|_0^T$$

thus, the identity () becomes

$$X + \frac{n}{2}\int_Q \left[\left|\phi'\right|^2 - \left|\nabla \phi\right|^2\right] dxdt + \int_Q \left|\nabla \phi\right|^2 dxdt = \frac{1}{2}\int_\Sigma m_k \nu_k \left(\frac{\partial \phi}{\partial \nu}\right)^2 d\Gamma dt. \tag{2.42}$$

At $\Sigma\left(x^0\right)$, thanks to the Cauchy-Schwarz inequality, we have

$$0 < m\left(x\right) \cdot \nu\left(x\right) = \sum_{k=1}^n m_k \nu_k \tag{2.43}$$

$$\leq \left(\sum_{k=1}^n m_k^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^n \nu_k^2\right)^{\frac{1}{2}} \tag{2.44}$$

$$= \left\|m\left(x\right)\right\| \leq R\left(x^0\right) \tag{2.45}$$

and on the other hand, since $m(x)\cdot\nu(x)\leq 0$ on $\Sigma_*\left(x^0\right)$, we can infer that

$$\int_\Sigma m_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \leq \int_{\Sigma(x^0)} m_k\nu_k\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \tag{2.46}$$

$$\overset{(2.53)}{\leq} R\left(x^0\right)\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt \tag{2.47}$$

Using (2.54) in (2.52), we get

$$X+\frac{n}{2}\int_Q\left[\left|\phi'\right|^2-\left|\nabla\phi\right|^2\right]dxdt+\int_Q\left|\nabla\phi\right|^2 dxdt\leq\frac{R\left(x^0\right)}{2}\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt. \tag{2.48}$$

In addition, we have

$$\begin{aligned}X+\frac{n}{2}\int_Q\left[\left|\phi'\right|^2-\left|\nabla\phi\right|^2\right]dxdt+\int_Q\left|\nabla\phi\right|^2 dxdt &= X+\frac{n}{2}\int_Q\left|\phi'\right|^2 dxdt-\frac{n}{2}\int_Q\left|\nabla\phi\right|^2 dxdt\\ &+\int_Q\left|\nabla\phi\right|^2 dxdt+\frac{1}{2}\int_Q\left|\phi'\right|^2 dxdt-\frac{1}{2}\int_Q\left|\phi'\right|^2 dxdt\end{aligned}$$

$$\begin{aligned}&= X+\frac{n-1}{2}\int_Q\left|\phi'\right|^2 dxdt+\frac{2-n}{2}\int_Q\left|\nabla\phi\right|^2 dxdt+\frac{1}{2}\int_Q\left|\phi'\right|^2 dxdt\\ &= X+\frac{n-1}{2}\int_Q\left|\phi'\right|^2 dxdt-\frac{n-2}{2}\int_Q\left|\nabla\phi\right|^2 dxdt+\frac{1}{2}\int_Q\left|\phi'\right|^2 dxdt\\ &= X+\frac{n-1}{2}\int_Q\left|\phi'\right|^2 dxdt-\frac{n-1}{2}\int_Q\left|\nabla\phi\right|^2 dxdt+\frac{1}{2}\int_Q\left|\phi'\right|^2 dxdt+\frac{1}{2}\int_Q\left|\nabla\phi\right|^2 dxdt\\ &= X+\frac{n-1}{2}\int_Q\left[\left|\phi'\right|^2-\left|\nabla\phi\right|^2\right]dxdt+\frac{1}{2}\int_Q\left[\left|\phi'\right|^2+\left|\nabla\phi\right|^2\right]dxdt.\end{aligned}$$

We pose

$$Y=\int_Q\left[\left|\phi'\right|^2-\left|\nabla\phi\right|^2\right]dxdt.$$

Moreover, the conservation of energy () gives us

$$\begin{aligned}\frac{1}{2}\int_Q\left[\left|\phi'\right|^2+\left|\nabla\phi\right|^2\right]dxdt &= \int_0^T\frac{1}{2}\int_\Omega\left[\left|\phi'\right|^2+\left|\nabla\phi\right|^2\right]dxdt\\ &= \int_0^T E(0)\,dt\\ &= TE(0),\end{aligned}$$

so the inequality () becomes

$$X+\frac{n-1}{2}Y+TE(0)\leq\frac{R\left(x^0\right)}{2}\int_{\Sigma(x^0)}\left|\frac{\partial\phi}{\partial\nu}\right|^2 d\Gamma dt. \tag{2.49}$$

□

Now, we estimate Y by the

Lemma 2.7 *For any weak solution ϕ of equation () we have*

$$\boxed{Y = \left(\phi'(t), \phi(t)\right)\Big|_0^T.}$$

Proof. We multiply equation (2.6) by ϕ and integrating over Q, we obtain

$$\int_Q \phi'' \phi dxdt - \int_Q \Delta\phi\phi dxdt = 0. \tag{2.50}$$

We have

$$\begin{aligned}\left(\phi'(t), \phi(t)\right)\Big|_0^T &= \int_0^T \frac{\partial}{\partial t}\left(\phi'(t), \phi(t)\right) dt \\ &= \int_Q \phi''\phi dxdt + \int_Q \left|\phi'\right|^2 dxdt\end{aligned}$$

then

$$\int_Q \phi''\phi dxdt = \left(\phi'(t), \phi(t)\right)\Big|_0^T - \int_Q \left|\phi'\right|^2 dxdt \tag{2.51}$$

on the other hand, from Green's identity () and the fact that ($\phi = 0$ on Σ) we have

$$\int_Q \phi\Delta\phi dxdt = \int_\Sigma \phi\frac{\partial\phi}{\partial x_i}\nu_k d\Gamma dt - \int_Q |\nabla\phi|^2 dxdt \tag{2.52}$$

$$= -\int_Q |\nabla\phi|^2 dxdt. \tag{2.53}$$

From (2.57), (2.58) and (2.59) we deduce

$$\left(\phi'(t), \phi(t)\right)\Big|_0^T - \int_Q \left(\left|\phi'\right|^2 - |\nabla\phi|^2\right) dxdt = 0,$$

and therefore

$$Y = \left(\phi'(t), \phi(t)\right)\Big|_0^T.$$

Let us return to the proof of theorem (2.4)
by lemma (2.7), we obtain

$$\begin{aligned}X + \frac{n-1}{2}Y &= \left(\phi'(t), m_k\frac{\partial\phi(t)}{\partial x_k}\right)\Big|_0^T + \frac{n-1}{2}\left(\phi'(t), \phi(t)\right)\Big|_0^T \\ &= \left(\phi'(t), m_k\frac{\partial\phi(t)}{\partial x_k} + \frac{n-1}{2}\phi(t)\right)\Big|_0^T.\end{aligned}$$

then

$$X+\frac{n-1}{2}Y=\int_{\Omega}\phi'(t)\left(m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)dx\Big|_0^T. \tag{2.60}$$

Thanks to the Cauchy inequality with ε we have for all $\varepsilon>0$

$$\int_{\Omega}\phi'(t)\left(m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)dx\leq\frac{\varepsilon}{2}\int_{\Omega}\left|\phi'(t)\right|^2dx+ \tag{2.61}$$

$$\frac{1}{2\varepsilon}\int_{\Omega}\left(m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)^2dx \tag{2.54}$$

on the other hand

$$\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right)^2dx=\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}\right)^2dx+ \tag{2.62}$$

$$\frac{(n-1)^2}{4}\int_{\Omega}\phi^2dx+(n-1)\underbrace{\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}\right)\phi dx}_{(*)} \tag{2.55}$$

and

$$(*)\Leftrightarrow\int_{\Omega}\sum_{k=1}^{n}m_k\frac{\partial\phi}{\partial x_k}\phi dx=\frac{1}{2}\int_{\Omega}\sum_{k=1}^{n}m_k\frac{\partial}{\partial x_k}\phi^2dx.$$

Moreover, from the Gaussian divergence formula (1.1) and since ($\phi=0$ on Σ) we have

$$\int_{\Omega}\frac{\partial}{\partial x_k}\left(m_k\phi^2\right)dx=\int_{\Gamma}m_k\nu_k\phi^2d\Gamma=0,$$

and

$$\frac{1}{2}\int_{\Omega}\sum_{k=1}^{n}m_k\frac{\partial}{\partial x_k}\phi^2dx=-\frac{1}{2}\int_{\Omega}\sum_{k=1}^{n}\frac{\partial m_k}{\partial x_k}\phi^2dx \tag{2.56}$$

$$=-\frac{n}{2}\int_{\Omega}\phi^2dx. \tag{2.57}$$

Using (2.63) in (2.62), we obtain

$$\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}+\frac{n-1}{2}\phi\right)^2dx=\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}\right)^2dx+\frac{(n-1)^2}{4}\int_{\Omega}\phi^2dx-\frac{n(n-1)}{2}\int_{\Omega}\phi^2dx$$

$$=\int_{\Omega}\left(m_k\frac{\partial\phi}{\partial x_k}\right)^2dx+\left[\frac{(n-1)^2}{4}-\frac{n(n-1)}{2}\right]\int_{\Omega}\phi^2dx.$$

We notice that

$$\left(m_k \frac{\partial \phi}{\partial x_k}\right)^2 = \left(\sum_{k=1}^{n} m_k \frac{\partial \phi}{\partial x_k}\right)^2 \overset{C.Schwarz}{\leq} \left(\sum_{k=1}^{n} m_k^2\right)\left(\sum_{k=1}^{n}\left(\frac{\partial \phi}{\partial x_k}\right)^2\right),$$

this implies that

$$\left(m_k \frac{\partial \phi}{\partial x_k}\right)^2 \leq \|m(x)\|^2 |\nabla \phi|^2,$$

then

$$\int_\Omega \left(m_k \frac{\partial \phi}{\partial x_k}\right)^2 dx \leq \int_\Omega \|m(x)\|^2 |\nabla \phi|^2 \leq R^2\left(x^0\right) \int_\Omega |\nabla \phi|^2 dx.$$

On the other hand, we have

$$\left[\frac{(n-1)^2}{4} - \frac{n(n-1)}{2}\right] = -\frac{\left(n^2-1\right)}{4} \leq 0,$$

with all this, we deduce that

$$\int_\Omega \left(m_k \frac{\partial \phi}{\partial x_k} + \frac{n-1}{2}\phi\right)^2 dx \leq \int_\Omega \left(m_k \frac{\partial \phi}{\partial x_k}\right)^2 dx \tag{2.58}$$

$$\leq R^2\left(x^0\right) \int_\Omega |\nabla \phi|^2 dx. \tag{2.59}$$

Substituting (2.64) into (2.61) with the choice of $\varepsilon = R\left(x^0\right)$, we obtain

$$\int_\Omega \phi'(t)\left(m_k \frac{\partial \phi(t)}{\partial x_k} + \frac{n-1}{2}\phi(t)\right) dx \leq \frac{R\left(x^0\right)}{2} \int_\Omega \left|\phi'(t)\right|^2 dx + \frac{R\left(x^0\right)}{2} \int_\Omega |\nabla \phi(t)|^2 dx \leq \frac{R\left(x^0\right)}{2} \int_\Omega \left[\left|\phi'(t)\right|^2 + |\nabla \phi(t)|^2\right] dx.$$

then

$$\int_\Omega \phi'(t)\left(m_k \frac{\partial \phi(t)}{\partial x_k} + \frac{n-1}{2}\phi(t)\right) dx \leq R\left(x^0\right) E(0). \tag{2.65}$$

From (2.60) and (2.65), we deduce

$$\begin{aligned}\left|X+\frac{n-1}{2}Y\right| &= \left|\left(\phi'(t), m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)\Big|_0^T\right| \\ &\le 2\sup_{0\le t\le T}\left|\left(\phi'(t), m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)\right| \\ &\le 2\left\|\left(\phi'(t), m_k\frac{\partial\phi(t)}{\partial x_k}+\frac{n-1}{2}\phi(t)\right)\right\|_{L^\infty(0,T)} \\ &\le 2R\left(x^0\right)E(0).\end{aligned}$$

We take $T\left(x^0\right)=2R\left(x^0\right)$, we get

$$\left|X+\frac{n-1}{2}Y\right|\le T\left(x^0\right)E(0)$$

thus

$$\begin{aligned}TE(0)-T\left(x^0\right)E(0) &\le TE(0)-\left|X+\frac{n-1}{2}Y\right| \\ &\le X+\frac{n-1}{2}Y+TE(0) \\ &\le \frac{R\left(x^0\right)}{2}\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt\end{aligned}$$

where

$$\left(T-T\left(x^0\right)\right)E(0)\le\frac{R\left(x^0\right)}{2}\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt$$

and therefore

$$\int_\Omega\left(\left|\phi'\right|^2+|\nabla\phi|^2\right)dx\le\frac{R\left(x^0\right)}{\left(T-T\left(x^0\right)\right)}\int_{\Sigma(x^0)}\left(\frac{\partial\phi}{\partial\nu}\right)^2 d\Gamma dt.$$

□

2.5 Additional theorems for controllability

Theorem 2.12 ***(Russell's Principle for Hyperbolic Systems)***

For the wave equation $\frac{\partial^2 y}{\partial t^2}-\Delta y=u(t)\chi_\omega$, exact controllability holds if:
The control time $T>2diam\left(\Omega\backslash\omega\right)$,
The control region ω satisfies the Geometric Control Condition (GCC).

Proof. Use microlocal analysis to show all bicharacteristics enter ■×$[0,T]$■ × $[0,T]$. □

Theorem 2.13 ***(Null Controllability for Parabolic Systems)***

For the heat equation $\frac{\partial y}{\partial t} - \Delta y = u(t)\chi_\omega$, null controllability holds for any $T > 0$ if ω is a nonempty open subset.

Proof. Proof Tool: □

Carleman Estimates: Weighted energy inequalities to bound solutions.

Theorem 2.14 *(Ingham's Inequality for Observability)*

For the Schrödinger equation $\frac{\partial y}{\partial t} = i\Delta y$, observability holds if the sensor region ω satisfies GCC, with the estimate:

$$\int_0^T \int_\omega |y|^2 \, dxdt \geq C \|y(0)\|_{L^2(\Omega)}$$

Example:

If ω is a neighborhood of the boundary Γ, observability holds for T sufficiently large.

2.6 Second chapter exercises

Exercise 1:**(Heat Equation with Distributed Control)**

Consider the following one dimensional heat equation on $(0,1)$:

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \chi_{[a,b]} u(t) \text{ in } (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

where

$u(t)$ is the control acting on the subinterval $[a,b] \subset (0,1)$.

$\chi_{[a,b]}$ is the carachteristic function of $[a,b]$

y_0 is the initial state in $L^2(0,1)$

Show that the system is approximately controllable on the state space $Y = L^2(0,1)$ for any $T > 0$ if $[a,b]$ is a nonempty subinterval.

Hint: Use the fact that eigenfunctions of $-\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary conditions form a basis, and show that the control can influence all modes.

Exercise 2: (Wave Equation with Boundary Control)

Consider the following one dimensional wave equation on $(0,1)$:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \qquad \text{in } (0,1) \times (0,T) \\ y(0,t) = 0, \ y(1,t) = u(t) \qquad \text{in } (0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) \qquad \text{in } (0,1) \end{cases}$$

with boundary control $u(t) \in L^2(0,T)$.

$(y_0, y_1) \in L^2(0,1) \times H_0^1(0,T)$

Prove exact controllability for $T > 2$ using the Hilbert Uniqueness Method (HUM).

Hint: Use the duality between controllability and observability, and the fact that the wave propagates with speed 1.

Exercise 3: (Schrödinger Equation Observability)

Consider the following one dimensional Schrödinger equation on $(0,1)$:

$$\begin{cases} \frac{\partial y}{\partial t} = i\frac{\partial^2 y}{\partial x^2} \qquad \text{in } (0,1) \times (0,T), \\ y(0,t) = 0, \ y(1,t) = 0 \qquad \text{in } (0,T), \\ y(x,0) = y_0(x), \qquad \text{in } (0,1), \end{cases}$$

with observation at a point $x_0 \in (0,1)$

$$z(t) = y(x_0, t)$$

Determine whether the system is observable in time $T > 0$.

Hint: Check if the observability map

$$y_0 \to z(t)$$

is injective using spectral decomposition.

Exercise 4: **(Null Controllability of the Heat Equation)**
Consider the following one dimensional heat equation on $(0,1)$:

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \chi_{[a,b]} u(t) \text{ in } (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

Prove null controllability (i.e., driving the state to zero at time $T > 0$) using Carleman estimates.
Hint: Construct a weighted energy estimate and use duality.

Exercise 5: (Geometric Control Condition for Waves)
For the two dimensional wave equation on a disk $\Omega = \left\{x \in \mathbb{R}^2 : \|x\| \leq 1\right\}$:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \Delta y & \text{in } \Omega \times (0,T) \\ y = u & \text{in } \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega \end{cases}$$

with boundary control $u(t)$.
Explain why exact controllability holds if $T > 2$ and the control region $\partial\Omega$ satisfies GCC.
Hint: Use the finite speed of propagation and bicharacteristic covering.

Exercise 6: (Beam Equation Observability)
For the Euler-Bernoulli beam equation on $(0,1)$:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^4 y}{\partial x^4} & \text{in } \Omega \times (0,T) \\ y(0,t) = y(1,t) = 0 & \text{in } \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega \end{cases}$$

with observation

$$z(t) = \frac{\partial^2 y}{\partial x^2}(x_0, t)$$

Determine if the system is exactly observable in time $T > 0$.
Hint: Analyze the spectral properties of the beam operator.

Exercise 7: (Counter example for Controllability)

Consider the heat equation on $(0,1)$:

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \chi_{[a,b]} u(t) \text{ in } (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

with Dirichlet boundary conditions.

Show that the system is not approximately controllable if the control region $[a,b]$ is too small.

Hint: Identify a mode (eigenfunction) orthogonal to the control subspace.

Exercise 8: (Delay Systems)

Consider the scalar delay equation:

$$\begin{aligned} x'(t) &= ax(t) + bx(t-1) + u(t), \\ x(t) &= 0 \end{aligned}$$

with $x(t) \in \mathbb{R}$, $u \in L^2(0,T)$.

Characterize the controllability of this system using the spectral approach.

Hint: Study the roots of the characteristic equation $\lambda - a - be^{-\lambda} = 0$.

Exercise 9: (Non-Homogeneous Boundary Conditions)

For the heat equation on on $(0,1)$:

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} \qquad \text{in } (0,1) \times (0,T) \\ y(0,t) = 0, y(1,t) = u(t) \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

with boundary control $u(t)$.

Transform the system into one with homogeneous boundary conditions and analyze its controllability.

Hint: Use the lifting method (e.g., $y = v + u(t)\,x$).

Exercise 10:(Wave Equation with Boundary Control)

Consider the 1D wave equation

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} \qquad \text{in } (0,1) \times (0,T) \\ y(0,t) = u(t), y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) \qquad \text{in } (0,1) \end{cases}$$

Prove that the system is exactly controllable in time T■2.

Hint: Use d'Alembert's formula and the method of characteristics.

Exercise 11: (Null Controllability of the Heat Equation 2)

Investigate the null controllability of

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + \chi_{[a,b]} u(t) \text{ in } (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

Use the moment method to construct a control u(t) that drives the solution to zero at time T>0.

Hint: Expand solutions in eigenfunctions of the Laplacian.

Exercise 12:(Delay Differential Equation)
Analyze the controllability of

$$x'(t) = Ax(t) + Bu(t-h)$$

with delay h>0. Transform the system into an infinite-dimensional product space and derive conditions on A and B for controllability.

Exercise 13: (Boundary vs. Distributed Control)
Compare the null controllability time for the heat equation

$$\frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + u(t)$$

with:
(i) Distributed control $\chi_{[a,b]} u(t)$,
(ii) Boundary control $y_x(t,0) = u(t)$.
Which case requires a shorter time T?

Exercise 14:(Coupled Hyperbolic System)
Study the exact controllability of two coupled wave equations

$$\begin{aligned} \frac{\partial^2 y}{\partial t^2} &= \frac{\partial^2 y}{\partial x^2} + z \\ \frac{\partial^2 z}{\partial t^2} &= \frac{\partial^2 z}{\partial x^2} + y \end{aligned}$$

with boundary controls.

Use characteristic methods to determine controllability time.

Exercise 15:(Schrödinger Equation Observability)
Prove an observability inequality for

$$\begin{cases} \frac{\partial y}{\partial t} = i \frac{\partial^2 y}{\partial x^2} \qquad \text{in } (0,1) \times (0,T), \\ y(0,t) = 0, \ y(1,t) = 0 \qquad \text{in } (0,T), \\ y(x,0) = y_0(x), \qquad \text{in } (0,1), \end{cases}$$

and sensor y

$$z = y(t, 0)$$

Discuss the minimal time T required.

Exercise 16:(Duality Principle)

Show that exact controllability of the wave equation

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2} + u(t)\,\delta(x) & \text{in } (0,1)\times(0,T) \\ y(0,t) = 0, y(1,t) = 0 & \text{in } (0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } (0,1) \end{cases}$$

implies observability of its adjoint system

$$\begin{cases} \frac{\partial^2 \phi}{\partial t^2} = \frac{\partial^2 \phi}{\partial x^2} & \text{in } (0,1)\times(0,T) \\ \phi(0,t) = 0, \phi(1,t) = 0 & \text{in } (0,T) \\ \phi(x,T) = 0, \frac{\partial \phi}{\partial t}(x,T) = 0 & \text{in } (0,1) \end{cases}$$

with measurements at $x = 0$.

Exercise 17:(Gradient Observability for the Heat Equation)

Determine if the heat equation

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} & \text{in } (0,1)\times(0,T) \\ y(0,t) = y(1,t) = 0 & \text{in } (0,T) \\ y(x,0) = y_0(x) & \text{in } (0,1) \end{cases}$$

is observable when measuring the flux

$$\frac{\partial y}{\partial x}(0,t)$$

instead of the state $y(0,t)$.

Exercise 18: (Coupled Beam Equation)

Consider the Euler-Bernoulli beam

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^4 y}{\partial x^4} & \text{in } \Omega\times(0,T) \\ y(0,t) = y(1,t) = 0 & \text{in } \partial\Omega\times(0,T) \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega \end{cases}$$

Prove observability using measurements of the shear force

$$\frac{\partial^3 y}{\partial x^3}(1,t)$$

Exercise 19:(Inverse Observability Problem)

Suppose the heat equation is observable with a sensor at an unknown location x_0.

Can x_0 be uniquely determined from the sensor data? Justify your answer.

Exercise 20:(Semigroup Approach to Controllability)

For a system

$$x'(t) = Ax(t) + Bu(t)$$
$$x(0) = x_0$$

governed by a C_0-semigroup on a Hilbert space, derive conditions for exact controllability in terms of the operators A and B.

Exercise 21:(Spectral Conditions for the Heat Equation)
Analyze the approximate controllability of

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + u(t)\,\delta(x - x_0) \text{ in } (0,1) \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

using the eigenfunctions of the Dirichlet Laplacian.
Are all modes reachable?

Exercise 22:(Retarded System in Function Spaces)
Consider the system

$$x'(t) = Lx(t) + Mu(t)$$

(where $Lx(t) = Ax(t) + Bx(t-\tau)$)
Represent the delay systemas an abstract ODE on C([-■,0];$\mathbb{R}^n$).
Check controllability using the Kalman rank condition.
Exercise 23:(Non-Homogeneous Boundary Control)
Convert the heat equation

$$\begin{cases} \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial x^2} + u(t)\,\delta(x - x_0) \text{ in } (0,1) \times (0,T) \\ y(0,t) = u(t)\,; y(1,t) = 0 \qquad \text{in } (0,T) \\ y(x,0) = y_0(x) \qquad \text{in } (0,1) \end{cases}$$

into a distributed control problem via lifting.
Show equivalence of controllability.

Exercise 24:(Beam Equation with Piezoelectric Actuators)
Study the exact controllability of a beam equation

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} = \frac{\partial^4 y}{\partial x^4} + u(t,x) \qquad \text{in } \Omega \times (0,T) \\ y(0,t) = y(1,t) = 0 \qquad \text{in } \partial\Omega \times (0,T) \\ y(x,0) = y_0(x)\,, \frac{\partial y}{\partial t}(x,0) = y_1(x) \qquad \text{in } \Omega \end{cases}$$

with piezoelectric actuators modeled by

$$u(t,x) = \sum_{k=1}^{N} \chi_{[a_k,\,b_k]}(x)\,u_k(t)$$

Use modal analysis.

Chapter 3

Optimal control of linear PDEs

Optimal control transcends mere controllability—it seeks the best way to steer a system, balancing performance and cost. In this chapter, we ascend from controllability to optimization, marrying PDE theory with calculus of variations. Through adjoint methods and Pontryagin's Maximum Principle, we derive optimality systems that couple forward-state and backward-adjoint PDEs, unlocking control laws for heat, wave, and Schrödinger equations. You will confront constraints.

During this chapter, we will study the classical theory of optimal control for linear PDEs in distributed system. We begin our chapter by the presentation of our optimal control problem and we give it characterization then we finish by some examples.

3.1 Optimal Control Problem

A control problem involves manipulating a system defined by an input-output relationship. The input, known as the control, can take various forms, such as a function related to boundary conditions, initial conditions, coefficients in a partial differential equation (PDE) that models the system, or any parameter within the equation. The output represents the state or solution of the system, or any related information.

An optimal control problem is an optimization challenge where the objective is to minimize a criterion based on the observation of the state and the control variable, subject to constraints defined by PDEs.

The theory of optimal control requires the following conditions:

- A control variable u that belongs to the set of admissible controls.
- A state $y(u)$ that is to be controlled.
- An observation $z(u)$.
- A cost function $J(u)$ that should be minimized.

Among the objectives of this theory are to establish necessary conditions for u to be a minimum and to develop algorithms that approximate the minimum control u. These conditions are referred to as "quasi-variational inequalities."

3.2 Preleminaries

Let $\mathcal{Y}, \mathcal{U}$ and $\mathcal{Z}$ be infinite dimensional Hilbert spaces of states, controls and observation resp. $\mathcal{U}_{ad} \subset \mathcal{U}$ is a subset of admissible controls supposed non empty, closed and convex.

Consider the following well-posed state equation related to the control $v \in \mathcal{U}_{ad}$

$$\mathcal{A}y = f + \mathcal{B}v. \tag{3.1}$$

where

$\mathcal{A} \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ is a linear partial differential operator stationary or evolutionary (elliptic, parabolic and hyperbolic) modeling a distributed system makes an isomorphism on $\mathcal{Y}$ identified to $\mathcal{Y}$

$\mathcal{B} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$ is the control operator.

Our optimal control problem consists in looking for a control function $u \in \mathcal{U}_{ad}$ which minimizes the following cost function

$$J(v) = \|\mathcal{C}y(v) - y_d\|^2_{\mathcal{Z}} + N\|v\|^2_{\mathcal{U}} \qquad \forall v \in \mathcal{U}_{ad}, \tag{3.2}$$

where

J is convex function from $\mathbb{R}$ to $\mathbb{R} \cup \{+\infty\}$,

$\mathcal{C} \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ the observation operator and $N > 0$,

y_d is the fixed observation in $\mathcal{Z}$.

i.e, our optimal control problem is

$$\begin{cases} \text{find } u \in \mathcal{U}_{ad} \text{ such that} \\ J(u, y(u)) = \inf\limits_{v \in \mathcal{U}_{ad}} J(v). \end{cases} \tag{3.3}$$

Theorem 3.1 ***(Existence and uniqueness of optimal control)***

Let $\mathcal{U}_{ad} \subset \mathcal{U}$ closed and nonempty, J is lower semicontinuous, bounded from below and coercive on $\mathcal{U}_{ad}$. Then there exists a minimizer for J on $\mathcal{U}_{ad}$. Moreover if J is strictly convex, the minimizer is unique.

Proof. 1.**Existence**

Since J is bounded from below,

$$m = \inf_{v \in \mathcal{U}_{ad}} J(v),$$

Let (v_n) be a minimizing sequence[1] in $\mathcal{U}_{ad}$ since J is coercive, (v_n) is converge

[1] a minimizing sequence of the criterion J on the set U_{ad} is a sequence (v_n) such that $\lim\limits_{n \to +\infty} J(v_n) = \inf\limits_{v \in U_{ad}} J(v)$

to $u \in \mathcal{U}_{ad}$, and J is lower semicontinuous, then

$$J(u) \leq \lim_{n \to +\infty} \inf_{v \in \mathcal{U}_{ad}} J(v_n) = \inf_{v \in \mathcal{U}_{ad}} J(v) = m,$$

then u is a minimizer of J on $\mathcal{U}_{ad}$.

2.**Uniqueness**

Suppose that the problem (1.3) has two solution u_1, u_2 such that $J(u_1) = \inf_{v \in \mathcal{U}_{ad}} J(v) = J(u_2)$.

We have that J is strictly convex then

$$J\left(\frac{u_1 + u_2}{2}\right) < \frac{1}{2} J(u_1) + \frac{1}{2} J(u_2) = m,$$

Unless $u_1 = u_2$, we obtain a contradiction with the assumption that u_1 is solution of (1.3) .Thus $u_1 \neq u_2$ there (1.3) admits a unique solution. □

3.3 Characterization of the optimal control (optimality system)

A first order optimality condition gives:

$$J'(u).(v-u) \geq 0 \qquad \forall v \in \mathcal{U}_{ad},$$

we know that the cost function J is Gateaux-differentiable with

$$J'(u).(v-u) = \lim_{t \to 0} \frac{J(u + t(v-u)) - J(u)}{t} \text{ for every } v \in \mathcal{U}_{ad},$$

by a simple calculus we get

$$\begin{aligned} J(u + t(v-u)) = J(u) + t^2 \|\mathcal{C}y(v-u)\|^2_{\mathcal{Z}} + 2t(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}} + t^2 N \|v-u\|^2_{\mathcal{U}} \\ +2tN(u, v-u)_{\mathcal{U}}, \end{aligned}$$

which gives

$$\begin{aligned} \frac{J(u + t(v-u)) - J(u)}{t} = t \|\mathcal{C}y(v-u)\|^2_{\mathcal{Z}} + 2(\mathcal{C}y(u) - y_d, \mathcal{C}y(v-u))_{\mathcal{Z}} + tN \|v-u\|^2_{\mathcal{U}} + \\ 2N(u, v-u)_{\mathcal{U}}, \end{aligned}$$

make t tends to a 0 to get

$$J'(u).(v-u) = 2(\mathcal{C}^*(\mathcal{C}y(u) - y_d), y(v-u))_{\mathcal{Y}} + 2N(u, v-u)_{\mathcal{U}} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

Let's introduce the adjoint state $p = p(u)$

$$\mathcal{A}^* p(u) = \mathcal{C}^*(\mathcal{C}y\,(u) - y_d),$$

where A^* is the adjoint operator of A, then

$$\begin{aligned}(\mathcal{C}^*(\mathcal{C}y\,(u) - y_d), y(v-u))_{\mathcal{Y}} &= (\mathcal{A}^* p(u), y(v-u))_{\mathcal{Y}} = (p(u), \mathcal{A}y(v-u))_{\mathcal{Y}} \\ &= (p(u), \mathcal{B}(v-u))_{\mathcal{Y}} = (\mathcal{B}^* p(u), (v-u))_{\mathcal{U}},\end{aligned}$$

Hence,

$$\begin{aligned}J'\,(u)\,.\,(v-u) &= (\mathcal{B}^* p(u), (v-u))_{\mathcal{U}} + N(u, v-u)_{\mathcal{U}} \\ &= (\mathcal{B}^* p(u) + Nu, v-u)_{\mathcal{U}} \quad \forall v \in \mathcal{U}_{ad},\end{aligned}$$

Therefore the optimal control u is characterized by the following optimality system :

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^* p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^* p(u) + Nu, v-u)_{\mathcal{U}} \geq 0 \qquad \forall v \in \mathcal{U}_{ad}. \end{cases}$$

the first two equation must be associated with some appropriate boundary and initial condition and the pair $(u, p(u))$ called optimal pair.

Remark 3.1 *if $\mathcal{U}_{ad} = \mathcal{U}$ we have also*

$$J'\,(u)\,.\,(v-u) \leq 0 \quad \forall v \in \mathcal{U},$$

and with the previous condition we get

$$J'\,(u)\,.\,(v-u) = 0 \quad \forall v \in \mathcal{U},$$

therefore the optimality system become as following

$$\begin{cases} \mathcal{A}y(u) = f + \mathcal{B}u, \\ \mathcal{A}^* p(u) = \mathcal{C}^*(\mathcal{C}y(u) - y_d), \\ (\mathcal{B}^* p(u) + Nu, v-u)_{\mathcal{U}} = 0 \qquad \forall v \in \mathcal{U}. \end{cases}$$

3.3.1 *Examples*

3.3.1.1 *1-Optimal control of an elliptic distributed system :*

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, consider the following Laplace equation with Newman boundary condition

$$\begin{cases} -\Delta y + y = f \text{ in } \Omega, \\ \frac{\partial y}{\partial \nu} = v \qquad \text{on } \Gamma, \end{cases} \tag{3.4}$$

where $f \in L^2(\Omega)$ is a source function and $v \in L^2(\Gamma)$ is a control function.

Associate to (3.4) the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(\Omega)} + N \|v\|^2_{L^2(\Gamma)},$$

where $y(v)$ is the solution of (3.4), $y_d \in L^2(\Omega)$ is a fixed observation and $N > 0$.

Our optimal control consists a determine $u \in \mathcal{U}_{ad}$ that minimizing $J(v)$ .That's why we have to search u solution of

$$\inf \left\{ J(v,y) : (v,y) \in \mathcal{U}_{ad} \times H^1(\Omega) \text{ verifies (3.4)} \right\}.$$

In this case :

the state space $\mathcal{Y} = H^1(\Omega)$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(\Gamma)$.

the observation operator is canonical injection

$$\begin{aligned} \mathcal{C} : H^1(\Omega) &\longrightarrow L^2(\Gamma) \\ y &\longrightarrow y, \end{aligned}$$

An optimality condition gives us

$$\begin{aligned} J'(u).(v-u) &\geq 0 \forall v \in \mathcal{U}_{ad} \\ &\iff 2(y(u) - y_d, y(v-u))_{L^2(\Omega)} + 2N(u, v-u)_{L^2(\Gamma)} \geq 0 \forall v \in \mathcal{U}_{ad}. \end{aligned}$$

Now, we introduce an adjoint state $p = p(u)$

$$\begin{cases} -\Delta p + p = y(u) - y_d & \text{in } \Omega, \\ \frac{\partial p}{\partial \eta} = 0 & \text{on } \Gamma. \end{cases}$$

and using second Green formula (see appendix) we get

$$\begin{aligned} (y(u) - y_d, y(v-u))_{L^2(\Omega)} &= (-\Delta p + p, y(v-u))_{L^2(\Omega)} \\ &= (p, -\Delta y(v-u) + y(v-u))_{L^2(\Omega)} \\ &\quad + \int_\Gamma (p \frac{\partial y}{\partial \nu}(v-u) - y(v-u) \frac{\partial p}{\partial \nu}) dx \\ &= \int_\Gamma p(v-u) dx. \end{aligned}$$

Finally, the solution of (3.4) is characterized by the following optimality system:

$$\begin{cases} -\Delta y(u) + y(u) = f & \text{in}\Omega, \\ \frac{\partial y}{\partial \nu}(u) = u & \text{on } \Gamma, \\ -\Delta p + p = y(u) - y_d & \text{in } \Omega, \\ \frac{\partial p}{\partial \nu} = 0 & \text{on}\Gamma, \\ \int_\Gamma (p + Nu)(v - u)dx \geq 0 \ \forall v \in \mathcal{U}_{ad}. \end{cases}$$

2-Optimal control of a parabolic distributed system:

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, $T > 0$, Consider time space cylinder $Q = \Omega \times [0, T]$, $\Sigma = \Gamma \times [0, T]$ and the Heat equation with Dirichlet boundary condition

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega v \text{ in } Q, \\ y = 0 \quad \text{on } \Sigma, \\ y(x, 0) = y_0(x) \quad \text{in } \Omega, \end{cases} \tag{3.5}$$

where $f \in L^2(Q), v \in L^2(0, T, L^2(w))$, χ_ω is the characteristic function of ω a bounded open of Ω and $y_0 \in L^2(\Omega)$.

Our optimal control problem consists in looking for a control function $u \in L^2(0, T, L^2(\omega))$ which minimizes the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(Q)} + N \|v\|^2_{L^2(0,T,L^2(\omega))},$$

with $y(v)$is the solution of (3.5) $y_d \in L^2(Q), N > 0$. So we want to characterize the solution of the following optimal control

$$\inf \left\{ J(v, y) : (v, y) \in \mathcal{U}_{ad} \times L^2(0, T, H_0^1(\Omega)) \text{ verifies } (3.5) \right\}.$$

In this case

the state space $\mathcal{Y} = L^2(0, T, H_0^1(\Omega))$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(0, T, L^2(\omega))$.

the observation operator is canonical injection

$$\begin{aligned} \mathcal{C} : L^2(0, T, H_0^1(\Omega)) &\longrightarrow L^2(\Omega) \\ y &\longrightarrow y, \end{aligned}$$

An optimality condition gives us

$$J'(u).(v-u) = 2(y(u)-y_d, y(v-u))_{L^2(Q)} + 2N(u, v-u)_{L^2(0,T,L^2(\omega))} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

by introducing the adjoint state $p = p(u)$

$$\begin{cases} \frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \text{ in } Q, \\ p = 0 \qquad\qquad \text{on } \Sigma, \\ p(x,T) = 0 \qquad \text{in } \Omega, \end{cases}$$

and using second Green formula (see appendix) we get

$$(y(u) - y_d, y(v-u))_{L^2(Q)} = (p, v-u)_{L^2(0,T,L^2(\omega))}.$$

Finally, the solution of (3.4) is characterized by the following optimality system:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f + \chi_\omega u & \text{in} Q, \\ y = 0 & \text{on } \Gamma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \\ \frac{\partial p}{\partial t} - \Delta p = y(u) - y_d & \text{in} Q, \\ p = 0 & \text{on } \Gamma, \\ p(x,T) = 0 & \text{in } \Omega, \\ \int_0^T \int_\omega (p + Nu)(v-u)dx = 0 \ \forall v \in \mathcal{U}_{ad}. \end{cases}$$

3-Optimal control of a hyperbolic distributed system:

Hyperbolic systems, such as the wave equation or linear elasticity models, involve time-dependent phenomena with finite propagation speeds. Controlling such systems requires addressing challenges like energy conservation, time reversibility, and compatibility with boundary conditions.

Let Ω a bounded domain of $\mathbb{R}^n$ with boundary Γ of class C^2, $T > 0$, Consider time space cylinder $Q = \Omega \times [0,T]$, $\Sigma = \Gamma \times [0,T]$ and the Heat equation with Dirichlet boundary condition.

Consider the controlled wave equation

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = v & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{3.6}$$

where

$v \in L^2(Q)$ is the control function,

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Our optimal control problem consists in looking for a control function $u \in L^2(Q)$ which minimizes the following cost function

$$J(v) = \|y(v) - y_d\|^2_{L^2(Q)} + N \|v\|^2_{L^2(Q))},$$

with $y(v)$is the solution of (3.5) $y_d \in L^2(Q), N > 0$. So we want to characterize the solution of the following optimal control

$$\inf\left\{J(v,y) : (v,y) \in \mathcal{U}_{ad} \times L^2(0,T,H_0^1(\Omega)) \text{ verifies } (3.5)\right\}.$$

In this case

the state space $\mathcal{Y} = L^2(0,T,H_0^1(\Omega))$, the observation space $\mathcal{Z} = L^2(\Omega)$ and the control space $\mathcal{U} = L^2(Q)$.

the observation operator is canonical injection

$$\begin{aligned} \mathcal{C} :\ L^2(0,T,H_0^1(\Omega)) &\longrightarrow L^2(Q) \\ y &\longrightarrow y, \end{aligned}$$

An optimality condition gives us

$$J'(u).(v-u) = 2(y(u)-y_d, y(v-u))_{L^2(Q)} + 2N(u, v-u)_{L^2(Q)} \geq 0 \quad \forall v \in \mathcal{U}_{ad}.$$

by introducing the adjoint state $p = p(u)$

Adjoint PDE: The adjoint state p satisfies a backward wave equation:

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} - \Delta p = y(u) - y_d & \text{in } Q, \\ p = 0 & \text{on } \Sigma, \\ p(x,T) = 0, \frac{\partial p}{\partial t}(x,T) = 0 & \text{in } \Omega, \end{cases}$$

The terminal conditions $p(x,T) = 0, \frac{\partial p}{\partial t}(x,T) = 0$ reflect time reversibility.
and using second Green formula, we get

$$(y(u)-y_d, y(v-u))_{L^2(Q)} = (p, v-u)_{L^2(Q)}.$$

Finally, the solution of (3.4) is characterized by the following optimality system:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = u & \text{in} Q, \\ y = 0 & \text{on } \Gamma, \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \\ \frac{\partial^2 p}{\partial t^2} - \Delta p = y(u) - y_d & \text{in} Q, \\ p = 0 & \text{on } \Gamma, \\ p(x,T) = 0, \frac{\partial p}{\partial t}(x,T) = 0 & \text{in } \Omega, \\ \int_0^T \int_\Omega (p + Nu)(v-u)dx = 0 \ \forall v \in \mathcal{U}. \end{cases}$$

3.4 Third chapter exercises

Exercise 1: (Distributed Control for the Laplace Equation)
Minimize the cost functional:

$$J(u) = \frac{1}{2}\int_{\Omega} |y(u) - y_d|^2 \, dx + \frac{N}{2}\int_{\Omega} |u|^2 \, dx$$

subject to the following Laplace equation:

$$\begin{cases} -\Delta y = u \text{ in } \Omega, \\ y = 0 \quad \text{on } \Gamma \end{cases}$$

where
Ω is a bounded smooth domain in $\mathbb{R}^n$ with a regular boundary Γ,
u is a distributed control in $L^2(\Omega)$.
$y \in H_0^1(\Omega)$ is the state
1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.
2)Derive the adjoint equation and its boundary conditions.
3)Formulate the optimality system (state, adjoint, and control equations).
4)Express the optimal control uu in terms of the adjoint state p.
Exercise 2: (Boundary Control for the Laplace Equation)
Minimize:

$$J(u) = \frac{1}{2}\int_{\Omega} |y(u) - y_d|^2 \, dx + \frac{N}{2}\int_{\Gamma} |u|^2 \, d\Gamma$$

subject to:

$$\begin{cases} -\Delta y = 0 \text{ in } \Omega, \\ \frac{\partial y}{\partial \eta} = u \quad \text{on } \Gamma \end{cases}$$

where
Ω is a bounded smooth domain in $\mathbb{R}^n$ with a regular boundary Γ,
u is a boundary control in $L^2(\Gamma)$.
$y \in H^1(\Omega)$ is the state
$y_d \in L^2(\Omega)$
1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.
2)Derive the adjoint PDE and its boundary conditions.
3)Find the relationship between uu and the adjoint state p.
4)Write the full optimality system.
Exercise 3: (Distributed Control for the Heat Equation)
Minimize:

$$J(u) = \frac{1}{2}\int_0^T \int_{\Omega} |y(u) - y_d|^2 \, dxdt + \frac{N}{2}\int_0^T \int_{\Omega} |u|^2 \, dxdt$$

subject to:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases}$$

where

Ω is a bounded smooth domain in $\mathbb{R}^n$ with a regular boundary Γ, $Q = \Gamma \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $T > 0$. With

u is a distributed control in $L^2(Q)$,

$y \in L^2\left(0,T; H_0^1(\Omega)\right)$ is the state.

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.

2)Derive the adjoint equation (backward heat equation) and its terminal condition.

3)Express the optimal control u in terms of p.

4)State the optimality system.

Exercise 4: (Initial Control for the Heat Equation)

Minimize:

$$J(u) = \frac{1}{2}\int_0^T \int_\Omega |y(x,T) - y_d|^2 \, dxdt + \frac{N}{2}\int_\Omega |u|^2 \, dx$$

subject to:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = u(x) & \text{in } \Omega, \end{cases}$$

where

Ω is a bounded smooth domain in $\mathbb{R}^n$ with a regular boundary Γ, $Q = \Gamma \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $T > 0$.

u is a initial control in $L^2(\Omega)$.

$y \in L^2\left(0,T; H_0^1(\Omega)\right)$ is the state.

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.

2)Derive the adjoint equation and its terminal condition.

3)Find the optimality condition linking u and p.

4)Explain how the control u depends on the adjoint solution p.

Exercise5: (Distributed Control for the Wave Equation)

Minimize:

$$J(u) = \frac{1}{2}\int_0^T \int_\Omega |y - y_d|^2 \, dxdt + \frac{N}{2}\int_0^T \int_\Omega |u|^2 \, dx$$

subject to:

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases}$$

where:

Ω is a bounded smooth domain in $\mathbb{R}^n$ with a regular boundary Γ, $Q = \Gamma \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $T > 0$.

u is a distributed control in $L^2(Q)$.

$(y_0, y_1) \in H_0^1(\Omega) \times L^2(\Omega)$

$y \in C\left(0,T; H_0^1(\Omega)\right) \cap C^1\left(0,T; L^2(\Omega)\right)$ is the state.

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.

2)Derive the adjoint wave equation and its terminal conditions.

3)State the optimality condition for uu.

4)Discuss the time direction of the adjoint equation.

Exercise 6: (Parameter Identification (Linearized))

Let $a(x)$ be known and $u(x)$ a small perturbation both in $L^\infty(\Omega)$. The state satisfies:

$$\begin{cases} -\operatorname{div}((a+u)\nabla y) = f \text{ in } \Omega, \\ y = 0 \quad \text{on } \Gamma \end{cases}$$

Minimize:

$$J(u) = \frac{1}{2}\int_\Omega |y - y_d|^2 \, dxdt + \frac{N}{2}\int_\Omega |u|^2 \, dx$$

1)Linearize the PDE around $u = 0$ to get

$$-\operatorname{div}(a\nabla\delta y) = \operatorname{div}(u\nabla y_0)$$

2)Derive the adjoint equation for the linearized problem.

3)Find the optimality condition for u.

Exercise 7:(Pointwise Observations in Time)

Minimize:

$$J(u) = \frac{1}{2}\sum_{i=0}^{m} \left|y(x_i,T) - y_d^i\right|^2 + \frac{N}{2}\int_0^T \int_\Omega |u|^2 \, dxdt$$

subject to the heat equation:

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = u & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(x,0) = u(x) & \text{in } \Omega, \end{cases}$$

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.
2)Derive the adjoint equation with Dirac delta sources at x_i.
3)Express u in terms of p.
4)Explain the effect of point observations on the adjoint source term.
Exercise 8: (Neumann Boundary Conditions)
Minimize:

$$J(u) = \frac{1}{2}\int_{\Omega} |y - y_d|^2 \, dxdt + \frac{N}{2}\int_{\Omega} |u|^2 \, dx$$

subject to:

$$\begin{cases} -\Delta y + y = u \text{ in } \Omega, \\ \frac{\partial y}{\partial \eta} = 0 \qquad \text{on } \Gamma \end{cases}$$

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.
2)Derive the adjoint equation and its boundary conditions.
3)Find the optimality condition.
4)Solve the system analytically if $\Omega = (0, 1)$.

Exercise 9: (Coupled Elliptic System)
Consider the system:

$$\begin{cases} -\Delta y_1 + y_2 = u \\ -\Delta y_2 + y_1 = u \text{ in } \Omega, \\ y_1 = 0 \\ y_2 = 0 \qquad \text{on } \Gamma \end{cases}$$

Minimize:

$$J(u) = \frac{1}{2}\int_{\Omega} |y_1 - y_d|^2 \, dxdt + \frac{N}{2}\int_{\Omega} |u|^2 \, dx$$

1) Fix the state space, obsevation space, control space and the operator $\mathcal{C}$.
2)Derive the adjoint system (coupled PDEs).
3)State the optimality condition for u.
4)Write the full optimality system.
Exercise 10: (Sparse Control with L1 Regularization)
Revisit Exercise 1 but with:

$$J(u) = \frac{1}{2}\int_{\Omega} |y - y_d|^2 \, dxdt + \frac{N}{2}\int_{\Omega} |u|^2 \, dx$$

1)Explain why the L^1 term complicates the optimality conditions.
2)Compare the adjoint equation to the L^2-regularized case.

3) Research: How is the optimal control characterized? (Hint: Subdifferential calculus.)

Exercise 34 **(Observation in $H^1(\Omega)$)**

Let Ω be a bounded domain of R^n, Γ is its boundary which is of class C^2.

Consider the controlled Laplace equation with Neumann condition be

$$\begin{cases} -\Delta y + y = f & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Gamma, \end{cases} \tag{L}$$

where $f \in L^2(\Omega), v \in L^2(\Gamma)$, associate to (L) the following cost function

$$J(y,v) = \|\nabla y(v) - \nabla y_d\|^2_{(L^2(\Omega))^n} + N\|v\|^2_{L^2(\Gamma)},$$

with $y_d \in H^1(\Omega)$ is a desired state, $N > 0$.

Consider the following optimal control problem.

$$\inf\{J(y,v) : (y,v) \in H^1(\Omega) \times L^2(\Gamma) \text{ and } (y,v) \text{ checks } (L)\}$$

1) Show that (L) admits a weak solution in $H^1(\Omega)$. (Apply the Lax-Milgram theorem).

2) Fix the state space, the observation space, the control space and the operator C.

3) Consider the variational equation:

$$\begin{cases} \text{Find } p \in H^1(\Omega) \text{ s.t.} \\ \int\limits_\Omega (\nabla p.\nabla \phi + p\phi)\, dx = \int\limits_\Omega (\nabla y(v) - \nabla y_d)\,.\nabla \phi dx \text{ for all } \phi \in H^1(\Omega)\ . \end{cases} \tag{WF}$$

Show that (WF) admits a unique weak solution $p \in H^1(\Omega)$.

4) Show that the optimal pair $(u, y(u))$ is characterized by the following optimality system:

$$\begin{cases} -\Delta y\,(u) + y\,(u) = f, & \\ -\Delta p + p = -\operatorname{div}\left((\nabla y(v) - \nabla y_d)\right) & \text{in } \Omega, \\ \frac{\partial y(u)}{\partial \nu} = u, & \\ \frac{\partial p}{\partial \nu} = (\nabla y(v) - \nabla y_d)\,.\nu & \\ p + Nu = 0 \ \text{ a.e. on } \Gamma & \text{on } \Gamma, \\ (y\,(u)\,, p) \in H^1\,(\Omega) \times H^1\,(\Omega) & \end{cases}$$

(Remember that $\int_\Omega \nabla\varphi.F dx = \int_{\partial\Omega} \varphi F.\eta d\sigma - \int_\Omega \varphi div F dx$ for all $\varphi \in H^1(\Omega)$, $F \in (H^1(\Omega))^n$)

Exercise 35 **(Newmann boundary control)**

Let Ω be a bounded domain of R^n, Γ is its boundary which is of class C^2,$Q = \Omega\times]0,T[$ and $\Sigma = \Gamma\times]0,T[$.

Consider the controlled heat equation on the boundary

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f & \text{in } Q, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Sigma, \\ y(x,0) = y_0(x) & \text{in } \Omega, \end{cases} \tag{H}$$

with $f \in L^2(Q), v \in L^2(\Sigma)$ and $y_0 \in L^2(\Omega)$.
(H) has a weak solution in $L^2(0,T;H^1(\Omega))$.
Associate to (H) the cost function :

$$J(y,v) = \|y - y_d\|^2_{L^2(Q)} + \|y(T) - y_d(T)\|^2_{L^2(\Omega)} + N\|v\|^2_{L^2(\Sigma)}$$

with $y_d \in C([0,T], L^2(\Omega))$ is the desired state, $N > 0$. Consider the following optimal control problem.

$$\inf\{J(y,v) : (y,v) \in L^2(0,T;H^1(\Omega)) \times L^2(\Sigma) \text{ and } (y,v) \text{ checks } (H)\}.$$

1) Fix the state space Y, the observation space Z, the control space U and the operator C.
2) Show that the optimal pair $(u, y(u))$ is characterized by

$$\begin{cases} \frac{\partial y}{\partial t} - \Delta y = f, \\ -\frac{\partial p}{\partial t} - \Delta p = y(u) - y_d \\ \frac{\partial y}{\partial \nu} = u, \\ \frac{\partial p}{\partial \nu} = 0 \\ y(x,0) = y_0(x), \\ p(x,T) = y(u)(T) - y_d(T) \\ u = -\frac{1}{N} \text{ a.e. on } \Sigma. \end{cases} \begin{array}{l} \\ \text{in } Q, \\ \\ \text{on } \Gamma, \\ \\ \text{in } \Omega, \\ \end{array}$$

Exercise 36 **(Newmann boundary control)**
Let Ω be a bounded domain of R^n, Γ is its boundary which is of class C^2,$Q = \Omega\times]0,T[$ and $\Sigma = \Gamma\times]0,T[$.
Consider the controlled wave equation on the boundary

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = f & \text{in } Q, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Sigma, \\ y(x,0) = y_0(x), \frac{\partial y}{\partial t}(x,0) = y_1(x) & \text{in } \Omega, \end{cases} \tag{W}$$

where $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$, $f \in L^2(Q)$ and $v \in U_{ad}$, where U_{ad} is a convex closed nonempty subset of $L^2(\Sigma)$.
(W) has a weak solution in $C(0,T;H^1(\Omega) \cap C^1(0,T;L^2(\Omega))$.
Associate with (W) the cost function :

$$J(y,v) = \|y - y_d\|^2_{L^2(Q)} + \|y(T) - y_d(T)\|^2_{L^2(\Omega)} + N\|v\|^2_{L^2(\Sigma)}$$

with $y_d \in C([0,T], L^2(\Omega))$.

Consider the following optimal control problem :

$$\inf\{J(y,v) : (y,v) \in L^2(0,T;H^1(\Omega)) \times L^2(\Sigma) \text{ and } (y,v) \text{ checks } (W)\}.$$

1) Fix the state space Y, the observation space Z, the control space U and the operator C.

2) Show that the optimal pair $(u, y(u))$ is characterized by :

$$\begin{cases} \frac{\partial^2 y}{\partial t^2} - \Delta y = f; & \\ \frac{\partial^2 p}{\partial t^2} - \Delta p = y(u) - y_d & \text{in } Q, \\ \frac{\partial y}{\partial \nu} = u, & \\ \frac{\partial p}{\partial \nu} = 0 & \text{on } \Gamma, \\ y(u)(x,0) = y_0(x), & \\ \frac{\partial y(u)}{\partial t}(x,0) = y_1(x); & \\ p(x,T) = 0, & \\ \frac{\partial p}{\partial t}(x,T) = y(u)(T) - y_d(T) & \text{in } \Omega, \\ (p + Nu, v - u)_{L^2(0,T;L^2(\Sigma))} \geq 0 \ \forall v \in \mathcal{U}_{ad} & \end{cases}$$

Exercise 37 (**Boundary observation and distributed control, Dirichlet conditions**)

Let Ω be a bounded domain of R^n, Γ is its boundary which is of class C^2.

Consider the following control problem:

$$\begin{cases} -\Delta y + c.\nabla y = f + v \text{ in } \Omega, \\ y = 0 \qquad\qquad \text{on } \Gamma, \end{cases} \tag{T}$$

where c a constant vector and $f \in L^2(\Omega)$.

Associate with (T) the cost function

$$J(y,v) = \int_\Gamma \left| \frac{\partial y}{\partial \nu} - y_d \right|^2 d\Gamma + N \int_\Omega |v|^2 \, dx$$

with

$$\inf\{J(y,v) : (y,v) \in H_0^1(\Omega)) \times L^2(\Omega) \text{ and } (y,v) \text{ checks } (T)\}.$$

1) Show that, by elliptic regularity, $J(y,v)$ is well defined and that there exists a unique minimizer.

2) Fix the state space Y, the observation space Z, the control space U and the operator C.

3) Show that the optimal pair $(u, y(u))$ is characterized by the following opti-

mality system

$$\begin{cases} -\triangle y + c.\nabla y = f + v; \\ -\Delta p - c.\nabla p - p \operatorname{div} c = 0 & \text{in } Q, \\ y = 0, \\ p = \frac{\partial y}{\partial \nu} - y_d & \text{on } \Gamma, \\ \int_0^T \int_\Omega (p + Nu, v - u)_{L^2(\Omega)} \geq 0 \ \forall v \in \mathcal{U}_{ad} & \text{in } \Omega, \end{cases}$$

4) Write the optimality conditions: adjoint problem and Euler equations.

Exercise 38 **(Boundary control)**

Let Ω be a bounded domain in $\mathbb{R}^n$, with a Lipschitz boundary Γ. Let Γ_1 be a closed subset in Γ, and set $\Gamma_2 = \Gamma \backslash \Gamma_1$.

We consider the following elliptic equation

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = v & \text{on } \Gamma_1, \\ y = 0 \text{ on } \Gamma_2, \end{cases} \tag{LB}$$

The function $f \in L^2(\Omega)$ is a given source term and the function $v \in L^2(\Gamma_1)$ is a control variable. Denote by $H^1_{\Gamma_2}(\Omega)$ the space

$$H^1_{\Gamma_2}(\Omega) = \left\{ y \in H^1(\Omega) \text{ s.t. } y = 0 \text{ on } \Gamma_2 \right\}.$$

1) Prove that equation (LB) admits a unique weak solution in $H^1_{\Gamma_2}(\Omega)$ (give the precise definition of a weak solution).

We want to study the control problem

$$\inf\{J(y,v) : (y,v) \in H^1_{\Gamma_2}(\Omega) \times L^2(\Gamma_1) \text{ and } (y,v) \text{ checks } (LB)\}.$$

with

$$J(y,v) = \int_\Omega |y - y_d|^2 \, d\Gamma + N \int_{\Gamma_1} |v|^2 \, dx$$

the function y_d belongs to $L^2(\Omega)$.

2) Let (v_n) be a sequence in $L^2(\Gamma_1)$ converging to u for the weak topology of $L^2(\Gamma_1)$.

Let $y(v_n)$ be the solution to equation (LB) corresponding to v_n.

What can we say about the sequence $y(v_n)$?

Prove that problem (P6) admits a unique weak solution.

Characterize the optimal control by writing first order optimality conditions.

Exercise 39 **(Distributed control)**

Let Ω be a bounded domain in $\mathbb{R}^n$, with a Lipschitz boundary ■.

Consider the elliptic equation

$$\begin{cases} -\Delta y + y = f + \chi_\omega v & \text{in } \Omega, \\ \frac{\partial y}{\partial \nu} = 0 & \text{on } \Gamma. \end{cases} \tag{LN}$$

The function f belongs to $f \in L^2(\Omega)$, the control $v \in L^2(\omega)$, and ω is an open subset in Ω.

We want to study the optimal control problem

$$\inf\{J(y,v) : (y,v) \in H^1(\Omega) \times L^2(\omega) \text{ and } (y,v) \text{ checks } (LN)\}.$$

with

$$J(y,v) = \int_\Gamma |y - y_d|^2 \, d\Gamma + N \int_\omega |v|^2 \, dx$$

the function y_d belongs to $H^1(\Omega)$.

1)Prove that problem (P7) admits a unique weak solution.

2)Characterize the optimal control by writing first order optimality conditions.

Exercise 40 Let Ω be a bounded domain of R^n, Γ is its boundary which is of class C^2,$Q = \Omega \times (0,T)$ and $\Sigma = \Gamma \times (0,T)$. We consider linear thermoelastic coupled systems

$$\begin{cases} y_{tt} - \alpha^2 \Delta y + \gamma_1 \nabla \theta = 0 & \\ \theta_t - k\Delta\theta + \gamma_2 \operatorname{div} y_t = 0 & \text{in } Q, \\ y = 0 & \\ \frac{\partial \theta}{\partial \nu} = v & \text{on } \Sigma, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \qquad ((\text{TE}))$$

with $\alpha > 0$, $k > 0$, $\gamma_1 > 0$ and $\gamma_2 > 0$, the last are the coupling parameters. Also, $y_0 \in H_0^1(\Omega), y_1 \in L^2(\Omega)$, $\theta_0 \in L^2(\Omega)$ and $v \in L^2(\Sigma)$. The system (TE) has a unique solution $(y,\theta) \in C(0,T; H_0^1(\Omega) \times L^2(\Omega))$

Consider the following optimal control problem

$$\inf\{J(y,\theta,v) : (y,\theta,v) \in C(0,T; H_0^1(\Omega) \times L^2(\Omega)) \times L^2(\Sigma) \text{ and } (y,v) \text{ checks } (TE)\},$$

with

$$J(y,\theta,v) = \int_Q \left(|y - y_d|^2 + |\theta - \theta_d|^2 \right) dxdt + N \int_\Sigma |v|^2 \, d\Gamma dt,$$

where $N > 0$.

Characterize this solution by establishing first order optimality conditions.

$$\begin{cases} y_{tt} - \alpha^2 \Delta y + \gamma_1 \nabla \theta = 0 & \\ \theta_t - k\Delta\theta + \gamma_2 \operatorname{div} y_t = 0 & \text{in } Q, \\ y = 0 & \\ \frac{\partial \theta}{\partial \nu} = u & \text{on } \Sigma, \\ y(x,0) = y_0(x), y_t(x,0) = y_1(x) & \\ \theta(x,0) = \theta_0(x) & \text{in } \Omega, \end{cases} \qquad ((\text{TE}))$$

$$\begin{cases} \varphi_{tt} - \alpha^2 \Delta \varphi + \gamma_2 \nabla \psi_t = y(u) - y_d \\ -\psi_t - k\Delta\psi - \gamma_1 \operatorname{div} \varphi_t = \theta(u) - \theta_d \text{ in } Q, \\ \varphi = 0 \\ \frac{\partial \psi}{\partial \nu} = 0 \qquad \text{on } \Sigma, \\ \varphi(x,T) = 0, \varphi_t(x,T) = 0 \\ \psi(x,T) = 0 \qquad \text{in } \Omega, \end{cases}$$

with the optimality condition

$$(p + Nu, v - u)_{L^2(\Sigma)} \geq 0 \ \forall v \in \mathcal{U}_{ad}.$$

Appendices

Definition 3.1 (**Exponential of a matrix**)

The series of matrices of general term $\frac{A^k}{k!}$ converges normally on any bounded part of $\mathcal{M}_n(\mathbb{R})$.

We call then exponential the application

$$exp \ : \ \mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$$
$$A \to \exp A = \sum_{k=o}^{\infty} \frac{A^k}{k!}.$$

Theorem 3.2 ***(Cayley-Hamilton)***

Any square matrix A satisfies the equation of its characteristic polynomial, i.e if $\chi_A(\lambda) = \det(\lambda I - A)$ *then*

$$\chi_A(A) = 0.$$

Definition 3.2 (**Rank of a matrix**)

The rank of a matrix A is the dimension of the vector space generated (or spanned) by its columns. This corresponds to the maximal number of linearly independent columns of A.

Definition 3.3 **Transition matrix**

the resolvant is the solution of

$$\begin{cases} \Phi'(t, t_0) = A(t)\,\Phi(t, t_0), \ t > 0, \\ \Phi(t_0, t_0) = I, \end{cases}$$

with $\Phi(t, t_0) \in \mathcal{M}_n(\mathbb{R})$

has the property

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\,\Phi(t_1, t_0).$$

If $A(t) = A$ is time-invariant $\Phi(t, s) = \sum_{k=o}^{\infty} \frac{A^k}{k!}(t-s)^k$

Then, the solution of

$$\begin{cases} z'(t) = A(t)\,z(t) + B(t)\,u(t), \ t > 0, \\ z(t_0) = z_0, \end{cases}$$

is given by

$$z(t) = \Phi(t, t_0)\,z_0 + \int_{t_0}^{t} \Phi(t, s)\,Bu(s)\,ds.$$

When $t_0 = 0$, $\Phi(t, 0) = M(t)$ and

$$z(t) = M(t)\,z_0 + M(t) \int_0^t M^{-1}(s)\,Bu(s)\,ds.$$

Theorem 3.3 ***(Exponontial bound of an exponentional matrix)***

Let $A \in \mathcal{M}_n(\mathbb{R})$ with distinct eigenvalues distinct eigenvalues $\lambda_1, ..., \lambda_k$ $(k \leq n)$ of

algebraic multiplicities $m_1, ..., m_k$, respectively, so that $m_1 + ... + m_k = n$.

If ω is such that

$$\omega > \max\{\operatorname{Re}\lambda_i, i = 1, ..., k\},$$

then there exists a constant $M \geq 1$ such that

$$\left\|e^{tA}\right\| \leq Me^{\omega t}.$$

Definition 3.4 (Positive definite matrix, positive semidefinite matrix, negative matrix and negative semidefinite matrix)

Let $A \in \mathcal{M}_n(\mathbb{R})$ be a symmetric matrix and $x \in \mathbb{R}^n$.

A is positive definite if $x^T Ax > 0$ for all $x \neq 0$

A is positive semidefinite if $x^T Ax \geq 0$ for all x.

A is negative definite if $-A$ is positive definite.

A is negative semidefinite if $-A$ is positive semidefinite.

Definition 3.5 (**Banach space**)

A normed vector space or normed space E, is a vector space over the real or complex numbers, on which a norm is defined. A norm is a real-valued function defined on the vector space that is commonly denoted $\|.\|$, and has the following properties:

i) It is nonnegative, meaning that $\|x\| \geq 0$ for every $x \in E$.

ii) It is positive on nonzero vectors, that is,

$\|x\| = 0$ implies $x = 0$. for very vector $x \in E$,

iii) For every scalar λ , $\|\lambda x\| = |\lambda|\|x\|$.

iv) The triangle inequality holds; that is, for every vectors x and y,

$$\|x + y\| \leq \|x\| + \|y\|.$$

.

Definition 3.6 (**Hilbert space**)

A Hilbert space H is a real or complex inner product space that is also a complete metric space with respect to the distance function induced by the inner product.

To say that H is a complex inner product space means that H is a complex vector space on which there is an inner product (x, y) associating a complex number to each pair of elements x, y of H that satisfies the following properties:

i) The inner product is conjugate symmetric; that is, the inner product of a pair of elements is equal to the complex conjugate of the inner product of the swapped elements

$$(y, x) = \overline{(x, y)}.$$

Importantly, this implies that (x, x) is a real number.

The inner product is linear in its first argument.

For all complex numbers a and b

$$(ax_1 + bx_2, y) = a\,(x_1, y) + b\,(x_2, y)$$

ii)The inner product of an element with itself is positive definite i.e.

$(x, x) > 0$ if $x \neq 0$, $(x, x) = 1$ iff $x = 0$.

It follows from properties *i*) and *ii*) that a complex inner product is antilinear, also called conjugate linear, in its second argument, meaning that

$$(x, ay_1 + by_2) = \overline{a}\,(x, y_1) + \overline{b}\,(x, y_2)$$

A real inner product space is defined in the same way, except that H is a real vector space and the inner product takes real values. Such an inner product will be a bilinear map.

The norm induced by the inner product is

$$\|x\| = \sqrt{(x, x)}.$$

Definition 3.7 (C_0-**Semigroup**)

A strongly continuous semigroup on a Banach space E or C_0-semigroup $\{S\,(t)\}_{t\geq 0}$, is a map

$$S\,(t) : \mathbb{R}_+ \to L(E)$$

such that

i) $S\,(0) = I$ (identity operator on E)

ii) For all $t, s \geq 0 :\ S(t + s) = S\ (t)\ S\ (s)$

iii) For all $x \in E :\ \|S(t)x - x\| \to 0$ when $t \to 0$.

Definition 3.8 (**The infinitesimal generator**)

The infinitesimal generator A of a strongly continuous semigroup $\{S\,(t)\}_{t\geq 0}$ is defined by

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t}$$

whenever the limit exists.

The domain of A, $D(A)$, is the set of $x \in E$ for which this limit does exist; $D(A)$ is a linear subspace and A is linear on this domain.

The operator A is closed, although not necessarily bounded, and the domain is dense in E.

Definition 3.9 (L^p **spaces**)

The set of L^p-functions (where $p \geq 1$) generalizes L^2-space. Instead of square integrable, the measurable function f must be p-integrable for f to be in L^p.

On a measure space X, the L^p norm of a function f is

$$\|f\|_{L^p(X)} = \left(\int_X |f(x)|^p \, dx \right)^{\frac{1}{p}}.$$

The L^p-functions are the functions for which this integral converges.

For $p = 2$, the space of L^p-functions is a Banach space which is not a Hilbert space.

For $p > 1$, the dual vector space to L^p is given by integrating against functions in L^q, where $\frac{1}{p} + \frac{1}{q} = 1$. This makes sense because of Hölder's inequality for integrals. In particular, the only L^p-space which is self-dual is L^2.

Definition 3.10 (**Sobolev spaces**)

Let $k \in \mathbb{N}$, $1 \leq p \leq \infty$. The Sobolev space $W^{k,p}(\Omega)$ is defined to be the set of all functions f on Ω such that for every multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ with $|\alpha| \leq k$ the mixed partial derivative

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial^{\alpha_1} x_1 ... \partial^{\alpha_n} x_n}$$

exists in the weak sense and is in $L^p(\Omega)$, i.e.

$$\|D^\alpha f\|_{L^p(\Omega)} < \infty.$$

That is, the Sobolev space $W^{k,p}(\Omega)$ is defined as

$$W^{k,p}(\Omega) = \{u \in L^p(\Omega) \text{ such that } D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leq k\}.$$

The natural number k is called the order of the Sobolev space $W^{k,p}(\Omega)$.

There are several choices for a norm for $W^{k,p}(\Omega)$. The following two are common and are equivalent in the sense of equivalence of norms:

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha| \leq k} \|D^\alpha u\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}} & 1 \leq p < \infty; \\ \max_{|\alpha| \leq k} \|D^\alpha u\|_{L^\infty(\Omega)} & p = \infty. \end{cases}$$

Theorem 3.4 ***(Hill-Yoshida theorem)***

Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space E, ω a real number, and $M > 0$. Then A generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$ that satisfies

$$\|S(t)\| \leq M e^{\omega t}$$

if and only if

i) A is closed and $D(A)$ is dense in E,

ii) every real $\lambda > \omega$ belongs to the resolvent set of A and for such λ and for all positive integers n,

$$\left\|(\lambda I - A)^{-n}\right\| \leq \frac{M}{(\lambda - \omega)^n}.$$

Theorem 3.5 ***(Lumer–Phillips theorem)***

Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space E. Then A generates a contraction semigroup if and only if

i)$D(A)$ is dense in E,

ii)A is dissipative, and

iii)$A - \lambda I$ is surjective for some $\lambda > 0$, where I denotes the identity operator.

An operator satisfying the last two conditions is called maximally dissipative.

Definition 3.11 (**Directional derivative 1**)

Let $J : U \subset X \to Y$ be an operator, X, Y be two Banach spaces and $U \neq \emptyset$ be an open. J is said to be directionally differentiable $x \in U$ if the limit

$$dJ(x,h) = \lim_{t \to 0^+} \frac{J(x+th) - J(x)}{t} \in Y$$

exists for all $h \in X$.

J is called Gateaux-differentiable cake in $x \in U$ if J is directionally differentiable in x and its directional derivative is

$$J'(x) : h \in X \to dJ(x,h) \in Y$$

is linear and bounded,i.e., $J(x) \in \mathcal{L}(X,Y)$.

Definition 3.12 (**Directional derivative 2)**

Let X, Y be two Banach spaces and $J : U \subset X \to Y$ s.t. U a nonempty open. J is directionally differentiable at $x \in U$ if the limit

$$dJ(x,h) = \lim_{t \to 0^+} \frac{J(x+th) - J(x)}{t} \in Y,$$

Theorem 3.6 ***(First order optimality condition)***

Let E be a Banach space and $U \subset E$ a nonempty convex. Moreover, let $J : U \to \mathbb{R}$ defined on an open neighborhood of U. Let u be a local solution of

$$\inf_{v \in U} J(v),$$

where J is cake-derivable. Then, the following optimality condition is verified

$$\langle J'(u), v - u \rangle_{E',E} \geq 0 \quad \forall v \in U.$$

If J is convex on U this condition is necessary and sufficient for global optimality.

Theorem 3.7 *(Brezis, 2010, p.47)*

If E, F two Banach spaces $A:\mathcal{D}(A)\subset E\to F$ an unbounded operator of dense closed domain.

Then we have the equivalence of:

i) A is surjective.

ii) $\exists\gamma>0:\|v\|\leq\gamma\|A^*v\|_E$, $\forall v\in\mathcal{D}(A^*)$.

iii) $\ker(A^*)=\{0\}$ and $\operatorname{Im}(A^*)$ is closed.

Proposition 3.1 *(Infinite Dimensional Linear Systems Theory)*

Let X,Y and Z be three Banach spaces and $F\in\mathcal{L}(X,Y)$ and $G\in\mathcal{L}(Y,Z)$ then

$$\operatorname{Im}F\subset\operatorname{Im}G\Leftrightarrow\exists\gamma>0,\forall z\in Z:\|G^*z\|_{X^*}\geq\gamma\|F^*z\|_{Y^*}.$$

Proposition 3.2 *(Zabczyk, J., 2009;p 211)*

If $F\in\mathcal{L}(X,Y)$ and $G\in\mathcal{L}(Y,Z)$, where X,Y and Z are separable Hilbert spaces s.t. $\|F^*f\|=\|G^*f\|$ for any $f\in Z^*$, then $\operatorname{Im}F\subset\operatorname{Im}G$ and $\left\|F^{-1}f\right\|=\left\|G^{-1}f\right\|$ for any $f\in\operatorname{Im}F$.

Lemma 3.1 *Let H_1, H_2 be two Hilbert spaces $F:H_1\to H_2$ be a bounded linear operator. then*

$$\overline{\operatorname{Im}F}=(\ker F^*)^{\perp}$$

Theorem 3.8 ***(Green formula)***

Let $\Omega\subset\mathbb{R}^n$ be a bounded and regular domain, and ν the unit normal vector outward on $\Gamma=\partial\Omega$. Then we have

For $u\in H^1(\Omega)$ and $v\in H^2(\Omega)$ we have the first Green formula

$$\int_\Omega u\Delta v dx=-\int_\Omega\nabla u\nabla v dx+\int_\Gamma u\frac{\partial v}{\partial\nu}d\Gamma.$$

For $u,v\in H^2(\Omega)$ we have the second Green formula

$$\int_\Omega(u\Delta v-v\Delta u)\,dx=\int_\Gamma\left(u\frac{\partial v}{\partial\nu}-v\frac{\partial u}{\partial\nu}\right)d\Gamma.$$

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