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قســـم الهندســـة الكهربائيـــة

Echahid Cheikh Larbi Tebessi University
Faculty of Sciences and Technology
Department of Electrical Engineering

Course

In accordance with LMD programs Academic

3rd Bachelor's Degree / Automatics

Control of Linear Systems

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January 2025

Semester: 5
Teaching Unit: UEF 3.1.1
Subject: Control of Linear Systems
Total Hours: 45h00 (Lecture: 1h30, DW: 1h30)
Credits: 4
Coefficient: 2

Objectifs :

This module consolidates the knowledge acquired in the second year and equips students with the ability to represent dynamic systems and analyze their properties in the state-space framework. It also introduces key methods for the analysis and synthesis of control systems.

Recommended Prerequisites: Basic mathematics , Continuous and sampled linear systems.

Course Content:

Chapter 1: Controller Design in the Frequency Domain (4 Weeks)
Frequency responses and properties of controllers (P, PI, PID, PD, phase advance, phase lag), Frequency domain specifications (gain margin, phase margin, resonance factor, bandwidth, and interpretations), Controller design using Bode diagrams, Tuning methods with Black-Nichols charts.

Chapter 2: State Representation of Systems (2 Weeks)
Introduction and key concepts (state, state variables, etc.), State-space representation of continuous linear systems, State-space representation of discrete systems, Canonical forms State-space representation of nonlinear systems and linearization.

Chapter 3: Analysis of systems in State Space (3 Weeks)
Solving state equations and transition matrices, Methods for computing transition matrices Modal analysis (diagonalization), Stability analysis, Controllability and observability (definitions and testing methods)

Chapter 4: State Feedback Control (3 Weeks)
Problem formulation for pole placement via state feedback, Calculation methods for single-variable systems, Multi-variable system cases, Implementation.

Chapter 5: State Observer Design (3 Weeks)
Introduction, Deterministic observers (Luenberger) and calculation methods. Reduced-order observers, Stochastic observers (Kalman filter).

Evaluation Method: Continuous Assessment: 40% ; Final Exam: 60%

Bibliographic References:

1. Philippe de Larminat, « Automatique : Commande des systèmes linéaires », Hermès Lavoisier, 1996.

2. Hubert Egon, « Asservissement linéaires échantillonnés et représentation d'état », Méthodes, 2001.
3. Luc Jaulin, « Représentation d'état pour la modélisation et la commande des systèmes », Lavoisier, 2005.
4. Robert L. Williams, Douglas A, «Lawrence, Linear State-Space Control Systems », Edition John Wiley & Sons, 2007.
5. R. Longchamp, « Commande numérique de systèmes dynamiques », Presses Polytechniques et Universitaires Romandes, 1995.
6. G. F. Franklin, J. D. Powell, L. M. Workman, «Digital control of dynamic systems», Addison-Wesley Series in Electrical and Computer Engineering: Control Engineering, 1990.
7. K. J. Aström, B. Wittenmark, « Computer controlled systems: theory and design», Prentice-Hall, 1984.
8. R. H. Middleton, G. C. Goodwin, « Digital control and estimation: a unified approach», Prentice Hall, 1990.

Preface

This document is a comprehensive course companion for undergraduate students in **Control of Linear Systems**. It is specifically designed for a 45-hour course at Echahid Cheikh Larbi Tebessi University in Tebessa, Algeria. The primary objective is to build upon the knowledge acquired in the second year and to equip students with the skills necessary to represent dynamic systems and analyse their properties within the state-space framework. The course also introduces fundamental methods for the analysis and synthesis of control systems.

To fully benefit from this course, students are expected to have a strong foundation in basic engineering principles, including mechanics, electronics, and industrial process knowledge. Building on these fundamentals, the course explores core topics such as the principles of linear systems, controller design strategies, stability analysis, and performance evaluation in both the frequency and time domains.

The course is organized into five chapters, structured as follows: The first chapter focuses on the synthesis of controllers in the frequency domain, while the second chapter delves into the representation of systems using state-space models. The third chapter provides a detailed analysis of systems in the state-space framework. The fourth chapter explores state feedback control, and the final chapter covers the synthesis of state observers.

By the end of the course, students will have acquired the theoretical knowledge and practical skills required to analyse, design, and implement effective control strategies for linear systems. This prepares them to tackle the challenges of modern engineering in this ever-evolving field.

Abdelhamid DJARI

LECTURE NOTES

Control of Linear Systems

Carried out by:
Dr. Abdelhami Djari

Echahid Cheikh Larbi Tébessi University
December 2024

This lecture notes is intended for third-year undergraduate students in Automation.

Abstract

This lecture notes offers a detailed exploration of control systems, focusing on both frequency-domain methods and state-space representations. Designed to equip students and engineers with the knowledge and tools needed for system analysis and design, the material covers key topics essential for mastering control systems.

The frequency-domain section introduces controller design techniques using Bode plots, Nyquist plots, and root locus methods to ensure stability and optimize performance. State-space representation is presented as a versatile framework, particularly suited for analyzing and modeling complex MIMO systems.

System analysis in state-space is explored through concepts such as controllability, observability, stability, and response, providing a deeper understanding of dynamic behavior. State feedback control techniques are discussed to enable precise pole placement for improved stability and performance. Finally, the design of state observers, including Luenberger observers and Kalman filters, is examined to facilitate accurate state estimation in scenarios with unmeasured variables.

This lecture notes bridges theoretical principles with practical applications, offering a comprehensive guide for analyzing, controlling, and optimizing dynamic systems in fields like automation, robotics, and control engineering.

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Chapter 1

Controller design in the frequency domain

1.1 Introduction

This section focuses on the frequency responses and characteristics of various types of controllers, including Proportional (P), Proportional-Integral (PI), Proportional-Derivative (PD), Proportional-Integral-Derivative (PID), as well as phase lead and phase lag controllers. Key specifications in the frequency domain, such as gain and phase margins, resonance factor, and bandwidth, will be examined alongside their interpretations. Additionally, the process of designing controllers using Bode plots will be discussed, highlighting the practical application of these techniques. Finally, tuning methods will be explored using the Black-Nichols chart, providing a systematic approach to optimizing controller performance in dynamic systems.

1.2 Overview of Linear System Stability

This section provides a refresher on the stability of linear systems, focusing on the criteria and principles that determine whether a system will remain stable under various conditions. It covers fundamental concepts, such as the analysis of system poles, and explores methods like the Routh-Hurwitz criterion and root locus techniques, which are commonly used to assess system stability. Understanding these principles is crucial for ensuring that a system performs predictably without exhibiting undesired oscillations or divergence over time.

1.2.1 Stability of a Linear System

Most physical systems can be described as operators that map inputs R to corresponding outputs S (Figure 1.1).

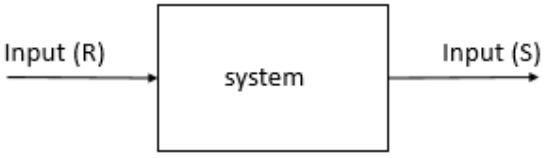


FIGURE 1.1: General model of a system

- **Definition 1:** A linear time-invariant (LTI) system is a system whose behavior and response to inputs are both linear and consistent over time. This means that the system's properties do not change with time, and the output is directly proportional to the input. Additionally, the system adheres to the principles of superposition (additivity) and homogeneity (scaling). Its time-invariance ensures that its behavior is the same regardless of when an input is applied, meaning the system's response to a given input will not vary over time.

- **Definition 2:** A linear system is considered stable if its output remains bounded for all bounded inputs. Otherwise, it is classified as unstable.

Every LTI system is governed by one or more linear differential equations with constant coefficients.

- Exemple 1: Let's consider a mass-spring-damper dynamic system depicted in the following figure:

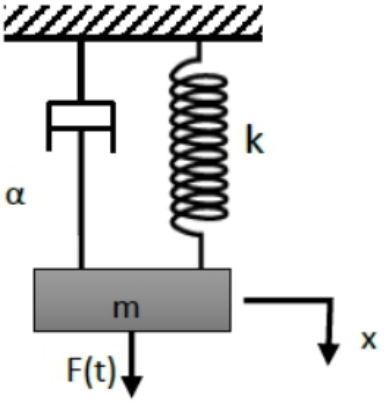


FIGURE 1.2: mass-spring-damper system

According to the general law of dynamics, this system is governed by the following dynamic equation:

$$\sum(F_i) = m.\frac{\partial^2}{\partial t^2}x(t) \tag{1.1}$$

This equation becomes:

$$m\ddot{x}(t) + \alpha\dot{x}(t) + kx(t) = F(t) \tag{1.2}$$

Thus, this system is governed by the equation (Equation 1.2), which is a linear differential equation with constant coefficients.

1.2.2 The Laplace transform

To simplify the analysis of dynamic systems, various transformation techniques, such as the Laplace transform, are included. Considering a real signal $x(t)$ where $x(t) = 0$ for $t < 0$, the Laplace transform $L(x(t))$ is defined as the function X of the complex variable s such that:

$$X(s) = L(x(t)) = \int_0^{+\infty} x(t)e^{-st}dt \tag{1.3}$$

Applying this transformation to the mass-spring-damper system described above yields:

$$ms^2X(s) + \alpha sX(s) + kX(s) = U(s) \tag{1.4}$$

with U(s)=L(F(t)) This implies that:

$$\frac{X(s)}{U(s)} = \frac{1}{ms^2 + \alpha s + k} \tag{1.5}$$

The ratio $G(s) = \frac{X(s)}{U(s)}$ is defined as the transfer function (TF) of the system, where $U(s)$ represents the input and $X(s)$ represents the output. A system is stable if and only if all the poles of its transfer function (the roots of the denominator) have strictly negative real parts. In the previous example, when $m = 1$, $\alpha = 0.5$, and $k = 5/16$, the system has two complex poles with negative real parts ($s_1 = -0.25 - 0.5j$ and $s_2 = -0.25 + 0.5j$), which indicates that the system is stable. When calculating the poles of the transfer function (TF) is challenging, various criteria are employed to simplify stability analysis, such as the Routh, Nyquist, and Hurwitz criteria.

1.2.3 Routh Criterion

The Routh criterion is a method used to determine the stability of a system by analyzing the characteristic equation without having to calculate the roots. It provides

a_n	a_{n-2}	a_{n-4}	...	a_1
a_{n-1}	a_{n-3}	a_{n-5}	...	a_0
$b_m = \frac{a_{n-1}a_{n-2}-a_n a_{n-3}}{a_{n-1}}$	$b_{m-1} = \frac{a_{n-1}a_{n-4}-a_n a_{n-5}}{a_{n-1}}$	...	$b_0 = \frac{a_{n-1}a_1-a_n a_0}{a_{n-1}}$	0
$\frac{b_m a_{n-3}-a_{n-1}b_m-1}{b_m}$	$\frac{b_m a_{n-5}-a_{n-1}b_{m-2}}{b_m}$	...	0	0

TABLE 1.1: Routh Table

a systematic way to assess whether all the roots of the polynomial have negative real parts, ensuring system stability. The Routh algebraic criterion does not provide a concept of safety margin, but it allows for the diagnosis of stability in higher-order systems. Let $H(s)$ represent the closed-loop transfer function, and let $D(s)$ be the denominator of $H(s)$. $D(s)$ is a polynomial of degree n.

$$D(s) = a_n s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \tag{1.6}$$

The Routh criterion is applied by arranging the sequence of coefficients a_i in a table, across two rows, in descending order of n, alternating between the two rows. A calculation is then performed to generate an additional row, following the algorithm illustrated in the diagram below. We then have a table with three rows, where the third row contains fewer terms than the previous ones. The remaining positions in the third row are filled with zeros on the right.

The same calculation is repeated on the last two rows to create a fourth row.

The process is iterated until only zeros remain in the row. The number of poles with positive real parts in the transfer function $H(s)$ is equal to the number of sign changes in the first column. Consequently, the closed-loop system is stable if all the coefficients in the first column have the same sign.

- **Note:** The maximum number of rows is equal to the number of terms in the polynomial $D(s)$, which corresponds to the order of the system, plus one.

Example: Consider a system with a transfer function: $G(s) = \frac{k}{s^3+s^2+2s}$, placed in a unity feedback control loop. The closed-loop transfer function will be: $H(s) = \frac{G(s)}{1+G(s)} = \frac{k}{s^3+s^2+2s+k}$ Let's apply the Routh criterion by constructing the following table.

s^3	1	2
s^2	1	k
s^1	2-k	0
s^0	k	0

TABLE 1.2: Routh Table for $H(s)$

For the system to be stable, there must be no sign changes in the first column, meaning that $2 - K > 0$. Therefore, the system is stable if $K < 2$.

1.2.4 Reverse Criterion

The reverse criterion is essentially a simplified version of the Nyquist criterion for the common case where the open-loop transfer function $G(s)$ of the system has no poles with a positive real part.

Under these conditions, the stability criterion can be stated as follows:

- If the open-loop transfer function $G(s)$ of a control system has no poles with positive real parts, then the system is stable in closed-loop if, when traversing the Nyquist plot of the open-loop transfer function in the direction of increasing pulsations ω , the critical point C always remains to the left of the curve.

Figure 1.3 illustrates this inverse criterion:

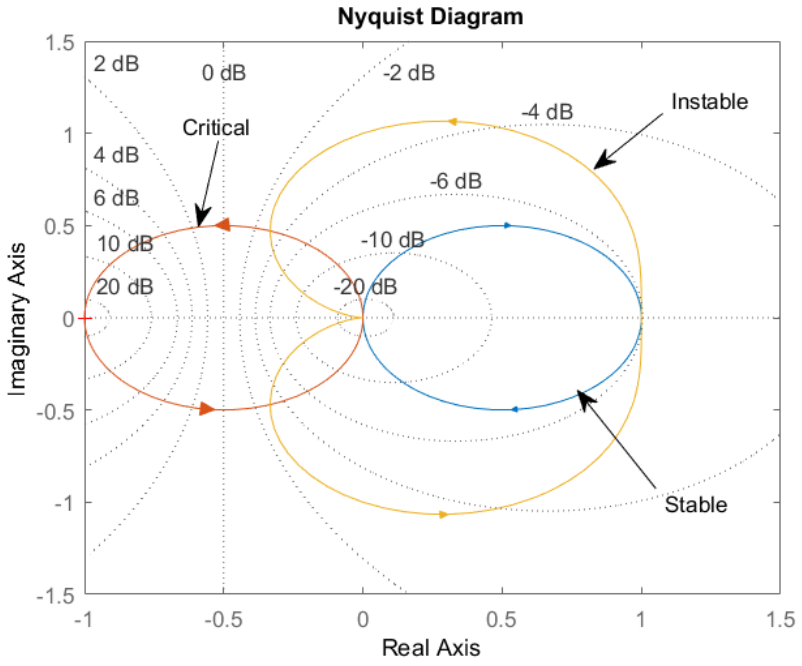


FIGURE 1.3: Reverse criterion.

- **Note:** Always remember that the Nyquist plot is drawn for the open-loop system in order to assess the stability of the closed-loop system.

1.2.5 Stability margins

1.2.5.1 Concept of Stability Margin

The stability criteria we've discussed so far help determine whether a closed-loop system is stable or not. The outcome is always binary: the system is either stable or unstable. However, some of these criteria give insight into the notion of a safety margin concerning the system's stability.

For instance, when the Routh criterion indicates that a system is stable (e.g., if $K < 4$), we can intuitively understand that, for the sake of stability, it is likely better to set the gain K at 2 rather than pushing it close to the upper limit, such as 3.99. Similarly, we tend to consider a system whose Nyquist plot passes far to the right of the critical point as being "more" stable than one that passes very close to this point. These considerations lead us to define the concept of stability margin, which relates to the "distance" from the critical instability threshold. We will define the gain margin and phase margin based on simple geometric observations from the system's Nyquist plot.

1.2.5.2 Gain margin

Gain margin is a measure used in control systems to assess the stability of a system. Specifically, it indicates how much gain can be increased before the system becomes unstable. It is defined as the difference in decibels between the gain at the phase crossover frequency (where the phase angle is -180 degrees) and 0 dB (unity gain).

Mathematically, it can be expressed as:

$$GM_{dB} = -20log_{10}\left(|G(j\omega)|\right) \tag{1.7}$$

where $G(j)$ is the magnitude of the open-loop transfer function at the phase crossover frequency. A positive gain margin indicates a stable system, while a negative gain margin suggests potential instability.

- **Interpretation:** A larger gain margin indicates a more stable system, as it can tolerate greater increases in gain before reaching instability. Conversely, a smaller or negative gain margin implies that the system is closer to instability.

In summary, the gain margin graphically represents the stability margin of a control system, helping engineers assess how changes in gain affect system behavior.

Example: Let's consider a system with an open-loop transfer function $G(s)$ placed in a feedback loop with unit feedback, where:

$$G(s) = \frac{5 \times 10^6}{(s+100)^3} \tag{1.8}$$

Calculating the gain margin involves first determining the value of ω_π and then performing the calculation: $GM_{dB} = -20log_{10}\left(|G(j\omega_\pi)|\right)$.

ω_π is such that: $\varphi\left(\omega_\pi\right) = -\pi$; where: $\varphi\left(\omega\right) = Arg(G(j\omega)) = 0 - 3Arctan(\frac{\omega}{100})$
$\varphi(\omega_\pi) = -\pi \Rightarrow -3Arctan(\frac{\omega_\pi}{100}) = -\pi$

$\Rightarrow \omega_\pi = 173rad/s$ We then have: $GM_{dB} = -20log_{10}\left(|G(173j)|\right) = 12dB$

1.2.5.3 Phase margin

The phase margin is a measure of the stability of a control system in the frequency domain. It is defined as the amount of additional phase lag at the gain crossover frequency (the frequency at which the gain of the open-loop transfer function equals one or 0 dB) that can be tolerated before the system becomes unstable. The phase margin is calculated as:

$$PM = \pi + Arg(G(j\omega_0)) \tag{1.9}$$

This represents the difference between the actual phase at the gain crossover frequency and $-180°$.

Interpretation:A positive phase margin indicates that the system is stable, while a negative phase margin suggests instability. The larger the phase margin, the more stable the system is considered to be. Let's revisit the system from the previous example: Calculating the phase margin involves first determining the value of ω_0, and then proceeding to calculate the margin $PM = \pi + Arg(G(j\omega_0))$.

ω_π is such that: $|G(j\omega_0)| = 1$ In addition:

$$|G(j\omega_0)| = \frac{5 \times 10^6}{\sqrt{\left((j\omega_0)^2 + 100^2\right)^3}} = 1 \tag{1.10}$$

From which: $\omega_0 = 76.6rad/s$ We then have:$PM = \pi + Arg(G(j\omega_0))$ We also have: $Arg(G(j\omega_0)) = -3Arctan(\frac{\omega_0}{100})$

$\Rightarrow PM = =67.6°$

The gain and phase margins can be graphically illustrated using a Bode plot (Figure 1.4), a Nyquist plot (Figure 1.5) or a Black-Nichols diagram (Figure 1.6) .

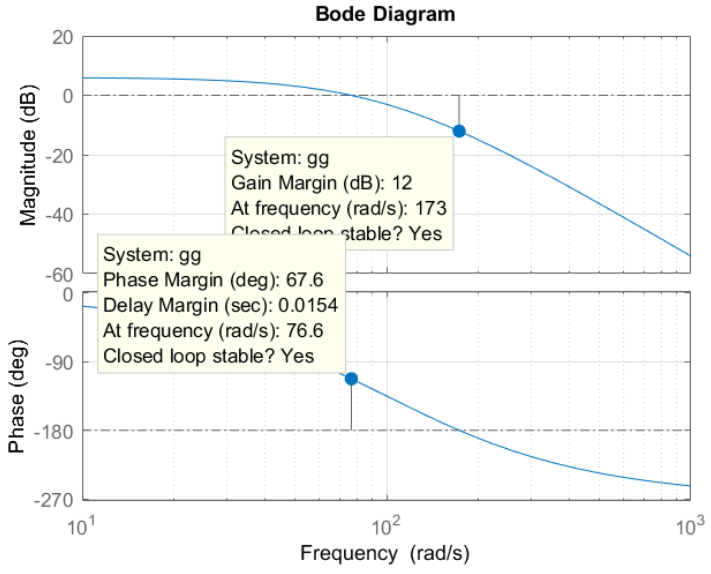


FIGURE 1.4: Gain and Phase Margin in the Bode Diagram.

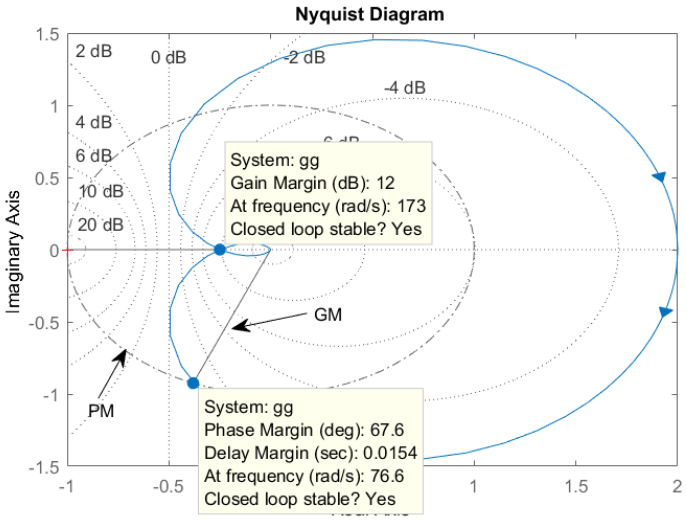


FIGURE 1.5: Gain and Phase Margin in the Nyquist Diagram.

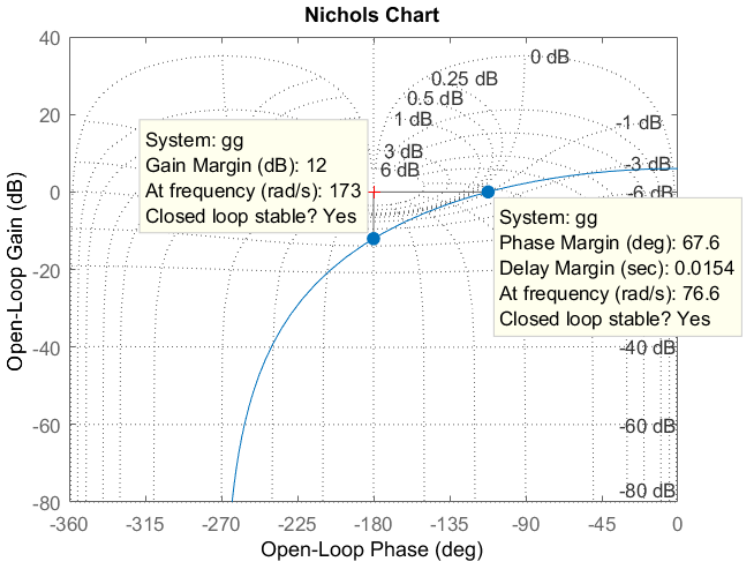


FIGURE 1.6: Gain and Phase Margin in the Black-Nichols diagram.

1.3 Performance Metrics for Control of Linear Systems

The performance of linear control systems is typically evaluated using key metrics that describe how well the system responds to inputs and how effectively it meets stability and accuracy requirements. These metrics focus on system dynamics in both the time and frequency domains. Below is a brief description of the main performance measures with corresponding equations:

- **Stability:** Stability ensures that the system's output remains bounded for any bounded input. For linear time-invariant (LTI) systems, stability is determined by the location of poles in the system's transfer function. The system is stable if all poles of $G(s) = \frac{Y(s)}{U(s)}$ have negative real parts (i.e., lie in the left half of the complex plane).

- **Precision:** characterized by Steady-state error witch measures the difference between the desired output and the actual output as time approaches infinity. It depends on the system type and input signals:

$$e_{ss} = \lim_{t \to \infty} (r(t) - y(t)) \tag{1.11}$$

- **Rapidity:** materialized by Rise time and settling time. Rise time t_r is the time taken for the system output to rise from 10 to 90 percent of its final value for a step input. The following approximate relation is more commonly used:

$$t_r = \frac{3}{\omega_0} = \frac{3}{\omega_{nBF}} \tag{1.12}$$

where: ω_0 and ω_{nBF} are the cutoff frequencies at 0 dB for the open-loop and closed-loop systems, respectively. Settling time t_s is the time required for the system's output to remain within a certain percentage (typically 2 or 5 percent) of its final value y_∞ after a disturbance or input change.

$$y(t_s) = \pm y_\infty \tag{1.13}$$

They provide a measure of the system's speed of response.

- **Overshoot:** Maximum overshoot M_p refers to how much the system output exceeds the desired response in a step response. It is typically expressed as a percentage:

$$M_p = \frac{(y_{max} - y_\infty)}{y_\infty} \times 100 \tag{1.14}$$

where y_{max} is the maximum peak and y_∞ is the steady-state value:

- **Bandwidth:** Bandwidth is a frequency domain performance measure that defines the range of frequencies over which the system can accurately respond to input signals. It is usually measured at the frequency where the magnitude of the frequency response drops to $-3dB$ from its peak value.

1.4 Frequency responses and frequency characteristics of controllers

The frequency response of a system describes how the output of the system responds to sinusoidal inputs of varying frequencies. It provides insight into the system's behavior in the frequency domain, revealing how different frequencies are amplified, attenuated, or shifted in phase as they pass through the system.

- **Magnitude Response:** This part of the frequency response indicates how much the output amplitude of the system changes relative to the input amplitude across different frequencies. It is typically expressed in decibels (dB) and can show whether the system acts as a low-pass, high-pass, band-pass, or band-stop filter.

- **Phase Response:** This component shows the phase shift introduced by the system as a function of frequency. Phase shifts can impact the timing of the output signal relative to the input signal and are crucial for understanding the stability and dynamic behavior of the system.

The frequency response can be represented graphically using:

- **Bode Plots:** These plots show both magnitude and phase response on logarithmic scales, allowing for easy visualization of how the system behaves over a wide range of frequencies.

- **Nyquist Plots:** These provide a polar plot representation of the complex frequency response, illustrating both magnitude and phase simultaneously.

The frequency response is crucial in control system design and analysis for several reasons:

- **Stability Analysis:** It helps assess the stability of the system based on the phase and gain margins.

- **Filter Design:** It guides the design of filters and controllers by revealing how the system interacts with various frequency components.

- **System Characterization:** It allows engineers to characterize the dynamic properties of a system, helping to predict how it will behave under different operating conditions.

Overall, understanding the frequency response is essential for designing effective control systems and ensuring desired performance in various applications.

1.4.1 Proportional (P) controller

he proportional controller is essentially a gain amplifier represented as $C(s) = K$, which adjusts the system's static gain. The effect of static gain on system performance has already been studied in the previous chapter, and its influence is summarized as follows:

- If $K < 1$, meaning the controller acts as an attenuator, the system's stability is improved, and the overshoot in closed-loop response is reduced. However, this comes at the cost of slower response time and reduced accuracy.
- If $K > 1$, the system's speed and accuracy in the closed-loop response are enhanced, but stability is decreased, which can lead to instability and an increased overshoot.

Note: The influence of static gain variation on closed-loop performance can be visualized using the Bode plot of an open-loop system (Figure 1.7).

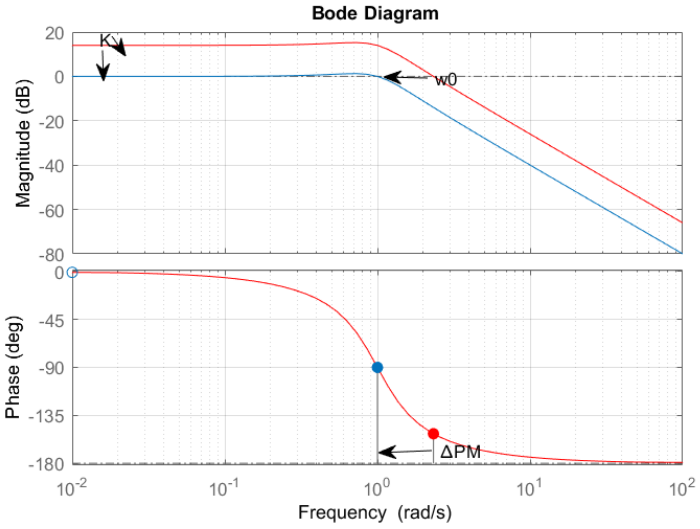


FIGURE 1.7: Impact of a Decrease in Static Gain on Performance

1.4.2 Integrator (I) controller

The integral action is typically part of a Proportional-Integral-Derivative (PID) controller and can be represented mathematically as:

$$u(t) = K_i \int_0^t e(\tau)d\tau \Rightarrow R(s) = \frac{U(s)}{E(s)} = \frac{1}{s} \tag{1.15}$$

where: $u(t)$ is the control signal, K_i is the integral gain, $e(\tau)$ is the error signal (difference between desired and actual output) and t is time.

Performance:

The integral action improves the performance of the control system in the following ways:

- Elimination of Steady-State Error: The integral term ensures that even small, persistent errors are accumulated and corrected over time, eliminating steady-state error (offset) in the system.
- Improved Accuracy: By addressing long-term discrepancies between the set-point and the actual output, the integral action enhances the accuracy of the system.

However, too much integral action can have drawbacks:

- It may slow down the system's response, as the accumulated error may cause overshoot or oscillations.
- It can reduce system stability, particularly in systems with significant time delays.

Integral action is essential for reducing steady-state error in control systems but must be carefully tuned to balance accuracy and stability Figure 1.8 .

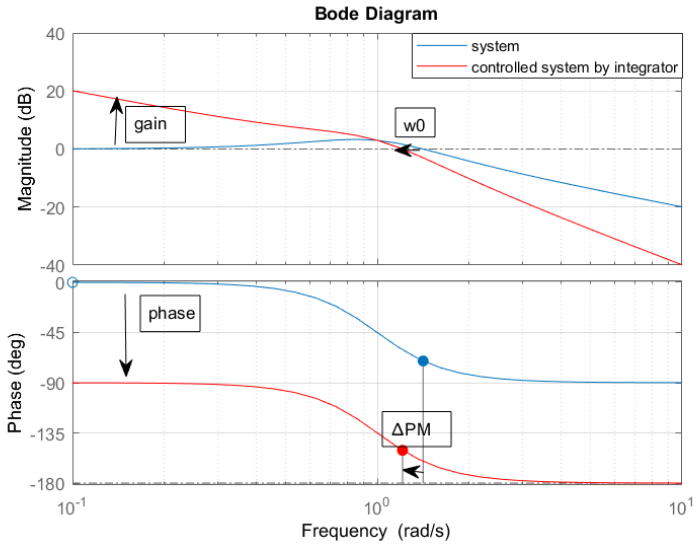


FIGURE 1.8: Influence of an integrator on performance.

1.4.3 Derivative (D) controller

This controller is a derivative of the transfer function:

$$u(t) = \frac{\mathrm{d}e(t)}{\mathrm{d}x} \Rightarrow R(s) = \frac{U(s)}{E(s)} = s \tag{1.16}$$

Its role is to add a zero to the open-loop transfer function. Intuitively, we can think of its action as the opposite of that of an integrator. Let's verify this with a Bode plot Figure 1.9. Notably, at the frequency $\omega = 10$, the gain has increased by $20dB$,

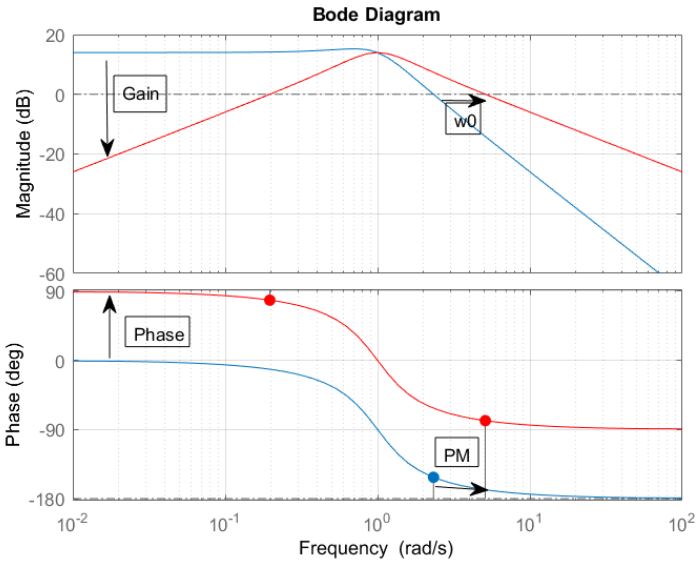


FIGURE 1.9: Influence of an Derivative controller on performance.

which allows us to immediately plot the corresponding graph for G in dB. As for the phase diagram, it is shifted upward by $\pi/2$. We also observe that the cutoff frequency at 0 dB increases as well. Considering that $t_r = \frac{3}{\omega_0}$; we can deduce that the rise time decreases. Therefore, the differentiator will tend to accelerate the closed-loop system. The increase in ω_0 also affects the phase margin, but this influence depends on the system's order.

1.4.4 Proportional-Integral-Derivative (PID) controller

A PID regulator, or Proportional-Integral-Derivative controller, is a control loop mechanism widely used in industrial control systems. It adjusts the output to minimize the difference between a desired setpoint and the actual process variable by calculating and applying corrective actions based on three parameters: proportional (P), integral (I), and derivative (D). The relationship between the control signal $u(t)$ and the error signal $e(t)$ is given by the following expression:

$$u(t) = K_p e(t) + K_i \int e(t)dt + K_d \frac{\mathrm{d}}{\mathrm{d}t} e(t) \Rightarrow R(s) = \frac{U(s)}{E(s)} = K_p + \frac{K_i}{s} + K_d s \quad (1.17)$$

1.4.5 Phase lead controller

A **phase lead controller** is a device used in control systems to enhance stability and responsiveness. It works by increasing the system's phase margin, helping to

offset phase delays caused by certain system components. It speeds up the system's response and improves its dynamic performance, particularly by increasing speed and accuracy while reducing settling time.

The transfer function of this controller is given by:

$$R(s) = \frac{1 + \alpha\tau s}{1 + \tau s} \tag{1.18}$$

with $\alpha > 1$; To better understand the action of this controller, let's plot its Bode diagram Figure 1.10. There are two cutoff frequencies: $1/\tau$ and $1/\alpha\tau$, where:$1/\tau > 1/\alpha\tau$ The magnitude of this controller becomes:

$$R(\omega) = |R(j\omega)| = \sqrt{\frac{1 + (\alpha\tau\omega)^2}{1 + (\tau\omega)^2}} \tag{1.19}$$

and:

$$\varphi(\omega) = Arg(R(j\omega)) = arctan(\alpha\tau\omega) - arctan(\tau\omega) \tag{1.20}$$

he advantage of this controller is evident in its phase diagram: at the frequency $\omega_{max} = \frac{1}{\tau\sqrt{\alpha}}$, the phase shift reaches a maximum, which can be easily calculated.

$$\varphi_{max} = arcsin(\frac{\alpha - 1}{\alpha + 1}) \tag{1.21}$$

The principle of corrective action involves aligning ω_{max} with the cutoff frequency

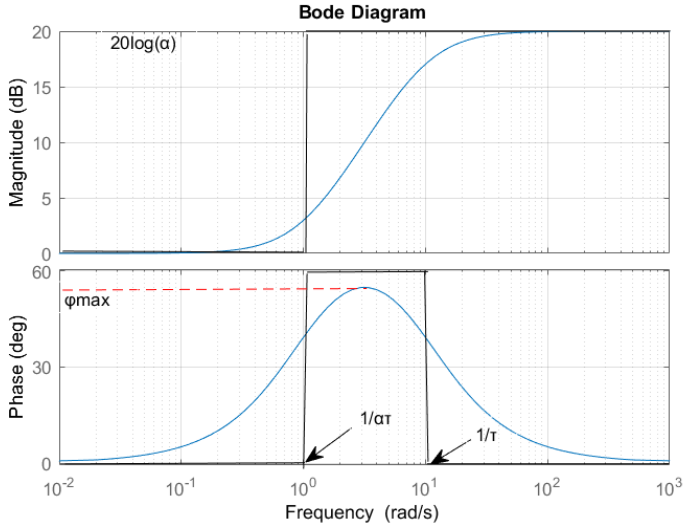


FIGURE 1.10: Bode diagram of a phase lead controller.

at 0 dB, ω_0, of the system to be corrected and adjusting φ_{max}, known as the phase boost, in order to achieve the desired phase margin.

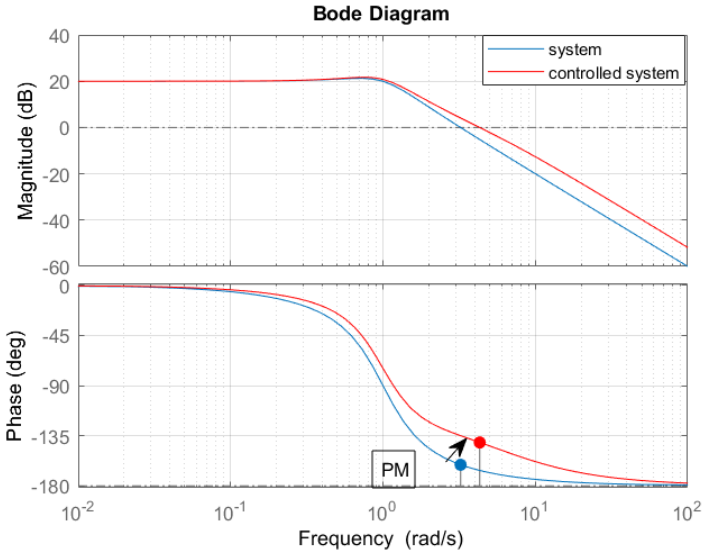


FIGURE 1.11: Bode diagram of the system corrected by a phase lead controller.

Example: We consider the transfer function of a linear system $G(s)$ defined as follows:

$$G(s) = \frac{10}{s^2 + s + 1} \tag{1.22}$$

We want to adjust this system so that its phase margin is equal to 45°.The phase margin before the adjustment is 18.9° at frequency $\omega_0 = 3.23 rad/s$: The phase margin is insufficient Figure 1.11. To correct it, we need to increase the phase by $45° - 18.9° = 26.1°$ at the frequency ω_0. Therefore, we introduce a phase-lead controller, which is adjusted in such a way that:$\frac{1}{\tau\sqrt{\alpha}} = \omega_0 = 3.23$ rad/s and $\varphi_{max} = 26.1° = arcsin\frac{\alpha-1}{\alpha+1}$ Therefore, we have: $\alpha = 2.5710$ and $\tau = 0.1931$.

Finally:

$$[h!]R(s) = \frac{1 + 0.4965s}{1 + 0.1931s} \tag{1.23}$$

1.4.6 Phase lag controller

A Phase Lag controller (in French: Correcteur à retard de phase) is a control system component designed to improve the stability and steady-state performance of a system by reducing the system's phase margin. It essentially slows down the system's response to changes by adding a phase lag, which delays the output response to an input signal.

The primary function of a phase lag controller is to reduce steady-state error while maintaining a level of stability. It works by reducing the gain of the system at high frequencies, helping to achieve better accuracy in tracking the desired output without sacrificing too much responsiveness. This type of controller is often used in

systems where precision is crucial but doesn't require a fast reaction time. Its transfer function is:

$$R(s) = \frac{\alpha(1+\tau s)}{1+\alpha\tau s} \tag{1.24}$$

where $\alpha > 1$; To better understand the action of this controller, let's plot its Bode diagram Figure 1.12. There are two cutoff frequencies: $1/\tau$ and $1/\alpha\tau$, where:$1/\tau > 1/\alpha\tau$ The magnitude of this controller becomes:

$$R(\omega) = |R(j\omega)| = \frac{\alpha\sqrt{1+(\tau\omega)^2}}{\sqrt{1+(\alpha\tau\omega)^2}} \tag{1.25}$$

and:

$$\varphi(\omega) = Arg(R(j\omega)) = arctan(\tau\omega) - arctan(\alpha\tau\omega) \tag{1.26}$$

When $\omega \to 0$, we have: $R(\omega) \to 20log(\alpha)$ dB.

When $\omega \to \infty$, we have: $R(\omega) \to 0$ dB.

The analysis of the Bode diagram Figure 1.12 allows us to predict the action of this controller. When it is placed in series with the system to be corrected, in the forward path, the two Bode diagrams will combine. As a result, the static gain increases by $20log(\alpha)$, which improves accuracy. By setting the parameter τ to a sufficiently small value, this correction only affects low frequencies, leaving the high-frequency gain virtually unchanged. The additional negative phase shift introduced by the controller also occurs at low frequencies, so it does not impact the stability margin, as the 0 dB cutoff frequencies are generally found in higher frequency ranges.

In any case, to adjust the phase-lag controller, one would choose the value of a that provides the desired resulting static gain and then select τ such that $1/\tau << \omega_0$.
Example: Consider a system with a transfer function $G(s)$ placed in a unity feedback loop.

$$G(s) = \frac{2800}{s^3+30s^2+300s+1000} \tag{1.27}$$

We want this closed-loop system to have a steady-state (position) error of 5 percent. We have : $G(s) = \frac{2800}{(10+s)^3}$; $\omega_0 = 10$ rad/s; static gain $K = 2.8$ and $PM = 45°$. Let's now calculate the steady-state error obtained in closed-loop under these conditions. either: $e_{ss} = \frac{1}{1+2.8}$ The observed accuracy does not meet the specifications. To achieve a position error of 5 percent, it is necessary to have a static gain, $\hat{K}$, such that: $e_{ss} = \frac{1}{1+\hat{K}} = 0.05 \Rightarrow \hat{K} = 19$.

The new static gain is: $\hat{K} = 2.8\alpha$ The new open-loop transfer function is:

$$G_c(s) = R(s)G(s) = \frac{\alpha(1+\tau s)}{1+\alpha\tau s}.\frac{2800}{s^3+30s^2+300s+1000} \tag{1.28}$$

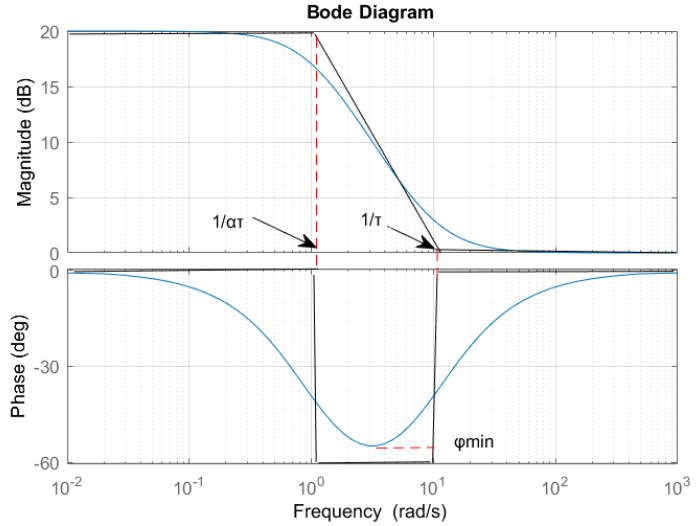


FIGURE 1.12: Bode diagram of a phase lag controller.

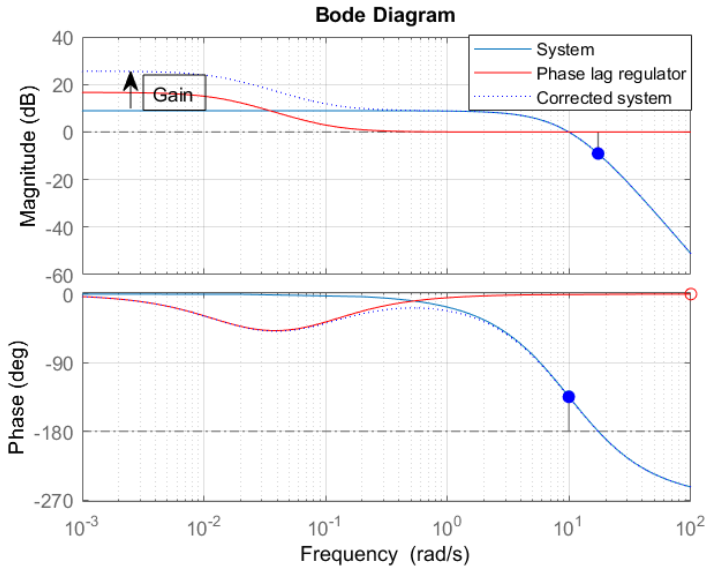


FIGURE 1.13: Corrected Bode Diagram at Low Frequencies.

Therefore, it is necessary to adjust the parameter α such that: $\alpha = \frac{19}{2.8} = 6.8 \Rightarrow 20log(\alpha) = 16.7$ dB To conclude, it is sufficient to choose τ such that $\frac{1}{\tau}$ is much lower than the cutoff frequency at 0 dB ω_0. For example, we can take $\tau = 10$ seconds. Ultimately, we have:

$$R(s) = \frac{6.8(10s+1)}{68s+1} \tag{1.29}$$

Figure 1.13 shows the compared Bode diagrams of the initial system and the corrected system. It is important to note that the Bode diagrams of $G(s)$ and $R(s)$ combine to form the diagram of the corrected system $G_c(s)$.

1.5 Designing controllers using the Bode plot

In frequency representation, PD (Proportional-Derivative), PI (Proportional-Integral), and PID (Proportional-Integral-Derivative) controllers each have distinct effects on system dynamics. A PD controller enhances stability by increasing high-frequency gain, helping to reduce overshoot and improve response speed. A PI controller, by adding a pole at the origin, improves steady-state accuracy for tracking and eliminates steady-state error, especially effective at low frequencies. A PID controller combines both, balancing steady-state accuracy and transient response, which broadens the system's frequency range effectiveness by adding both low- and high-frequency gain adjustments.

1.5.1 PD controller

Its transfer function is given by:

$$R(j\omega) = K_p(1 + jT_d\omega) \tag{1.30}$$

$$|R(j\omega)| = K_p\sqrt{1 + (T_d\omega)^2} \tag{1.31}$$

if $\omega \to 0 : |R(j\omega)|_{dB} \to 20log(K_p)$
if $\omega \to \infty$: $|R(j\omega)|_{dB} \to 20log(\omega)$ This controller shows a line with a slope of 0 (parallel to the x-axis) at low frequencies ($\omega \to 0$) and a line with a slope of 20 dB at high frequencies ($\omega \to \infty$). The frequency representation of this controller is illustrated by the Bode plot in Figure 1.14. D-action: adds phase → improves phase margin (damping).

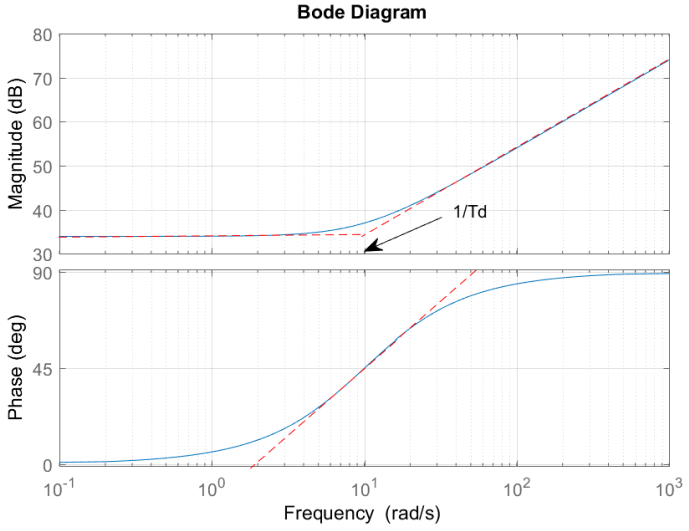


FIGURE 1.14: Bode plot of PD regulator with $K_p = 50$ and $T_d = 0.1$.

1.5.2 PI controller

Its transfer function is given by:

$$R(j\omega) = K_p(1 + \frac{1}{jT_i\omega}) \tag{1.32}$$

$$|R(j\omega)| = K_p\sqrt{1 + \left(\frac{1}{T_i\omega}\right)^2} \tag{1.33}$$

if $\omega \to 0 : |R(j\omega)|_{dB} \to -20log(\omega)$
if $\omega \to \infty$: $|R(j\omega)|_{dB} \to 20log(K_p)$ This controller shows a line with a slope of -20 dB at low frequencies ($\omega \to 0$) and a line with a slope of 0 dB (parallel to the x-axis) at high frequencies ($\omega \to \infty$). The frequency representation of this controller is illustrated by the Bode plot in Figure 1.15. I-action: adds gain → improves steady-state behavior.

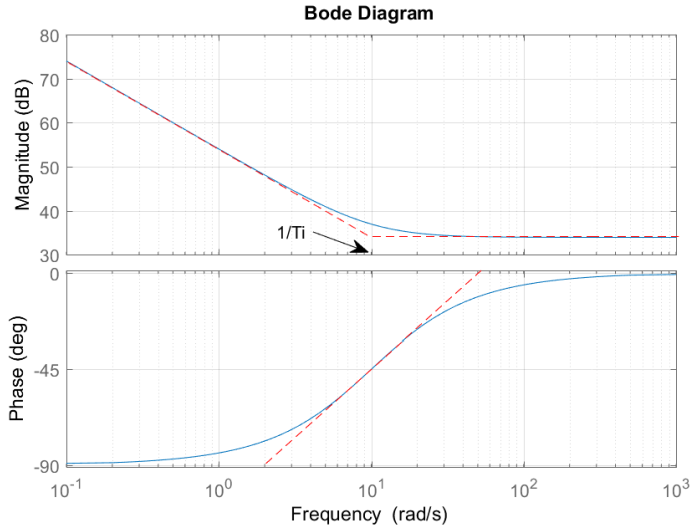


FIGURE 1.15: Bode plot of PID regulator with $K_p = 50$ and $T_i = 0.1$

1.5.3 PID controller

Its transfer function is given by:

$$R(j\omega) = K_p(1 + jT_d\omega + \frac{1}{jT_i\omega}) \tag{1.34}$$

$$|R(j\omega)| = K_p\frac{\sqrt{(1 - T_iT_d\omega^2)^2 + (T_i\omega)^2}}{T_i\omega} \tag{1.35}$$

if $\omega \to 0 : |R(j\omega)|_{dB} \to -20log(\omega)$
if $\frac{1}{T_d} < \omega < \frac{1}{T_i}(withT_i < T_d)$: $|R(j\omega)|_{dB} \to 20log(K_p)$

if $\omega \to 0 : |R(j\omega)|_{dB} \to 20log(\omega)$
This controller shows a line with a slope of -20 dB at low frequencies ($\omega \to 0$) and a line with a slope of 0 dB (parallel to the x-axis) where $\frac{1}{T_d} < \omega < \frac{1}{T_i}$ and a line with a slope of 20 dB at high frequencies ($\omega \to \infty$). The frequency representation of this controller is illustrated by the Bode plot in Figure 1.16.

- Adjust the proportional gain to get the required crossover frequency and/or steady-state tracking error.
- If needed, use the derivative action to add phase in the neighborhood of ω_0 in order to increase the phase margin.
- If needed, use the integral action to increase the gain at low frequencies in order to guarantee the required steady-state tracking error.

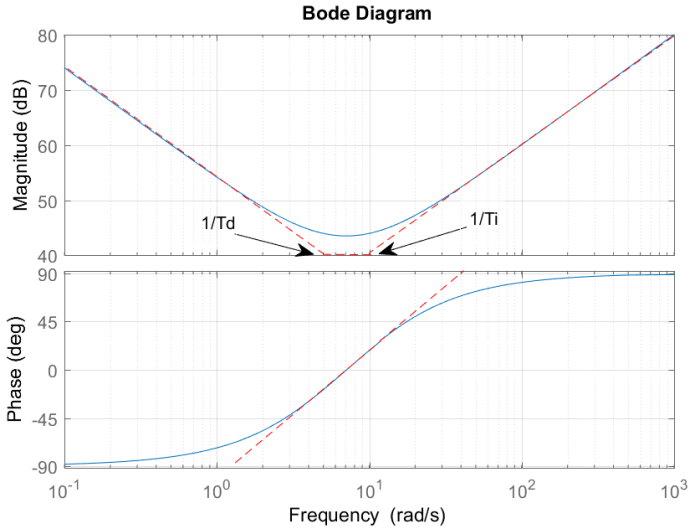


FIGURE 1.16: Bode plot of PID regulator with $K_p = 50$ and $T_i = 0.1$

1.5.4 Quick method for plotting the Bode diagram

This refers to a simplified technique for sketching a Bode diagram efficiently. It provides a fast approach for approximating the amplitude and phase plots of a system's frequency response, often using asymptotic lines to represent gain and phase changes at specific breakpoints or slopes.

- If the transfer function has no integrator or differentiator, the Bode plot begins with a 0 dB slope asymptote and a phase of 0 rad. Then, at the frequency of the next breakpoint in the transfer function (ordered by increasing frequency), the slope changes by +n decades (or $+20n$ dB) and phase shifts by $+n\frac{\pi}{2}$ rad if

the term is in the numerator of the transfer function with order n, or decreases by $-n$ decades (or $-20n$ dB) and phase of $-n\frac{\pi}{2}$ rad if it's in the denominator, continuing this way through all breakpoints.

- If the transfer function includes an integrator of order p, the plot starts with a slope of $-p$ decades (or $-20p$ dB) and phase of $-p\frac{\pi}{2}$ rad and then follows the above steps.

- If it includes a differentiator of order q, the Bode plot begins at $+q$ decades (or $+20q$ dB) and phase of $-q\frac{\pi}{2}$ rad, after which it follows the steps as if there were no integrator or differentiator.

Examples: Let's consider the systems of the following transfer functions:

- System without integrator or differentiator:

$$G_1 = \frac{555\,(s+10)}{(s+100)\,(s+700)^2} \tag{1.36}$$

- The system contains a second-order integrator:

$$G_2 = \frac{(s+10)^2\,(s+100)}{s^2\,(s+1000)} \tag{1.37}$$

- The system contains a third-order differentiator:

$$G_3 = \frac{s^3\,(s+1000)}{(s+10)^2\,(s+100)^3} \tag{1.38}$$

The Bode plots of $G_1(s)$; $G_2(s)$ and $G_3(s)$ are illustrated in Figure 1.17; Figure 1.18 and Figure 1.19 respectively.

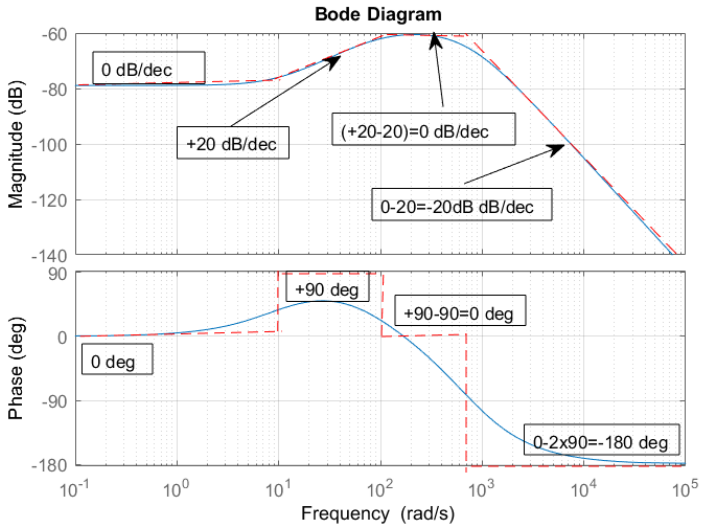


FIGURE 1.17: Bode plot of a system without integrator or differentiator.

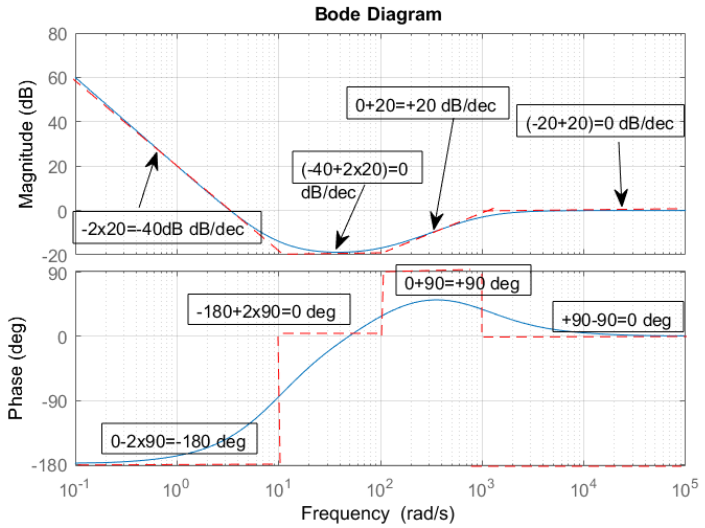


FIGURE 1.18: Bode plot of a system contains a third-order integrator.

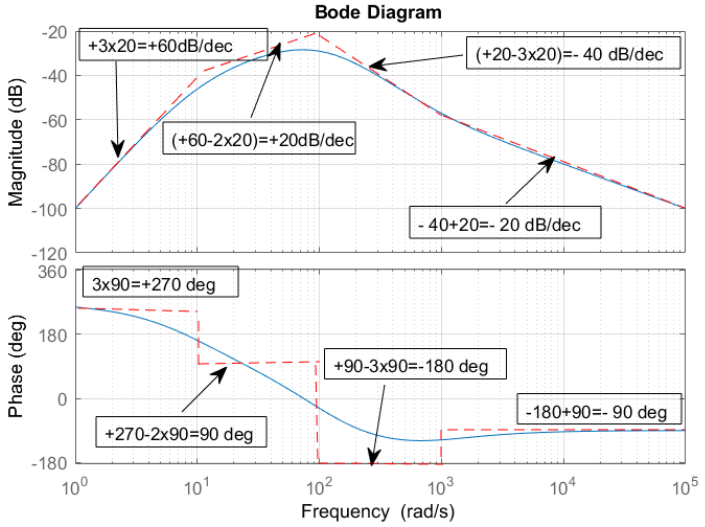


FIGURE 1.19: Bode plot of a system contains a second-order differentiator.

1.5.5 Determination of Minimum-Phase Transfer Functions from Bode Plots

To determine if a system is minimum phase, we analyze the frequency-response curves, focusing on the high-frequency behavior.

The process begins by drawing asymptotic approximations to the experimentally obtained log-magnitude plot. These asymptotes should have slopes in multiples of $-20\,\text{dB/decade}$. For example, if the slope transitions from $-20\,\text{dB/decade}$ to $-40\,\text{dB/decade}$ at a frequency $\omega = \omega_1$, the transfer function must include a term of the form $\frac{1}{1+j\omega/\omega_1}$. Similarly, if the slope changes by $-40\,\text{dB/decade}$ at $\omega = \omega_n$, the transfer function will include a quadratic factor. The undamped natural frequency of this quadratic factor corresponds to the corner frequency ω_n.

$$H(s) = \frac{1}{1 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2} \tag{1.39}$$

The damping ratio ζ for this quadratic factor can be determined by examining the resonant peak near ω_n in the log-magnitude plot. The height of this peak is compared to standard curves, such as those shown in Figure 1.20, to estimate ζ.

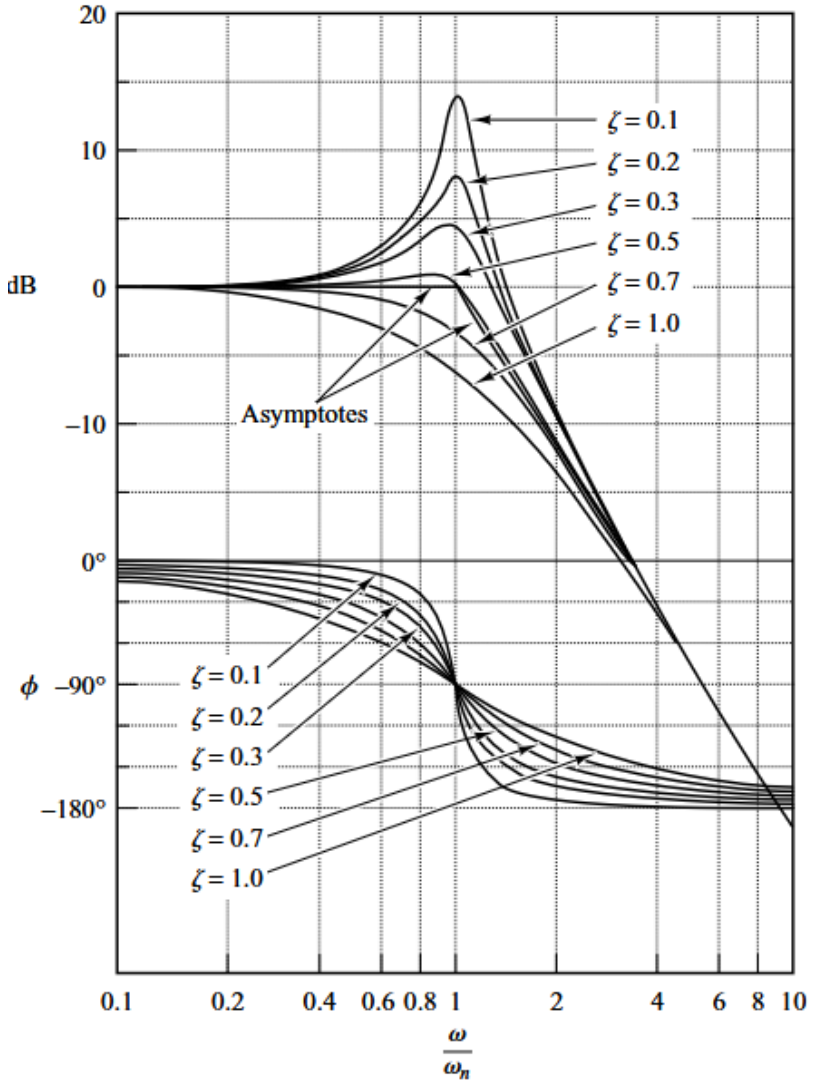


FIGURE 1.20: Log-magnitude curves, together with the asymptotes, and phase-angle curves of the quadratic transfer function given by Equation 1.39.

After identifying all the factors contributing to the transfer function $G(j\omega)$, the overall gain can be determined from the low-frequency portion of the log-magnitude plot.

Since terms like $1 + j\frac{A\omega}{\omega_1}$ and $1 + 2\zeta j\frac{\omega}{\omega_2} + \left(j\frac{\omega}{\omega_2}\right)^2$ approach unity as ω approaches zero, the sinusoidal transfer function $G(j\omega)$ can be expressed at very low frequencies as:

$$\lim_{\omega \to 0} G(j\omega) = \frac{K}{(j\omega)^{\beta}}$$

In many practical systems, the value of β is typically 0, 1, or 2.

a. For $\beta = 0$, or type 0 systems, $G(j\omega) = K$ when $\omega \ll 1$, or equivalently, $20 \log |G(j\omega)| = 20 \log K$ when $\omega \ll 1$.

The low-frequency asymptote is a flat line at $20 \log K$ dB, allowing the value of K to be determined from this horizontal line.

b. For $\beta = 1$, or type 1 systems, for $\omega \ll 1$, $G(j\omega) = \frac{K}{j\omega}$, or the magnitude of the transfer function is given by:

$$20 \log |G(j\omega)| = 20 \log K - 20 \log \omega$$

This indicates that the low-frequency asymptote has a slope of -20 dB/decade. The frequency at which this asymptote (or its extension) crosses the 0 dB line is numerically equal to K.

c. For $\beta = 2$, or type 2 systems, for $\omega \ll 1$, the transfer function is given by:

$$G(j\omega) = \frac{K}{(j\omega)^2}$$

, the magnitude of the transfer function is given by:

$$20 \log |G(j\omega)| = 20 \log K - 40 \log \omega$$

The slope of the low-frequency asymptote is -40 dB/decade. The frequency at which this asymptote (or its extension) crosses the 0 dB line is numerically equal to $\sqrt{K}$.

Figure 1.21 illustrates examples of log-magnitude curves for type 0, type 1, and type 2 systems, along with the corresponding frequency at which the gain K is related.

- **Example:**

To determine the transfer function of the system based on the experimental frequency-response curves shown in Figure 1.22.

The first step is to approximate the log-magnitude curve using asymptotes with slopes of -20 dB/decade and its multiples, as illustrated in Figure 1.22. Next, we estimate the corner frequencies. For the system depicted in Figure 1.22, the transfer function is approximated in the following form:

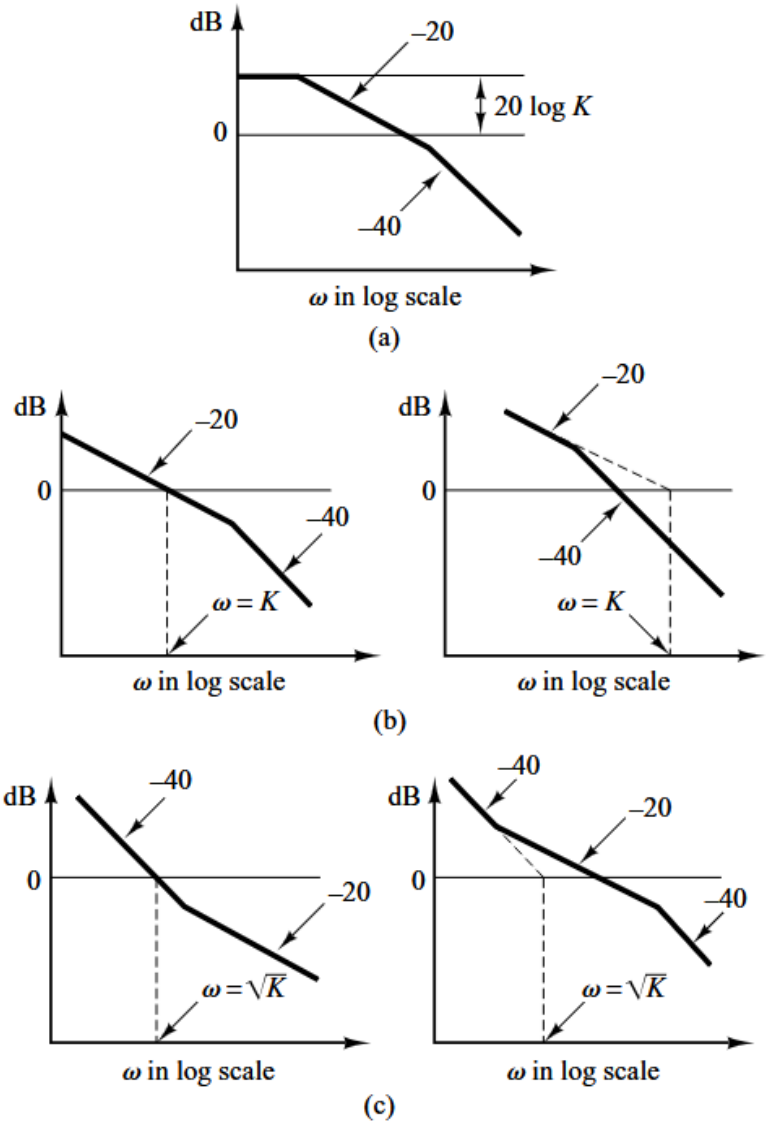


FIGURE 1.21: (a) Log-magnitude curve for a type 0 system; (b) log-magnitude curves for type 1 systems; (c) log-magnitude curves for type 2 systems. (The slopes displayed are in dB/decade.)

1.6 Tuning using the Black-Nichols chart

Tuning using the Black-Nichols chart refers to the process of adjusting the parameters of a control system based on the graphical representation provided by the Black-Nichols chart. This chart helps in visualizing the system's stability and response characteristics, allowing engineers to determine optimal gain and phase adjustments for effective system performance.

For a given transfer function calculated in the table of points below Table 1.3, we have the open-loop (OL) transfer function representation in the Black plot Figure 1.22.

We can identify:

- The low-frequency range

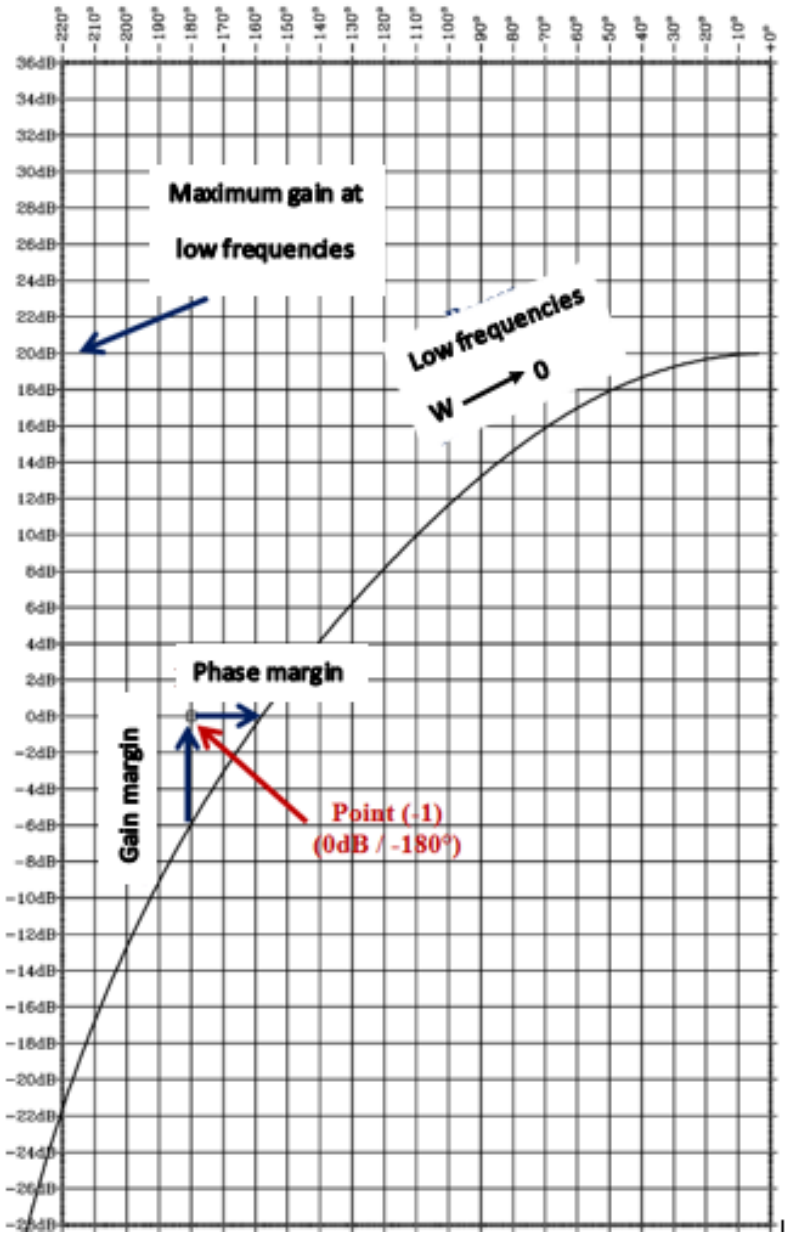


FIGURE 1.22: Black-Nichols diagram.

- The maximum gain at low frequencies (20 dB)
- The phase margin (22°)
- The gain margin (6 dB)

The system will be stable in closed-loop since the curve passes to the right of the (-1) point. However, this stability is limited, as both the gain and phase margins are low (the curve is fairly close to the -1 point). Consequently, the system is likely to oscillate for an extended period in response to command changes.

ω	Gain (dB)	Phase (deg)
0.01	19.9886	-3.72182
0.05	19.7232	-18.3307
0.075	19.3981	-26.9928
0.1	18.9768	-35.138
0.12	18.5869	-41.2402
0.13	18.3784	-44.1498
0.15	17.9408	-49.6898
0.2	16.7762	-62.0205
0.5	10.1642	-108.8
1	1.87087	-150.255
1.5	-4.63479	-175.585
2	-10.0432	-192.724
2.5	-14.6559	-204.965
4	-25.3256	-226.536
5	-30.7189	-234.598

TABLE 1.3: Gain and phase over the frequency of a given transfer function.

1.7 Solved exercises

Exercise 01: Consider a system governed by the transfer function G(s), placed in a control loop as shown in Figure 1.23. Analyze the stability of this closed-loop system. $G(s) = \frac{1}{s(s^2+2s+4)}$

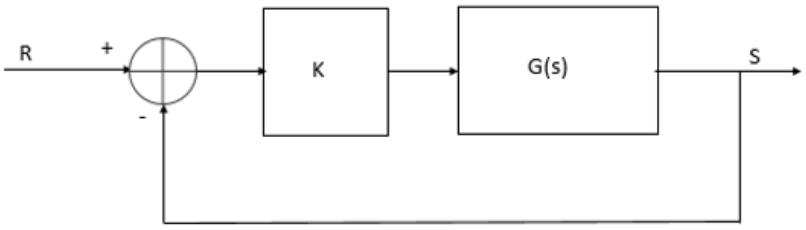


FIGURE 1.23: Simple feedback loop.

Solution of exercise 01: To analyze the stability of a closed-loop system with the transfer function $G(s) = \frac{1}{s(s^2+2s+4)}$, let us proceed step by step:

For a unity feedback system, the closed-loop transfer function $T(s)$ is given by:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{k}{s(s^2+2s+4)}}{1+\frac{k}{s(s^2+2s+4)}} = \frac{k}{s(s^2+2s+4)+k}.$$

The characteristic equation of the closed-loop system is:

$$s(s^2 + 2s + 4) + k = 0.$$

Expanding this, we get:

$$s^3 + 2s^2 + 4s + k = 0.$$

To analyze stability, we check the roots of the characteristic equation $s^3 + 2s^2 + 4s + 1 = 0$. This can be done using the **Routh-Hurwitz** criterion.

Construct the Routh table:

$$\begin{array}{ccc} s^3 & 1 & 4 \\ s^2 & 2 & \text{k} \\ s^1 & \frac{8-k}{2} & 0 \\ s^0 & \text{k} & \end{array}$$

- Stability Conditions:

1. All the coefficients in the first column are strictly positive if: $\frac{8-k}{2} > 0$ implies that $k < 8$.

2. No sign changes occur in the first column.

Since these conditions are satisfied, the system is stable in closed loop.

Exercise 02: Consider a system characterized by the open-loop transfer function:

$$G(s) = \frac{k}{(s+10)^3}, \quad k > 0$$

1. Examine the stability of the closed-loop system when $G(s)$ is used in a unity feedback configuration.

2. Determine the value of k that ensures a gain margin of 6 dB.

3. Calculate the phase margin for the determined value of k.

4. Generate Bode plots of the open-loop system and indicate the gain and phase margins on the diagrams.

Solution of exercise 02:

1. Let's calculate the closed-loop transfer function:

The closed-loop transfer function is given by:

$$H(s) = \frac{G(s)}{1 + G(s)} = \frac{K}{(s+10)^3 + K}.$$

The denominator of the transfer function is expressed as:

$$D(s) = (s+10)^3 + K = s^3 + 30s^2 + 300s + K.$$

To analyze the stability condition, we establish the Routh table as follows:

$$\begin{array}{ccc} s^3 & 1 & 300 \\ s^2 & 30 & K \\ s^1 & 9000 - K & 0 \\ s^0 & K & 0 \end{array}$$

The system is stable in closed-loop if there is no sign change in the first column. This requires $K < 9000$.

To achieve a gain margin of 6 dB, we use:

$$\Delta G = -20 \log G(\omega_p) = 6\,\text{dB}, \text{ with } \omega_p = -\pi.$$

The phase condition is:

$$\omega(\omega_p) = -3 \arctan\left(\frac{\omega_p}{10}\right) = -\pi,$$

leading to:

$$\arctan\left(\frac{\omega_p}{10}\right) = \frac{\pi}{3} \implies \omega_p = 17.3\,\text{rad/s}.$$

At this frequency:

$$G(\omega_p) = \frac{K}{(\sqrt{\omega_p^2 + 100})^3} = \frac{K}{8000}.$$

Thus:

$$\Delta G = -20 \log G(\omega_p) = -20 \log \frac{K}{8000} = -20 \log K + 78\,\text{dB}.$$

2. To achieve a gain margin of 6 dB:

$$\Delta G = 6\,\text{dB} \implies -20 \log K = -72\,\text{dB} \implies K = 3981.$$

3. Now, calculate the phase margin for this K value. At the crossover frequency:

$$G(\omega_{c0}) = \frac{3981}{(\sqrt{\omega_{c0}^2 + 100})^3} = 1.$$

Solving:

$$\sqrt{\omega_{c0}^2 + 100} = 15.85 \implies \omega_{c0} = 12.3\,\text{rad/s}.$$

The phase margin is:

$$\Delta\omega = \pi + \omega(\omega_{c0}) = \pi - 3\arctan\left(\frac{\omega_{c0}}{10}\right) = 0.48\,\text{rad} \approx 27°.$$

4. The plot of the Bode diagram. For $\omega \to 0$:

$$G(s) \approx \frac{K}{1000} \implies G(\omega) \approx \frac{3981}{1000} \approx 3.98,$$

resulting in a static gain of $20\log 3.98 = 12\,\text{dB}$.

The gain Bode plot starts with a horizontal segment at 12 dB until the cutoff frequency $\omega = 10\,\text{rad/s}$. Beyond this, the slope decreases by 3 units, transitioning to $-60\,\text{dB/decade}$. This segment passes through the point $(100, -48\,\text{dB})$. The asymptotic phase plot follows the slopes of the gain plot, enabling the identification of ω_{c0}, where the real gain curve intersects the x-axis. The phase margin is then measured as the difference between $-\pi$ and the phase at ω_{c0} on the phase diagram.

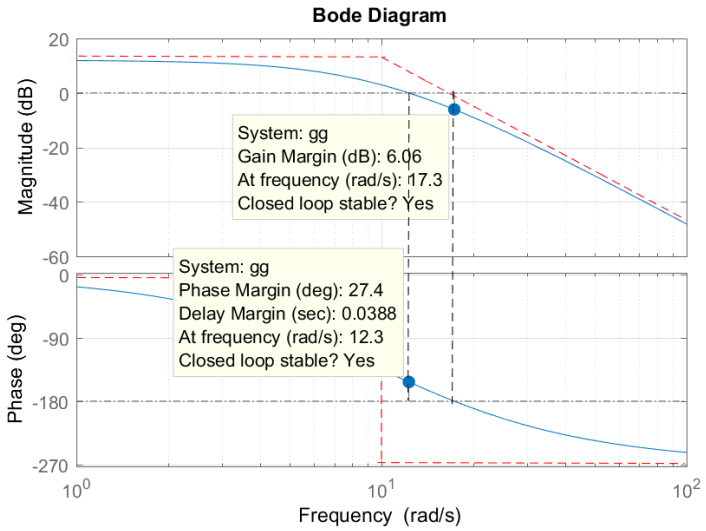


FIGURE 1.24: Highlighting the stability margins on the Bode diagram.

Similarly, the frequency ω_p can be easily located on the phase diagram as the point where the real phase curve intersects the line $\omega = -\pi$. At this frequency, the gain margin is then measured on the gain diagram as the difference between the real curve and the x-axis (Figure 1.24).

Exercise 03: Consider a system with an open-loop transfer function $G(s)$ given by:

$$G(s) = \frac{k}{s(s+100)^2}, \quad k > 0$$

Determine the conditions on the value of k such that the closed-loop system with unit feedback has a phase margin greater than 45° and a gain margin greater than 6 dB.

Solution of exercise 03:

The condition for the phase margin is expressed as follows:

$$\Delta w = \pi + w(v_{c0}) = \pi - \frac{\pi}{2} - 2\arctan\left(\frac{v_{c0}}{100}\right) = \frac{\pi}{4}$$

Therefore, we have:

$$2\arctan\left(\frac{v_{c0}}{100}\right) = \frac{\pi}{4}$$

Solving for v_{c0}, we get:

$$v_{c0} = 100\tan\left(\frac{\pi}{8}\right) = 41.4\,\text{rad/s}$$

Next, at this frequency, the open-loop transfer function satisfies:

$$G(v_{c0}) = \frac{K\,vc0}{v_{c0}^2 + 10^4} = 1$$

Thus, the value of K is:

$$K = 485 \times 10^3$$

In conclusion, a phase margin of $\Delta w > 45°$ is obtained for $K < 485 \times 10^3$.
Now, let's calculate the gain margin. By definition:

$$\Delta G = -20\log G(v_p) \quad \text{with} \quad w(v_p) = -\pi$$

At the gain crossover frequency v_p, the phase is:

$$w(v_p) = -\frac{\pi}{2} - 2\arctan\left(\frac{v_p}{100}\right) = -\pi$$

Solving for v_p, we find:

$$\arctan\left(\frac{v_p}{100}\right) = \frac{\pi}{4} \quad \Rightarrow \quad v_p = 100\,\text{rad/s}$$

At this frequency, the magnitude of the transfer function is:

$$G(v_p) = \frac{K\,v_p}{v_p^2 + 10^4} = K \times 2 \times 10^6$$

Thus, the gain margin is:

$$\Delta G = -20\log G(vp) = -20\log\left(K \times 2 \times 10^6\right) = -20\log K + 126\,\text{dB}$$

For the gain margin to be greater than 6 dB:

$$\Delta G > 6\,\text{dB} \quad \Rightarrow \quad -20\log K > -120\,\text{dB} \quad \Rightarrow \quad \log K < 6 \quad \Rightarrow \quad K < 10^6$$

Since both conditions $\Delta w > 45°$ and $\Delta G > 6\,\text{dB}$ must be satisfied simultaneously, the most restrictive condition on K is:

$$K < 485 \times 10^3$$

Exercise 04: Consider a system governed by the transfer function G(s), placed in a control loop:

$$G(s) = \frac{k}{(s+1)(s+8)^2}, \quad k > 0$$

a. Determine the stability conditions of this system placed in a unity feedback loop.
b. To eliminate the steady-state error, an integrator is introduced into the forward path. Determine the new stability conditions and draw conclusions.

Solution of exercise 04:

a. The given open-loop transfer function is:

$$G(s) = \frac{k}{(s+1)(s+8)^2}, \quad k > 0$$

The closed-loop transfer function with unity feedback is:

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{\frac{k}{(s+1)(s+8)^2}}{1+\frac{k}{(s+1)(s+8)^2}} = \frac{k}{(s+1)(s+8)^2+k}$$

The characteristic equation is:

$$D(s) = (s+1)(s+8)^2 + k$$

To ensure stability, we analyze the roots of the characteristic equation $D(s)$:

$$D(s) = s^3 + 17s^2 + 80s + (64+k)$$

Using the Routh-Hurwitz Criterion, the system is stable if all the coefficients of the characteristic polynomial are positive, and there are no sign changes in the first column of the Routh table.

Routh Table

$$\begin{array}{c|cc} s^3 & 1 & 80 \\ s^2 & 17 & 64+k \\ s^1 & \frac{17.80-1.(64+k)}{17} & 0 \\ s^0 & 64+k & \end{array}$$

The stability conditions are:

1. $64 + k > 0$, or $k > -64$. Since $k > 0$, this condition is satisfied. 2. The first-order coefficient:

$$\frac{17.80 - 1.(64 + k)}{17} > 0$$

Simplify:

$$17.80 - (64 + k) > 0 \quad \Rightarrow \quad k < 1296$$

the system is stable for all $k < 1296$.

b. Introducing an integrator in the forward path adds an $\frac{1}{s}$ term to the transfer function:

$$G(s) = \frac{k}{s(s + 1)(s + 8)^2}$$

The new characteristic equation becomes:

$$D(s) = s(s + 1)(s + 8)^2 + k$$

Expanding $(s + 8)^2$:

$$D(s) = s(s^2 + 16s + 64) + k$$

$$D(s) = s^3 + 16s^2 + 64s + k$$

The characteristic polynomial is:

$$D(s) = s^3 + 16s^2 + 64s + k$$

Routh Table for the Modified System

$$\begin{array}{c|cc} s^3 & 1 & 64 \\ s^2 & 16 & k \\ s^1 & \frac{16.64-1.k}{16} & 0 \\ s^0 & k & \end{array}$$

The stability conditions are:

1. $k > 0$. 2. The first-order coefficient:

$$\frac{16.64 - 1.k}{16} > 0$$

Simplify:

$$\Rightarrow \quad k < 1024$$

The system with Integrator is stable if $k < 1024$. The inclusion of the integrator eliminates the steady-state error but narrows the range for adjusting the gain k, thereby affecting the stability margin.

Exercise 05: Consider a system with an open-loop transfer function G(s) defined by:

$$G(s) = \frac{1}{(s+1)(s+4)}$$

We place this system in a unit feedback control loop by preceding it with a proportional controller C(s)=K with $K > 0$.

a. Calculate the value of K that ensures the system has a phase margin of 45°. Then, calculate the closed-loop rise time.

b. Determine the value of K that ensures a rise time of 0.2s. Calculate the new value of the phase margin and conclude.

Solution of exercise 05: The open-loop transfer function is given by:

$$KG(p) = \frac{K}{(p+1)(p+4)},$$

or equivalently:

$$KG(j\omega) = \frac{K}{(1+j\omega)(4+j\omega)}.$$

To achieve a phase margin of 45°, the condition is:

$$\Delta\phi = \pi - \arctan(\omega_{c0}) - \arctan\left(\frac{\omega_{c0}}{4}\right) = \frac{\pi}{4}.$$

This simplifies to:

$$\arctan(\omega_{c0}) + \arctan\left(\frac{\omega_{c0}}{4}\right) = \frac{3\pi}{4}.$$

Next, calculate the tangent of both sides of this equation:

$$\tan\left[\arctan(\omega_{c0}) + \arctan\left(\frac{\omega_{c0}}{4}\right)\right] = \tan\left(\frac{3\pi}{4}\right) = -1.$$

From this, we find:

$$\frac{5\omega_{c0}}{4(1-\frac{\omega_{c0}^2}{4})} = -1 \quad \Rightarrow \quad \omega_{c0}^2 - 5\omega_{c0} - 4 = 0.$$

Solving this quadratic equation:

$$\Delta = b^2 - 4ac = 25 + 16 = 41.$$

The only positive solution is:

$$\omega_{c0} = \frac{5+\sqrt{41}}{2} = 5.7\,\text{rad/s}.$$

By definition:

$$KG(\omega_{c0}) = \frac{K}{\sqrt{1+\omega_{c0}^2}\sqrt{16+\omega_{c0}^2}} = 1.$$

Thus,

$$K = \frac{\sqrt{1+\omega_{c0}^2}}{\sqrt{16+\omega_{c0}^2}} = 40.3.$$

Setting K to this value ensures a phase margin of 45°. The closed-loop rise time can be estimated using the approximate relation:

$$\omega_{c0}t_m \approx 3 \quad \Rightarrow \quad t_m \approx \frac{3}{\omega_{c0}} = \frac{3}{5.7} = 0.53\,\text{s}.$$

If the rise time is set to 0.2s, then:

$$\omega_{c0}t_m \approx 3 \quad \Rightarrow \quad \omega_{c0} \approx \frac{3}{0.2} = 15\,\text{rad/s}.$$

To achieve this cutoff frequency at 0 dB, the gain K must be adjusted:

$$KG(\omega_{c0}) = \frac{K}{\sqrt{1+\omega_{c0}^2}\sqrt{16+\omega_{c0}^2}} = 1,$$

so

$$K = \frac{\sqrt{1+\omega_{c0}^2}}{\sqrt{16+\omega_{c0}^2}} = 233.4.$$

The phase margin is then recalculated:

$$\Delta\phi = \pi - \arctan(\omega_{c0}) - \arctan\left(\frac{\omega_{c0}}{4}\right) = 0.326\,\text{rad} = 18.7°.$$

This result demonstrates that increasing the speed of a system with proportional control degrades its stability.

Exercise 06: Consider a system with an open-loop transfer function G(s) defined by:

$$G(s) = \frac{k}{(s+1)^3} \quad k > 0$$

which is placed in a unit feedback control loop. The goal is to achieve both a phase margin greater than 45° and a rise time of less than 0.5 seconds.

a. Calculate the value of k that ensures a closed-loop rise time of 2.15 seconds.

b. For this value of k, calculate the phase margin.

c. Deduce the expression for the phase lead regulator transfer function that needs to be introduced into the forward path to achieve the desired specifications.

Solution of exercise 06:

The open-loop transfer function is:

$$G(s) = \frac{k}{(s+1)^3}.$$

a. Calculate k for a rise time $t_m = 2.15\,\text{s}$ The rise time is related to the cutoff frequency ω_c approximately as:

$$\omega_c t_m \approx 3.$$

For $t_m = 2.15\,\text{s}$, the cutoff frequency is:

$$\omega_c = \frac{3}{t_m} = \frac{3}{2.15} \approx 1.395\,\text{rad/s}.$$

At the cutoff frequency, the open-loop gain magnitude is unity ($|KG(j\omega_c)| = 1$):

$$\left| \frac{k}{(j\omega_c + 1)^3} \right| = 1.$$

Taking the magnitude:

$$\frac{k}{\sqrt{(1+\omega_c^2)^3}} = 1.$$

Substitute $\omega_c = 1.395$:

$$k = \sqrt{(1 + 1.395^2)^3}.$$

Calculate k:

$$k = \sqrt{(1+1.945)^3} = \sqrt{(2.945)^3} \approx \sqrt{25.55} \approx 5.05.$$

Thus, $k \approx 5.05$.

b. Calculate the phase margin for $k = 5.05$ The open-loop phase margin is given by:

$$\phi(\omega) = -3\arctan(\omega) \quad \text{(since the pole at } s+1 \text{ repeats 3 times)}.$$

At the cutoff frequency $\omega_c = 1.395$:

$$\phi(\omega_c) = -3 \arctan(1.395).$$

Calculate $\arctan(1.395)$:

$$\arctan(1.395) \approx 0.95 \,\text{rad}.$$

Thus,

$$\phi(\omega_c) = -3 \times 0.95 = -2.85 \,\text{rad} \approx -163.4^\circ.$$

The phase margin is:

$$\Delta\phi = 180^\circ + \phi(\omega_c) = 180^\circ - 163.4^\circ = 16.6^\circ.$$

So, the phase margin for $k = 5.05$ is approximately 16.6°.

c. Design a phase lead regulator to achieve the desired specifications To meet the desired specifications ($\Delta\phi > 45^\circ$): 1. The regulator must provide an additional phase lead to compensate for the deficit. 2. A phase lead regulator has the form:

$$C_{\text{lead}}(s) = \frac{Ts + 1}{\alpha Ts + 1}, \quad 0 < \alpha < 1.$$

The maximum phase lead occurs at the frequency $\omega_m = \frac{1}{T\sqrt{\alpha}}$ and is given by:

$$\phi_{\max} = \sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right).$$

The required additional phase lead is:

$$\phi_{\text{add}} = 45^\circ - 16.6^\circ = 28.4^\circ.$$

Set $\phi_{\max} \geq 28.4^\circ$:

$$\sin^{-1}\left(\frac{1-\alpha}{1+\alpha}\right) \geq 28.4^\circ \quad \Rightarrow \quad \frac{1-\alpha}{1+\alpha} \geq \sin(28.4^\circ).$$

Calculate $\sin(28.4^\circ)$:

$$\sin(28.4^\circ) \approx 0.476.$$

Thus:

$$\frac{1-\alpha}{1+\alpha} \geq 0.476 \quad \Rightarrow \quad 1 - \alpha \geq 0.476(1+\alpha).$$

Expand and solve for α:

$$1 - \alpha \geq 0.476 + 0.476\alpha \quad \Rightarrow \quad 1 - 0.476 \geq \alpha(1 + 0.476).$$

$$\alpha \leq \frac{1-0.476}{1+0.476} \approx \frac{0.524}{1.476} \approx 0.355.$$

The frequency ω_m should align with $\omega_c = 1.395$:

$$\omega_m = \frac{1}{T\sqrt{\alpha}} \quad \Rightarrow \quad T = \frac{1}{\omega_m\sqrt{\alpha}}.$$

Substitute $\alpha = 0.355$ and $\omega_m = 1.395$:

$$T = \frac{1}{1.395\sqrt{0.355}} \approx \frac{1}{1.395 \times 0.596} \approx 1.195.$$

The phase lead regulator is:

$$C_{\text{lead}}(s) = \frac{1.195s+1}{0.424s+1}.$$

- The proportional gain $k = 5.05$ gives a rise time of 2.15 seconds but a low phase margin of 16.6°. - A phase lead regulator is required to achieve the desired phase margin and rise time, with the transfer function:

$$C_{\text{lead}}(s) = \frac{1.195s+1}{0.424s+1}.$$

1.8 Conclusion

Understanding frequency response characteristics and tuning techniques is essential for designing effective controllers. By examining specifications and applying design methods like Bode plots and the Black-Nichols chart, this chapter provides a comprehensive guide to enhancing controller performance in various applications.

Chapter 2

State space representation of systems

2.1 Introduction

This chapter provides an in-depth exploration of foundational concepts and various methods for representing dynamic systems. Key concepts, including state, state variable, control input, and output, will first be introduced to establish a basis for understanding system dynamics. The chapter will then discuss state-space representations, beginning with continuous linear systems and extending to discrete systems. Canonical forms will be examined to illustrate standard representations, followed by state-space modeling for nonlinear systems and methods of linearization to approximate their behavior under certain conditions. All the methods studied so far—whether for linear or nonlinear systems, in continuous or discrete time—remain effective until system complexity reaches a point where solely relying on input-output relationships no longer allows for adequate control. Similarly, these models become challenging to implement when dealing with systems that have multiple inputs and outputs. State-space representation serves as a powerful tool for modeling the behavior of both linear and nonlinear systems in either continuous or discrete time, with the added benefit of preserving the temporal representation of system dynamics.

2.2 State-space representations of continuous linear systems

State-space representations of continuous linear systems provide a mathematical model that describes a system's behavior in terms of a set of first-order differential equations. These equations use state variables to represent the internal dynamics of the

system, incorporating inputs, outputs, and the state itself. This framework captures the evolution of the system's state over time, making it a powerful tool for analyzing and controlling complex systems in engineering and science

2.2.1 System state and state variables

Any linear dynamic system can be modeled by linear differential equations. However, as the order of these differential equations increases, analyzing these systems becomes more complex. To address this, the state-space representation method simplifies the process by converting the system into a single first-order differential equation in matrix form. This approach makes it easier to handle complex systems by reducing them to a manageable, structured format suitable for analysis and control design. The state of a system is typically represented by a state vector, denoted as:

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \text{ or alternatively : } x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_i(t); (i = 1...n) \text{ are state variables.}$$

- **Note:** It is important to emphasize that the state-space representation of a system is not unique and largely depends on the choice of state variables we select.

Let's consider a system modeled by the following linear differential equation:

$$\overset{(3)}{z}(t) + 2\overset{(2)}{z}(t) - 3\overset{(1)}{z}(t) + z(t) = 7u(t) \tag{2.1}$$

where: $\overset{(i)}{z}(t) = \frac{d^i}{dz^i}$; $u(t)$ is the input of the system and $y(t) = 100z(t)$ is the output of the system. Let's assume: $x_1 = z(t)$; $x_2 = \dot{z}(t)$; $x_3 = \ddot{z}(t)$
with: $(\overset{(1)}{z}(t) = \dot{z}(t); \overset{(2)}{z}(t) = \ddot{z}(t); \overset{(3)}{z}(t) = \dddot{z})$
Equation 2.1 becomes:

$$\begin{cases} \dot{x}_1 = \dot{z} = x_2 \\ \dot{x}_2 = \ddot{z} = x_3 \\ \dot{x}_3 = \dddot{z} = -x_1 + 3x_2 - 2x_3 + 7u(t) \\ y(t) = 100z(t) = 100x_1 \end{cases} \tag{2.2}$$

Equation 2.6 can be expressed in the following matrix form:

$$\begin{cases} \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix} u(t) \\ y(t) = 7x_1 \end{cases} \tag{2.3}$$

This representation can be expressed in the following state model:

$$\begin{cases} \dot{X}(t) = A.X(t) + B.U(t) \\ Y(t) = C.X(t) + D.U(t) \end{cases} \tag{2.4}$$

where: where, A is state matrix of size $(n \times n)$; B is the input matrix of size $(n \times p)$; $X(t)$ is the state vector of size $(n \times 1)$; $U(t)$ is the input vector of size $(p \times 1)$; $Y(t)$ is the output vector of size $(q \text{x} n)$. Any direct relationship between inputs and outputs must be considered. The D matrix, with dimensions $p \text{x} q$, represents this connection. The state representation of such a linear system can be illustrated as shown in Figure 2.1.

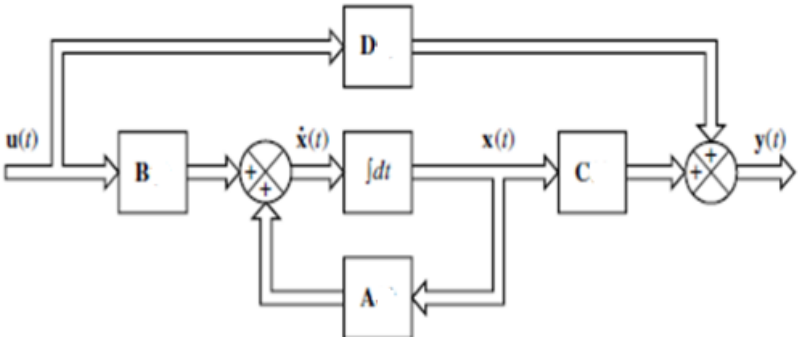


FIGURE 2.1: Schematic representation of state model.

2.3 State-space representations of discrete linear systems

In continuous-time state-space representation, we introduced functional diagrams of systems that include only integrators and gains, demonstrating that it is possible to reconstruct any transfer function from these elements. In discrete time, a system can similarly be decomposed using gains and shift operators, specifically with transfer functions involving z^{-1} operators. In a discrete-time representation, the ability to express the system's state at a given moment based on the input signal and its "past," or its previous state, is particularly meaningful. The general form for state equations of such a linear system in discrete time is therefore:

$$\begin{cases} x(k+1) = A.x(k) + B.e(k) \\ s(k) = C.x(k) + D.e(k) \end{cases} \tag{2.5}$$

- **Note:** For a system of order n, meaning it has n state variables, the first equation is referred to as the control equation, and the second as the observation equation.

The schematic representation in Figure 2.2 is often used to illustrate this modeling approach. Note that, in this representation, the signals are actually vectors of discrete-time signals.

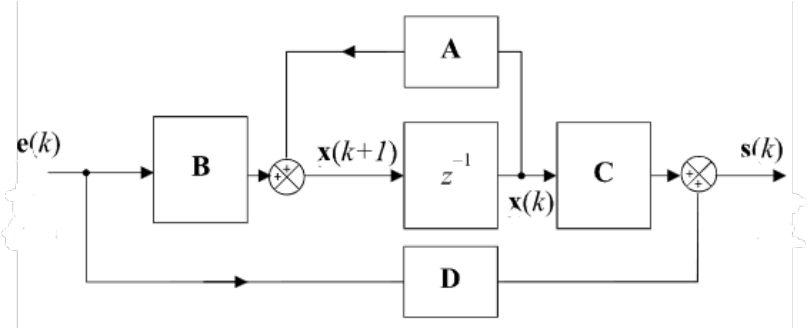


FIGURE 2.2: Schematic representation of a state-space model in discrete time.

- **Example:**Consider a system governed by the control equation:

$$\begin{cases} x(k+1) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} .x(k) + \begin{bmatrix} 2 \\ 5 \end{bmatrix} .u(k) \end{cases} \tag{2.6}$$

Let us assume that this system is subjected to a unit step input, where $u(k) = 1$ for all $k0$, and we will determine the sequence of the first five samples corresponding to the two signals of the state vector, namely $x_1(k)$ and $x_2(k)$. We assume that the initial state is characterized by $x(0) = 0$. By performing successive iterations, we can easily obtain the sequence corresponding to the state vector at each sampling instant.

The 05 state vectors are obtained below:

$$\begin{cases} x(1) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \\ x(2) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -15 \end{bmatrix} \\ x(3) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -15 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -16 \\ 30 \end{bmatrix} \\ x(4) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} -16 \\ 30 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 80 \\ 25 \end{bmatrix} \\ x(5) = \begin{bmatrix} -3 & 1 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} 80 \\ 25 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} -213 \\ -445 \end{bmatrix} \end{cases} \tag{2.7}$$

2.4 Canonical forms

As previously mentioned, the state-space representation of a system is not unique. In the following sections, we will introduce several types of state-space representations that can be derived from a transfer function G(p)G(p). The choice of representation may depend on the available form of the transfer function or the specific type of analysis intended from the model. It is also important to reiterate that the state-space representation of a system is not unique, and the system's properties are not inherently tied to the chosen representation.

2.4.1 Modal representation

This type of representation, also known as parallel representation, is particularly suitable for modeling a system whose transfer function is expressed as a sum. Let $G(s)$ be its transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\alpha_1}{1-p_1} + \frac{\alpha_2}{1-p_2} + ... + \frac{\alpha_n}{1-p_n} \tag{2.8}$$

In this case, the state-space representation is provided as follows:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 0 & p_2 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & p_n \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix} x(t) \end{cases} \tag{2.9}$$

The matrix A is diagonal, and its eigenvalues correspond to the poles of the transfer function.

2.4.2 Series representation

Let's consider the following transfer function:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\alpha}{(1-p_1)(1-p_2)...(1-p_n)} \tag{2.10}$$

The state-space of $G(s)$ in series representation is provided as follows:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} p_1 & 0 & \cdots & 0 \\ 1 & p_2 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & p_n \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & \cdots & \alpha \end{bmatrix} x(t) \end{cases} \tag{2.11}$$

2.4.3 Controllable canonical representation

Here, we assume that the transfer function is not factored:

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_1 s + a_0} \tag{2.12}$$

where: $m < n$.

Let's choose $a_n = 1$ (which is equivalent to dividing both the numerator and the denominator by a_n) and divide the numerator and denominator by s^n. We get:

$$G(s) = \frac{b_m s^{m-n} + b_{m-1} s^{m-n-1} + ... + b_1 s^{-n+1} + ... + b_0 s^{-n}}{1 + a_{n-1} s^{-1} + a_{n-2} s^{-2} + ... + a_1 s^{-n+1} + ... + a_0 s^{-n}} \tag{2.13}$$

The controllable canonical representation of this system can be given by:

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0 & a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} b_0 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix} x(t) \end{cases} \tag{2.14}$$

This type of representation is known as the companion form because the system's control matrix includes, in a single row, all the coefficients of the transfer function's denominator. It is referred to as controllable since, although only the variable x_n is directly influenced by the input signal, all state variables are affected through successive integrations. If a system is fully controllable, it can be transformed into a controllable companion form.

2.4.4 Observable canonical representation

Easier to understand, the observable companion form can be highlighted using the previously transformed form of $G(s)$ from the preceding paragraph (Equation 3.13).

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & 0 & \vdots \\ 0 & \cdots & 0 & 0 & 1 & -a_n \end{bmatrix} x(t) + \begin{bmatrix} b_0 \\ \vdots \\ b_m \\ \vdots \\ 0 \\ 0 \end{bmatrix} u(t) \\ y(t) = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} x(t) \end{cases} \tag{2.15}$$

This form of representation is known as the observable companion form because the system's control matrix consolidates all the transfer function's denominator coefficients into a single column. Additionally, the system's output is equal to x_n, which is influenced by all state variables through successive integrations. If a system is fully observable, it can be represented in this observable companion form.

2.5 Calculation of the transfer function from the state-space representation

If we apply the Laplace transform to the state equations, we obtain:

$$\begin{cases} \dot{x}(t) = A.x(t) + B.u(t) \\ y(t) = C.x(t) \end{cases} \tag{2.16}$$

$$\Rightarrow \begin{cases} sX(s) = A.X(s) + B.U(s) \\ Y(s) = C.X(s) \end{cases} \Rightarrow \begin{cases} sX(s) - A.X(s) = B.U(s) \\ Y(s) = C.X(s) \end{cases} \Rightarrow \begin{cases} (s.I - A).X(s) = B.U(s) \\ Y(s) = C.X(s) \end{cases}$$

$$\Rightarrow \begin{cases} X(s) = (s.I - A).^{-}1B.U(s) \\ Y(s) = C.X(s) \end{cases} \Rightarrow \begin{cases} Y(s) = C.[((s.I - A)^{-}1)B].U(s) \end{cases}$$

The transfer function can be written as: :

$$G(s) = \frac{Y(s)}{U(s)} = C.[((s.I - A)^{-}1)B] \tag{2.17}$$

The inverse of a square matrix is equal to its adjoint matrix divided by its determinant. From this, we can deduce that the poles of the transfer function are the values of s that are solutions to the equation: $det(s.I - A) = 0$. These are the eigenvalues of the matrix [A].

- **Note:** To obtain a transfer function that models the actual behavior of the system, it must be both controllable and observable. Otherwise, the transfer function obtained will only correspond to its controllable and observable part.

2.6 State-space representations of nonlinear linear systems

In nonlinear systems, the fundamental physical behavior is represented by nonlinear differential equations. While the state-space representation can manage these systems, transfer function methods are not effective. A general nn-th order nonlinear, continuous-time, SISO system is represented by a nonlinear, scalar ordinary differential equation with constant coefficients:

$$\frac{d^n x(t)}{dt^n} = f(x(t), \frac{dx(t)}{dt}, \frac{d^2x(t)}{dt^2}, ..., \frac{d^{n-1}x(t)}{dt^{n-1}}, u(t), \frac{du(t)}{dt}, ..., \frac{d^m u(t)}{dt^m}) \tag{2.18}$$

where: $n \geq m$ and, if we put the following variables change, Equation 2.19 can be obtained:

$$\begin{cases} x_1 = x(t) \\ x_2 = \frac{dx(t)}{dt} \\ \quad \vdots \\ x_n = \frac{d^n x(t)}{dt^n} \end{cases} \text{ and } \begin{cases} u_1 = u(t) \\ u_2 = \frac{du(t)}{dt} \\ \quad \vdots \\ u_m = \frac{d^m u(t)}{dt^m} \end{cases}$$

$$\dot{X} = f(X(t), U(t)) \tag{2.19}$$

where: $X(t) = [x_1, x_2, ..., x_n]^T$, $U(t) = [u_1, u_2, ..., u_m]^T$

f is a nonlinear vector function. If we choose the output $y(t) = x(t)$, then:

$$y(t) = C.X(t) \tag{2.20}$$

with: $C = [1, 0, 0, ..., 0]$

2.7 linearization techniques

Here are some common linearization techniques widely used in the literature:

- Taylor Series Expansion (Small Signal Linearization): This technique involves expanding the nonlinear function around an operating point using a Taylor series and neglecting higher-order terms, making it suitable for systems with small deviations around the equilibrium.

- Jacobian Linearization: For systems represented by nonlinear differential equations, the Jacobian matrix of partial derivatives around a steady state is used to approximate the system with a linear model.

- Describing Function Method:This approximate method is used primarily for nonlinear systems with periodic inputs. It replaces the nonlinear element with a quasi-linear transfer function, allowing analysis in the frequency domain.

- Feedback Linearization:This technique applies a nonlinear feedback transformation to convert a nonlinear system into a fully or partially linearized system. It is commonly used in control applications where the system's structure allows exact linearization.

- Input-Output Linearization: Also known as Lie Derivative-based linearization, this method linearizes the input-output relationship of a system by transforming state variables, often used for nonlinear control design.

- State Transformation Linearization: By transforming the state space, this approach linearizes a system near a point or along a trajectory, often using tools from differential geometry.

- Extended Kalman Filter (EKF) Linearization: Used in estimation, this technique approximates nonlinear dynamics by linearizing about the current estimate at each time step, widely applied in signal processing and control systems.

Each technique has specific applications depending on the type and structure of the nonlinear system as well as the desired accuracy and computational requirements.

- **Example** (Jacobian Linearization):

A physical system is in state-space representation when we have a mathematical model of it as a set of input, output and state variables related by first-order differential equations only.

The system is given by:

$$m\ddot{y} + b\dot{y} + k_1 y + k_2 y^3 = u \tag{2.21}$$

By introducing $x_1 = \dot{y}; x_2 = y$, Equation 2.23 can be obtained:

$$\dot{x} = \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\dot{y}\\ y\end{pmatrix} = \begin{bmatrix}\ddot{y}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\frac{1}{m}(-b\dot{y} - (k_1 + k_2 y^2)y)\\ \dot{y}\end{bmatrix} + \begin{bmatrix}\frac{u}{m}\\ 0\end{bmatrix} \tag{2.22}$$

$\Rightarrow$

$$\dot{x} = \begin{bmatrix}\frac{-b}{m} & -(k_1 + k_2 y^2) + \\ 1 & 0\end{bmatrix}\begin{bmatrix}\dot{y}\\ y\end{bmatrix} + \begin{bmatrix}\frac{1}{m}\\ 0\end{bmatrix} u \tag{2.23}$$

or:

$$\dot{X} = A(X).X + B.u \tag{2.24}$$

Note that the system matrix A is not constant; however, this is still a state-space representation.

Given a non-linear system:

$$\dot{X} = f(X) + B.u \tag{2.25}$$

The idea of linearization is to work out of a current set point x_0, so that:

$$f(X_0 + d) \approx f(X_0) + J_f(X_0).d \tag{2.26}$$

Here, we have : $f_1(x_1, x_2) = \frac{1}{m}(-bx_1 - (k_1 + k_2x_2^2)x_2)$ and $f_2(x_1, x_2) = x_1$ So that the Jacobian matrix is:

$$J_f(X) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{-b}{m} & \frac{-k_1-3k_2x^2}{m} \\ 1 & 0 \end{bmatrix} \tag{2.27}$$

Note that : $\dot{X} = \frac{\mathrm{d}X}{\mathrm{d}t} = \frac{\mathrm{d}(X_0+d)}{\mathrm{d}t} = \frac{\mathrm{d}d}{\mathrm{d}t} = \dot{d}$ so:

$$\dot{d} = J_f(X_0).d + f(X_0) + Bu \tag{2.28}$$

or,

$$\dot{d} = Ad + Bu + c \tag{2.29}$$

where: $A = J_f(X_0)$ and $c = f(X_0)$

2.8 Solved exercises

Exercise 01:

We consider a system with a transfer function G(s) given by:

$$G(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n}s + 1}.$$

a. Propose a state-space representation of this system in observable canonical form.

b. Propose a state-space representation of this system in controllable canonical form.

Solution of exercise 01:

The transfer function of the system is given as:

$$G(s) = \frac{K}{\frac{s^2}{\omega_n^2} + \frac{2\xi}{\omega_n}s + 1}.$$

Rewriting for clarity:

$$G(s) = \frac{K\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

The denominator is the characteristic polynomial of the system:

$$P(s) = s^2 + 2\xi\omega_n s + \omega_n^2.$$

a. State-Space Representation in Observable Canonical Form:

For observable canonical form, the state-space matrices are structured to directly represent the transfer function from input to output. The state-space model is:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u, \quad y = \mathbf{C}\mathbf{x} + \mathbf{D}u.$$

- The characteristic polynomial is:

$$P(s) = s^2 + 2\xi\omega_n s + \omega_n^2.$$

- The corresponding observable canonical form is:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix}.$$

- The output matrices are:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \mathbf{D} = 0.$$

Thus, the observable canonical form is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}.$$

b. State-Space Representation in Controllable Canonical Form:

For controllable canonical form, the state-space matrices are structured to focus on the input-output controllability of the system.

- The characteristic polynomial $P(s)$ is written as:

$$P(s) = s^2 + 2\xi\omega_n s + \omega_n^2.$$

In controllable canonical form, the state-space representation is derived directly from this polynomial.

- The state-space matrices are:

$$\mathbf{A} = \begin{bmatrix} -2\xi\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- The output matrices depend on the numerator of $G(s)$:

$$\mathbf{C} = \begin{bmatrix} K & 0 \end{bmatrix}, \quad \mathbf{D} = 0.$$

Thus, the controllable canonical form is:

$$\dot{x} = \begin{bmatrix} -2\xi\omega_n & -\omega_n^2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad y = \begin{bmatrix} K & 0 \end{bmatrix} x.$$

Exercise 02:

We consider a system with a transfer function G(s) given by:

$$G(s) = \frac{s+1}{s^3 + 2s^2 + 4s + 8}.$$

- Propose a state-space representation of this system in observable canonical form.

Solution of exercise 02:

The given transfer function is:

$$G(s) = \frac{s+1}{s^3 + 2s^2 + 4s + 8}.$$

The denominator of $G(s)$, $P(s) = s^3 + 2s^2 + 4s + 8$, is the characteristic polynomial, and the numerator, $N(s) = s + 1$, determines the output equation.

In the observable canonical form, the state-space representation is structured directly to reflect the relationship between the input and output. The general form is:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du.$$

- Step 1: Write the Characteristic Polynomial and Denominator Dynamics The characteristic polynomial is:

$$P(s) = s^3 + 2s^2 + 4s + 8.$$

This corresponds to the following differential equation:

$$y^{(3)} + 2y^{(2)} + 4\dot{y} + 8y = u.$$

- Step 2: Define the State Variables Define the state variables as:

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}.$$

Then:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = -8x_1 - 4x_2 - 2x_3 + u.$$

- Step 3: Construct the State-Space Matrices From the above equations, the state-space matrices are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -4 & -2 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

- Step 4: Derive the Output Equation The numerator $N(s) = s + 1$ determines the output equation. Using the state variables, the output is:

$$y = x_3 + x_1.$$

Thus:

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = 0.$$

The state-space representation in observable canonical form is:

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -8 & -4 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \mathbf{x}.$$

Exercise 03:

a. Determine the transfer function of the system defined by its state equations:

$$\dot{x}(t) = A \cdot x(t) + B \cdot e(t)$$

$$s(t) = C \cdot x(t)$$

where the matrices are defined as:

$$A = \begin{bmatrix} -2 & -4 \\ -2 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

b. Calculate the poles of the transfer function and derive the expression for its output in response to a unit step input.

c. Assume that the system's initial state is defined by a zero state vector.

Solution of exercise 03:

The transfer function of the system is expressed as:

$$G(p) = \mathbf{C} \cdot (p\mathbf{I} - \mathbf{A})^{-1} \cdot \mathbf{B}.$$

We have:

$$p\mathbf{I} - \mathbf{A} = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix} - \begin{bmatrix} -2 & -4 \\ -2 & -9 \end{bmatrix} = \begin{bmatrix} p+2 & 4 \\ 2 & p+9 \end{bmatrix}.$$

Thus, the inverse is given by:

$$(p\mathbf{I} - \mathbf{A})^{-1} = \frac{1}{\det(p\mathbf{I} - \mathbf{A})} \begin{bmatrix} p+9 & -4 \\ -2 & p+2 \end{bmatrix},$$

where:

$$\det(p\mathbf{I} - \mathbf{A}) = (p+2)(p+9) - 8.$$

The inverse of $(pI - A)$ is given by:

$$(pI - A)^{-1} = \frac{1}{p^2 + 11p + 10} \begin{bmatrix} p+9 & -4 \\ -2 & p+2 \end{bmatrix}.$$

Thus, the transfer function is:

$$G(p) = \frac{1}{p^2 + 11p + 10} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} p+9 & -4 \\ -2 & p+2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

which simplifies to:

$$G(p) = \frac{-4}{p^2 + 11p + 10} = \frac{-4}{(p+1)(p+10)}.$$

The poles of $G(p)$ are clearly at $p = -1$ and $p = -10$.

When a unit step input is applied to the system, the Laplace transform of the response is:

$$S(p) = \frac{-4}{p(p+1)(p+10)}.$$

Taking the inverse Laplace transform gives the time-domain response:

$$s(t) = \frac{-4}{10}\left[1 + \frac{10}{1 - 10e^{-t}} - \frac{1}{1 - 10e^{-10t}}\right] = \frac{-4}{10} + \frac{4}{9}e^{-t} - \frac{4}{90}e^{-10t}.$$

Exercise 04:

Determinate the expression for the transition matrix $\Phi(t)$ of the system defined by the state equations:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

where:

$$A = \begin{bmatrix} -3 & -6 \\ 0 & 9 \end{bmatrix}.$$

Solution of exercise 04: We are asked to calculate the expression for the transition matrix $\Phi(t)$ of the system defined by the state equations:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

where:

$$A = \begin{bmatrix} -3 & -6 \\ 0 & 9 \end{bmatrix}.$$

We need to compute the matrix exponential e^{At}, which can be done by finding the eigenvalues and eigenvectors of A or by using other methods for diagonalizable or block matrices.

First, we calculate the eigenvalues of A by solving the characteristic equation:

$$\det(A - \lambda I) = 0.$$

For the given matrix A:

$$A - \lambda I = \begin{bmatrix} -3 - \lambda & -6 \\ 0 & 9 - \lambda \end{bmatrix}.$$

The determinant is:

$$\det(A - \lambda I) = (-3 - \lambda)(9 - \lambda) - (0)(-6) = (\lambda + 3)(\lambda - 9).$$

Setting the determinant equal to zero:

$$(\lambda + 3)(\lambda - 9) = 0,$$

which gives the eigenvalues:

$$\lambda_1 = -3, \quad \lambda_2 = 9.$$

- Eigenvectors of A

For $\lambda_1 = -3$, solve:

$$(A-(-3)I)v=0 \quad \Rightarrow \quad \begin{bmatrix} 0 & -6 \\ 0 & 12 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

From this, we get the eigenvector:

$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

For $\lambda_2 = 9$, solve:

$$(A-9I)v=0 \quad \Rightarrow \quad \begin{bmatrix} -12 & -6 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0.$$

From this, we get the eigenvector:

$$v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The general solution for the state transition matrix is:

$$\Phi(t) = Pe^{\Lambda t}P^{-1},$$

where P is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. We have:

$$P = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}, \quad \Lambda = \begin{bmatrix} -3 & 0 \\ 0 & 9 \end{bmatrix}.$$

The matrix exponential $e^{\Lambda t}$ is:

$$e^{\Lambda t} = \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{9t} \end{bmatrix}.$$

Now, compute P^{-1}:

$$P^{-1} = \frac{1}{-2} \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Now, we can compute $\Phi(t)$:

$$\Phi(t) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{9t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Or:

$$\Phi(t) = \begin{bmatrix} e^{-3t} & e^{9t} \\ -2e^{-3t} & -2e^{9t} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

The result is:

$$\Phi(t) = \begin{bmatrix} e^{-3t} - e^{9t} \cdot \frac{1}{2} & -\frac{e^{9t}}{2} + \frac{e^{-3t}}{2} \\ -2e^{-3t} + e^{9t} & e^{9t} - e^{-3t} \end{bmatrix}.$$

This is the transition matrix $\Phi(t)$ for the given system.

2.9 Conclusion

In this chapter, we have thoroughly examined essential concepts and methods for representing dynamic systems, establishing a solid understanding of states, state variables, control inputs, and outputs. Through the introduction of state-space representation, we analyzed both continuous and discrete linear systems and explored canonical forms to standardize these representations. Furthermore, we delved into state-space modeling for nonlinear systems and discussed linearization techniques to simplify these models under certain conditions.

The methods covered provide robust tools for analyzing and controlling systems across a wide range of complexity, from simple linear models to more complex nonlinear systems. However, as system complexity increases, these models can become limited, particularly for systems with multiple inputs and outputs. Here, state-space representation proves invaluable, offering a comprehensive framework that maintains the system's temporal characteristics and provides an effective basis for understanding and controlling both linear and nonlinear dynamics in continuous or discrete time.

This chapter has laid the groundwork for advanced control strategies and sets the stage for further exploration of techniques that enhance the controllability and observability of complex dynamic systems

Chapter 3

Systems analysis in state-space

3.1 Introduction

State-space representation is a powerful framework for analyzing and designing dynamic systems. This chapter focuses on key aspects of state-space analysis, providing a systematic approach to understanding and solving systems described in this form. We begin by addressing the solution of state equations, which define the evolution of a system's state variables over time. These equations serve as the foundation for predicting system behavior and assessing performance in response to inputs. Next, we explore various methods for computing the transition matrix, a crucial tool in state-space analysis. The transition matrix provides a direct way to solve the state equations, linking the system's initial state to its future states over time. The chapter also delves into modal analysis and stability, examining how the diagonalization of system matrices can simplify the study of system dynamics. Through this approach, we gain insights into stability criteria and the role of eigenvalues and eigenvectors in shaping system behavior. Finally, we introduce the concepts of controllability and observability, which are fundamental to evaluating a system's ability to be controlled and observed through its inputs and outputs. These properties are essential in determining whether a system is well-posed for control design and implementation.

3.2 Solution of the state equations

It refers to the process of finding the state variables of a dynamic system over time, given initial conditions and system inputs. This typically includes determining the system's response using mathematical methods such as matrix exponentials, transition matrices, or numerical techniques. The state equation of a LTI system can be given by:

$$\dot{x}(t) = A.x(t) + B.u(t) \tag{3.1}$$

Let's apply the Laplace transform to the system of differential equations in the state-space representation.

$$PX(s) - x(0) = A.X(s) + B.U(s) \tag{3.2}$$

$\Rightarrow (sI - A)X(s) = x(0) + B.U(s)$(3.3) where I represents the square identity matrix of dimension n. Thus:

$$X(s) = (sI - A)^{-1}x(0) + (sI - A)^{-1}B.U(s) \tag{3.4}$$

The inverse Laplace transform of Equation 3.5 provides the following time-domain solution for $x(t)$:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}B.u(\tau)d\tau \tag{3.5}$$

where: e^{At} is the transition matrix.

3.3 Methods for computing the transition matrix

In the resolution of state-space equations, one of the most delicate operations is calculating the transition matrix. This matrix plays a crucial role in determining the system's behavior over time. Various methods have been developed to compute this matrix, each with its own advantages and challenges. In this section, we will explore some of these approaches, highlighting their applications and the mathematical principles that underpin them.

3.3.1 Use of the Laplace transform

The transition matrix e^{At} has the Laplace transform given by the matrix $(sI - A)^{-1}$. Let's transform the expression of $(sI - A)^{-1}$:

$$(sI - A)^{-1} = \frac{1}{s}(I - \frac{A}{s})^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + ... + \frac{A^n}{s^{n+1}} + ... \tag{3.6}$$

By applying the inverse Laplace transform, we obtain:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + ... + \frac{A^n}{n!}t^n + ... \tag{3.7}$$

This result will come as no surprise, as it reflects the expression of the Taylor series expansion.

3.3.2 Application to the direct calculation of the transition matrix

The Taylor series expansion can be used directly, provided that the control matrix is nilpotent (there exists a positive integer k such that$A^k = 0$ for all $n > k$). Naturally, the smaller k is, the simpler and faster the direct computation becomes.

- **Example:** Let us now consider the control matrix: $A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}$

$$A^2 = A \times A = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \times \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \Rightarrow A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore, for $n \geq 2$:

$$A^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We have thus:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + ... + \frac{A^n}{n!}t^n + ... = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix}.t = \begin{bmatrix} 1+2t & -4t \\ t & 1-2t \end{bmatrix} \tag{3.8}$$

3.3.3 Diagonalization method for the transition matrix

It is easy to observe that the calculation of the transition matrix is straightforward when it is diagonal. Indeed:

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix} \tag{3.9}$$

This observation naturally leads us to devise a relatively simple method for calculating e^{At}: it is enough to diagonalize the matrix A. Let us consider an arbitrary control matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \tag{3.10}$$

The eigenvectors and eigenvalues of this matrix are defined by:

$$A.V_i = \lambda_i V_i \tag{3.11}$$

The non-zero vectors V_i are the eigenvectors of A, and the λ_i are its eigenvalues. These quantities are very easy to determine, as the λ_i are the roots of the characteristic equation of the matrix A, defined by: $det(A) = 0$.

If we denote D as the diagonal matrix formed by the eigenvalues of A, and T as the modal matrix formed by its eigenvectors, we have:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix} \tag{3.12}$$

and:

$$T = \begin{bmatrix} (V_1) & (V_2) & \cdots & (V_n) \end{bmatrix} \tag{3.13}$$

Under these conditions: $A = T.D.T^{-1}$

It is then easy to verify that:

$$A^n = TDT^{-1} \times TDT^{-1} \times TDT^{-1} \times TDT^{-1} \cdots TDT^{-1} = TD^nT^{-1} \tag{3.14}$$

Since:

$$e^{At} = I + At + \frac{A^2}{2!}t^2 + ... + \frac{A^n}{n!}t^n + ... \tag{3.15}$$

Equation 3.15 becomes:

$$e^{At} = T(I + Dt + \frac{D^2}{2!}t^2 + ... + \frac{D^n}{n!}t^n + ...)T^{-1} = Te^{Dt}T^{-1} \tag{3.16}$$

Therefore:

$$e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & e^{\lambda_n t} \end{bmatrix} T^{-1} \tag{3.17}$$

Consequently, the functions $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, ... and $e^{\lambda_n t}$, which form the basis of the elementary functions of the state vector, and thus the internal signals of the system, are called the modes of the system.

- **Example:** Let us consider the following control matrix: $A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}$

Let us calculate the transition matrix using the diagonalization method.

The eigenvalues of this matrix are determined as the solutions of its characteristic equation: $det(\lambda I - A) = 0 \Rightarrow det(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}) = 0 \Rightarrow det \begin{bmatrix} \lambda + 2 & 2 \\ 1 & \lambda + 3 \end{bmatrix} = 0$

Let it be: $(\lambda+2)(\lambda+3)-2=0 \Rightarrow \lambda_1=-1$ and $\lambda_2=-4$ Let us calculate the first eigenvector: $A=\begin{bmatrix}-2 & -2\\ -1 & -3\end{bmatrix}.\begin{bmatrix}x_1\\ y_1\end{bmatrix}=\lambda_1\begin{bmatrix}x_1\\ y_1\end{bmatrix}=-\begin{bmatrix}x_1\\ y_1\end{bmatrix}$

$$\Rightarrow \begin{cases}-2x_1-2y_1=-x_1\\ -x_1-3y_1=-y_1\end{cases} \Rightarrow x_1=-2y_1$$

- **Note:** This result should not be surprising, as there are infinitely many eigenvectors, all collinear, associated with a given eigenvalue.

For example, let us take: $V_1=\begin{bmatrix}-2\\ 1\end{bmatrix}$

Similarly, the second eigenvector can be calculated just as easily: $A=\begin{bmatrix}-2 & -2\\ -1 & -3\end{bmatrix}.\begin{bmatrix}x_2\\ y_2\end{bmatrix}=\lambda_2\begin{bmatrix}x_2\\ y_2\end{bmatrix}=-4\begin{bmatrix}x_2\\ y_2\end{bmatrix}$

$$\Rightarrow \begin{cases}-2x_2-2y_2=-4x_2\\ -x_2-3y_2=-4y_2\end{cases} \Rightarrow x_2=y_2$$

let us take: $V_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ Thus, we have: $T=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}$ and : $T^{-1}=\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix}$

Furthermore: $D=\begin{bmatrix}-1 & 0\\ 0 & -4\end{bmatrix}$ Let us now calculate the transition matrix:

$$e^{At}=\begin{bmatrix}-2 & 1\\ 1 & 1\end{bmatrix}.\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-4t}\end{bmatrix}.\begin{bmatrix}-\frac{1}{3} & \frac{1}{3}\\ \frac{1}{3} & \frac{2}{3}\end{bmatrix} \tag{3.18}$$

From which:

$$e^{At}=\begin{bmatrix}\frac{2}{3}e^{-t}+\frac{1}{3}e^{-4t} & -\frac{2}{3}e^{-t}+\frac{1}{3}e^{-4t}\\ -\frac{1}{3}e^{-t}+\frac{1}{3}e^{-4t} & \frac{1}{3}e^{-t}+\frac{2}{3}e^{-4t}\end{bmatrix} \tag{3.19}$$

3.4 Modal analysis and stability (diagonalization)

In state-space analysis, when the system matrix A has multiple poles, its eigenvalues result in a Jordan form with one or more Jordan blocks. To compute the state-transition matrix e^{At}, we diagonalize A if possible, or use its Jordan decomposition. The matrix A can be written as $A=PJP^{-1}$, where J is the Jordan matrix and P is the modal matrix of eigenvectors and generalized eigenvectors. The exponential of A

becomes $ee^{At} = Pe^{Jt}P^{-1}$. For a Jordan block corresponding to an eigenvalue λ, say:

$$J_k = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \tag{3.20}$$

the matrix exponential $e^{J_k t}$ is computed as:

$$e^{J_k t} = e^{\lambda t}. \begin{bmatrix} 1 & t & \frac{t^2}{2!} & \cdots & \frac{t^{(n-1)}}{(n-1)!} \\ 0 & 1 & t & \cdots & \frac{t^{(n-2)}}{(n-2)!} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & t \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.21}$$

This approach handles the coupling between repeated eigenvalues via the structure of the Jordan block. Finally, e^{At} is constructed by assembling all Jordan block exponentials into e^{Jt} and transforming it back with P.

- **Calculation of the matrix P:** Let us describe the general algorithm. Suppose that A has an eigenvalue λ of multiplicity m. We find vectors such that: $(A-\lambda I)^k \overrightarrow{v} = \overrightarrow{0}$ but $(A-\lambda I)^{k-1} \overrightarrow{v} \neq \overrightarrow{0}$ Such vectors are called generalized eigenvectors. For every eigenvector $\overrightarrow{v_1}$ we find a chain of generalized eigenvectors $\overrightarrow{v_2}$ through $\overrightarrow{v_k}$ such that: $(A-\lambda I)\overrightarrow{v_1} = \overrightarrow{0}$;$(A-\lambda I)\overrightarrow{v_2} = \overrightarrow{v_1}$; ...; $(A-\lambda I)\overrightarrow{v_k} = \overrightarrow{v_{k-1}}$ Finlay, the matrix P can be given by: $P = [(v_1)(v_2)...(v_m)]$

- **Example:** As an example, consider the following matrix: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix}$

  The characteristic equation is: $|\lambda I - A| = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3$.

  The matrix A has an eigenvalue $\lambda = 1$ of multiplicity 3. The Jordan matrix is: $J = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and the matrix $P = [(v_1)(v_2)(v_3)]$ where:

  $(A-\lambda I) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & -3 & 2 \end{bmatrix}$ and : $(A-\lambda I)\overrightarrow{v_1} = \overrightarrow{0}$; $(A-\lambda I)\overrightarrow{v_2} = \overrightarrow{v_1}$; and $(A-\lambda I)\overrightarrow{v_3} = \overrightarrow{v_2}$

$$(A-\lambda I)\,\overrightarrow{v_1}=\overrightarrow{0}\Rightarrow\begin{bmatrix}-1&1&0\\0&-1&1\\1&-3&2\end{bmatrix}.\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\Rightarrow\begin{cases}y-x=0\\z-y=0\\x-3y+2z=0\end{cases}$$

Taken $x=1$ then the calculated values of y and z are $y=1$ and $z=1$, so: $v_1=\begin{bmatrix}1\\1\\1\end{bmatrix}$

$(A-\lambda I)\,\overrightarrow{v_2}=\overrightarrow{v_1}\Rightarrow\begin{cases}y-x=1\\z-y=1\\x-3y+2z=1\end{cases}$ Applying the same above procedure, the vector v_2 is obtained as: $v_2=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$(A-\lambda I)\,\overrightarrow{v_3}=\overrightarrow{v_2}\Rightarrow\begin{cases}y-x=1\\z-y=2\\x-3y+2z=3\end{cases}$

Applying the same above procedure, the vector v_3 is obtained as: $v_3=\begin{bmatrix}0\\1\\3\end{bmatrix}$

Finally, the matrix P is given by: $\begin{bmatrix}1&1&0\\1&2&1\\1&3&3\end{bmatrix}$

Then it can be seen that: $P^{-1}AP=\begin{bmatrix}1&1&0\\0&1&1\\0&0&1\end{bmatrix}=J$

Noting that: $e^{Jt}=\begin{bmatrix}e^t&te^t&\frac{1}{2}t^2e^t\\0&e^t&te^t\\0&0&e^t\end{bmatrix}$

we find: $e^{At}=P^{-1}e^{Jt}P=\begin{bmatrix}1&1&0\\1&2&1\\1&3&3\end{bmatrix}.\begin{bmatrix}e^t&te^t&\frac{1}{2}t^2e^t\\0&e^t&te^t\\0&0&e^t\end{bmatrix}.\begin{bmatrix}3&-3&1\\-2&3&-1\\1&-2&1\end{bmatrix}$

$$\Rightarrow e^{At}=\begin{bmatrix}e^t(1-t+\frac{1}{2}t^2)&te^t(1-t)&\frac{1}{2}t^2e^t\\\frac{1}{2}t^2e^t&e^t(1-t-t^2)&e^t(t+\frac{1}{2}t^2)\\e^t(t+\frac{1}{2}t^2)&e^t(-3t-t^2)&e^t(1+2t+\frac{1}{2}t^2)\end{bmatrix}$$

3.5 Concepts of controllability and observability

system is said to be controllable at time t_0 if it is possible by means of an unconstrained control vector to transfer the system from any initial state $x(t_0)$ to any other state in a finite interval of time. A system is said to be observable at time t_0 if, with the system in state $x(t_0)$, it is possible to determine this state from the observation of the output over a finite time interval. The concepts of controllability and observability were introduced by Kalman. They play an important role in the design of control systems in state space. In fact, the conditions of controllability and observability may govern the existence of a complete solution to the control system design problem. The solution to this problem may no exist if the system considered is not controllable. Although most physical systems are controllable and observable, corresponding mathematical models may not possess the property of controllability and observability. Then it is necessary to know the conditions under which a system is controllable and observable.

3.5.1 Controllability

The general problem in control theory involves controlling a system so that it transitions from a given initial state to a specified final state. In state-space representation, this entails determining the control signal $e(t)$ over a given time interval, $\begin{bmatrix} t_1 & t_2 \end{bmatrix}$, to drive the system from the state $x(t_1)$ to the desired state $x(t_2)$.

There are many criteria for controllability or non-controllability. The Kalman criterion is one of the most commonly used.

A system is fully controllable and completely reachable if and only if the vectors $B, AB, A^2B, \ldots, A^{n-1}B$ are linearly independent. This statement can also be expressed as follows: the controllability matrix is defined as the matrix formed by the n column vectors $B, AB, A^2B, \ldots, A^{n-1}B$.

The pair $[A], (B)$ is completely controllable if and only if the controllability matrix is regular, meaning its determinant is nonzero.

- **Example:** Let us revisit the example of the system governed by the following state equations:

$$\dot{x}(t) = [A]x(t) + (B)e(t)$$

where:

$$[A] = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}, \quad (B) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

The controllability matrix is composed of two vectors:

$$[C] = [(B) \,|\, [A](B)].$$

Now, calculate:

$$[A](B) = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}.$$

Hence:

$$[C]([A](B)) = \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}$$

This matrix is of rank 2, as its determinant is non-zero:

$$\det \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} = 9$$

Therefore, the system is fully controllable.

3.5.1.1 Calculation of the control input of a system

- **Principle of Kalman's construction:**

The objective here is to compute the control (or input) signal of the system to drive it from its initial state $x(0)$ at $t = 0$ to a desired state $x(t)$ at any given time t. If the system is fully controllable (or reachable), it is possible to calculate this control signal. The state of the system is given by:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Be(\tau)\,d\tau,$$

or equivalently:

$$x(t) - e^{At}x(0) = \int_0^t e^{A(t-\tau)}Be(\tau)\,d\tau.$$

The task is to determine the expression for $e(\tau)$ over the interval $[0, t]$ to achieve this goal.

Kalman demonstrated that the control signal $e(t)$ can be constructed using the following expression:

$$e(t) = B^T e^{A^T(t-\tau)} W^{-1} \big[x(t) - e^{At}x(0)\big],$$

where the matrix W is defined as:

$$W = \int_0^t e^{A\tau} B B^T e^{A^T\tau}\,d\tau.$$

- **Validation of the Construction:**

To validate this result, substitute $e(t)$ into the integral:

$$I = \int_0^t e^{A(t-\tau)} B e(\tau)\, d\tau.$$

Substituting the expression for $e(\tau)$, we get:

$$I = \int_0^t e^{A(t-\tau)} B \big\{ B^T e^{A^T(t-\tau)} W^{-1} \big[x(t) - e^{At} x(0) \big] \big\}\, d\tau.$$

Simplifying:

$$I = \big[x(t) - e^{At} x(0) \big] \int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)}\, d\tau\, W^{-1}.$$

Consider the integral:

$$\int_0^t e^{A(t-\tau)} B B^T e^{A^T(t-\tau)}\, d\tau.$$

Now, let us make a change of variable: $t - \tau \to \tau$. We obtain:

$$\int_0^t e^{A\tau} B B^T e^{A^T\tau} (-d\tau) = \int_t^0 e^{A\tau} B B^T e^{A^T\tau}\, d\tau.$$

Thus:

$$I = \big[x(t) - e^{At} x(0) \big] W W^{-1} = x(t) - e^{At} x(0).$$

This verifies the solution of the state equation:

$$x(t) - e^{At} x(0) = \int_0^t e^{A(t-\tau)} B e(\tau)\, d\tau.$$

- **Example:**

We aim to drive the system defined by:

$$\dot{x}(t) = Ax(t) + Be(t),$$

where:

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

from the initial state:

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

to the final state:

$$x(t) = \begin{bmatrix} 2 \\ -3 \end{bmatrix},$$

at $t = 0.5\,\mathrm{s}$.

Since A is a diagonal matrix, its transpose equals itself. Therefore:

$$e^{At} = e^{A^T t} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix}.$$

Thus, the controllability Gramian W is:

$$W = \int_0^t \begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-2\tau} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau.$$

After performing the calculations, for instance using Mathematica, we find:

$$W = \begin{bmatrix} -\frac{1}{4}(e^{-4t} - 1) & \frac{2}{3}(e^{-3t} - 1) \\ \frac{2}{3}(e^{-3t} - 1) & -2(e^{-2t} - 1) \end{bmatrix}.$$

Consequently:

$$W^{-1} = \frac{1}{\det(W)} \begin{bmatrix} -2(e^{-2t} - 1) & -\frac{2}{3}(e^{-3t} - 1) \\ -\frac{2}{3}(e^{-3t} - 1) & -\frac{1}{4}(e^{-4t} - 1) \end{bmatrix}.$$

For $t = 0.5\,\mathrm{s}$, we have:

$$\det(W) = \begin{vmatrix} 0.216 & -0.518 \\ -0.518 & 1.264 \end{vmatrix} = 0.0047.$$

And:

$$W^{-1} = \begin{bmatrix} 269 & 110 \\ 110 & 46 \end{bmatrix}.$$

Thus, we have:

$$e(t) = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} e^{-2(0.5-t)} & 0 \\ 0 & e^{-(0.5-t)} \end{bmatrix} \begin{bmatrix} 269 & 110 \\ 110 & 46 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 208e^{-2(0.5-t)} - 164e^{-(0.5-t)}.$$

It is straightforward to verify that this control signal indeed drives the system to the desired state within a time $t = 0.5\,\mathrm{s}$. Specifically:

$$x(0.5) = \int_0^{0.5} \begin{bmatrix} e^{-2(0.5-\tau)} & 0 \\ 0 & e^{-(0.5-\tau)} \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} e(\tau)\, d\tau \approx \begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

3.5.2 Observability

In state-space representation, it is essential to be able to determine the state of the system at any given time, meaning the ability to compute the state vector $x(t)$. Some state variables are easy to measure: a sensor placed in the right location within the system can provide access to the desired information. In such cases, the state variable is said to be **measurable**.

In other cases, direct measurement may not be possible, and the variable is referred to as **non-measurable**. However, even if it is not directly measurable, it may still influence the system's output $s(t)$. In this scenario, it becomes possible to infer the value of the variable from the measured output. Such a variable is then said to be **observable**.

3.5.2.1 Definition

A system is said to be observable at a time t_1 if knowing the input signal and the output signal over the time interval $[t_1, t_2]$ allows the determination of the system's state at time t_1.

If a system is observable for any time t_1, it is referred to as completely observable.

3.5.2.2 Observability Criterion

Consider a system defined as:

$$\begin{cases} \dot{x}(t) = [A]x(t) + (B)e(t), \\ s(t) = (C)x(t) + D \cdot e(t). \end{cases}$$

This system is completely observable if and only if the row vectors::
$(C), (C)[A], (C)[A]^2, \ldots, (C)[A]^{n-1}$ are linearly independent.

This statement can also be expressed as follows: the *observability matrix* is defined as the matrix formed by the n row vectors $(C), (C)[A], (C)[A]^2, \ldots, (C)[A]^{n-1}$:

$$[O]([A], C) = \begin{bmatrix} (C) \\ (C)[A] \\ (C)[A]^2 \\ \vdots \\ (C)[A]^{n-1} \end{bmatrix}.$$

The pair $[A], (C)$ is completely observable if and only if the observability matrix is nonsingular, meaning its determinant is non-zero.

- **Example:**

Consider a system governed by the state equation:

$$\begin{cases} \dot{x}(t) = [A]x(t) + (B)e(t), \\ s(t) = (C)x(t), \end{cases}$$

where:

$$[A] = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}, \quad (C) = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Is this system completely observable?

To test the observability condition, examine the rank of: $\begin{bmatrix} C \\ CA \end{bmatrix}$. Since $\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ the rank of $\begin{bmatrix} C \\ CA \end{bmatrix}$ is 2. Hence, the system is completely observable.

3.6 Solved exercises

Exercise 01: Consider a system governed by the state equation:

$$\dot{x}(t) = Ax(t) + Be(t)$$

where:

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

and assume that at time $t = 0$, the system is in the state:

$$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Given that the system is subjected to a unit step input, determine the expression for the state vector $x(t)$ at any time t, noting that the control matrix is nilpotent.

Solution of exercise 01:

It is easy to verify that the control matrix is nilpotent. Indeed:

$$[A]^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Therefore, for any $n \geq 2$, we have:

$$[A]^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, the matrix exponential $e^{[A]t}$ can be expressed as:

$$e^{[A]t} = \left(I + [A]t + \frac{[A]^2 t^2}{2!} + \cdots + \frac{[A]^n t^n}{n!} + \cdots \right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -t & -t \\ t & t \end{bmatrix} = \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix}.$$

The general form of the solution to the state equation is:

$$x(t) = e^{[A]t} x(0) + \int_0^t e^{[A](t-\tau)} B e(\tau)\, d\tau.$$

Since the system is subjected to a step input $e(t) = 1$, we have:

$$x(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Therefore, the state vector is:

$$x(t) = \begin{bmatrix} 1-t & -t \\ t & 1+t \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 1-(t-\tau) & -(t-\tau) \\ (t-\tau) & 1+(t-\tau) \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} d\tau.$$

This simplifies to:

$$x(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 2-t+\tau \\ t-\tau-1 \end{bmatrix} d\tau.$$

Finally, solving the integrals:

$$x(t) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} \int_0^t (2-t+\tau)\, d\tau \\ \int_0^t (t-\tau-1)\, d\tau \end{bmatrix}.$$

This results in:

$$x(t) = \begin{bmatrix} -1 + 2t - \frac{t^2}{2} \\ 1 - t + \frac{t^2}{2} \end{bmatrix}.$$

Exercise 02: Let's consider a system governed by the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with:

$$A = \begin{bmatrix} 1 & 2 \\ a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Determine the condition on the parameter a for the system to be fully controllable

Solution of exercise 02:

To determine the condition on the parameter a for the system to be fully controllable, we need to check the controllability matrix. A system is fully controllable if and only if the controllability matrix has full rank, which means it must be of full column rank (in this case, rank 2 for a 2-state system).

The state equation is given as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with:

$$A = \begin{bmatrix} 1 & 2 \\ a & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The controllability matrix $\mathcal{C}$ is defined as:

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix}$$

First, we calculate AB:

$$AB = \begin{bmatrix} 1 & 2 \\ a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+2 \\ a+1 \end{bmatrix} = \begin{bmatrix} 3 \\ a+1 \end{bmatrix}$$

So, the controllability matrix becomes:

$$\mathcal{C} = \begin{bmatrix} 1 & 3 \\ 1 & a+1 \end{bmatrix}$$

For the system to be fully controllable, the determinant of the controllability matrix must be non-zero:

$$\det(\mathcal{C}) = \det \begin{bmatrix} 1 & 3 \\ 1 & a+1 \end{bmatrix} = 1(a+1) - 3(1) = a + 1 - 3 = a - 2$$

For full controllability, we require:

$$a - 2 \neq 0 \quad \Rightarrow \quad a \neq 2$$

Thus, the system is fully controllable if $a \neq 2$.

Exercise 03: Let's consider a system governed by the state equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with:

$$A = \begin{bmatrix} a & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

Determine the condition on the parameter a for the system to be fully observable.

Solution of exercise 03:

To determine the condition on the parameter a for the system to be fully observable, we need to check the observability matrix. A system is fully observable if and only if the observability matrix has full rank, meaning it must have rank 2 for a 2-state system.

The state equation is given as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with:

$$A = \begin{bmatrix} a & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -2 \end{bmatrix}$$

The observability matrix $\mathcal{O}$ is defined as:

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix}$$

First, we compute CA:

$$CA = \begin{bmatrix} 1 & -2 \end{bmatrix} \begin{bmatrix} a & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1(a) + (-2)(2) & 1(-1) + (-2)(1) \end{bmatrix} = \begin{bmatrix} a-4 & -1-2 \end{bmatrix} = \begin{bmatrix} a-4 & -3 \end{bmatrix}$$

So, the observability matrix becomes:

$$\mathcal{O} = \begin{bmatrix} 1 & -2 \\ a-4 & -3 \end{bmatrix}$$

For the system to be fully observable, the determinant of the observability matrix must be non-zero:

$$\det(\mathcal{O}) = \det \begin{bmatrix} 1 & -2 \\ a-4 & -3 \end{bmatrix}$$

We calculate the determinant:

$$\det(\mathcal{O}) = 1(-3) - (-2)(a-4) = -3 + 2(a-4) = -3 + 2a - 8 = 2a - 11$$

For full observability, we require:

$$2a - 11 \neq 0 \quad \Rightarrow \quad a \neq \frac{11}{2}$$

Thus, the system is fully observable if $a \neq \frac{11}{2}$.

Exercise 04: We want to drive the system described by the state equation $\dot{x}(t) = A \cdot x(t) + B \cdot u(t)$ with $A = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & -\frac{1}{8} \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, from an initial state $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ to a final state $x(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ at $t = 2$ seconds.

- Determine the expression for the control signal $u(t)$ that ensures the system is properly controlled.

Solution of exercise 04:

The desired control signal is expressed as:

$$e(t) = B^T e^{A^T(t-t_0)} W^{-1} \left[x(t) - e^{At} x(0) \right]$$

where:

$$W = \int_0^t e^{A^T t} B B^T e^{At} \, dt$$

Since A is a diagonal matrix, its transpose is equal to itself. Therefore:

$$e^{At} = e^{A^T t} = \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/8} \end{bmatrix}$$

Thus:

$$W = \int_0^t \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/8} \end{bmatrix} \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\begin{bmatrix} 1 & 2 \end{bmatrix} \right) \right) \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/8} \end{bmatrix} dt$$

Simplifying:

$$W = \int_0^t \begin{bmatrix} e^{-t/2} & 2e^{-t/8} \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-t/2} & 0 \\ 0 & e^{-t/8} \end{bmatrix} dt$$

$$W = \int_0^t \begin{bmatrix} e^{-t} & 2e^{-5t/8} \\ 2e^{-5t/8} & 4e^{-t/4} \end{bmatrix} dt$$

$$W = \int_0^t \begin{bmatrix} -e^{-t} & 16e^{-5t/8} \\ 16e^{-5t/8} & -16e^{-t/4} \end{bmatrix} dt$$

This leads to:

$$W = \begin{bmatrix} 1 - e^{-t} & \frac{16}{5}(1 - e^{-5t/8}) \\ \frac{16}{5}(1 - e^{-5t/8}) & 16(1 - e^{-t/4}) \end{bmatrix}$$

Therefore, we have:

$$[W]^{-1} = \frac{1}{\det[W]} \begin{bmatrix} 16(1 - e^{-t/4}) & -\frac{16}{5}(1 - e^{-5t/8}) \\ -\frac{16}{5}(1 - e^{-5t/8}) & 1 - e^{-t} \end{bmatrix}$$

For $t = 2\,\mathrm{s}$, we calculate the determinant of $[W]$ as:

$$\det[W] = \begin{vmatrix} 0.865 & -2.283 \\ -2.283 & 6.296 \end{vmatrix} = 0.234$$

And thus:

$$[W]^{-1} = \begin{bmatrix} 26.91 & -9.76 \\ -9.76 & 3.70 \end{bmatrix}$$

Since the control signal is given by:

$$e(t) = B^T e^{A^T(t-t_0)} [W]^{-1} \left[x(t) - e^{At} x(0) \right]$$

we substitute the values to get:

$$e(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} e^{-(2-t)/2} & 0 \\ 0 & e^{-(2-t)/8} \end{bmatrix} \begin{bmatrix} 26.91 & -9.76 \\ -9.76 & 3.70 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This simplifies to:

$$e(t) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 44.06e^{-(2-t)/2} \\ -15.82e^{-(2-t)/8} \end{bmatrix}$$

Finally, we get the control signal expression:

$$e(t) = 44.06e^{(t-2)/2} - 31.64e^{(t-2)/8}$$

3.7 Conclusion

This chapter has provided a comprehensive exploration of state-space representation, emphasizing its critical role in the analysis and design of dynamic systems. By solving state equations, we established a foundational understanding of how system states evolve over time, enabling the prediction of behavior and performance. The transition matrix emerged as a key analytical tool, offering a direct link between initial conditions and future states. Modal analysis further enhanced our grasp of system dynamics, showing how diagonalization simplifies the study of stability and the influence of eigenvalues and eigenvectors. Additionally, the discussion on controllability and observability highlighted their significance in determining whether a system can be effectively controlled and observed through its inputs and outputs. Together, these concepts form a robust framework for evaluating, designing, and implementing control strategies, ensuring a deeper understanding of the dynamic systems they govern.

Chapter 4

State feedback control

4.1 Introduction

State feedback control is a fundamental technique in modern control theory, enabling the design of controllers that influence the behavior of dynamic systems by modifying their state variables. One of the key concepts in state feedback control is pole placement, which involves selecting the system's poles to achieve desired dynamics in the closed-loop system. This approach allows for precise control over the system's stability, speed of response, and damping.

The first step in implementing state feedback is the (formulation of the pole placement problem). By choosing appropriate feedback gains, it is possible to position the poles of the closed-loop system at specific locations in the complex plane, ensuring the system behaves according to performance specifications such as settling time, overshoot, and damping ratio.

For single-input single-output (SISO) systems, pole placement techniques are relatively straightforward. The design process typically involves solving a set of equations to determine the state feedback gains that place the poles at the desired locations. For more complex multiple-input multiple-output (MIMO) systems, the pole placement becomes more challenging due to the interaction between multiple inputs and outputs. Advanced methods and algorithms, such as the use of the Ackermann's formula or control allocation, are employed to calculate the required feedback gains.

Finally, the implementation of state feedback control in practical systems involves the development of algorithms and hardware that apply the calculated feedback gains to the system in real time. The implementation must take into account factors such as sensor noise, computation time, and actuator limitations, ensuring that the theoretical design translates into reliable and effective control in real-world applications.

This chapter will provide an in-depth exploration of the state feedback control design process, covering both the theory and practical aspects of implementing pole

placement in SISO and MIMO systems.

4.2 Formulation of the pole placement problem using state feedback

In state feedback control, the goal is to design a feedback law that modifies the closed-loop dynamics of a system to achieve desired performance. For a system represented in state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $x(t)$ is the state vector, $u(t)$ is the control input, A is the state matrix, and B is the input matrix, the state feedback control law is given as:

$$u(t) = -Kx(t)$$

where K is the state feedback gain matrix. Substituting $u(t)$ into the system equation gives the closed-loop dynamics:

$$\dot{x}(t) = (A - BK)x(t)$$

The eigenvalues of the matrix $A - BK$ determine the poles of the closed-loop system. The pole placement problem involves selecting K such that the eigenvalues of $A - BK$ are placed at desired locations in the complex plane, ensuring the closed-loop system meets stability and performance criteria (Figure 4.1).

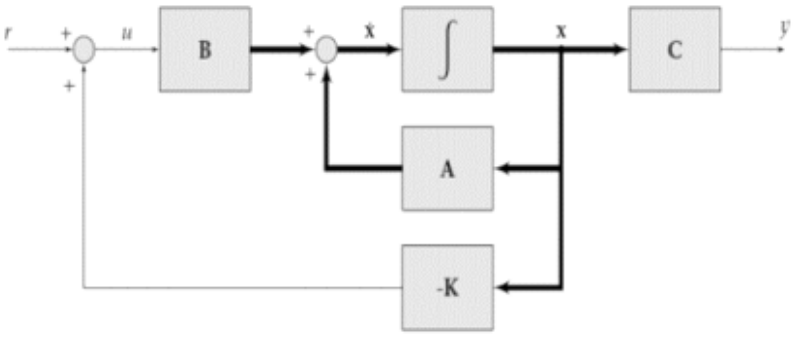


FIGURE 4.1: State feedback control Design in State Space.

4.2.1 Closed-Loop Transfer Function

The closed-loop transfer function of a system can be derived from its state-space representation. Assume the open-loop system is governed by the following state equations:

$$\dot{x}(t) = [A]x(t) + Be(t), \quad s(t) = Cx(t)$$

In the closed-loop configuration with state feedback, the state equations become:

$$\dot{x}(t) = [A]x(t) + B\{e(t) - Kx(t)\}$$

or equivalently:

$$\dot{x}(t) = \{[A] - BK\}x(t) + Be(t)$$

Applying the Laplace transform to these equations, we get:

$$pX(p) = \{[A] - BK\}X(p) + BE(p)$$

$$S(p) = CX(p)$$

From this, we derive:

$$\{pI - [A] + BK\}X(p) = BE(p)$$

and solving for $X(p)$:

$$X(p) = \{pI - [A] + BK\}^{-1}BE(p)$$

The closed-loop transfer function is then:

$$H(p) = \frac{S(p)}{E(p)} = C\{pI - [A] + BK\}^{-1}B$$

4.2.2 Determining the State Vector

State feedback control assumes that the system is fully understood and, more importantly, that the state vector is accessible. Three scenarios may arise:

- All internal signals forming the state vector are measurable: In this case, the state variables can be directly measured using appropriately placed sensors, providing the necessary information for state feedback.
- Not all state variables are measurable, but the system is fully observable: Here, the state vector can be reconstructed at any given moment by using the output

and input signals of the system over a preceding time interval. This reconstruction is achieved using a *state observer*.

- The system is not fully observable: In such cases, the state vector must be estimated using a (state estimator).

- **Note:** The terms "observer" and "estimator" are often used interchangeably, but this distinction is minor. The key point is to have a method for reconstructing the state vector at each instant to enable state feedback control.

- **Example:** Pole Placement for a SISO System

Consider the following system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

1. System Matrices: - $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ - $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
2. Desired Poles: Assume we want the closed-loop poles at $\lambda_1 = -2$ and $\lambda_2 = -4$.
3. Characteristic Polynomial: The desired characteristic polynomial is:

$$p(s) = (s+2)(s+4) = s^2 + 6s + 8$$

4. Feedback Gain Calculation: The characteristic equation of the closed-loop system is determined by:

$$\det(sI - (A - BK)) = 0$$

Here, $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, so:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 - k_1 & -3 - k_2 \end{bmatrix}$$

The characteristic polynomial of $A - BK$ is:

$$\det(sI - (A - BK)) = s^2 + (3 + k_2)s + (2 + k_1) = s^2 + 6s + 8$$

Matching coefficients with the desired polynomial:

$$3 + k_2 = 6 \implies k_2 = 3$$

$$2 + k_1 = 8 \implies k_1 = 6$$

Thus, the feedback gain is:

$$K = \begin{bmatrix} 6 & 3 \end{bmatrix}$$

5. Verification: The closed-loop system matrix is:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}$$

The eigenvalues of $A - BK$ are -2 and -4, confirming the poles are placed as desired.

This example illustrates how state feedback can be used to place the poles of a system at specific locations to achieve desired dynamics. The process involves formulating the problem mathematically, solving for the feedback gains, and verifying the closed-loop pole placement.

4.3 Calculation methods for single-input single-output (SISO) systems

State feedback control aims to design a feedback gain K for a single-input single-output (SISO) system to place the closed-loop poles at desired locations for stability and performance. The method involves algebraic computation or tools like the Ackermann formula.

4.3.0.1 Using Pole Placement by Comparing Characteristic Polynomials

The system is given in state-space form:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

For state feedback control:

$$u(t) = -Kx(t)$$

The closed-loop dynamics become:

$$\dot{x}(t) = (A - BK)x(t)$$

Procedure:

1. Compute the Open-Loop Characteristic Polynomial: The open-loop characteristic polynomial is $\det(sI - A)$.

2. Define the Desired Characteristic Polynomial: The desired poles are specified by a polynomial $p(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_0$.

3. Equate and Solve: By ensuring $\det(sI - (A - BK)) = p(s)$, the feedback gain K is calculated by solving the polynomial equations.

- **Example:**

Consider the system:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Desired poles: $\lambda_1 = -2$, $\lambda_2 = -3$.

1. Open-loop characteristic polynomial:

$$\det(sI - A) = \det \begin{bmatrix} s & -1 \\ 5 & s+6 \end{bmatrix} = s^2 + 6s + 5$$

2. Desired characteristic polynomial:

$$p(s) = (s+2)(s+3) = s^2 + 5s + 6$$

3. Comparing coefficients:

The closed-loop characteristic polynomial is $\det(sI - (A - BK))$. Assume $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$:

$$A - BK = \begin{bmatrix} 0 & 1 \\ -5 - k_1 & -6 - k_2 \end{bmatrix}$$

Equating coefficients with $s^2 + 5s + 6$:

$$-6 - k_2 = 5 \implies k_2 = -11$$

$$-5 - k_1 = 6 \implies k_1 = -11$$

Thus, $K = \begin{bmatrix} -11 & -11 \end{bmatrix}$.

4.3.0.2 Using the Ackermann Formula

The Ackermann formula provides a systematic way to compute K using controllability. For an n-state system:

$$K = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} C^{-1} p(A)$$

where: - C is the controllability matrix: $C = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$, - $p(A)$ is the desired characteristic polynomial applied to A.

Example: For the previous system:

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

1. Controllability matrix:

$$C = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -6 \end{bmatrix}$$

$$\det(C) = 0 \times -6 - 1 \times 1 = -1 \quad \text{(System is controllable.)}$$

2. Desired characteristic polynomial:

$$p(s) = s^2 + 5s + 6 \quad \Longrightarrow \quad p(A) = A^2 + 5A + 6I$$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix}^2 = \begin{bmatrix} -5 & -6 \\ 30 & 31 \end{bmatrix}$$

$$p(A) = \begin{bmatrix} -5 & -6 \\ 30 & 31 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 \\ -5 & -6 \end{bmatrix} + 6 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3. Compute K:

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} C^{-1} p(A)$$

The result will be the same as the first method: $K = \begin{bmatrix} -11 & -11 \end{bmatrix}$.

- Comparison of characteristic polynomials: Manual and straightforward for small systems. - Ackermann formula: Systematic and useful for larger systems with controllability.

Both methods ensure that the poles of the closed-loop system are placed as desired, enabling precise control of the system's dynamics.

4.4 Calculation methods for multiple-input multiple-output (MIMO) systems

In Multiple-Input Multiple-Output (MIMO) systems, state feedback control involves designing a feedback matrix K to achieve desired closed-loop performance by placing the eigenvalues of the closed-loop system at specific locations. The state-space representation for a MIMO system is given as:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where: - $x(t)$ is the n-dimensional state vector, - $u(t)$ is the m-dimensional input vector, - A is the $n \times n$ state matrix, - B is the $n \times m$ input matrix.

The control law is:

$$u(t) = -Kx(t)$$

where K is an $m \times n$ feedback gain matrix. The closed-loop system dynamics become:

$$\dot{x}(t) = (A - BK)x(t)$$

The eigenvalues of $A-BK$ determine the stability and performance of the closed-loop system.

4.4.1 Steps to Design K for MIMO Systems

1. Check Controllability: Ensure the pair (A, B) is controllable by verifying the rank of the controllability matrix:

$$\mathcal{C} = [B\, AB\, A^2B\, \ldots\, A^{n-1}B]$$

If $\text{rank}(\mathcal{C}) = n$, the system is controllable.

2. Define Desired Poles: Specify the desired eigenvalues of $A - BK$ based on performance requirements (stability, speed of response, damping).

3. Solve for K: Use methods like: - **Eigenstructure Assignment**: Assign desired eigenvalues and eigenvectors to $A - BK$. - **Numerical Pole Placement Algorithms**: Implement computational tools like MATLAB's 'place' function or the Ackermann's formula for pole placement.

—

- **Example:** Pole Placement in a MIMO System

- Given System:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

- Step 1: Controllability Check:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & -1 & -6 & -11 & 10 \end{bmatrix}$$

The rank of $\mathcal{C}$ is 3, so the system is controllable.

- Step 2: Desired Poles:

Let the desired eigenvalues be $\{-2, -3, -4\}$.

- Step 3: Calculation of K:

Using MATLAB's 'place' function:

$$K = \text{place}(A, B, [-2, -3, -4])$$

Alternatively, solve numerically or symbolically for K to ensure $\det(sI-(A-BK))$ matches the desired characteristic polynomial:

$$\det(sI - (A - BK)) = (s + 2)(s + 3)(s + 4)$$

- Step 4: Verify the Closed-Loop System:

Substitute K into $A - BK$ and compute its eigenvalues to confirm they match the desired values.

4.4.2 Tools for MIMO State Feedback Design

- **MATLAB Commands:**

- 'place(A, B, desired$_p oles$)' – '$acker(A, B, desired_p oles)$'

- **Eigenstructure Assignment:**

For systems where both eigenvalues and eigenvectors need control, eigenstructure assignment algorithms are used, often involving numerical solvers.

The calculation of state feedback for MIMO systems requires understanding the controllability of the system, defining performance goals through desired pole locations, and using numerical or symbolic tools to compute K. By carefully designing K, the closed-loop system can meet stability and performance requirements, even in the presence of multiple inputs and outputs.

4.5 Controllability in Continuous-Time Modes

4.5.1 Definition

We assume that the state vector is known. The expression of the closed-loop transfer function allows us to calculate its poles. The transfer function is given by:

$$H(p) = C \cdot \text{adj}(pI - [A] + BK) \cdot B \big/ \det(pI - [A] + BK)$$

The poles of the closed-loop transfer function are the roots of the characteristic equation:

$$\det(pI - [A] + BK) = 0$$

These are the eigenvalues of the matrix $[A] - BK$. The ability to choose the gain vector K to place these poles at desired locations, thus ensuring specific closed-loop performance, is called *modal controllability*.

4.5.2 Modal Controllability Criterion

A system is controllable in modes in closed-loop if it is fully controllable in open-loop. To verify modal controllability for the closed-loop system, the controllability matrix is calculated as:

$$\mathcal{C}(A, B) = [B\, AB\, A^2B\, \ldots\, A^{n-1}B]$$

The system is controllable if $\mathcal{C}$ has full rank (n), i.e., its determinant is non-zero.

4.5.3 Non-Controllable Systems

If a system is not fully controllable, its state equations can be transformed into a new basis to separate the controllable and uncontrollable subsystems. The transformed state-space representation becomes:

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}e(t), \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}$$

Here, $(\tilde{A}_{11}, \tilde{B}_1)$ represents the controllable subsystem. The feedback law becomes:

$$\dot{\tilde{x}}(t) = \begin{bmatrix} \tilde{A}_{11} - \tilde{B}_1\tilde{k}_1 & \tilde{A}_{12} - \tilde{B}_1\tilde{k}_2 \\ 0 & \tilde{A}_{22} \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix} e(t)$$

This demonstrates that the gain vector can only influence the controllable subsystem. The non-controllable modes remain unaffected.

A system that is not fully controllable is stable in closed-loop if and only if its non-controllable modes are stable. Stability requires the eigenvalues corresponding to these modes to have negative real parts.

- **Example:** Non-Controllable System

Given:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Controllability Matrix:

$$\mathcal{C}(A,B) = [B, AB] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The determinant:

$$\det(\mathcal{C}(A,B)) = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

Since the determinant is zero, the system is not fully controllable.

- Eigenvalues: Compute the eigenvalues of A:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 3\lambda + 2 = 0$$

Roots are $\lambda_1 = 1, \lambda_2 = 2$.

- Change of Basis: Diagonalize A with eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Transformation matrix:

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \quad T^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Transformed system:

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \tilde{B} = T^{-1}B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The transformed system shows that only the first mode is controllable. The second mode is uncontrollable and corresponds to an unstable eigenvalue ($2 > 0$). Therefore, the system cannot be stabilized via state feedback.

4.6 Solved exercises

Exercise 01: Consider the control system depicted in the following figure.

The plant is described by:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad \text{with:} \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The system implements state feedback control with $u = -Kx(t)$ (Figure 4.2).

We aim to determine the state feedback gain vector K such that the closed-loop poles are located at:

$$s_1 = -10, \quad s_2 = -2 + 4j, \quad s_3 = -2 - 4j.$$

Calculate the state feedback gain K in this scenario.

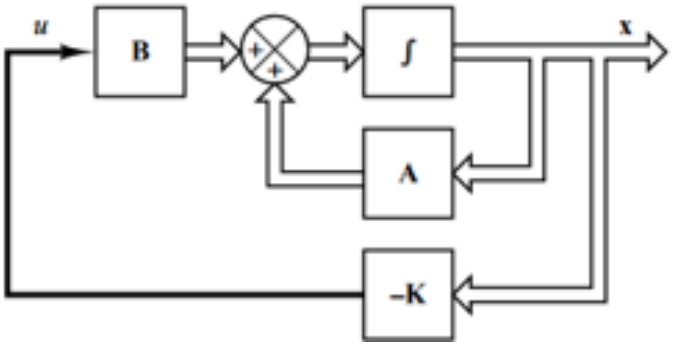


FIGURE 4.2: State feedback control Design in State Space.

Solution of exercise 01:

To calculate the state feedback gain vector K, we use the pole placement method. The system is governed by the equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where $u(t) = -Kx(t)$. Substituting $u(t)$, the closed-loop dynamics become:

$$\dot{x}(t) = (A - BK)x(t)$$

The goal is to choose K such that the eigenvalues of $A - BK$ match the desired closed-loop poles: $s_1 = -10$, $s_2 = -2 + 4j$, $s_3 = -2 - 4j$.

The characteristic polynomial corresponding to the desired poles is:

$$p(s) = (s + 10)(s + 2 - 4j)(s + 2 + 4j)$$

Expanding:

$$p(s) = (s + 10)\left[s^2 + 4s + (4^2 + 0^2)\right]$$

$$p(s) = (s + 10)(s^2 + 4s + 16)$$

$$p(s) = s^3 + 14s^2 + 56s + 160$$

The desired characteristic equation is:

$$\det(sI - (A - BK)) = s^3 + 14s^2 + 56s + 160$$

The controllability matrix is:

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B \end{bmatrix}$$

Calculate AB and A^2B to construct $\mathcal{C}$.

Given:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \\ -6 \\ -6 \end{bmatrix}, \quad A^2B = A(AB) = \begin{bmatrix} -6 \\ -6 \\ 11 \end{bmatrix}$$

Thus:

$$\mathcal{C} = \begin{bmatrix} 0 & 1 & -6 \\ 0 & -6 & -6 \\ 1 & -6 & 11 \end{bmatrix}$$

- Verify controllability: Compute $\det(\mathcal{C})$. If $\det(\mathcal{C}) \neq 0$, the system is controllable.

$$\det(\mathcal{C}) = \begin{vmatrix} 0 & 1 & -6 \\ 0 & -6 & -6 \\ 1 & -6 & 11 \end{vmatrix} = -42 \neq 0$$

The system is controllable.

- Determine K using pole placement: Let $K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$. The closed-loop characteristic equation is:

$$\det(sI - (A - BK)) = \det(sI - A + BK)$$

Match the coefficients of the desired polynomial $s^3 + 14s^2 + 56s + 160$ with the characteristic polynomial of $A - BK$. Using MATLAB or manual calculations (such as Ackermann's formula), the result is:

$$K = \begin{bmatrix} 72 & 96 & 36 \end{bmatrix}$$

Exercise 02:

Consider a system defined by the state equation:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

where:

$$A = \begin{bmatrix} -1 & -2 \\ -1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

a. Prove the system is fully controllable.

b. Analyze the dynamic performance of the system (rise time and overshoot) under state feedback control with a gain vector $k = \begin{bmatrix} -55 & -168 \end{bmatrix}$, assuming a unit step reference input.

Solution of exercise 02:

a. The controllability of a system is determined using the controllability matrix:

$$W_c = \begin{bmatrix} B & AB \end{bmatrix}.$$

Given:

$$A = \begin{bmatrix} -1 & -2 \\ -1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

Calculate AB:

$$AB = \begin{bmatrix} -1 & -2 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 3-4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Construct W_c:

$$W_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}.$$

Check the determinant of W_c:

$$\det(W_c) = (-3)(-1) - (1)(1) = 3 - 1 = 2.$$

Since $\det(W_c) \neq 0$, the controllability matrix is full rank, meaning the system is fully controllable.

b. The closed-loop system dynamics with state feedback $u = -kx + r$ are governed by:

$$\dot{x}(t) = (A - BK)x(t) + Br,$$

where $K = \begin{bmatrix} -55 & -168 \end{bmatrix}$.

- Calculate $A - BK$:

$$BK = \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} -55 & -168 \end{bmatrix} = \begin{bmatrix} 165 & 504 \\ -55 & -168 \end{bmatrix}.$$

$$A - BK = \begin{bmatrix} -1 & -2 \\ -1 & -4 \end{bmatrix} - \begin{bmatrix} 165 & 504 \\ -55 & -168 \end{bmatrix} = \begin{bmatrix} -166 & -506 \\ 54 & 164 \end{bmatrix}.$$

The characteristic equation of $A - BK$ determines the system's poles:

$$\det(sI - (A - BK)) = 0.$$

$$\det\left(\begin{bmatrix} s+166 & 506 \\ -54 & s-164 \end{bmatrix}\right) = (s+166)(s-164) + 54 \cdot 506.$$

$$(s+166)(s-164) + 27324 = s^2 + 2s - 166 \cdot 164 + 27324.$$

$$= s^2 + 2s - 27224 + 27324 = s^2 + 2s + 100.$$

The poles are:

$$s = -1 \pm j10.$$

- The rise time for a second-order system with natural frequency ω_n and damping ratio ζ is approximated as:

$$t_r \approx \frac{\pi - \arccos(\zeta)}{\omega_d},$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$, $\omega_n = \sqrt{100} = 10$, and $\zeta = \frac{-\text{Real part of pole}}{\omega_n} = \frac{1}{10} = 0.1$.

Substituting values:

$$\omega_d = 10\sqrt{1-0.1^2} = 10\sqrt{0.99}.$$

$$t_r \approx \frac{\pi - \arccos(0.1)}{10\sqrt{0.99}}.$$

Approximate t_r numerically:

$$t_r \approx 0.3\,\text{seconds}.$$

- Overshoot is determined by:

$$M_p = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}.$$

Substituting $\zeta = 0.1$:

$$M_p = e^{-\frac{0.1\pi}{\sqrt{1-0.1^2}}}.$$

Approximate M_p:

$$M_p \approx 0.74\,(74\%).$$

Exercise 03:

Consider a system described by the state equation:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

where:

$$A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

a. Demonstrate that the system is fully controllable.

b. Determine the state feedback gain vector $k = [k_1, k_2]$ to design a feedback loop such that the system meets the following specifications: - A phase margin of 60°. - A rise time of 3 seconds.

Solution of exercise 03:

The state equation of the system is:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t),$$

where:

$$A = \begin{bmatrix} -2 & -2 \\ -1 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

a. The controllability of a system is determined by the controllability matrix:

$$\mathcal{C} = \begin{bmatrix} B & A \cdot B \end{bmatrix}.$$

Substitute A and B:

$$\mathcal{C} = \begin{bmatrix} B & A \cdot B \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -2 & -4 \end{bmatrix}.$$

Check the rank of $\mathcal{C}$: If $\text{rank}(\mathcal{C}) = n$, where n is the order of the system ($n = 2$ here), the system is fully controllable.

$$\det(\mathcal{C}) = \begin{vmatrix} 1 & -2 \\ -2 & -4 \end{vmatrix} = (1)(-4) - (-2)(-2) = -4 - 4 = -8 \neq 0.$$

Since $\det(\mathcal{C}) \neq 0$, the controllability matrix is full rank (rank = 2). Thus, the system is fully controllable.

b. We design the state feedback $u(t) = -k \cdot x(t)$ such that the system achieves: - hase margin of 60°. - Rise time (T_r) of 3 seconds.

From the rise time, the dominant closed-loop poles are determined using the approximation:

$$T_r \approx \frac{3}{\omega_n},$$

where ω_n is the natural frequency.

Substitute $T_r = 3$:

$$\omega_n = \frac{3}{3} = 1.$$

The desired damping ratio ζ is derived from the phase margin. For a phase margin of 60°, $\zeta \approx 0.5$.

The desired poles are given by:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}.$$

Substitute $\zeta = 0.5$ and $\omega_n = 1$:

$$s_{1,2} = -0.5 \pm j\sqrt{1-0.5^2},$$

$$s_{1,2} = -0.5 \pm j0.866.$$

Thus, the desired closed-loop poles are approximately:

$$s_{1,2} = -0.5 \pm j0.87.$$

The characteristic equation of the closed-loop system is:

$$\det(sI - (A - BK)) = 0,$$

where $K = [k_1\ k_2]$.

The desired characteristic equation is:

$$(s + 0.5)^2 + 0.87^2 = 0.$$

Expand the equation:

$$s^2 + s + 1.007 = 0.$$

The closed-loop characteristic equation must match:

$$\det(sI - (A - BK)) = s^2 + (2 + k_2)s + (3 + 2k_2 - k_1) = 0.$$

Equating coefficients: - Coefficient of s: $2 + k_2 = 1 \implies k_2 = 1$. - Constant term: $3 + 2k_2 - k_1 = 1.007$. Substitute $k_2 = 1$:

$$3 + 2(1) - k_1 = 1.007,$$

$$8 - k_1 = 1.007 \implies k_1 = 4.$$

Thus, the state feedback gain vector is:

$$k = \begin{bmatrix} 5 & 1.007 \end{bmatrix}.$$

4.7 Conclusion

State feedback control serves as a powerful and versatile tool in the design of modern control systems, offering precise manipulation of system dynamics through pole placement. By tailoring the location of poles in the closed-loop system, engineers can ensure stability and achieve specific performance criteria such as fast response times, minimal overshoot, and adequate damping.

For single-input single-output (SISO) systems, the pole placement process is relatively straightforward, leveraging simple computation techniques to determine feedback gains. However, in the case of multiple-input multiple-output (MIMO) systems, the complexity increases due to the interplay between multiple control channels. Advanced computational methods, including Ackermann's formula and other optimization techniques, become indispensable for achieving desired performance.

This chapter has provided a comprehensive exploration of the principles, methods, and implementation strategies of state feedback control, from fundamental concepts to advanced applications in MIMO systems. With this foundation, readers are well-equipped to design and analyze control systems that meet diverse performance requirements in practical scenarios.

Chapter 5

State Observer Design

5.1 Introduction

The ability to estimate the internal states of a dynamic system plays a critical role in control and monitoring applications. In many practical scenarios, not all state variables can be measured directly due to limitations in sensor availability, cost, or reliability. Observers provide a powerful tool to reconstruct these states using the measurable outputs and a mathematical model of the system. This chapter delves into the theory and design of observers, presenting both deterministic and stochastic approaches.

We begin by exploring "deterministic observers", which operate under the assumption of a perfectly known system model and noise-free measurements. A key example in this category is the "Luenberger observer", a widely used technique for linear systems that guarantees asymptotic state estimation under suitable conditions. Following this, we discuss "methods for designing observers", including practical considerations for tuning observer gains and ensuring stability and robustness. For systems where full-order observers are computationally expensive or unnecessary, we introduce "reduced-order observers", which estimate a subset of the state variables to reduce complexity. The chapter then transitions to "stochastic observers", which account for the presence of process and measurement noise. The "Kalman filter", an optimal stochastic observer, is presented as a cornerstone of modern control theory, with applications ranging from navigation systems to signal processing. By the end of this chapter, readers will have a comprehensive understanding of the principles and methods for state estimation, equipping them with the tools to address a wide range of practical control and monitoring challenges.

5.2 Deterministic observers

In the pole-placement method for control system design, it is typically assumed that all state variables are accessible for feedback.

However, in practical applications, not all state variables may be measurable. This limitation could arise due to the component being located in an inaccessible area or the high cost of sensors.

To address this challenge, the estimation of unmeasurable state variables, known as **observation**, is employed. A device or program that performs this estimation is referred to as a **state estimator**, **state observer**, or simply an **observer**.

State observers are classified into two main types:

- **Full-Order State Observer**: Observes all state variables of the system, regardless of whether some are directly measurable or not.
- **Reduced-Order State Observer**: Focuses solely on estimating the state variables that are not directly measurable.

This chapter also discusses the integration of state feedback control with an observer-based approach, where a state observer is used in conjunction with state feedback to enhance system performance.

- **Remarks:**
- A state observer estimates the state variables based on the measurements of the output and control variables.
- Here the concept of observability plays an important role.
- State observers can be designed if and only if the observability condition is satisfied.

5.2.1 Luenberger observer

Consider a plant described by the following equations:

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{5.1}$$

The observer's mathematical model closely resembles that of the plant but includes an additional term to account for estimation errors. This term compensates for inaccuracies in the matrices **A** and **B**, as well as the initial estimation error.

The *estimation error* (or observation error) is defined as the difference between the measured output and the estimated output. The *initial error* refers to the difference between the actual initial state and the estimated initial state(Figure 5.1).

The mathematical model of the observer is defined as follows: $\hat{\mathbf{x}}$ represents the estimated state vector, $\hat{\mathbf{C}}\mathbf{x}$ denotes the estimated output, and $\mathbf{K}_e$ is the observer gain matrix.

The observer's dynamic equation is expressed as:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e\,(y - C\hat{x}) \tag{5.2}$$

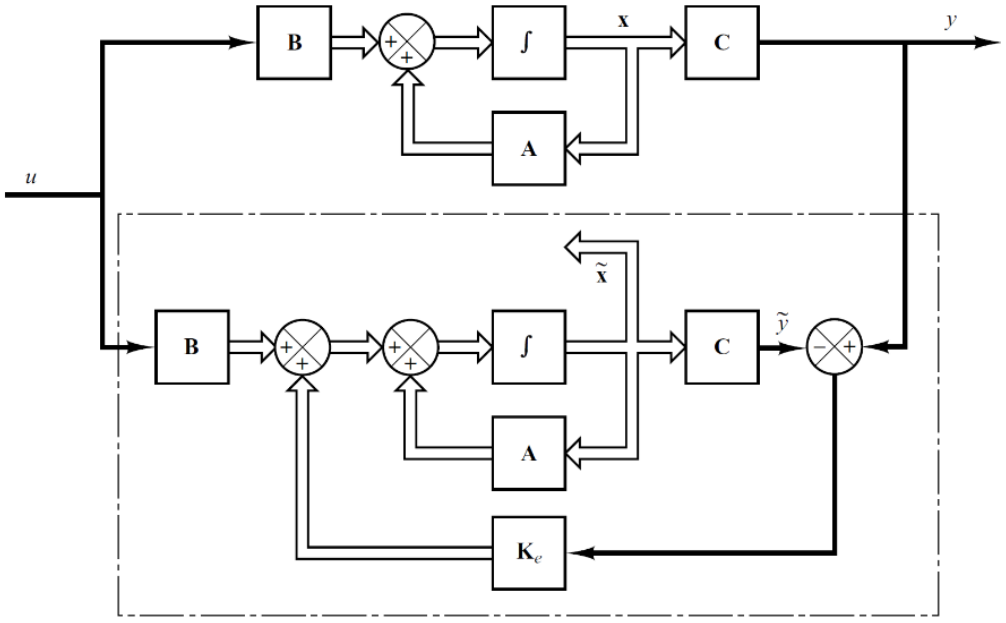


FIGURE 5.1: The Full-State Luenberger Observer.

5.2.2 Full Order State Observer

The state observer discussed here has the same order as the plant.

Consider a plant represented by the following equations:

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{5.3}$$

- The equation for the state observer is expressed as:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x}) \tag{5.4}$$

- To derive the observer error equation, subtract Equation (Equation 5.4) from (Equation 5.3):

$$\dot{x} - \dot{\hat{x}} = Ax + Bu - [A\hat{x} + Bu + K_e(y - C\hat{x})] \tag{5.5}$$

$$\Rightarrow \dot{x} - \dot{\hat{x}} = Ax + Bu - A\hat{x} - Bu - K_e(Cx - C\hat{x}) \tag{5.6}$$

- By simplifying the above equation, we can express it as:

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) - K_eC(x - \hat{x}) \tag{5.7}$$

- Define the error vector e as the difference between x and $\hat{x}$:

$$e = x - \hat{x} \tag{5.8}$$

- Substituting e into (Equation 5.7), the error dynamics can be written as:

$$\dot{e} = Afe - K_eCe \tag{5.9}$$

- Finally, the error equation simplifies to:

$$\dot{e} = (A - K_eC)e \tag{5.10}$$

From the equation above, we observe that the dynamic behavior of the error vector $\mathbf{e}$ is governed by the eigenvalues of the matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.

If $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ is a stable matrix, the error vector $\mathbf{e}$ will converge to zero for any initial error vector $\mathbf{e}(0)$.

This implies that $\hat{\mathbf{x}}(t)$ will approach $\mathbf{x}(t)$ regardless of the initial state $\mathbf{x}(0)$.

Furthermore, if the eigenvalues of $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ are selected such that the error dynamics are asymptotically stable and sufficiently fast, the error vector $\mathbf{e}$ will converge to zero (the origin) at an adequate speed.

If the plant is fully observable, it can be demonstrated that it is possible to select the matrix $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has any desired eigenvalues.

In other words, the observer gain matrix $\mathbf{K}_e$ can be chosen to produce the desired matrix $\mathbf{A} - \mathbf{K}_e\mathbf{C}$.

5.2.3 Reduced-order State Observer

We consider a linear, time-invariant system described by:

$$\dot{x} = Ax + Bu, \quad y = Cx + Du. \tag{5.11}$$

Assume that p of the n states can be directly measured. Let the state vector x be partitioned as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where $x_1 \in \mathbb{R}^p$ and $x_2 \in \mathbb{R}^{n-p}$.

The system dynamics can then be expressed as:

$$\dot{x}_1 = A_{11}x_1 + A_{12}x_2 + B_1u, \quad \dot{x}_2 = A_{21}x_1 + A_{22}x_2 + B_2u. \tag{5.12}$$

The system's observation equation is given by:

$$y = C_1x_1 \implies x_1 = C_1^{-1}y. \tag{5.13}$$

Since $x_1 \in \mathbb{R}^p$ is measurable, it is sufficient to estimate $x_2 \in \mathbb{R}^{n-p}$. We propose the estimate:

$$\hat{x}_2 = Ly + z, \tag{5.14}$$

where z evolves according to:

$$\dot{z} = Fz + Gy + Hu. \tag{5.15}$$

If we select the matrices as follows:

$$F = A_{22} - LC_1A_{12}, \quad H = B_2 - LC_1B_1, \quad GC_1 = A_{21} - LC_1A_{11} + FLC_1, \tag{5.16}$$

then the dynamics of the estimation error are governed by:

$$e_2 = Fe_2, \quad e_2 = x_2 - \hat{x}_2. \tag{5.17}$$

- **Example:**

Consider a linear time-invariant (LTI) system represented as:

$$\dot{x} = Ax + Bu, \quad y = Cx,$$

where:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

We aim to design a reduced state observer that estimates the unmeasured states $x_2, x_3 \in \mathbb{R}^2$, assuming $x_1 \in \mathbb{R}$ is directly measurable.

- *Step 1:* Partition the State Space

Partition the state vector x into:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

where:

$$x_1 = x_1 \in \mathbb{R}, \quad x_2 = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^2.$$

The system matrices are partitioned as:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where:

$$A_{11} = 0, \quad A_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 \\ -6 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}, \quad B_1 = 0, \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

- *Step 2:* Reduced State Observer Dynamics

The reduced state observer is designed to estimate $x_2 \in \mathbb{R}^2$ using the following structure:

$$\hat{x}_2 = Ly + z, \quad \dot{z} = Fz + Gy + Hu,$$

where:

$$e_2 = x_2 - \hat{x}_2, \quad \dot{e}_2 = Fe_2.$$

- *Step 3:* Design the Observer Matrices

a. Error Dynamics: The observer gain matrix L is chosen such that the error dynamics:

$$\dot{e}_2 = Fe_2,$$

have stable eigenvalues. The matrix F is given by:

$$F = A_{22} - LC_1A_{12}.$$

b. Auxiliary Matrices:

$$H = B_2 - LC_1B_1, \quad GC_1 = A_{21} - LC_1A_{11} + FLC_1.$$

Eigenvalue Placement: Assume we desire the eigenvalues of F to be -2 and -3. The characteristic polynomial is:

$$p(s) = (s+2)(s+3) = s^2 + 5s + 6.$$

We equate this to the determinant of $(sI - F)$:

$$\det(sI - F) = s^2 - \mathrm{tr}(F)s + \det(F).$$

Thus:

$$\mathrm{tr}(F) = -5, \quad \det(F) = 6.$$

- *Step 4:* Calculate Observer Gain

Matrix Equations:

$$F = A_{22} - LC_1A_{12}.$$

Substitute:

$$A_{22} = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Compute:

$$C_1A_{12} = 1, \quad LC_1A_{12} = L.$$

Let:

$$L = \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix}.$$

Then:

$$F = \begin{bmatrix} 0 & 1 \\ -11 & -6 \end{bmatrix} - \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \cdot 1 = \begin{bmatrix} -\ell_1 & 1 \\ -11 - \ell_2 & -6 \end{bmatrix}.$$

Equate the characteristic polynomial:

$$s^2 + 5s + 6 = \det\left(sI - \begin{bmatrix} -\ell_1 & 1 \\ -11 - \ell_2 & -6 \end{bmatrix} \right).$$

From F:

$$\mathrm{tr}(F) = -\ell_1 - 6 = -5 \implies \ell_1 = 1.$$

$$\det(F) = (-\ell_1)(-6) - (1)(-11 - \ell_2) = 6 \implies (1)(6) - (-11 - \ell_2) = 6 \implies \ell_2 = -11.$$

Thus:

$$L = \begin{bmatrix} 1 \\ -11 \end{bmatrix}.$$

Other Matrices:

$$H = B_2 - LC_1B_1 = \begin{bmatrix}0\\1\end{bmatrix} - \begin{bmatrix}1\\-11\end{bmatrix} \cdot 0 = \begin{bmatrix}0\\1\end{bmatrix}.$$

$$GC_1 = A_{21} - LC_1A_{11} + FLC_1 = A_{21} + F \cdot L = \begin{bmatrix}0\\-6\end{bmatrix} + \begin{bmatrix}-1 & 1\\-22 & -6\end{bmatrix} \cdot \begin{bmatrix}1\\-11\end{bmatrix}.$$

- *Step 5:* Observer Dynamics

The reduced observer becomes:

$$\dot{z} = \begin{bmatrix}-1 & 1\\-22 & -6\end{bmatrix} z + \begin{bmatrix}0\\1\end{bmatrix} y + \begin{bmatrix}0\\1\end{bmatrix} u,$$

and the state estimate:

$$\hat{x}_2 = Ly + z.$$

The reduced state observer provides an estimate for x_2 while using the measurable state x_1. The designed matrices ensure stability and accurate estimation based on the desired eigenvalues.

5.2.4 Duality Property

The design of the full-order observer involves selecting an appropriate $\mathbf{K}_e$ such that $\mathbf{A} - \mathbf{K}_e\mathbf{C}$ has the desired eigenvalues.

Therefore, the problem essentially becomes equivalent to the pole-placement problem.

In fact, the two problems are mathematically identical. This property is referred to as **duality**.

Consider the system described by Equation (Equation 5.3). In designing the full-order state observer, we can approach the problem by solving the dual problem, which means solving the pole-placement problem for the dual system. Assuming the control signal v is given, the system is represented as:

$$\mathbf{v} = -\mathbf{K}\mathbf{z} \tag{5.18}$$

For the dual system, we have:

$$\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{C}^*\mathbf{v}, \quad \mathbf{n} = \mathbf{B}^*\mathbf{z} \tag{5.19}$$

If the dual system is fully state controllable, the state feedback gain matrix K can be chosen such that the matrix $A^* - C^*K$ achieves the desired eigenvalues.

- Let $\mu_1, \mu_2, \ldots, \mu_n$ represent the desired eigenvalues for the state observer matrix. By selecting these same μ_i values as the desired eigenvalues for the state feedback gain matrix of the dual system, we achieve the desired result.

- Since the eigenvalues of $A^* - C^*K$ and $A - K^*C$ are identical, we can express:

$$|sI - (A^* - C^*K)| = (s - \mu_1)(s - \mu_2)\ldots(s - \mu_n) \tag{5.20}$$

Given that the eigenvalues of $A^* - C^*K$ and $A - K^*C$ are identical, we can express the relationship as follows:

$$\Rightarrow |sI - (A^* - C^*K)| = |sI - (A - K^*C)| \tag{5.21}$$

This implies that the placement of the eigenvalues for the dual system directly corresponds to the eigenvalues of the original system.

- By comparing the characteristic polynomial $sI - (A - K^*C)$ with that of the observer system $sI - (A - K_eC)$, it becomes clear that K_e and K^* are connected.

- Consequently, the observer gain matrix K_e for the original system can be obtained by utilizing the matrix K derived through the pole-placement method in the dual system, based on the relationship $K_e = K^*$.

5.3 Methods for designing observers

The design of a full-order observer is essentially reduced to determining the observer gain matrix K_e such that the error dynamics become asymptotically stable. Therefore, the task involves finding an appropriate matrix K_e so that $[A - K_eC]$ has the desired eigenvalues. This makes the problem equivalent to the pole-placement problem, allowing similar techniques to be applied.

5.3.1 Direct Substitution Method

- The feedback signal provided by the observer gain matrix K_e acts as a corrective input to the plant model, compensating for the uncertainties present in the system.
- When the system contains substantial uncertainties, the feedback signal generated through K_e needs to be relatively strong.

- Conversely, if the output signal y is heavily affected by disturbances or measurement noise, it becomes less reliable, and the feedback signal through K_e should be reduced accordingly.

- The observer gain matrix K_e is determined by the desired characteristic equation: $(s-\beta_1)(s-\beta_2)\cdots(s-\beta_n)=0$. - To ensure that the estimation error (observation error) converges to zero rapidly, the observer poles must be two to five times faster than the controller poles. - This means that the observer's estimation error decreases two to five times faster than the state vector x itself. - Such a faster decay of the observer error, compared to the desired system dynamics, ensures that the controller poles dominate the system's response.

It is crucial to mention that if sensor noise is significant, we might opt to set the observer poles slower than twice the speed of the controller poles. This adjustment will reduce the system's bandwidth and help smooth the noise. In this scenario, the system's response will be more strongly influenced by the observer poles. If the observer poles are positioned to the right of the controller poles in the left half of the s-plane, the system's behavior will be dominated by the observer poles rather than the control poles.

In state observer design, it is preferable to determine multiple observer gain matrices Ke based on different desired characteristic equations. For each of these matrices, simulation tests should be conducted to assess the resulting system performance. The best Ke is then chosen based on the overall system's performance. In many practical situations, selecting the optimal matrix Ke involves a trade-off between fast response and the system's sensitivity to disturbances and noise.

- **Example:**

Consider the system:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 20.6\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u(t)$$

The output equation is:

$$y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

We employ an observer-based approach to design state feedback control such that: $u=-K\hat{x}$

Design a full-order state observer, assuming the desired eigenvalues of the observer matrix are $\beta_1=-10$ and $\beta_2=-10$.

Let us first analyze the observability matrix.

The observability matrix is:

$$O=\begin{bmatrix}C\\ CA\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

Since the rank of the observability matrix O is 2, the given system is fully state observable, and the determination of the desired observer gain matrix is feasible.

The characteristic equation for the observer error matrix is given by:

$$|sI - (A - K_eC)| = 0$$

Assuming that:

$$K_e = \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix}$$

We can express the equation as:

$$|sI - (A - K_eC)| = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} k_{e1} \\ k_{e2} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}$$

This simplifies to:

$$s^2 + k_{e2}s - 20.6 + k_{e1} = 0$$

This represents the characteristic equation for the observer error matrix, which relates the observer gain parameters k_{e1} and k_{e2} to the system dynamics.

The desired characteristic polynomial is:

$$s^2 + 20s + 100 = s^2 + k_{e2}s - 20.6 + k_{e1}$$

By comparing the coefficients of the powers of s, we get:

$$(s - \beta_1)(s - \beta_2) = s^2 + 20s + 100$$

From this comparison, the observer gain matrix K_e is determined as:

$$K_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

We obtain the same K_e regardless of the method used.

The full-order state observer equation is given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + K_e(y - C\hat{x})$$

Expanding this:

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} 120.6 \\ 20 \end{bmatrix} (y - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix})$$

Simplifying the equation:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & 20.6\\ 1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) + \begin{bmatrix}120.6\\ 20\end{bmatrix}y - \begin{bmatrix}120.6\\ 20\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$$

Which results in:

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}0 & -100\\ 1 & -20\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) + \begin{bmatrix}120.6\\ 20\end{bmatrix}y$$

5.3.2 Using Transformation Matrix Q

The observable canonical form method is called alsow (transformation matrix Q). The observation form of the system's state equations is given by: The state-space form is given by:

$$\dot{x} = \begin{bmatrix}0 & 0 & 0 & \cdots & -a_0\\ 1 & 0 & 0 & \cdots & -a_1\\ \vdots & 1 & \ddots & \cdots & \\ 0 & \cdots & \cdots & 0 & a_{n-2}\\ 0 & 0 & \cdots & 1 & -a_{n-1}\end{bmatrix}x + \begin{bmatrix}b_0\\ b_1\\ \vdots\\ b_{n-2}\\ b_{n-1}\end{bmatrix}u$$

$$y = [0\ 0\ \cdots\ 0\ 1]x$$

The value of the observer gain matrix K_e can be calculated by:

$$K_e = Q\begin{bmatrix}\alpha_0 - a_0\\ \alpha_1 - a_1\\ \vdots\\ \alpha_{n-1} - a_{n-1}\end{bmatrix}$$

Where Q is the transformation matrix that converts the system state equation into the observable canonical form:

$$Q = (W.Ob)^{-1}$$

Here, W is the matrix defined previously, and Ob is the observability matrix:

$$Ob = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Note that $Q = I$ if the system is already in observable canonical form.

- **Example:**

Consider the identical system from the previous example. The system provided is already in observable canonical form, so the transformation matrix Q is the identity matrix (I).

The characteristic equation is:

$$|sI - A| = s^2 - 20.6 = 0$$

From this, we obtain:

$$a_1 = 0, \quad a_2 = -20.6$$

The desired characteristic polynomial is:

$$(s - \beta_1)(s - \beta_2) = (s + 10)(s + 10) = s^2 + 20s + 100$$

we have:

$$\alpha_1 = 20, \quad \alpha_2 = 100$$

The observer gain matrix K_e can be calculated using the following formula:

$$K_e = \begin{bmatrix} \alpha_2 - a_2 \\ \alpha_1 - a_1 \end{bmatrix}$$

Where:

$$\alpha_1 = 20, \quad \alpha_2 = 100$$

And:

$$a_1 = 0, \quad a_2 = -20.6$$

Thus:

$$K_e = \begin{bmatrix} 100 - (-20.6) \\ 20 - 0 \end{bmatrix}$$

Finally:

$$K_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

5.3.3 Ackermann's Formula

In this method, the observer gain matrix can be calculated as:

$$K_e = \phi(A)\, O_b^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

or:

$$K_e = \phi(A) \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

where $\phi(A)$ is defined previously.

- **Example:**

Consider the same system from the previous example. Using Ackermann's formula:

Where

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Using Ackermann's formula:

$$K_e = \phi(A) \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where:

$$\phi(A) = A^2 + \alpha_1 A + \alpha_2 I$$

With $\alpha_1 = 20$ and $\alpha_2 = 100$, we have:

$$\phi(A) = A^2 + 20A + 100I$$

$$\phi(A) = \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix}^2 + 20 \begin{bmatrix} 0 & 20.6 \\ 1 & 0 \end{bmatrix} + 100 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\phi(A) = \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix}$$

Using Ackermann's formula:

$$K_e = \phi(A) \cdot \begin{bmatrix} C \\ CA \end{bmatrix}^{-1} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 120.6 & 412 \\ 20 & 120.6 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$K_e = \begin{bmatrix} 120.6 \\ 20 \end{bmatrix}$$

5.4 Stochastic observers and Kalman filter

The Kalman filter is the optimal stochastic observer for linear time-invariant (LTI) systems subjected to process and measurement noise. Below are the equations that model the system and the Kalman filter. The continuous-time LTI system is modeled as:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t) \tag{5.22}$$

$$y(t) = Cx(t) + v(t)$$

Where: - $x(t)$: State vector $(n \times 1)$. - $u(t)$: Input vector $(m \times 1)$. - $y(t)$: Measurement vector $(p \times 1)$. - $w(t)$: Process noise, assumed to be Gaussian with zero mean and covariance Q $(w(t) \sim \mathcal{N}(0, Q))$. - $v(t)$: Measurement noise, Gaussian with zero mean and covariance R $(v(t) \sim \mathcal{N}(0, R))$. - A, B, C: System matrices of appropriate dimensions.

The Kalman filter provides an optimal estimate $\hat{x}(t)$ of the state $x(t)$ based on noisy measurements $y(t)$.

The predicted state at time t is given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)\big(y(t) - C\hat{x}(t)\big) \tag{5.23}$$

Where: - $K(t)$: Kalman gain matrix.

The covariance of the estimation error $P(t)$ evolves as:

$$\dot{P}(t) = AP(t) + P(t)A^T + Q - P(t)C^T R^{-1} CP(t) \tag{5.24}$$

The Kalman gain is computed as:

$$K(t) = P(t)C^T R^{-1} \tag{5.25}$$

The estimation error $e(t) = x(t) - \hat{x}(t)$ dynamics are governed by:

$$\dot{e}(t) = (A - K(t)C)e(t) + w(t) - K(t)v(t) \tag{5.26}$$

For steady-state systems (time-invariant A, B, C, Q, R), the Kalman filter simplifies as follows:

- The steady-state error covariance P is the solution to the Algebraic Riccati Equation (ARE):

$$AP + PA^T + Q - PC^T R^{-1} CP = 0 \tag{5.27}$$

- The steady-state Kalman gain becomes:

$$K = PC^T R^{-1} \tag{5.28}$$

Summary of Kalman Filter Steps

1.Initialize:

$$\hat{x}(0) = \hat{x}_0, \quad P(0) = P_0$$

2. Prediction:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t)$$

$$\dot{P}(t) = AP(t) + P(t)A^T + Q$$

3. Update:

$$K(t) = P(t)C^T R^{-1}$$

$$\hat{x}(t) = \hat{x}(t) + K(t)\big(y(t) - C\hat{x}(t)\big)$$

$$P(t) = (I - K(t)C)P(t)$$

4. Iterate: Repeat steps for each time increment.

Applications: - State estimation for control systems. - Sensor fusion. - Signal denoising in stochastic environments.

These equations form the backbone of continuous-time stochastic state observers and Kalman filtering.

- **Example:**

Consider a continuous-time linear time-invariant (LTI) system given by the state-space equations:

$$\dot{x}(t) = Ax(t) + Bu(t) + w(t),$$

$$y(t) = Cx(t) + v(t),$$

where: - $x(t)$ is the state vector, - $u(t)$ is the control input, - $y(t)$ is the measured output, - $w(t)$ and $v(t)$ are process and measurement noise, assumed to be zero-mean white noise with covariances Q and R, respectively.

Let:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

Assume:

$$Q = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad R = 0.5.$$

The Kalman filter gain K is obtained by solving the continuous-time algebraic Riccati equation (CARE):

$$A^T P + PA - PC^T R^{-1} CP + Q = 0,$$

where P is the error covariance matrix. The Kalman gain is then given by:

$$K = PC^T R^{-1}.$$

1. Solve the CARE for P:

Using the given A, C, Q, and R:

$$A^T = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Substitute these into the CARE:

$$\begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} P + P \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - P \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{0.5} \begin{bmatrix} 1 & 0 \end{bmatrix} P + \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} = 0.$$

Solving this matrix equation (numerically or analytically) gives:

$$P = \begin{bmatrix} 0.311 & 0.123 \\ 0.123 & 0.289 \end{bmatrix}.$$

2. Compute the Kalman Gain K:
Using $K = PC^T R^{-1}$:

$$K = \begin{bmatrix} 0.311 & 0.123 \\ 0.123 & 0.289 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{1}{0.5}.$$

$$K = \begin{bmatrix} 0.622 \\ 0.246 \end{bmatrix}.$$

The Kalman gain for this system is:

$$K = \begin{bmatrix} 0.622 \\ 0.246 \end{bmatrix}.$$

This gain minimizes the estimation error for the states $x(t)$, considering the given noise characteristics.

5.5 Solved exercises

Exercise 01: Consider the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

The output equation is:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

We employ an observer-based approach to design state feedback control such that: $u = -K\hat{x}$

Design a full-order state observer, assuming the desired eigenvalues of the observer matrix are $\beta_1 = -10$, $\beta_2 = -10$ and $\beta_3 = -15$

Solution of exercise 01: - System Dynamics:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$y(t) = Cx(t), \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

We aim to design a full-order state observer such that the eigenvalues of the observer error dynamics are $\{-10, -10, -15\}$.

The observer equation is:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\big(y(t) - C\hat{x}(t)\big),$$

where $\hat{x}(t)$ is the estimated state vector, and L is the observer gain matrix.

Define the error $e(t) = x(t) - \hat{x}(t)$. The error dynamics are:

$$\dot{e}(t) = \big(A - LC\big)e(t).$$

The matrix $A - LC$ must have eigenvalues $\{-10, -10, -15\}$.

- Characteristic Equation of $A-LC$: The eigenvalues $\{-10, -10, -15\}$ correspond to the characteristic equation:

$$\det(sI - (A - LC)) = (s + 10)^2(s + 15).$$

Expanding this, we get:

$$s^3 + 35s^2 + 400s + 1500 = 0.$$

- Matrix $A - LC$:

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 & -6 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

Simplify $A - LC$:

$$A - LC = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -5 - l_1 & -6 - l_2 & -l_3 \end{bmatrix}.$$

- Find L Using the Desired Polynomial: The characteristic polynomial of $A-LC$ is determined by:

$$\det(sI-(A-LC)) = s^3 + (6+l_2)s^2 + (5+l_1+l_3)s + (5l_2+l_3).$$

Match this with $s^3 + 35s^2 + 400s + 1500$ to find l_1, l_2, l_3:

- From the s^2-term: $6 + l_2 = 35$, so $l_2 = 29$.

- From the s-term: $5 + l_1 + l_3 = 400$, so $l_1 + l_3 = 395$.

- From the constant term: $5l_2 + l_3 = 1500$, so $5(29) + l_3 = 1500$, giving $l_3 = 1355$.

Substitute $l_3 = 1355$ into $l_1 + l_3 = 395$ to find $l_1 = -960$.

Thus, the observer gain matrix is:

$$L = \begin{bmatrix} -960 \\ 29 \\ 1355 \end{bmatrix}.$$

Exercise 02:

Consider the double integrator system defined as follows. By selecting the state variables appropriately, the state-space representation of the system becomes:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

- The goal is to design a regulator for this system. - Using the pole-placement-with-observer approach, design an observer-based controller. The desired closed-loop poles for the pole-placement part are:

$$s = -0.7071 + j0.7071, \quad s = -0.7071 - j0.7071.$$

- For the observer design, assuming a minimum-order observer, choose the desired observer pole at:

$$s = -5.$$

Solution of exercise 02:

- State-Space Representation of the System: The double integrator system is represented in state-space form as:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x,$$

where: - $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is the state vector, - u is the control input, - y is the system output.

The closed-loop poles for the state feedback control are:

$$s_1 = -0.7071 + j0.7071, \quad s_2 = -0.7071 - j0.7071.$$

This corresponds to a characteristic equation:

$$(s + 0.7071)^2 + 0.7071^2 = s^2 + 1.4142s + 1.$$

The observer pole is chosen at:

$$s_o = -5.$$

a. To design the state feedback control $u = -Kx$, the system matrix A is modified as:

$$A_{cl} = A - BK.$$

The characteristic equation of A_{cl} is:

$$\det(sI - (A - BK)) = s^2 + 1.4142s + 1.$$

Substitute A and B:

$$A - BK = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix},$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}.$$

The characteristic polynomial is:

$$\det(sI - (A - BK)) = \det \begin{bmatrix} s & -1 \\ k_1 & s + k_2 \end{bmatrix} = s^2 + k_2 s + k_1.$$

Equating coefficients with $s^2 + 1.4142s + 1$, we get:

$$k_2 = 1.4142, \quad k_1 = 1.$$

Thus, the state feedback gain vector is:

$$K = \begin{bmatrix} 1 & 1.4142 \end{bmatrix}.$$

b. Observer Design (L): For the observer, the dynamics are given by:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}),$$

where the observer error dynamics are governed by:

$$\dot{e} = (A - LC)e.$$

The observer pole is at $s_o = -5$, corresponding to the characteristic equation:

$$\det(sI - (A - LC)) = s + 5.$$

Substitute A, L, and C:

$$A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix},$$

$$A - LC = \begin{bmatrix} -l_1 & 1 \\ -l_2 & 0 \end{bmatrix}.$$

The characteristic polynomial is:

$$\det(sI - (A - LC)) = \det \begin{bmatrix} s + l_1 & -1 \\ l_2 & s \end{bmatrix} = s^2 + l_1 s + l_2.$$

Equating coefficients with $s + 5$, we get:

$$l_1 = 5, \quad l_2 = 0.$$

Thus, the observer gain vector is:

$$L = \begin{bmatrix} 5 \\ 0 \end{bmatrix}.$$

Exercise 03: Consider the system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

The output equation is:

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Design a regulator system using the pole-placement-with-observer approach. Assume the desired closed-loop poles for the pole placement are at:

$$s = -1 + j, \quad s = -1 - j, \quad s = -5.$$

The desired observer poles are placed at:

$$s = -6, \quad s = -6, \quad s = -6.$$

Additionally, determine the transfer function of the observer-based controller.

Solution of exercise 03:

To design a regulator system using the pole-placement-with-observer approach, we need to determine the state feedback gain matrix K and the observer gain matrix L that achieve the desired closed-loop poles and observer poles. The process involves the following steps:

- Step 1: Define the system matrices The system is given by:

$$\dot{x}(t) = Ax(t) + Bu(t)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

The output is $y(t) = Cx(t)$.

- Step 2: Desired closed-loop poles The desired closed-loop poles for the pole placement are at:

$$s = -1 + j, \quad s = -1 - j, \quad s = -5.$$

These poles correspond to the eigenvalues of the closed-loop system.

- Step 3: Pole placement for state feedback K The goal is to place the poles of the system by selecting an appropriate state feedback gain matrix K such that the poles of the closed-loop system are located at the desired values. The closed-loop dynamics with state feedback control $u(t) = -Kx(t)$ are:

$$\dot{x}(t) = (A - BK)x(t).$$

The characteristic equation of the closed-loop system is determined by the eigenvalues of the matrix $(A - BK)$. By using the desired poles, we can compute the value of K.

- Step 4: Observer design for L The observer is designed such that the estimation error dynamics are stable with poles at the desired observer locations. The observer is given by:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)),$$

where $\hat{x}(t)$ is the estimated state, and the observer poles are specified to be at:

$$s = -6, \quad s = -6, \quad s = -6.$$

The observer gain matrix L is determined by placing the poles of the observer's error dynamics.

- Step 5: Solving for K and L

Using pole placement, we can compute the state feedback gain matrix K that places the poles at $-1+j, -1-j, -5$. This can be done using the Ackermann's formula or any pole-placement method available in control software such as MATLAB.

Similarly, the observer gain matrix L is computed by placing the observer poles at $-6, -6, -6$. This can be done using the LQR method or pole-placement techniques.

- Step 6: Controller design The total system controller consists of two parts: the state feedback and the observer.

- State feedback: $u(t) = -Kx(t)$. - Observer: $\hat{x}(t)$ is estimated using the observer equation, and then the control input is calculated as $u(t) = -K\hat{x}(t)$.

The transfer function of the observer controller can be derived by considering the closed-loop system with the observer and the state feedback.

- MATLAB Example: To calculate K and L using MATLAB, the following code can be used:

```
A = [0 1 0; 0 0 1; -6 -11 -6];
B = [0; 0; 1];
C = [1 0 0];
desired_poles = [-1+1j, -1-1j, -5];  % Desired closed-loop poles
observer_poles = [-6, -6, -6];  % Desired observer poles

% Compute state feedback gain K
K = place(A, B, desired_poles);

% Compute observer gain L
L = place(A', C', observer_poles)';

disp('State Feedback Gain K:');
disp(K);

disp('Observer Gain L:');
disp(L);
```

This code calculates the state feedback gain K and observer gain L, and places the poles at the desired locations.

5.6 Conclusion

In this chapter, we have explored the fundamental principles and methods for designing state observers, highlighting their crucial role in control and monitoring systems. Starting with deterministic observers, such as the Luenberger observer, we established the theoretical framework for state estimation in noise-free environments. Practical considerations for gain tuning, stability, and robustness were also addressed, providing a foundation for real-world implementation.

The discussion then extended to reduced-order observers, offering solutions for systems where computational efficiency or complexity reduction is paramount. Finally, we delved into stochastic observers, emphasizing the Kalman filter's optimality and versatility in handling noisy measurements and system uncertainties.

By synthesizing these concepts, the chapter equips students with a robust understanding of state observer design. This knowledge is essential for developing advanced control systems capable of operating reliably in diverse and challenging scenarios.

Bibliography

1. Philippe de Larminat, "Automatique : Commande des systèmes linéaires" Hermès Lavoisier, 1996.

2. Hubert Egon, "Asservissement linéaires échantillonnés et représentation d'état" Méthodes, 2001.

3. Luc Jaulin, " Représentation d'état pour la modélisation et la commande des systèmes," Lavoisier, 2005.

4. Robert L. Williams, Douglas A. Lawrence, "Linear State-Space Control Systems," John Wiley Sons, 2007.

5. R. Longchamp, "Commande numérique de systèmes dynamiques ," Presses Polytechniques et Universitaires Romandes, 1995.

6. G. F. Franklin, J. D. Powell, L. M. Workman, "Digital Control of Dynamic Systems," Addison-Wesley Series in Electrical and Computer Engineering: Control Engineering, 1990.

7. K. J. Aström, B. Wittenmark, "Computer-Controlled Systems: Theory and Design," Prentice-Hall, 1984.

8. R. H. Middleton, G. C. Goodwin, "Digital Control and Estimation: A Unified Approach," Prentice Hall, 1990.