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Department of Sciences and Technology

Algebra

for First-Year Science and Technology Students

(Licence & Engineer)

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Abstract

This course handout is intended for first-year university students enrolled in Science and Technology programmes (Licence and Engineer cycle). It provides a self-contained and progressive introduction to the fundamental concepts of **Algebra**, spanning seven chapters that cover the essential mathematical structures required for advanced study in mathematics, physics, computer science, and engineering.

The handout opens with the basic language of set theory, relations, and functions, before developing the algebraic and geometric theory of complex numbers. It then introduces vector spaces and linear maps, examining their properties and the relationship between linear transformations and matrices. A thorough treatment of matrix algebra and determinants follows, including methods for computing inverses and solving linear systems. The handout concludes with the reduction of matrices, covering eigenvalues, eigenvectors, and diagonalisation. Throughout, theoretical results are supported by detailed worked examples and exercises designed to reinforce understanding and build problem-solving skills.

Keywords:
relations, functions, complex numbers, vector spaces, linear independence, basis, dimension, linear maps, matrices, determinants, invertibility, eigenvalues, eigenvectors, systems of linear equations, Gaussian elimination, Cramer's rule.

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Preface

This course notes handout is intended for first-year university students (Science and Technology). It covers the foundational topics of linear algebra and mathematical structures that are essential not only for the study of pure mathematics but also for a wide range of applied sciences, including physics, computer science, and engineering.

The material presented here has been carefully structured to guide the student from abstract mathematical concepts toward their concrete applications, building rigour and intuition simultaneously. Each chapter opens with the necessary definitions and theoretical results, illustrated by worked examples and followed by exercises designed to consolidate understanding.

The handout is organised into seven chapters:

- **Chapter 1 — Sets, Relations and Applications** introduces the basic language of mathematics. The notions of sets, binary relations, equivalence and order relations, as well as functions and their properties (injectivity, surjectivity, bijectivity), form the logical backbone upon which all subsequent chapters rest.

- **Chapter 2 — Complex Numbers** extends the real number system to the complex plane. Students will study the algebraic and geometric representations of complex numbers, modulus and argument, de Moivre's theorem, and the extraction of roots, with emphasis on their role in solving polynomial equations.

- **Chapter 3 — Vector Spaces** introduces one of the central structures of linear algebra. The concepts of vector spaces over a field, subspaces, linear independence, bases, and dimension are developed rigorously and illustrated through numerous examples drawn from classical spaces such as $\mathbb{R}^n$, polynomial spaces, and spaces of matrices.

- **Chapter 4 — Linear Maps** studies structure-preserving transformations between vector spaces. The kernel, image, rank, and nullity of a linear map are examined, and the fundamental rank-nullity theorem is proved. The relationship between linear maps and matrices is explored in detail.

- **Chapter 5 — Matrices, Associated Matrices, and Determinants** provides a thorough treatment of matrix algebra, including operations on matrices, invertibility, and the computation of determinants via cofactor expansion and row

reduction. The notion of the matrix associated with a linear map with respect to a given basis is also developed.

- **Chapter 6 — Reduction of Matrices** addresses the problem of simplifying the representation of a linear operator. Eigenvalues, eigenvectors, characteristic polynomials, and the conditions for diagonalisation are studied, along with the construction of a diagonalising basis when one exists.

- **Chapter 7 — Systems of Linear Equations** presents the theory and methods for solving linear systems, including Gaussian elimination, Cramer's rule, and the inverse matrix method. The conditions for existence and uniqueness of solutions are discussed in the framework of the rank of the augmented matrix.

Students are expected to have a solid background in high-school mathematics, particularly in elementary algebra and geometry. No prior knowledge of university-level mathematics is assumed; the text is self-contained and progresses gradually from foundational ideas to more advanced topics.

It is strongly recommended that students attempt all exercises after studying each chapter. Mathematics is a discipline learned through active practice: reading alone is not sufficient. Solutions to selected exercises are provided at the end of the booklet to allow self-assessment.

The author hope that this handout will serve as a reliable and accessible companion throughout the academic year, and that the ideas encountered here will spark a deeper curiosity for the beauty and power of mathematics.

The Author

Chapter 1

Sets, Relations and Applications

This chapter introduces the fundamental concepts of mathematical logic, sets, relations, and applications. It provides the basic tools needed to understand logical reasoning, mathematical statements, and their structures. The chapter also presents the theory of sets and explores different types of relations and their applications in various mathematical contexts. These concepts form an essential foundation for further studies in mathematics and computer science.

1.1 Elements of Logic

1.1.1 Assertion

Definition 1.1.1

An assertion is a sentence that is either true or false but not both at the same time.
We denote the "True" by (T) or (1) and by (F) or (0) for the "False"

Examples:

1/ $2+2=4$ ..... (T)

2/ $3^2=6$ ..... (F)

3/ For all $x \in \mathbb{R}$: $x^2 \geq 0$ ..... (T)

4/ For all $z \in \mathbb{C}$: $|z|=1$ ..... (F)

1.2 Logical Operations

1.2.1 Negation ¬

Definition 1.2.1

Given a logical proposition P, we call the *negation of* P the logical proposition $\bar{P}$, also written $\neg P$, which is false when P is true and true when P is false.

Examples:

1/ (P): $4 \leq 5$ ..... (T)
$(\bar{P})$: $4 > 5$ ..... (F)

2/ (q): For all $x \in \mathbb{N}$: $(x-6)(x-5) = 0$ ..... (F)
$(\bar{q})$: There exists at least one $x \in \mathbb{N}$: $(x-6)(x-5) \neq 0$ ..... (T)

1.2.2 Conjunction ∧ (and)

Definition 1.2.2

We call the *conjunction of* P *and* Q the logical proposition $P \wedge Q$, which is **True** when P **and** Q are both true.

Truth table for "Conjunction"

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

1.2.3 Disjunction ∨ (or)

Definition 1.2.3

We call the *disjunction of* P *and* Q the logical proposition $P \vee Q$, which is **True** when **at least one** of the propositions P or Q is true.

Truth table for "Conjunction"

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

1.2.4 Implication $\Rightarrow$

Definition 1.2.4

The proposition $(P \Rightarrow Q)$ is **False** if P is True and Q is False.

Remark 1.2.1. The logical proposition $(\bar{P} \vee Q)$ is equivalent to $(P \Rightarrow Q)$.

Truth table for "Implication"

P	$\bar{P}$	Q	$\bar{P} \vee Q$	$P \Rightarrow Q$
1	0	1	1	1
1	0	0	0	0
0	1	1	1	1
0	1	0	1	1

Examples:

1/ $\Big(2 \in \mathbb{N}\Big) \Rightarrow \Big(-3 > 0\Big)$ ..... (F)

2/ $\Big(\sqrt[3]{27} = 3\Big) \Rightarrow \Big(\frac{1}{2} = 0.5\Big)$ ..... (T)

1.2.5 Equivalence $\Leftrightarrow$

Definition 1.2.5

We say that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is the logical proposition $(P \Leftrightarrow Q)$.

Properties of logical operations

1/ $\bar{\bar{P}} \Leftrightarrow P$

2/ $P \wedge Q \Leftrightarrow Q \wedge P$

3/ $P \vee Q \Leftrightarrow Q \vee P$

4/ $\overline{P \wedge Q} \Leftrightarrow \bar{P} \vee \bar{Q}$ (De Morgan)

5/ $\overline{P \vee Q} \Leftrightarrow \bar{P} \wedge \bar{Q}$ (De Morgan)

6/ $(P \Rightarrow Q) \Leftrightarrow (\bar{Q} \Rightarrow \bar{P})$ (Contraposition)

1.3 Open Sentences

Definition 1.3.1

A sentence P may depend on a parameter x; for example "$x \leq 1$": the sentence $P(x)$ is true or false depending on the value of x.

1.4 Quantifiers

We use the following symbols:

1. $\exists$ is the *existential quantifier*: we write $\exists x$ to read "There exists x".
2. $\forall$ is the *universal quantifier*: we write $\forall x$ to read "For all x".
3. We write $\exists ! \, x$ to read "There exists a unique x".

Examples:

- "$\forall x \in [1, +\infty[: \ (x \geq 1)$" is true.
- "$\exists x \in \mathbb{R} \ : \ (x^2 < 0)$" is false.
- "$\forall n \in \mathbb{N}$: $n(n+1)$ is divisible by 2" is true.

1.4.1 Negation of Quantifiers

- The negation of "$\forall x \in E : P(x)$" is "$\exists x \in E : \overline{P(x)}$".
- The negation of "$\exists x \in E : P(x)$" is "$\forall x \in E : \overline{P(x)}$".

Examples:

- The negation of "$\forall x \in [1, +\infty[: \ (x \geq 1)$" is "$\exists x \in [1, +\infty[: \ (x < 1)$".
- The negation of "$\forall x \in \mathbb{R}, \ \exists y > 0 : (x + y > 10)$" is "$\exists x \in \mathbb{R}, \ \forall y > 0 : (x + y \leq 10)$".

Remark 1.4.1. *The order of quantifiers is very important.*

$$\forall x \in \mathbb{R},\ \exists y \in \mathbb{R} : (x - y > 0) \quad \text{(true)}$$

$$\exists y \in \mathbb{R},\ \forall x \in \mathbb{R} : (x - y > 0) \quad \text{(false)}$$

1.5 Tutorial 1 – Logic and Sets

Exercise 1.5.1. Let be P, Q two statements. Write the truth tables of the following stetements

1) $P \vee \left(\bar{P} \wedge Q\right)$

1) $\overline{(P \wedge Q)} \Rightarrow P$.

2) $P \Longleftrightarrow (\bar{P} \wedge Q)$

Then write the negation

Exercise 1.5.2. 1. Write the following sentance using the Quantifiers

" For any real number x, its squar is positive" . Then write the negation

" For every integer n, there exist a real x such that exponentiel of x is equal to n". Then write the negation

2. Write the following negation

a) $\forall x \in \mathbb{R}, \exists y > 0 : (x + y > 0)$

b) **if** n^2 is even **then** n is even

Exercise 1.5.3. Let $a, b \in \mathbb{R}$. Demonstrate by Contraposition that

1) **if** $(a \neq -1) \wedge (b \neq -1)$ **then** $(a + b + ab + 1 \neq 0)$

2) $\forall n \in \mathbb{N}^*$ if $(n^2 - 1)$ is not divisible by 8, then n is even

3) if $b < \sqrt{ab} < a$ then $a > b$, $\qquad a, b \in \mathbb{R}_+$

4) $(a \leq b) \Rightarrow \left(\sqrt{a+1} - \sqrt{a} \geq \sqrt{b+1} - \sqrt{b}\right)$, $\qquad a, b \in \mathbb{R}_+$

Exercise 1.5.4. Let $A = \{-7, -5, -3, -1, 1, 3, 5, 7\}$, $B = \{x \in \mathbb{Z},\ \ / \ x^2 < 9\}$,

$C = \{2, 3, 4, 5, 6\}$, $D = \{x \in \mathbb{Z},\ \ / \ x^2 \leq 9\}$

In each of the following fill in the blank with most appropriate symbol from $\in, \notin, \subset, \subseteq, =, \neq$, so that resulting statement is true

A.....D, $\quad$ 3.....B, $\quad$ 9.....D, $\quad$ $\{2\}$ .....$\bar{C}$, $\quad$ $\varnothing$.....D, $\quad$ A.....C

B.....C, $\quad$ 0.....$A \cap D$, $\quad$ 0.....$A \cup D$

1.6 Set Theory

Definition 1.6.1

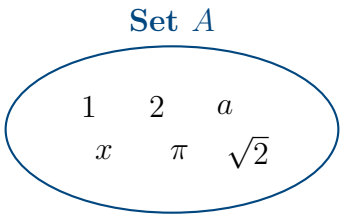
We call a *set* E any collection of objects. An object x is an *element* of E; we say that x *belongs to* E and write $x \in E$. If x is not an element of E, we write $x \notin E$.

Examples:

1. $A = \{1, 2, a, x, \pi, \sqrt{2}\}$ is given by liste; $\text{card}(A) = 6$.

2. $B = \{n \in \mathbb{N} \mid n \text{ is odd}\}$ is given by characteristic special.

3. A set can be illustrated by a **Venn diagram**:



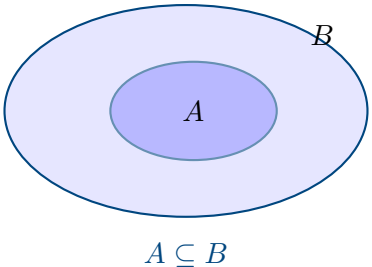
The **empty set** $\emptyset$ or $\{\ \}$ contains no element.

1.6.1 Subsets (Inclusion)

Definition 1.6.2

$A \subseteq B$ (read "A is a subset of B") iff every element of A is in B:

$$A \subseteq B \iff \forall x\,(x \in A \Rightarrow x \in B).$$



$A \subseteq B$

1.6.2 Power Set

Definition 1.6.3

The power set of a set A is the set of all subsets of A, including the empty set and A itself. It is usually denoted by $\mathcal{P}(A)$.

Examples: $A = \{a, 1, \pi\}$: $\mathcal{P}(A) = \{\emptyset, \{a\}, \{1\}, \{\pi\}, \{a, 1\}, \{a, \pi\}, \{1, \pi\}, A\}$.

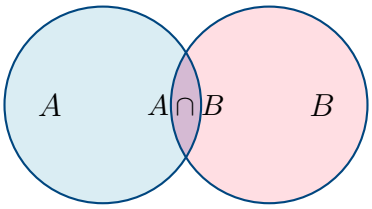
Definition 1.6.4 Equality of Sets

$A = B \iff (A \subseteq B) \wedge (B \subseteq A)$.

1.7 Operations on Sets

1.7.1 Intersection "$\cap$"

$$A \cap B = \{x \mid (x \in A) \wedge (x \in B)\}.$$

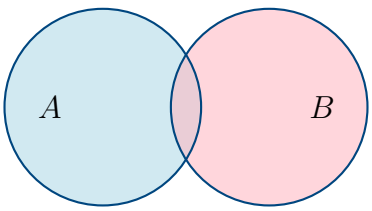


Intersection

Properties: (1) $E \cap F = F \cap E$; (2) $(E \cap F) \cap G = E \cap (F \cap G)$; (3) $E \cap \emptyset = \emptyset$.

1.7.2 Union "$\cup$"

$$A \cup B = \{x \mid (x \in A) \vee (x \in B)\}.$$

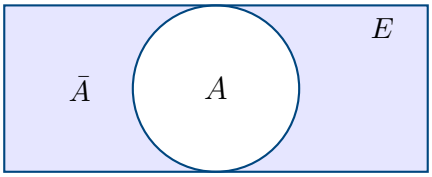


Union

Properties: (1) $E \cup F = F \cup E$; (2) $(E \cup F) \cup G = E \cup (F \cup G)$; (3) $E \cup \emptyset = E$.
Examples: $A = \{a, c, 5, x, y, \pi\}$, $B = \{a, x, 1, 4\}$: $A \cap B = \{a, x\}$, $A \cup B = \{a, c, 1, 5, x, y, \pi, 4\}$.

1.7.3 Complement "$\bar{A}$"

$\complement_E A = \{x \in E \mid x \notin A\} = \bar{A}$.



Complement $\bar{A}$

Properties:

1/ $A \subseteq B \Leftrightarrow \bar{B} \subseteq \bar{A}$

2/ $\bar{\bar{A}} = A$

3/ $\overline{A \cap B} = \bar{A} \cup \bar{B}$

4/ $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof. We have

$$\begin{aligned} A \subset B &\Leftrightarrow \forall x \in E,\ (x \in A) \Rightarrow (x \in B) \\ &\Leftrightarrow \forall x \in E,\ (x \notin B) \Rightarrow (x \notin A) \quad \text{(Contraposition of implication)} \\ &\Leftrightarrow \forall x \in E,\ (x \in \overline{B}) \Rightarrow (x \in \overline{A}) \\ &\Leftrightarrow \overline{B} \subset \overline{A}. \end{aligned}$$

Let $x \in E$. Then

$$\begin{aligned} x \in \overline{\overline{A}} &\Leftrightarrow x \notin \overline{A} \\ &\Leftrightarrow x \in A. \end{aligned}$$

Therefore,

$$\overline{\overline{A}} = A.$$

Let $x \in E$. Then

$$\begin{aligned} x \in \overline{A \cap B} &\Leftrightarrow x \notin A \cap B \\ &\Leftrightarrow (x \notin A) \vee (x \notin B) \\ &\Leftrightarrow (x \in \overline{A}) \vee (x \in \overline{B}) \\ &\Leftrightarrow x \in (\overline{A} \cup \overline{B}). \end{aligned}$$

Hence,

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

■

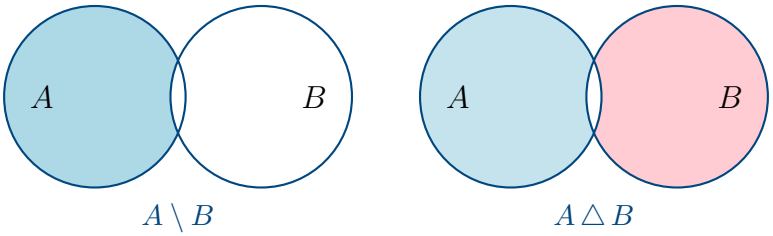
1.7.4 Difference "\"

$$A \setminus B = \{x \in A \text{ and } x \notin B\},$$

1.7.5 Symmetric Difference "△"

$$A \triangle B = (A \setminus B) \cup (B \setminus A).$$

$$A \triangle B = (A \cup B) \setminus (A \cap B).$$



1.7.6 Cartesian Product

Definition 1.7.1

The Cartesian product of two sets A and B is the set of all ordered pairs (x, y), where x is an element of A and y is an element of B. In other words, we have

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

Examples:, if

$$A = \{a, b, c\}, \qquad B = \{1, 2, 3, 4, 5\},$$

then

$$\begin{aligned} A \times B = \{&(a,1),(a,2),(a,3),(a,4),(a,5),\\ &(b,1),(b,2),(b,3),(b,4),(b,5),\\ &(c,1),(c,2),(c,3),(c,4),(c,5)\}. \end{aligned}$$

1.7.7 Partition of a Set

A *partition* of E is a set of one or more nonempty subsets of E: $E_1, E_2, \ldots$ such that
1) $E_1 \cup E_2 \cup \ldots = E$ 2) if $i \neq j$ then $E_i \cap E_j = \emptyset$

Examples: $\mathcal{F} = \Big\{\{a,\gamma\},\{d,\alpha,\beta\},\{c,1\},\{3,\ell\},\{b\}\Big\}$ is a partition of $E = \{1,a,\ell,3,b,c,d,\alpha,\beta,\gamma\}$.

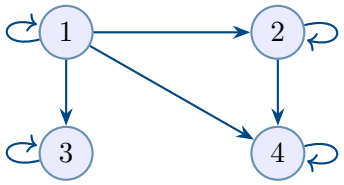
1.8 Relations

Definition 1.8.1 Binary Relation

A *binary relation* on E is an assertion xRy between two elements that may or may not hold.

Examples: On $A = \{1,2,3,4\}$, the divisibility relation $R = \{(a,b) \mid a \text{ divides } b\}$ gives:

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$



Divisibility on $\{1,2,3,4\}$

1.8.1 Reflexive Relation

$$R \textbf{ reflexive} \Leftrightarrow \forall x \in E,\ xRx.$$

Examples

1. The relation "equal to" $(=)$ is reflexive on $\mathbb{R}$.
2. The relation "divides" is reflexive on $\mathbb{N}^*$.
3. The relation "is a multiple of" is reflexive on $\mathbb{N}$.

1.8.2 Symmetric Relation

$$R \textbf{ symmetric} \Leftrightarrow \forall x,y,\ xRy \Rightarrow yRx.$$

Examples

1. The relation "perpendicular to" is symmetric in the set of lines.
2. The relation "parallel to" is symmetric in the set of lines.
3. The relation "equal to" is symmetric on $\mathbb{R}$.

1.8.3 Antisymmetric Relation

$$R \textbf{ anti-symmetric} \Leftrightarrow (xRy \wedge yRx) \Rightarrow x = y.$$

Examples

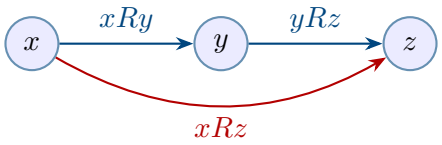
1. The relation "equal to" is antisymmetric on $\mathbb{R}$.
2. The relation "less than or equal to" $(\leq)$ is antisymmetric on $\mathbb{R}$.
3. The relation "is a multiple of" is antisymmetric on $\mathbb{N}$.

1.8.4 Transitive Relation

$$R \textbf{ transitive} \Leftrightarrow (xRy \wedge yRz) \Rightarrow xRz.$$

Examples

1. The relation "parallel to" is transitive in the set of lines.
2. The relation "less than or equal to" $(\leq)$ is transitive on $\mathbb{R}$.
3. The relation "is a multiple of" is transitive on $\mathbb{N}$.



1.8.5 Equivalence Relation

Definition 1.8.2 Equivalence Relation

R is an *equivalence relation* on E if it is reflexive, symmetric and transitive.
The *equivalence class* of a is $\dot{a} = \{x \in E \mid xRa\}$.
The set E/R of all classes is the *quotient set*.

Exercise

Let R be the relation defined on $\mathbb{R}$ by

$$\forall x, y \in \mathbb{R}, \quad xRy \iff x^3 - y^3 = x^2 - y^2.$$

Show that R is an equivalence relation.

Solution

Reflexivity For every $x \in \mathbb{R}$,

$$x^3 - x^3 = x^2 - x^2 = 0,$$

$$0 = 0, (true)$$

so xRx. Therefore R is reflexive.

Symmetry If xRy, then

$$x^3 - y^3 = x^2 - y^2.$$

Multiplying both sides by -1,

$$y^3 - x^3 = y^2 - x^2,$$

hence yRx. Therefore R is symmetric.

Transitivity Observe that

$$xRy \iff x^3 - y^3 = x^2 - y^2.$$

$$yRz \iff y^3 - z^3 = y^2 - z^2.$$

When the two equalities are added end-to-end, the result is

$$x^3 - z^3 = x^2 - z^2.$$

which implies xRz. Therefore R is transitive.
Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Exercise 1.8.1. Let A be the set $\{\alpha, \beta, \gamma, \sigma\}$. Let the relation $R = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta), (\gamma, \sigma), (\sigma, \sigma), (\gamma, \gamma), (\sigma, \gamma)\}$. Is this relationship reflexive ? symmetric? transitive

Solution

Let $A = \{\alpha, \beta, \gamma, \delta\}$
and $R = \{(\alpha, \alpha), (\alpha, \beta), (\beta, \alpha), (\beta, \beta), (\gamma, \sigma), (\sigma, \sigma), (\gamma, \gamma), (\sigma, \gamma)\}$.
We study the properties of R.
Reflexive

A relation R is reflexive if: $\forall x \in A, \quad xRx$

We have: $(\alpha, \alpha), (\beta, \beta), (\gamma, \gamma), (\delta, \delta) \in R$

Hence R is reflexive

Symmetric

R is symmetric if: $xRy \Rightarrow yRx$

For example:$\alpha R\beta$ and $\beta R\alpha$

Therefore: R is symmetric

Transitive

R is transitive if: xRy and $yRz \Rightarrow xRz$.

The required pairs are satisfied, therefore: R is transitive.

Hence R is an equivalence relation.

1.8.6 Order Relation

Definition 1.8.3 Order Relation

R is an *order relation* on E if it is reflexive, transitive and anti-symmetric.

Definition 1.8.4 Total Order

We say that a relation R is a total order on a set E if:

$$\forall x, y \in E : \quad xRy \ \vee \ yRx$$

Definition 1.8.5 partial Order

We say that a relation R is a partial order on a set E if:

$$\exists x, y \in E : \quad x\bar{R}y \ \wedge \ y\bar{R}x$$

Examples

1/ The relation "$\leq$" is a total order on $\mathbb{R}$.

2/ The relation "multiple" is a partial order on $\mathbb{N}$.

1.9 Applications and Functions

1.9.1 Basic Definitions

Definition 1.9.1 Application

Let E and F be two sets. An application from E to F is a rule that associates each element $x \in E$ with a unique element $y \in F$, denoted by
$f : E \to F, \qquad y = f(x).$
The element $y = f(x)$ is called the *image* of x, while x is called a *preimage* (or antecedent) of y.
The set E is called the *domain* of f, and F is called the *codomain*.

Definition 1.9.2 Function

A relation between two sets E and F is called a function if every element of E has at most one image in F.

1.9.2 Types of Applications

Definition 1.9.3 Injective Application

A mapping $f : E \to F$ is said to be injective if
$\forall x, y \in E, \qquad f(x) = f(y) \Longrightarrow x = y.$

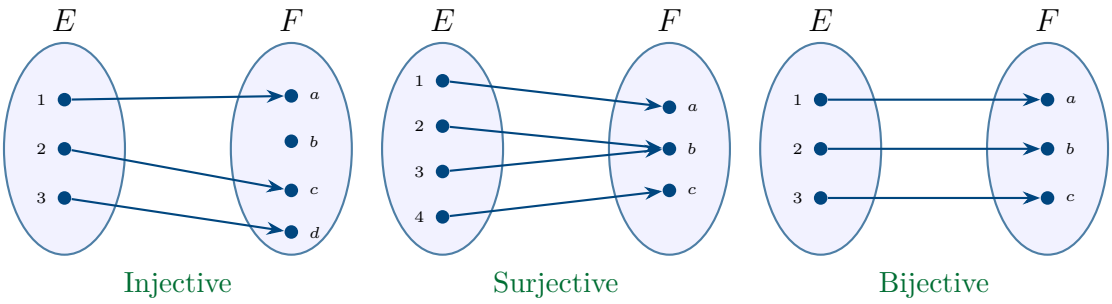
Definition 1.9.4 Surjective Application

A mapping $f : E \to F$ is said to be surjective if
$\forall y \in F, \qquad \exists x \in E$ such that $f(x) = y.$

Definition 1.9.5 Bijective Application

A mapping $f : E \to F$ is said to be bijective if it is both injective and surjective. Equivalently,
$\forall y \in F, \qquad \exists !, x \in E$ such that $f(x) = y.$

Examples



1.9.3 Inverse Application

Proposition 1.9.1

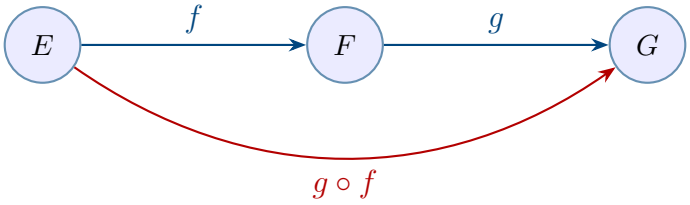
If $f : E \to F$ is bijective, then there exists an inverse mapping $f^{-1} : F \to E$, defined by $f^{-1}(y) = x \iff f(x) = y$.

Example Consider the function $f : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R} \setminus \{1\}, \qquad f(x) = \frac{x+5}{x-1}$.
Let $y = \frac{x+5}{x-1}$, then $y(x-1) = x+5$,
which gives $x(y-1) = y+5$.
Therefore,$x = \frac{y+5}{y-1}$, and consequently $f^{-1}(y) = \frac{y+5}{y-1}$.

1.9.4 Composition of Application

Definition 1.9.6

Let $f : E \to F, \qquad g : F \to G$. The composition of f and g is the application $g \circ f : E \to G$ defined by $(g \circ f)(x) = g(f(x))$.



1.10 Image of an Interval Under a Continuous Function

Theorem 1.10.1

Let I be an interval of $\mathbb{R}$. If f is continuous on I, then $f(I) = f(x) : x \in I$ is also an interval.

If f is continuous and increasing on $[a, b]$, then $f([a, b]) = [f(a), f(b)]$.

If f is continuous and increasing on $[a, b]$, then $f([a, b]) = [f(b), f(a)]$.

Example Let

$$f(x) = x^2.$$

Then

$$\begin{aligned} f([-2, 3]) &= f([-2, 0] \cup [0, 3]) \\ &= f([-2, 0]) \cup f([0, 3]). \end{aligned}$$

Since

$$f([-2, 0]) = [0, 4]$$

and

$$f([0, 3]) = [0, 9],$$

we obtain

$$f([-2, 3]) = [0, 9].$$

Remark 1.10.1. In general, $f(A \cap B) \neq f(A) \cap f(B)$.

1.11 Properties of Continuous and Monotone Functions

Proposition 1.11.1

Let (f) be a real-valued function.

1. If (f) is strictly monotone, then (f) is injective.
2. If (f) is continuous, then the image of every interval is an interval.

Theorem 1.11.1 Bijection Theorem

If $f : I \to \mathbb{R}$ is continuous and strictly monotone on an interval I, then f is a bijection onto $J = f(I)$, and $f^{-1} : J \to I$ is continuous and strictly monotone with the same sense of variation.

1.12 Tutorial 2 – Sets, Relations and Operations

Exercise 1.12.1. We consider the relation ρ defined on $\mathbb{R}$ by

$$\forall x, y \in \mathbb{R} : x \rho y \Leftrightarrow x^2 - y^2 = x - y$$

1) Show that ρ is an equivalence relation on $\mathbb{R}$.

2) Determine the equivalence class of an element a of $\mathbb{R}$

Exercise 1.12.2. Let ϕ be a relation defined on $\mathbb{Z}$ by

$$\forall x, y \in \mathbb{Z} : x \phi y \Leftrightarrow x - y \text{ multiple of } 2.$$

1) Show that ϕ is an equivalence relation on $\mathbb{Z}$.

2) Determine $\dot{0}, \dot{1}, \dot{2}, \dot{3}$

3) Determine the quotient set

Exercise 1.12.3. Let R be a relation defined on $\mathbb{R}$ by

$$\forall x, y \in \mathbb{R} : xRy \Leftrightarrow x \leq y \ .$$

1) Show that R is an order relation on $\mathbb{R}$.

2) Is the order total ?

Exercise 1.12.4. In $\mathbb{N}$, we define the relation R by

$$\forall n, m \in \mathbb{N} : nRm \Leftrightarrow \exists k \in \mathbb{N} : n = 4k + m$$

1) Check $1R9$ and $9R1$

2) Show that R is an order relation on $\mathbb{N}$

3) Is the order total ?

Exercise 1.12.5. Let $f : \{1, 2, 3, 4\} \to \{x, y, z\}$ given by $f(1) = x$, $f(2) = y$, $f(3) = z$, $f(4) = y$

Determine injectivity and surjectivity.

Exercise 1.12.6. Let $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n + 1$ Is it :injective, surjective, bijective?

Exercise 1.12.7. Let $f : \mathbb{R} \setminus \{2\} \to \mathbb{R} \setminus \{1\}$ defined by $f(x) = \frac{x}{x-2}$

Show that f is bijective by finding its inverse.

Given the expression $(f \circ f)(x)$. Find $(f^{-1})(x)$

Exercise 1.12.8. Let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^x$. Determine injectivity and surjectivity.

Would you need to change the domain for it to be bijective? If yes, what codomain?

Chapter 2

Complex Numbers

2.1 Historical Background

Complex numbers were born in the 16th century when the Italian *Gerolamo Cardano* (1501–1576) introduced $\sqrt{-15}$ to solve cubic equations. In 1572, *Rafaele Bombelli* (1526–1573) published numbers of the form $a + b\sqrt{-1}$ in his *Algebra*. At that time roots of negative numbers were called *imaginary*. The notation i appeared in 1777 with *Leonhard Euler* (1707–1783), who called them *impossible numbers*. In the 19th century, Gauss and Hamilton established the algebraic structure of $\mathbb{C}$; purely real numbers became a special case.

2.2 Definition

Definition 2.2.1 The Set $\mathbb{C}$

There exists a set of numbers, denoted $\mathbb{C}$ and called the **set of complex numbers**, with the following properties:

1. $\mathbb{C}$ contains $\mathbb{R}$.
2. In $\mathbb{C}$, addition and multiplication obey the same rules as in $\mathbb{R}$.
3. There exists in $\mathbb{C}$ a number i such that $i^2 = -1$.
4. Every element $z \in \mathbb{C}$ can be written *uniquely* in the form $z = a + ib$ with $a, b \in \mathbb{R}$.

Definition 2.2.2 Vocabulary

The writing $z = a + ib$ is called the **algebraic form** of z.

- $a = \mathrm{Re}(z)$ is the **real part** of z.
- $b = \mathrm{Im}(z)$ is the **imaginary part** of z.

Remark 2.2.1.

- If $b = 0$, then z is a **real number**.
- If $a = 0$, then z is a **purely imaginary number**.

Example 2.2.1. $3 + 4i$, $-2 - i$, $\dfrac{i}{3}$ are complex numbers. The reals 0, -2, $\sqrt{3}$ are also complex numbers (with $b = 0$).

2.3 Arithmetic in $\mathbb{C}$

Proposition 2.3.1 Equality and Zero

a) Two complex numbers are equal if and only if they have the same real part *and* the same imaginary part.

b) A complex number is zero if and only if both its real part and imaginary part are zero.

Proof. Immediate consequence of the uniqueness of the algebraic form. ■

Examples: Simplify each expression to algebraic form.

$$z_1 = 3 - 5i - (3i - 4), \quad z_2 = (3 - 2i)(-1 + 5i), \quad z_3 = (2 - 3i)^2$$

$$z_4 = (2i)^{13}, \quad z_5 = \frac{1}{4 - 2i}, \quad z_6 = \frac{1 + i}{2 - i}$$

Solutions:

$$\begin{aligned} z_1 &= 3 - 5i - 3i + 4 && = 7 - 8i \\ z_2 &= -3 + 15i + 2i - 10i^2 = -3 + 17i + 10 && = 7 + 17i \\ z_3 &= 4 - 12i + 9i^2 = 4 - 12i - 9 && = -5 - 12i \end{aligned}$$

$$z_4 = 2^{13} i^{13} = 8192\,(i^2)^6 \cdot i = 8192\,(-1)^6 \cdot i = 8192\,i$$

$$z_5 = \frac{1}{4-2i} \cdot \frac{4+2i}{4+2i} = \frac{4+2i}{16+4} = \tfrac{1}{5} + \tfrac{1}{10} i$$

$$z_6 = \frac{(1+i)(2+i)}{(2-i)(2+i)} = \frac{2+i+2i-1}{5} = \frac{1+3i}{5} = \tfrac{1}{5} + \tfrac{3}{5} i$$

2.4 Conjugate of a Complex Number

Definition 2.4.1 Conjugate

Let $z = a + ib$. The **conjugate** of z is the number $\overline{z} = a - ib$.

Examples: If $z = 4 + 5i$ then $\overline{z} = 4 - 5i$. $\overline{-3i} = 3i$. $\overline{5} = 5$.

2.4.1 Properties of Conjugation

Proposition 2.4.1 Conjugation Rules

For $z, z' \in \mathbb{C}$ and $n \in \mathbb{N}^*$:

a) $\overline{\overline{z}} = z$

b) $\overline{z + z'} = \overline{z} + \overline{z'}$

c) $\overline{z \cdot z'} = \overline{z} \cdot \overline{z'}$

d) $\overline{z^n} = \overline{z}^n$

e) $\overline{\dfrac{1}{z}} = \dfrac{1}{\overline{z}} \quad (z \neq 0)$

f) $\overline{\left(\dfrac{z}{z'}\right)} = \dfrac{\overline{z}}{\overline{z'}} \quad (z' \neq 0)$

Proof. Write $z = a + ib$ and $z' = a' + ib'$.

(a) $\overline{\overline{z}} = \overline{a - ib} = a + ib = z$.

(c) $\overline{zz'} = \overline{(a+ib)(a'+ib')} = \overline{aa' - bb' + i(ab' + a'b)} = aa' - bb' - i(ab' + a'b)$. And $\overline{z} \cdot \overline{z'} = (a - ib)(a' - ib') = aa' - bb' - i(ab' + a'b)$. Equal.

(d) By induction: base $n = 2$ uses (c). Inductive step: $\overline{z^{k+1}} = \overline{z^k \cdot z} = \overline{z^k} \cdot \overline{z} = \overline{z}^k \cdot \overline{z} = \overline{z}^{k+1}$.

(e) $\overline{\frac{1}{z}} = \overline{\frac{a-ib}{a^2+b^2}} = \frac{a+ib}{a^2+b^2} = \frac{1}{a-ib} = \frac{1}{\overline{z}}$. ∎

Proposition 2.4.2 Real and Purely Imaginary Characterisation

a) z is real $\iff z = \overline{z}$.

b) z is purely imaginary $\iff z = -\overline{z}$.

Proof. (a) $z = \overline{z} \Leftrightarrow a + ib = a - ib \Leftrightarrow 2ib = 0 \Leftrightarrow b = 0$.
(b) $z = -\overline{z} \Leftrightarrow a + ib = -a + ib \Leftrightarrow 2a = 0 \Leftrightarrow a = 0$. ■

Proposition 2.4.3 Product with Conjugate

For any $z = a + ib$: $\quad z\overline{z} = a^2 + b^2$.

Proof. $z\overline{z} = (a + ib)(a - ib) = a^2 - (ib)^2 = a^2 + b^2$. ■

Examples: Compute $\overline{z_1}$ and $\overline{z_2}$ for $z_1 = (2 - i)(i - 5)$, $z_2 = \dfrac{3 + 2i}{i}$.

$$\overline{z_1} = \overline{(2 - i)} \cdot \overline{(i - 5)} = (2 + i)(-i - 5) = -2i - 10 - i^2 - 5i = -9 - 7i.$$

$$\overline{z_2} = \frac{\overline{3 + 2i}}{\overline{i}} = \frac{3 - 2i}{-i} = \frac{(3 - 2i)i}{-i^2} = \frac{3i - 2i^2}{1} = 2 + 3i.$$

Exercise: Solve in $\mathbb{C}$: (a) $3z - 6 = 4i + z$, $\quad$ (b) $3z - 2 = \overline{z} + 1$.

Solution (a): $2z = 6 + 4i \Rightarrow z = 3 + 2i$.

Solution (b): Set $z = a + ib$:

$$3(a + ib) - 2 = (a - ib) + 1 \Rightarrow 3a + 3bi - 2 - a + ib - 1 = 0 \Rightarrow (2a - 3) + 4bi = 0.$$

So $2a - 3 = 0$ and $4b = 0$, giving $a = \frac{3}{2}$, $b = 0$, i.e. $z = \frac{3}{2}$.

2.5 Newton's Binomial Theorem

Theorem 2.5.1 Binomial Formula

For all complex numbers a, b and integer $n \geq 1$:

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n.$$

Proof. By induction on n.

Base $n = 1$: $(a + b)^1 = a + b = \binom{1}{0} a + \binom{1}{1} b$. ✓

Induction: Assume the formula holds for some $k \geq 1$. Then

$$\begin{aligned}(a+b)^{k+1} &= (a+b)(a+b)^k = (a+b)\sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^j \\ &= \sum_{j=0}^{k}\binom{k}{j}a^{k+1-j}b^j + \sum_{j=0}^{k}\binom{k}{j}a^{k-j}b^{j+1} \\ &= a^{k+1} + \sum_{j=1}^{k}\left[\binom{k}{j} + \binom{k}{j-1}\right]a^{k+1-j}b^j + b^{k+1}.\end{aligned}$$

By Pascal's rule $\binom{k}{j} + \binom{k}{j-1} = \binom{k+1}{j}$, which completes the induction. ■

Remark 2.5.1. The coefficients $\binom{n}{k}$ are read from Pascal's triangle:

$n\backslash k$	0	1	2	3	4	5	6
0	1						
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

Example: Expand $(z+5)^6$.

$$\begin{aligned}(z+5)^6 &= \binom{6}{0}z^6 + \binom{6}{1}z^5\cdot 5 + \binom{6}{2}z^4\cdot 25 + \binom{6}{3}z^3\cdot 125 \\ &\quad + \binom{6}{4}z^2\cdot 625 + \binom{6}{5}z\cdot 3125 + \binom{6}{6}\cdot 15625 \\ &= z^6 + 30z^5 + 375z^4 + 2500z^3 + 9375z^2 + 18750z + 15625.\end{aligned}$$

2.6 Representation in the Complex Plane

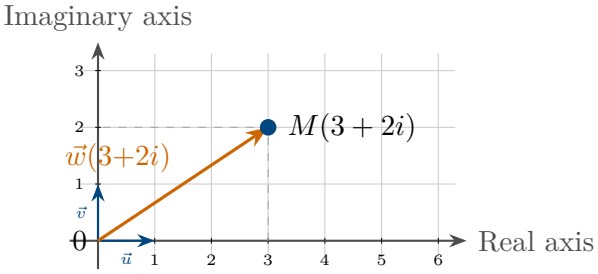
Definition 2.6.1 Affix, Image, Complex Plane

Fix a direct orthonormal frame $(O;\vec{u},\vec{v})$. For $z = a + ib$:

- The **image** of z is the point M with coordinates (a, b).
- The **affix** of a point $M(a, b)$ is $z = a + ib$.
- The **affix** of a vector $\vec{w}\binom{a}{b}$ is $z = a + ib$.

We write $M(z)$ and $\vec{w}(z)$.

Examples: The point $M\binom{3}{2}$ has affix $z = 3 + 2i$; the vector $\vec{w}\binom{3}{2}$ also has affix $3 + 2i$.



2.7 Properties

Proposition 2.7.1 Geometric Properties of Affixes

Let $M(z_M)$ and $N(z_N)$ be two points; $\vec{u}(z)$, $\vec{v}(z')$ two vectors.

a) $\overrightarrow{MN}$ has affix $z_N - z_M$.

b) $\vec{u} + \vec{v}$ has affix $z + z'$.

c) $k\vec{u}$ $(k \in \mathbb{R})$ has affix kz.

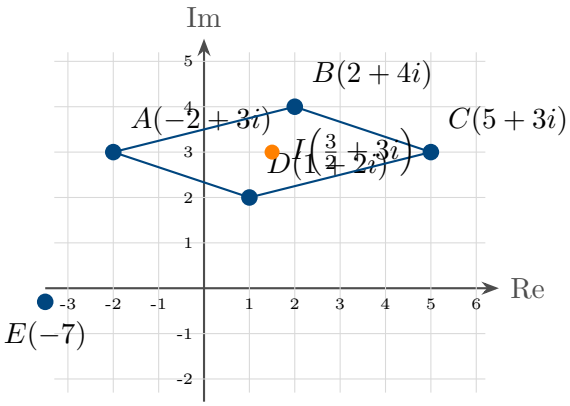
d) The midpoint I of $[MN]$ has affix $z_I = \dfrac{z_M + z_N}{2}$.

Proof. (a) $M = \binom{x_M}{y_M}$, $N = \binom{x_N}{y_N}$. Then $\overrightarrow{MN} = \binom{x_N - x_M}{y_N - y_M}$ and its affix is $(x_N - x_M) + i(y_N - y_M) = z_N - z_M$. Parts (b)–(d) follow analogously from vector coordinates. ■

Example 2.7.1. Consider $A(-2 + 3i)$, $B(2 + 4i)$, $C(5 + 3i)$, $D(1 + 2i)$, $E(-7)$.

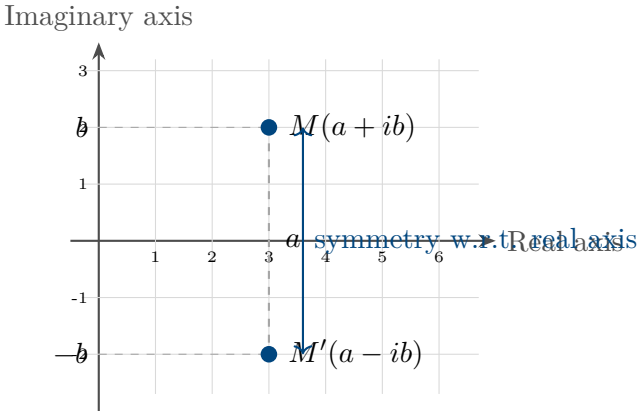
(a) $ABCD$ is a parallelogram: $z_{\overrightarrow{AB}} = z_B - z_A = (2 + 4i) - (-2 + 3i) = 4 + i$. $z_{\overrightarrow{DC}} = z_C - z_D = (5+3i) - (1+2i) = 4+i$. Since $\overrightarrow{AB} = \overrightarrow{DC}$, $ABCD$ is a parallelogram. Centre I: $z_I = \dfrac{z_A + z_C}{2} = \dfrac{-2 + 3i + 5 + 3i}{2} = \dfrac{3}{2} + 3i$.

(b) D, C, E are collinear: $z_{\overrightarrow{DC}} = 4 + i$; $z_{\overrightarrow{DE}} = z_E - z_D = -7 - (1 + 2i) = -8 - 2i = -2(4 + i)$. So $\overrightarrow{DE} = -2\overrightarrow{DC}$: collinear.



2.7.1 Image of a Conjugate

Remark 2.7.1. The images M and M' of z and $\overline{z}$ are **symmetric with respect to the real axis**.



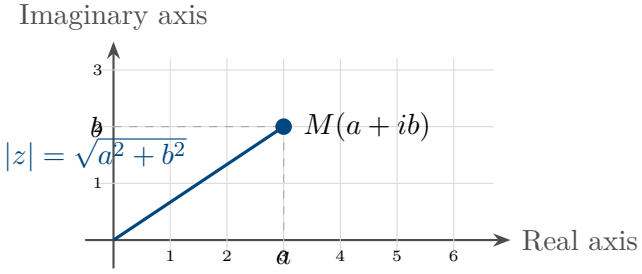
2.8 Modulus and Argument

Definition 2.8.1 Modulus

The **modulus** of $z = a + ib$ is the non-negative real number

$$|z| = \sqrt{a^2 + b^2}.$$

Geometrically, if M is the image of z, then $|z| = OM$.



Proposition 2.8.1 Basic Modulus Properties

For any $z \in \mathbb{C}$:

a) $|z|^2 = z\overline{z}$

b) $|\overline{z}| = |z|$

c) $|-z| = |z|$

Proof. (a) $z\overline{z} = (a+ib)(a-ib) = a^2 + b^2 = |z|^2$.
(b) $|\overline{z}| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2+b^2} = |z|$.
(c) $|-z| = \sqrt{(-a)^2 + (-b)^2} = |z|$. ■

Proposition 2.8.2 Modulus of Products and Powers

For non-zero $z, z' \in \mathbb{C}$ and $n \in \mathbb{N}^*$:

Operation	Rule
Product	$\lvert zz'\rvert = \lvert z\rvert\,\lvert z'\rvert$
Power	$\lvert z^n\rvert = \lvert z\rvert^n$
Inverse	$\left\lvert\frac{1}{z}\right\rvert = \frac{1}{\lvert z\rvert}$
Quotient	$\left\lvert\frac{z}{z'}\right\rvert = \frac{\lvert z\rvert}{\lvert z'\rvert}$

Proof. **Product:** $|zz'|^2 = zz' \cdot \overline{zz'} = zz'\overline{z}\overline{z'} = |z|^2\,|z'|^2$, so $|zz'| = |z|\,|z'|$.

Power: By induction; the base $n = 2$ follows from the product rule. The hereditary step: $|z^{k+1}| = |z^k \cdot z| = |z^k||z| = |z|^k|z| = |z|^{k+1}$. ■

Examples

a) $|3-2i| = \sqrt{9+4} = \sqrt{13}$

b) $|-3i| = |-3|\,|i| = 3$

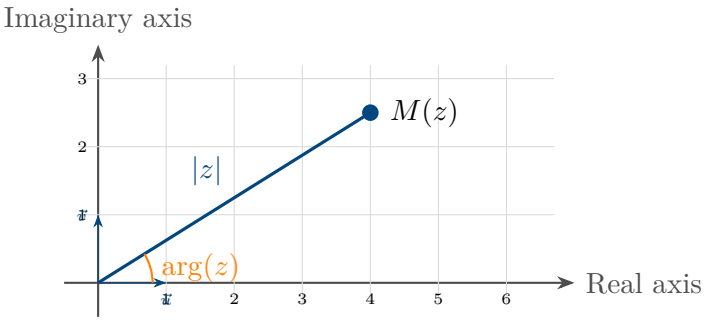
c) $\left|\sqrt{2}+i\right| = \sqrt{2+1} = \sqrt{3}$

d) $\left|\dfrac{-3i}{(\sqrt{2}+i)^2}\right| = \dfrac{|-3i|}{|\sqrt{2}+i|^2} = \dfrac{3}{3} = 1$

2.9 Argument

Definition 2.9.1 Argument

Let M be the image of a non-zero z. The **argument** of z, written $\arg(z)$, is any measure (in radians) of the angle $(\vec{u}\,;\overrightarrow{OM})$.



Remark 2.9.1. A non-zero complex number has infinitely many arguments of the form $\arg(z) + 2k\pi$, $k \in \mathbb{Z}$. We write $\arg(z) \pmod{2\pi}$. The number 0 has no argument.

Examples $z = 3 + 3i$: $|z| = \sqrt{18} = 3\sqrt{2}$, $\arg(z) = \dfrac{\pi}{4} \pmod{2\pi}$.

Proposition 2.9.1 Argument Properties

For non-zero $z \in \mathbb{C}$:

a) $z \in \mathbb{R} \Leftrightarrow \arg(z) = 0 \pmod{\pi}$

b) z purely imaginary $\Leftrightarrow \arg(z) = \frac{\pi}{2} \pmod{\pi}$

c) $\arg(\overline{z}) = -\arg(z)$

d) $\arg(-z) = \arg(z) + \pi$

For non-zero z, z' and $n \in \mathbb{N}^*$:

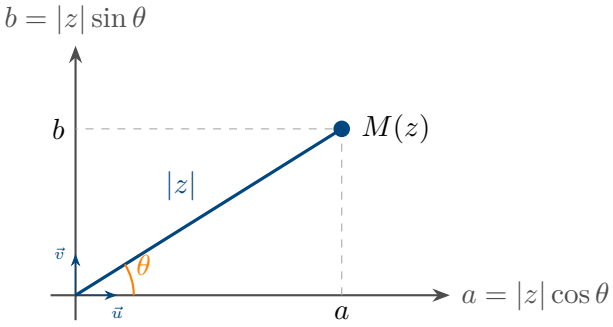
Operation	Rule
Product	$\arg(zz') = \arg(z) + \arg(z')$
Power	$\arg(z^n) = n\arg(z)$
Inverse	$\arg\left(\frac{1}{z}\right) = -\arg(z)$
Quotient	$\arg\left(\frac{z}{z'}\right) = \arg(z) - \arg(z')$

2.10 Trigonometric Form

Definition 2.10.1 Trigonometric Form

The **trigonometric form** of a non-zero z with $\theta = \arg(z)$ is

$$z = |z|(\cos\theta + i\sin\theta).$$



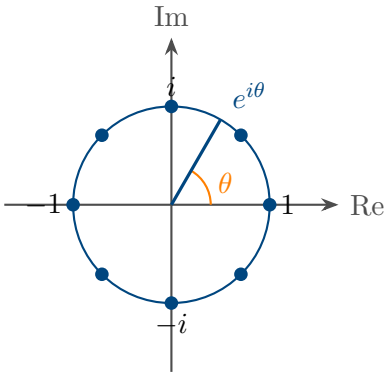
Examples

1. $z_1 = 3(\cos\pi + i\sin\pi) = 3(-1+0i) = -3$.
2. $z_2 = \sqrt{3}\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right) = \sqrt{3}\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) = \frac{3}{2} - \frac{\sqrt{3}}{2}i$.
3. $z_3 = -5i$: $|z_3| = 5$, $\arg(z_3) = -\frac{\pi}{2}$. Trig form: $5\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$.
4. $z_4 = \sqrt{3} + i$: $|z_4| = 2$, $\frac{z_4}{|z_4|} = \frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. So $z_4 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

2.11 The Unit Circle $\mathbb{U}$

Definition 2.11.1 Unit Circle

$\mathbb{U}$ is the set of complex numbers of modulus 1. Geometrically it is the unit circle (trigonometric circle). If $z = a + ib \in \mathbb{U}$, then $a^2 + b^2 = 1$.



2.12 Trigonometric Identities

2.12.1 Addition Formulas

Proposition 2.12.1 Addition Formulas

For all real a, b:

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$
$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$
$$\sin(a-b) = \sin a \cos b - \cos a \sin b$$
$$\sin(a+b) = \sin a \cos b + \cos a \sin b$$

Proof. Let $(O; \vec{i}, \vec{j})$ be an orthonormal frame and $\vec{u}, \vec{v}$ unit vectors with $(\vec{i}; \vec{u}) = a$ and $(\vec{i}; \vec{v}) = b$, so $\vec{u} = \begin{pmatrix} \cos a \\ \sin a \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} \cos b \\ \sin b \end{pmatrix}$.

First formula: $\vec{u} \cdot \vec{v} = \cos a \cos b + \sin a \sin b$. Also $\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\vec{u}; \vec{v}) = \cos(b-a) = \cos(a-b)$.

Second formula: $\cos(a+b) = \cos(a-(-b)) = \cos a \cos(-b) + \sin a \sin(-b) = \cos a \cos b - \sin a \sin b$.

Third formula: $\sin(a-b) = \cos\left(\frac{\pi}{2} - (a-b)\right) = \cos\left(\left(\frac{\pi}{2} - a\right) + b\right) = \cos\left(\frac{\pi}{2} - a\right)\cos b - \sin\left(\frac{\pi}{2} - a\right)\sin b = \sin a \cos b - \cos a \sin b$.

Fourth formula: $\sin(a+b) = \sin(a-(-b)) = \sin a \cos(-b) - \cos a \sin(-b) = \sin a \cos b + \cos a \sin b$. ∎

Examples Compute $\cos \frac{5\pi}{12}$ and $\sin \frac{5\pi}{12}$.

$$\cos \frac{5\pi}{12} = \cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}.$$

$$\sin \frac{5\pi}{12} = \sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}.$$

2.12.2 Double-Angle Formulas

Proposition 2.12.2 Double-Angle Formulas

For any real a:

$$\cos(2a) = \cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$$
$$\sin(2a) = 2\cos a \sin a$$

Proof. Set $b = a$ in the addition formulas: $\cos(2a) = \cos^2 a - \sin^2 a$ and $\sin(2a) = 2\cos a \sin a$. Using $\cos^2 a + \sin^2 a = 1$: $\cos^2 a - \sin^2 a = 2\cos^2 a - 1 = 1 - 2\sin^2 a$. ■

Example Compute $\cos\frac{\pi}{8}$ and $\sin\frac{\pi}{8}$.

$$\cos\frac{\pi}{4} = 2\cos^2\frac{\pi}{8} - 1 \;\Rightarrow\; \cos^2\frac{\pi}{8} = \frac{1+\cos\frac{\pi}{4}}{2} = \frac{2+\sqrt{2}}{4}, \quad \cos\frac{\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}}.$$

$$\sin^2\frac{\pi}{8} = 1 - \cos^2\frac{\pi}{8} = \frac{2-\sqrt{2}}{4}, \quad \sin\frac{\pi}{8} = \sqrt{\frac{2-\sqrt{2}}{4}}.$$

Example Solve $\cos(2x) = \sin x$ in $[0, 2\pi]$.

$$1 - 2\sin^2 x = \sin x \;\Rightarrow\; 2\sin^2 x + \sin x - 1 = 0.$$

Let $X = \sin x$: $\Delta = 9$, $X_1 = \frac{1}{2}$, $X_2 = -1$. $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$. $\sin x = -1 \Rightarrow x = \frac{3\pi}{2}$. Solution set: $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$.

2.13 Exponential Form

Definition 2.13.1 Exponential Form

Every non-zero complex z of modulus r and argument θ writes as

$$z = r\,e^{i\theta}.$$

Proposition 2.13.1 Euler's Identity

$$e^{i\pi} = -1.$$

Proof. $e^{i\pi} = \cos\pi + i\sin\pi = -1 + i\cdot 0 = -1$. ■

Remark 2.13.1. Euler's identity connects the five fundamental mathematical constants: e, i, π, 1, 0.

Examples

$e^{i0} = 1$. $\quad e^{i\pi/2} = i$. $\quad e^{-i\pi/2} = -i$

Proposition 2.13.2 Exponential Rules

For real θ, θ':

a) $e^{i\theta}\, e^{i\theta'} = e^{i(\theta+\theta')}$

b) $\dfrac{1}{e^{i\theta}} = e^{-i\theta}$

c) $\dfrac{e^{i\theta}}{e^{i\theta'}} = e^{i(\theta-\theta')}$

d) $\overline{e^{i\theta}} = e^{-i\theta}$

Examples Convert to exponential form:

1. $z_1 = -2i$: $|z_1| = 2$, $\arg(z_1) = -\frac{\pi}{2}$, so $z_1 = 2e^{-i\pi/2}$.

2. $z_2 = -3$: $|z_2| = 3$, $\arg(z_2) = \pi$, so $z_2 = 3e^{i\pi}$.

3. $z_3 = \sqrt{3} - 3i$: $|z_3| = 2\sqrt{3}$, $\dfrac{z_3}{|z_3|} = \frac{1}{2} - \frac{\sqrt{3}}{2}i = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)$, so $z_3 = 2\sqrt{3}\, e^{-i\pi/3}$.

Convert to algebraic form:

1. $z_4 = e^{i\pi/6} = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$.

2. $z_5 = 4e^{i\pi/4} = 4\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 2\sqrt{2} + 2i\sqrt{2}$.

Example $z = 1 + i\sqrt{3} = 2e^{i\pi/3}$. Deduce:

$$iz = 2ie^{i\pi/3} = 2e^{i\pi/2}e^{i\pi/3} = 2e^{i5\pi/6}.$$
$$i\overline{z} = 2ie^{-i\pi/3} = 2e^{i\pi/2}e^{-i\pi/3} = 2e^{i\pi/6}.$$
$$-\frac{2i}{z} = \frac{2(-i)}{2e^{i\pi/3}} = \frac{e^{-i\pi/2}}{e^{i\pi/3}} = e^{-i5\pi/6} = e^{i7\pi/6}.$$

2.13.1 De Moivre and Euler Formulas

Theorem 2.13.1 De Moivre's Formula

For $n \in \mathbb{N}^*$ and $\theta \in \mathbb{R}$:

$$\left(e^{i\theta}\right)^n = e^{in\theta}, \qquad \text{i.e.,} \quad (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

Example

Express $\cos(3x)$ in terms of $\cos x$.

$$\begin{aligned}\cos(3x) + i\sin(3x) &= (\cos x + i\sin x)^3 \\ &= \cos^3 x + 3i\cos^2 x \sin x - 3\cos x \sin^2 x - i\sin^3 x.\end{aligned}$$

Taking real parts: $\cos(3x) = \cos^3 x - 3\cos x \sin^2 x = 4\cos^3 x - 3\cos x$ (using $\sin^2 x = 1 - \cos^2 x$).

Proposition 2.13.3 Euler's Formulas

For $\theta \in \mathbb{R}$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Example Linearise $\cos^3 x$ and find a primitive of $x \mapsto \cos^3 x$.

$$\cos^3 x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^3 = \frac{e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-3ix}}{8} = \frac{\cos 3x + 3\cos x}{4}.$$

A primitive: $\dfrac{\sin 3x}{12} + \dfrac{3\sin x}{4}$. **Example** [Divertissement: proof of $\cos^2\theta + \sin^2\theta = 1$]

$$\cos^2\theta + \sin^2\theta = \cos^2\theta - i^2\sin^2\theta = (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta) = e^{-i\theta}e^{i\theta} = e^0 = 1.$$

2.14 Applications to Geometry

2.14.1 The logarithm of a complex number

Let

$$z = re^{i\theta},$$

where $r = |z| > 0$ is the modulus of z and $\theta = \arg(z)$ is an argument of z. The complex logarithm is defined by

$$\log z = \ln r + i(\theta + 2k\pi), \qquad k \in \mathbb{Z}.$$

Examples.

1. For the complex number $z = 1$, we have

$$|z| = 1, \qquad \arg(1) = 2k\pi, \quad k \in \mathbb{Z}.$$

Hence,

$$\log(1) = i\,2k\pi, \qquad k \in \mathbb{Z},$$

and the principal value is

$$\mathrm{Log}(1) = 0.$$

2. For $z = -1$, we have

$$|-1| = 1, \qquad \arg(-1) = \pi + 2k\pi, \quad k \in \mathbb{Z}.$$

Therefore,

$$\log(-1) = i(\pi + 2k\pi), \qquad k \in \mathbb{Z},$$

while the principal logarithm is

$$\mathrm{Log}(-1) = i\pi.$$

3. Consider $z = 1 + i$. Since

$$|1 + i| = \sqrt{2}, \qquad \arg(1 + i) = \frac{\pi}{4} + 2k\pi,$$

we obtain

$$\log(1 + i) = \ln(\sqrt{2}) + i\left(\frac{\pi}{4} + 2k\pi\right), \qquad k \in \mathbb{Z}.$$

Using $\ln(\sqrt{2}) = \frac{1}{2}\ln 2$, the principal value is

$$\mathrm{Log}(1 + i) = \frac{1}{2}\ln 2 + \frac{i\pi}{4}.$$

Proposition 2.14.1 Geometric Properties

Let $A(z_A)$, $B(z_B)$, $C(z_C)$ be distinct points.

a) $AB = |z_B - z_A|$

b) $\left(\vec{u}; \overrightarrow{AB}\right) = \arg(z_B - z_A)$

c) $\left(\overrightarrow{AB}; \overrightarrow{AC}\right) = \arg\left(\dfrac{z_C - z_A}{z_B - z_A}\right)$

Proof. (a) Let E with $\overrightarrow{OE} = \overrightarrow{AB}$, affix $e = z_B - z_A$. Then $AB = OE = |e| = |z_B - z_A|$.
(b) $\arg(z_B - z_A) = \arg(e) = (\vec{u}; \overrightarrow{OE}) = (\vec{u}; \overrightarrow{AB})$.
(c) $(\overrightarrow{AB}; \overrightarrow{AC}) = (\overrightarrow{AB}; \vec{u}) + (\vec{u}; \overrightarrow{AC}) = \arg(z_C - z_A) - \arg(z_B - z_A) = \arg\left(\dfrac{z_C - z_A}{z_B - z_A}\right)$. ■

Example $A(-2 - i)$, $B(1 - 2i)$, $C(-1 + 2i)$.

(a) **Isosceles in A:** $AB = |3 - i| = \sqrt{10}$, $AC = |1 + 3i| = \sqrt{10}$. So $AB = AC$. ✓

(b) **Right angle at A:** $\dfrac{z_C - z_A}{z_B - z_A} = \dfrac{1 + 3i}{3 - i} = \dfrac{(1 + 3i)(3 + i)}{10} = \dfrac{10i}{10} = i$. $\arg(i) = \frac{\pi}{2}$, so angle $\widehat{BAC} = \frac{\pi}{2}$. Right-angled at A. ✓**Example** Determine and sketch each set of points M with affix z:

(a) $|z - 2i| = 3$: circle of centre $A(2i)$, radius 3.

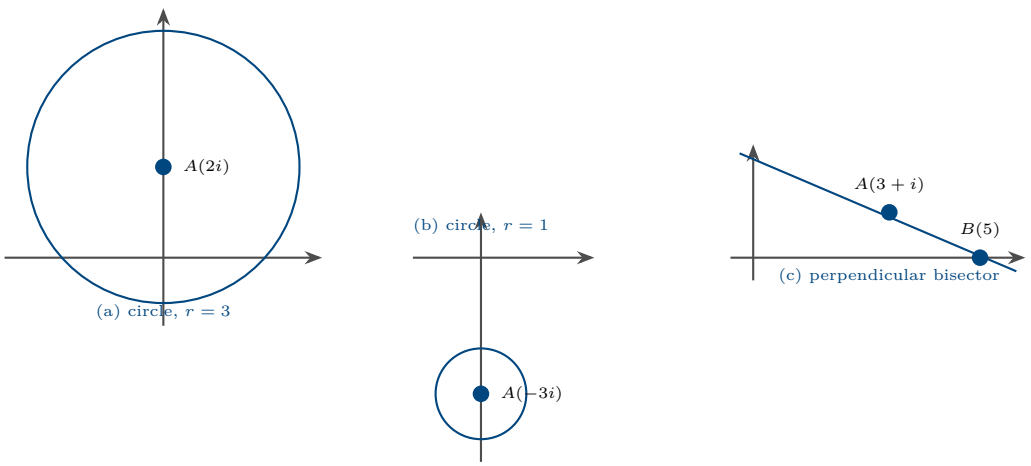
(b) $|iz - 3| = 1$: rewrite as $|z - (-3i)| = 1$, circle of centre $A(-3i)$, radius 1.

(c) $|\overline{z} - 3 + i| = |z - 5|$: $|\overline{z} - 3 + i| = |z - 3 - i| = |z - (3 + i)|$. Set of points equidistant from $A(3 + i)$ and $B(5)$: perpendicular bisector of $[AB]$.

(d) $\dfrac{|z - i|}{|z|} = 2$: $|z - i| = 2|z|$. Set $z = x + iy$: $x^2 + (y - 1)^2 = 4(x^2 + y^2) \Rightarrow 3x^2 + 3y^2 + 2y = 1 \Rightarrow x^2 + \left(y + \frac{1}{3}\right)^2 = \frac{4}{9}$. Circle of centre $A\left(-\frac{i}{3}\right)$, radius $\frac{2}{3}$.

(e) $\arg(z) = \frac{\pi}{4} \pmod{\pi}$: the bisector of the axes (without origin).

(f) $\arg(z - 2 + i) = \frac{\pi}{4} \pmod{2\pi}$: half-line from $A(2 - i)$ at angle $\frac{\pi}{4}$ (excluding A).



2.15 n-th Roots of Unity

Definition 2.15.1 n-th Root of Unity

A **n-th root of unity** is a complex z with $z^n = 1$, $n \in \mathbb{N}^*$.

Theorem 2.15.1 n-th Roots of Unity

The set $\mathbb{U}_n$ of n-th roots of unity has exactly n elements:

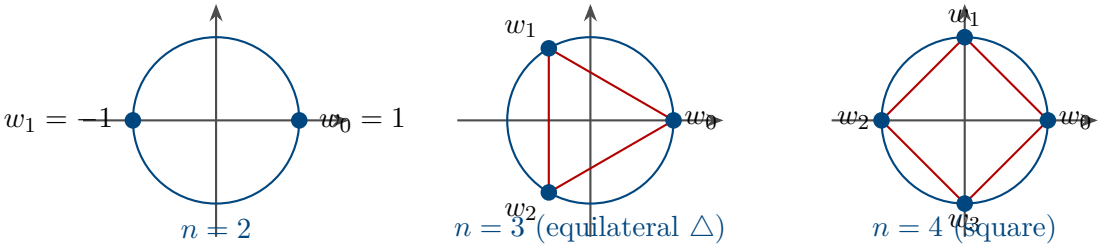
$$w_k = e^{i\frac{2k\pi}{n}}, \quad k = 0, 1, \ldots, n-1.$$

Proof. **Existence.** If $z^n = 1$ then $|z|^n = 1$ so $|z| = 1$, i.e. $z = e^{i\theta}$. $(e^{i\theta})^n = 1 \Rightarrow e^{in\theta} = 1 \Rightarrow n\theta = 2k\pi \Rightarrow \theta = \frac{2k\pi}{n}$. Restricting $k \in \{0, \ldots, n-1\}$ gives n roots.

Uniqueness. If $w_k = w_{k'}$ then $e^{i\frac{2k\pi}{n}} = e^{i\frac{2k'\pi}{n}}$, so $\frac{2k\pi}{n} = \frac{2k'\pi}{n} + 2l\pi$, giving $k - k' = ln$. Since $0 \leq k - k' < n$, we need $l = 0$, hence $k = k'$. ■

2.15.1 Geometric Representation

The images of the n-th roots of unity form a **regular n-gon** inscribed in the unit circle.



Example

Solve in $\mathbb{C}$:

(a) $(z-1)^3 = 1$: $z - 1$ is a 3rd root of unity: $z - 1 = e^{i\frac{2k\pi}{3}}$ for $k = 0, 1, 2$. $S = \left\{2,\ 1 + e^{i\frac{2\pi}{3}},\ 1 + e^{-i\frac{2\pi}{3}}\right\}$.

(b) $z^5 = -1$: $\left(\frac{z}{-1}\right)^5 = 1$, so $\frac{z}{-1} = e^{i\frac{2k\pi}{5}}$, $k = 0, \ldots, 4$. $z = -e^{i\frac{2k\pi}{5}} = e^{i\left(\pi + \frac{2k\pi}{5}\right)}$. $S = \left\{-1, e^{-i\frac{3\pi}{5}}, e^{-i\frac{\pi}{5}}, e^{i\frac{\pi}{5}}, e^{i\frac{3\pi}{5}}\right\}$.

2.15.2 Quadratic Equations in $\mathbb{C}$

Definition 2.15.2 Discriminant

The **discriminant** of $az^2 + bz + c$ $(a, b, c \in \mathbb{R},\ a \neq 0)$ is $\Delta = b^2 - 4ac$.

Proposition 2.15.1 Solutions of $az^2 + bz + c = 0$

- $\Delta > 0$: two distinct real roots $z_{1,2} = \dfrac{-b \pm \sqrt{\Delta}}{2a}$.
- $\Delta = 0$: one real root $z_0 = -\dfrac{b}{2a}$.
- $\Delta < 0$: two conjugate complex roots $z_{1,2} = \dfrac{-b \pm i\sqrt{-\Delta}}{2a}$.

Proof. Canonical form: $a\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a}$. If $\Delta < 0$: $\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2} = i^2\frac{-\Delta}{4a^2}$, so $z + \frac{b}{2a} = \pm i\frac{\sqrt{-\Delta}}{2a}$. ■

Proposition 2.15.2 Vieta's Formulas

The sum S and product P of the roots of $az^2 + bz + c = 0$ satisfy $S = -\dfrac{b}{a}$ and $P = \dfrac{c}{a}$.

Example

(a) $z^2 + 5 = 0$: $z^2 = -5 = 5i^2 \Rightarrow z = \pm i\sqrt{5}$. $S = 0$, $P = 5$.

(b) $z^2 + 3z + 4 = 0$: $\Delta = 9 - 16 = -7 < 0$. $z_{1,2} = \dfrac{-3 \pm i\sqrt{7}}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{7}}{2}i$.

2.15.3 Polynomial Equations of Degree n

Definition 2.15.3 Polynomial Function

A **polynomial** P over $\mathbb{C}$ is a function $P : \mathbb{C} \to \mathbb{C}$ of the form $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ with $a_n \neq 0$. The integer n is the **degree** of P.

Definition 2.15.4 Root

A complex number α is a **root** of P if $P(\alpha) = 0$.

Example

i and $-i$ are the roots of $P(z) = z^2 + 1$.

2.15.4 Factorisation Theorem

Theorem 2.15.2 Root Factorisation

Let $P(z) = z^n - a^n$. There exists a polynomial Q of degree $n-1$ such that $P(z) = (z-a)Q(z)$.

Proof. For $a = 1$: $(z-1)(z^{n-1} + z^{n-2} + \cdots + 1) = z^n - 1$. For general a, substitute $z \to z/a$ and multiply by a^n. ■

Corollary 2.15.1

If α is a root of a polynomial P of degree n, then there exists a polynomial Q of degree $n-1$ such that $P(z) = (z-\alpha)Q(z)$.

Corollary 2.15.2

A polynomial of degree n has **at most** n **roots**.

Proof. Suppose $\alpha_1, \ldots, \alpha_p$ are distinct roots. By repeated factorisation: $P(z) = (z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_p)Q_p(z)$. The degree of the right side is $p + \deg Q_p \geq p$, so $p \leq n$. ■

Example Factorise $P(z) = z^3 + z^2 + 4z + 4$ over $\mathbb{C}$. $P(-1) = -1 + 1 - 4 + 4 = 0$, so $z = -1$ is a root. Write $P(z) = (z+1)(az^2 + bz + c)$; by identification: $a = 1$, $b = 0$, $c = 4$. $Q(z) = z^2 + 4 = (z - 2i)(z + 2i)$. $\boxed{P(z) = (z+1)(z-2i)(z+2i)}$.

Example Solve $x^3 + x^2 - 3x + 1 = 0$ in $\mathbb{R}$. $P(1) = 1 + 1 - 3 + 1 = 0$, so $x = 1$ is a root. $P(x) = (x-1)(x^2 + 2x - 1)$. $x^2 + 2x - 1 = 0$: $\Delta = 8$, $x = -1 \pm \sqrt{2}$. $S = \{-1 - \sqrt{2},\ -1 + \sqrt{2},\ 1\}$.

2.16 Tutorial 3 – Complex numbers

Exercise 2.16.1. rite in the "algebraic" form $(a + ib)$ the following complex numbers $z_1 = (1 + 2i) - 2(3 - 2i), z_2 = (2 + i)\,\overline{(-i + 1)}, z_3 = \frac{1}{2+2i}, z_4 = \frac{1}{i^3}$.

Exercise 2.16.2. Solve the following equations, with unknown $z \in \mathbb{C}$

1) $z + 2i = iz - 3$ 2) $2\bar{z} + 5i = z + 4$ 3) $4z - 3\bar{z} = \frac{1-18i}{2-i}$

Exercise 2.16.3. Write the following numbers in trigonometric and exponential form. $z_1 = 1 + i\sqrt{3}, z_2 = -1 + i, z_3 = z_1 z_2, z_4 = \frac{z_1}{z_2}$

Exercise 2.16.4. Calculate 1) $\ln(1 - i)$ 2) $\ln(-i)$

Exercise 2.16.5. Using the complex logarithm, solve in $\mathbb{C}$: $e^{z-1} = -ie^3$

Exercise 2.16.6. Find the square roots of $z_1 = -1,\ z_2 = -7 + 6\sqrt{2}i$

Exercise 2.16.7. Calculate the 5th roots of complex numbers : $i, \qquad 2\sqrt{2} + 2\sqrt{2}i$

Exercise 2.16.8. Solve the equations:

$z^2 + z + 1 = 0,$

$z^2 - \sqrt{3}z - i = 0,$

Exercise 2.16.9. $z^3 - (1 + 4i)\, z^2 - 3\,(1 - 3i)\, z + 14 - 2i = 0, \qquad z \in \mathbb{C}$

Find the three solutions of the above equation given that one of these solutions is purely imaginary

Chapter 3

Vector Spaces

This chapter introduces the fundamental concepts of vector spaces and their basic properties. It presents the definitions of vectors, vector spaces, subspaces, and the operations defined on them. The chapter also discusses linear combinations, linear dependence, and bases, which are essential tools in linear algebra. These concepts provide the foundation for studying many mathematical and applied problems.

3.1 Vector space

Definition 3.1.1 Vector space

Let $\mathbb{K}$ be a field. A non-empty set E equipped with two operations, $(+)$ and $(\cdot)$, is called a *vector space over* $\mathbb{K}$ if the following axioms hold:

1. For all $x, y \in E$, $x + y \in E$.
2. For all $x, y \in E$, $x + y = y + x$.
3. For all $x \in E$, $x + 0_E = x$.
4. For all $x \in E$, $-x \in E$.
5. For all $x, y, z \in E$, $(x + y) + z = x + (y + z)$.
6. For all $\lambda \in K$ et $x \in E$, $\lambda x \in E$.
7. $\lambda(\mu x) = (\lambda\mu)x, \quad \forall \lambda, \mu \in K, \ \forall x \in E.$
8. $(\lambda + \mu)x = \lambda x + \mu x, \quad \forall \lambda, \mu \in K, \ \forall x \in E.$
9. $\lambda(x + y) = \lambda x + \lambda y, \quad \forall \lambda \in K, \ \forall x, y \in E.$

10. $1x = x, \quad \forall x \in E.$

The elements of E are called *vectors*; the elements of $\mathbb{K}$ are called *scalars*.

Examples

1. $\mathbb{R}[X]$ is a vector space over $\mathbb{R}$ endowed with the operations

$$(a_0 + a_1X + \cdots + a_nX^n) + (b_0 + b_1X + \cdots + b_nX^n)$$

$$= (a_0 + b_0) + (a_1 + b_1)X + \cdots + (a_n + b_n)X^n$$

and

$$\lambda(a_0 + a_1X + \cdots + a_nX^n) = \lambda a_0 + \lambda a_1X + \cdots + \lambda a_nX^n.$$

2. $E = \mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{n}$ is a vector space over the field $\mathbb{K} = \mathbb{R}$ with

$$x + y = (x_1 + y_1, \ldots, x_n + y_n),$$
$$\lambda x = (\lambda x_1, \ldots, \lambda x_n),$$

for $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

3. If E and F are vector spaces over the same field $\mathbb{K}$, then $E \times F$ is a vector space with

$$(x, y) + (x', y') = (x + x', y + y'), \quad \lambda(x, y) = (\lambda x, \lambda y).$$

Proposition 3.1.1

For all $\lambda \in \mathbb{K}$, $x \in E$:
(i) $\lambda \cdot 0 = 0$ and $0 \cdot x = 0$;
(ii) $\lambda x = 0 \Rightarrow \lambda = 0$ or $x = 0$;
(iii) $(-\lambda)x = -\lambda x$.

Proof.

1. $\lambda(0+0) = \lambda 0 + \lambda 0 = \lambda 0$ which implies $\lambda 0 = 0$. Moreover, $(0+0)x = 0x + 0x = 0x$ hence $0x = 0$.

2. Suppose that $\lambda x = 0$. If $\lambda \neq 0$, then $\lambda^{-1}(\lambda x) = 0$ and therefore $x = 0$.

3. $(\lambda + (-\lambda))x = \lambda x + (-\lambda)x = 0$ thus $(-\lambda)x = -(\lambda x)$. In the following, $(-\lambda)x$ will be denoted by $-\lambda x$ and $x + (-y)$ will be denoted by $x - y$.

3.2 Subspaces

Definition 3.2.1 Subspace

Let E be a vector space and let F be a nonempty subset of E. We say that F is a vector subspace of E if the restriction of the operations of E to F makes F a vector space.

Proposition

Let E be a vector space and let $F \subset E$. Then F is a vector subspace of E if and only if:

1. $F \neq \varnothing$.

2. (a) For all $u, v \in F$, $v + v \in F$.

   (b) For all $u \in F$ and $\lambda \in K$, $\lambda u \in F$.

Examples $E = \{(x, y, z) \in \mathbb{R}^3 \mid 2x - y - z = 0\}$.

- $2 \times 0 - 0 - 0 = 0 \Rightarrow (0, 0, 0) \in E \Rightarrow E \neq \varnothing$.

- For all $\alpha \in \mathbb{R}$, for all $a = (x, y, z) \in E$ and $a' = (x', y', z') \in E$,

  $\alpha a + a' = \alpha(x, y, z) + (x', y', z')$

  $= (\alpha x, \alpha y, \alpha z) + (x', y', z')$

  $= (\alpha x + x', \alpha y + y', \alpha z + z')$. Moreover, $2(\alpha x + x') - (\alpha y + y') - (\alpha z + z')$

  $= (2\alpha x - \alpha y - \alpha z) + (2x' - y' - z')$

  $= \alpha(2x - y - z) + (2x' - y' - z')$.

  Since $a \in E$ and $a' \in E$, $2x - y - z = 0 \quad$ and $\quad 2x' - y' - z' = 0$.

  Hence, $\alpha(2x - y - z) + (2x' - y' - z') = \alpha 0 + 0 = 0$. Therefore, $\alpha a + a' \in E$.

Thus, E is a vector subspace of $\mathbb{R}^3$. **Examples** Find which of the sets E_1, E_2, E_3, E_4 and E_5 are vector subspaces.

$E_1 = \{(x+y, x-y, 2y) \in \mathbb{R}^3 \mid x, y \in \mathbb{R}\}$

$E_2 = \{(x, y, z) \in \mathbb{R}^3 \mid x - y + z = 0\}$

$E_3 = \{(x, y) \in \mathbb{R}^2 \mid 2x + 3y = 1\}$

$E_4 = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$

$E_5 = \{(x, y) \in \mathbb{R}^2 \mid (x-y)^2 = x + y\}$

1) Study of E_1

- $(0,0,0) = (0+0, 0-0, 2\times 0) \in E_1$, thus $E_1 \neq \varnothing$.
- Let $u, v \in E_1$. Then there exist $x, y, x', y' \in \mathbb{R}$ such that $u = (x+y, x-y, 2y)$ and

  $v = (x'+y', x'-y', 2y')$.

  Hence,

  $u + v = (x+y, x-y, 2y) + (x'+y', x'-y', 2y')$

  $= (x + x' + y + y', x + x' - (y+y'), 2(y+y'))$.

  Let $X = x + x', \qquad Y = y + y'$. Then $u + v = (X + Y, X - Y, 2Y) \in E_1$. Therefore, E_1 is stable under addition.
- Let $\alpha \in \mathbb{R}$ and $u = (x+y, x-y, 2y) \in E_1$.

  Then $\alpha u = (\alpha x + \alpha y, \alpha x - \alpha y, 2\alpha y)$.

  Let $X = \alpha x, \qquad Y = \alpha y$.

  Thus, $\alpha u = (X + Y, X - Y, 2Y) \in E_1$.

  Therefore, E_1 is stable under scalar multiplication.

Hence, E_1 is a vector subspace of $\mathbb{R}^3$.

2) Study of E_2

$2(0) - 0 - 0 = 0$, we have $(0,0,0) \in E_2$.
Let $u = (x, y, z) \in E_2$ and $v = (x', y', z') \in E_2$.
Then $x - y + z = 0$ and $x' - y' + z' = 0$.
Moreover, $u + v = (x + x', y + y', z + z')$.
We compute: $(x + x') - (y + y') + (z + z') = (x - y + z) + (x' - y' + z') = 0 + 0 = 0$.
Thus, $u + v \in E_2$.
Now let $\alpha \in \mathbb{R}$. Then $\alpha u = (\alpha x, \alpha y, \alpha z)$,

and $\alpha x - \alpha y + \alpha z = \alpha(x - y + z) = \alpha 0 = 0$.

Hence, $\alpha u \in E_2$.

Therefore, E_2 is a vector subspace of $\mathbb{R}^3$.

3) Study of E_3

$E_3 = \{(x, y) \in \mathbb{R}^2 \mid 2x + 3y = 1\}$.

We test the zero vector:

$2(0) + 3(0) = 0 \neq 1$. Hence, $(0, 0) \notin E_3$.

Therefore, E_3 is not a vector subspace of $\mathbb{R}^2$.

4) Study of E_4

$E_4 = \{(x, y) \in \mathbb{R}^2 \mid y = x^2\}$.

We take $u = (2, 4) \in E_4$. Then $u + u = (4, 8)$.

But $8 \neq 4^2 = 16$. Thus,

$u + u \notin E_4$.

Hence, E_4 is not stable under addition.

Therefore, E_4 is not a vector subspace of $\mathbb{R}^2$.

5) Study of E_5

$E_5 = \{(x, y) \in \mathbb{R}^2 \mid (x - y)^2 = x + y\}$.

We take $u = (0, 1) \in E_5$ because $(0 - 1)^2 = 1 = 0 + 1$.

Let $\alpha = -1$. Then $\alpha u = (0, -1)$.

Now, $(0 - (-1))^2 = 1$ while $0 + (-1) = -1$. Thus, $1 \neq -1$.

Hence, $(0, -1) \notin E_5$.

Therefore, E_5 is not stable under scalar multiplication.

Consequently, E_5 is not a vector subspace of $\mathbb{R}^2$.

Proposition 3.2.1

If W_1 and W_2 are subspaces of V, then $W_1 \cap W_2$ is a subspace of V.

Proof. We verify the three conditions of the subspace criterion.

(1) Since W_1 and W_2 are subspaces, $\mathbf{0} \in W_1$ and $\mathbf{0} \in W_2$, hence $\mathbf{0} \in W_1 \cap W_2$.

(2) Let $u, v \in W_1 \cap W_2$. Then $u, v \in W_1$ and $u, v \in W_2$. Since both are subspaces, $u + v \in W_1$ and $u + v \in W_2$, hence $u + v \in W_1 \cap W_2$.

(3) Let $\alpha \in \mathbb{K}$ and $u \in W_1 \cap W_2$. Then $u \in W_1$ and $u \in W_2$. Since both are subspaces, $\alpha \cdot u \in W_1$ and $\alpha \cdot u \in W_2$, hence $\alpha \cdot u \in W_1 \cap W_2$.

All conditions are satisfied, so $W_1 \cap W_2$ is a subspace of V. ■

Remark 3.2.1. If W_1 and W_2 are subspaces of a vector space V, then $W_1 \cup W_2$ is **not** a subspace in general.

Examples Let $W_1 = \{(a, 0) \mid a \in \mathbb{R}\}$ and $W_2 = \{(0, b) \mid b \in \mathbb{R}\}$, both subspaces of $\mathbb{R}^2$. Then $(1, 0) \in W_1 \cup W_2$ and $(0, 1) \in W_1 \cup W_2$, but

$$(1, 0) + (0, 1) = (1, 1) \notin W_1 \cup W_2.$$

Hence $W_1 \cup W_2$ is not closed under addition, so it is not a subspace.

Proposition 3.2.2

Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V over $\mathbb{K}$. Then:

(1) $W_1 \cap W_2 \cap \cdots \cap W_n$ is a subspace of V.

(2) $W_1 + W_2 + \cdots + W_n = \{w_1 + w_2 + \cdots + w_n \mid w_i \in W_i,\ i = 1, \ldots, n\}$ is a subspace of V.

3.3 Span of a Subset

Definition 3.3.1

Let $v_1, v_2, \ldots, v_n$ be vectors in a vector space V over $\mathbb{K}$. A *linear combination* of these vectors is any expression of the form

$$\sum_{i=1}^{i=n} \lambda_i a_i = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$$

for some scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{K}$.

Examples 1) In the vector space $\mathbb{R}^2$, the vector $v = (-7, -13)$ is a linear combination of $v_1 = (-2, 1)$ and $v_2 = (1, 5)$:

$$v = 2v_1 + (-3)v_2.$$

.

2) In $\mathbb{R}^2$, the vector $v = (1, -3)$ is a linear combination of $v_1 = (0, 1)$, $v_2 = (2, -1)$,

$v_3 = (1, -2)$, $v_4 = (0, 3)$:

$$v = (-2)v_1 + (0)v_2 + 1 \cdot v_3 + \tfrac{1}{3}v_4.$$

Definition 3.3.2 Span

Let V be a vector space over $\mathbb{K}$ and $S = \{v_1, v_2, \ldots, v_n\}$ a subset of V. We say S *spans* V (or *generates* V) if every vector $v \in V$ can be written as a linear combination:

$$v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n.$$

The set of all such linear combinations is called the *span* of S. It is denoted

$$\text{span}(S) = \langle \{v_i\} \rangle.$$

The span is the set of all finite linear combinations of the generating family.

3.4 Linear Dependence, Independence, and Bases

Definition 3.4.1

$\{v_1, \ldots, v_p\}$ is *linearly independent* (free) if $\lambda_1 v_1 + \cdots + \lambda_p v_p = 0 \Rightarrow \lambda_i = 0$ for all i. A family that is both free and spanning is a *basis*.

Examples

1. $(1, 2, 1)$, $(-1, 3, 1)$, $(-1, 13, 5)$ are *dependent*: $2v_1 + 3v_2 - v_3 = 0$.

2. $(1, 1, -1)$, $(0, 2, 1)$, $(0, 0, 5)$ are *independent*.

Proposition 3.4.1

$\{v_1, \ldots, v_p\}$ is dependent iff at least one v_i is a linear combination of the others.

Definition 3.4.2

A basis of a vector space V is an ordered family of vectors B which has the property

1) B is free.

2) B to generate V.

Examples Let

$$B' = \{u_1 = (1, 1, 0),\ u_2 = (1, 0, 1),\ u_3 = (0, 2, 0)\}.$$

1) u_1, u_2, u_3 are linearly independent: by definition, we have to show that

$$au_1 + bu_2 + cu_3 = 0 \Longleftrightarrow a = b = c = 0.$$

We have

$$au_1 + bu_2 + cu_3 = (a, a, 0) + (b, 0, b) + (0, 2c, 0) = (a + b, a + 2c, b).$$

Hence

$$(a + b, a + 2c, b) = (0, 0, 0) \Longleftrightarrow \begin{cases} a + b = 0, \\ a + 2c = 0, \\ b = 0. \end{cases}$$

Introducing $b = 0$ in the first equation, we find $a = 0$, and introducing $a = 0$ in the second equation, we get $2c = 0$, i.e. $c = 0$.

So, a linear combination of the vectors of B' can be the null vector if and only if all the coefficients of the linear combination are 0; therefore B' is free.

Now we must prove that B' is a family of generators, i.e. that, given a generic vector

$$v = (x, y, z) \in \mathbb{R}^3,$$

there exist $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$ such that

$$v = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3.$$

Taking advantage of the calculation just made we have

$$(x, y, z) = (\lambda_1 + \lambda_2,\ \lambda_1 + 2\lambda_3,\ \lambda_2),$$

which corresponds to the linear system

$$\begin{cases} \lambda_1 + \lambda_2 = x, \\ \lambda_1 + 2\lambda_3 = y, \\ \lambda_2 = z. \end{cases}$$

From $\lambda_2 = z$ we obtain

$$\begin{cases} \lambda_2 = z, \\ \lambda_1 + z = x, \\ \lambda_1 + 2\lambda_3 = y. \end{cases}$$

Hence

$$\begin{cases} \lambda_2 = z, \\ \lambda_1 = x - z, \\ x - z + 2\lambda_3 = y. \end{cases}$$

Thus

$$\begin{cases} \lambda_1 = x - z, \\ \lambda_2 = z, \\ \lambda_3 = \dfrac{y - x + z}{2}. \end{cases}$$

It is a good exercise to verify that the linear combination of the vectors of B' with the coefficients just found gives back the generic vector $v = (x, y, z)$ of $\mathbb{R}^3$:

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = (x - z)(1, 1, 0) + z(1, 0, 1) + \frac{y - x + z}{2}(0, 2, 0)$$

$$= (x - z,\ x - z,\ 0) + (z,\ 0,\ z) + (0,\ y - x + z,\ 0)$$

$$= (x - z + z,\ x - z + y - x + z,\ z)$$

$$= (x, y, z).$$

Therefore, $v = (x, y, z)$, and consequently B' generates $\mathbb{R}^3$.

Since B' is linearly independent and generates $\mathbb{R}^3$, it is a basis of $\mathbb{R}^3$.

Proposition 3.4.2 Uniqueness of coordinates

If $B = \{v_1, \dots, v_n\}$ is a basis of E, every $x \in E$ writes uniquely as $x = \sum_{i=1}^{n} \lambda_i v_i$ (the λ_i are the *coordinates* of x in B), giving a bijection $\varphi_B \colon E \to \mathbb{K}^n$.

Examples

$$F = \left\{(x, y, z) \in \mathbb{R}^3 \ / \ 2x + y + 3z = 0\right\}.$$

F is a vector subspace of $\mathbb{R}^3$.

We have $v = (x, y, z) \in F \iff y = -2x - 3z$.

Thus, $v \in F \iff v = (x, -2x - 3z, z)$.

Hence, $v = x(1, -2, 0) + z(0, -3, 1)$.

Therefore, $(1, -2, 0)$ and $(0, -3, 1)$ generate F.

Therefore they form a basis of F.

3.5 Existence of Bases (Finite-Dimensional Case)

Theorem 3.5.1

Every non-zero finite-dimensional space has a basis. From every spanning family one can extract a basis; every free family can be extended to a basis (*Incomplete Basis Theorem*).

Examples Let

$$V = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4 \ : \ a - b = d - c \right\}.$$

Show that V is a subspace of $\mathbb{R}^4$, find a basis of V, and find $\dim(V)$.

Solution

The condition $a - b = d - c$ is equivalent to

$a = b - c + d$,

so we may write

$$V = \left\{ \begin{bmatrix} b-c+d \\ b \\ c \\ d \end{bmatrix} : b, c, d \in \mathbb{R} \right\}$$

$$= \left\{ b \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} : b, c, d \in \mathbb{R} \right\}.$$

This shows that

$$V = \operatorname{span}\{u, v, u_3\},$$

where

$$u = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad v = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, V is a subspace of $\mathbb{R}^4$.

Furthermore,

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is linearly independent, since the reduced row-echelon form of the matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since every column contains a pivot, the vectors are linearly independent.

Hence,

$$\{u, v, u_3\}$$

is a basis of V.

Therefore,

$$\dim(V) = 3.$$

3.6 Fundamental Dimension Theorems

Theorem 3.6.1

In a space spanned by n elements, every family with more than n elements is dependent.

Theorem 3.6.2 Uniqueness of dimension

All bases of a finite-dimensional space have the same cardinality, called the *dimension* $\dim_{\mathbb{K}} E$.

Corollary 3.6.1

- $\dim_{\mathbb{K}}\{0\} = 0; \quad \dim_{\mathbb{K}} \mathbb{K}^n = n; \quad \dim_{\mathbb{R}} \mathbb{R}_n[X] = n + 1.$
- $\dim_{\mathbb{R}} \mathbb{C} = 2; \quad \dim_{\mathbb{C}} \mathbb{C} = 1.$
- $\dim_{\mathbb{K}}(E_1 \times \cdots \times E_p) = \sum_i \dim_{\mathbb{K}} E_i.$
- A spanning family of exactly $n = \dim E$ elements is a basis; a free family of exactly n elements is a basis.

Theorem 3.6.3 Subspace dimension

If $F \subseteq E$ (both finite-dimensional): $\dim F \leq \dim E$, with equality iff $F = E$.

3.7 Sum, Direct Sum, and Complementary Subspaces

Definition 3.7.1

$E_1 + E_2 = \{x_1 + x_2 : x_i \in E_i\}$ is the *sum* of subspaces. The sum is *direct*, written $E_1 \oplus E_2$, when every element decomposes uniquely, which happens iff $E_1 \cap E_2 = \{0\}$. E_1 and E_2 are *complementary* in E when $E = E_1 \oplus E_2$.

Proposition 3.7.1

$E = E_1 \oplus E_2$ iff for any bases B_1 of E_1 and B_2 of E_2, $B_1 \cup B_2$ is a basis of E.

Theorem 3.7.1

(Finite-dimensional case) $E = E_1 \oplus E_2$ iff $E_1 \cap E_2 = \{0\}$ and $\dim E = \dim E_1 + \dim E_2$. Every subspace has a complementary subspace; all complementaries of a given subspace share the same dimension.

Proposition 3.7.2 Dimension formula

$\dim(E_1 + E_2) = \dim E_1 + \dim E_2 - \dim(E_1 \cap E_2)$.

3.8 Tutorial 4 – Vector Spaces and Subspaces

Exercise 3.8.1. Determine whether the following set are subspaces of $\mathbb{R}^3$?.

Justify your answers.

$E = \{(x, y, z) \in \mathbb{R}^3 : 2x - y + 4z = 0\}$

$F = \{(x, y, z) \in \mathbb{R}^3 : 3x - 2y - 6z = 5\}$

$G = \{(x, y) \in \mathbb{R}^2 : y = x^3\}$

$H = \{(x, y, z) \in \mathbb{R}^3 : x = 3y,\ z = -y\}$

Exercise 3.8.2. Show that the family $B = \{(1, 2); (-1, 1)\}$ is a basis of $\mathbb{R}^2$.

Show that B is a generating of $\mathbb{R}^2$.

Find the coordinates of the vector $v = (-3, 1)$ in the base B.

Which families are linearly independent among the following families :

a) $F_1 = \{(1, 1, 0); (1, 0, 0); (0, 1, 1)\}$ in $\mathbb{R}^3$

b) $F_2 = \{(0,1,1,0);(1,1,1,0);(2,1,1,0)\}$in $\mathbb{R}^4$.

Exercise 3.8.3. Let F and G two vector subspaces of $\mathbb{R}^3$ defined by :

$F = \{(x,y,z) \in \mathbb{R}^3; x - 2y + z = 0\}$ and $G = \{(x,y,z) \in \mathbb{R}^3; 2x - y + 2z = 0\}$

1. Give a basis for F , a basis for G, and deduce their respective dimensions.
2. Give a basis for $F \cap G$, and give its dimension.
3. Do we have $F \oplus G = \mathbb{R}^3$?

Exercise 3.8.4. We consider in $\mathbb{R}^3$, the subset F de ned by :

$F = \{(x - y, 2x + y + 4z, 3y + 2z)/x, y, z \in \mathbb{R}\}$

1. Show that F is a vector subspace of $\mathbb{R}^3$.
2. Give a basis for F, and give its dimension.

Chapter 4

Linear Maps

In this chapter, we study the notion of maps between vector spaces: linear maps.

4.1 Linear Maps

Definition 4.1.1

Let E and F be two K-vector spaces and let $f : E \to F$ be a map.
We say that f is linear if:

1. For all $u, v \in E, f(u+v) = f(u) + f(v)$.
2. For all $\lambda \in K$ and all $u \in E$, $f(\lambda u) = \lambda f(u)$.

Equivalently,
For all $u, v \in E$, for all $\lambda, \mu \in K$: $f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$,

Remark 4.1.1. 1) The set of all linear maps from E to F is $\mathcal{L}(E, F)$.

2) Every linear map satisfies $f(0) = 0$.

Example

Consider the map

$f : \mathbb{R}^2 \to \mathbb{R}^3, f(x, y) = (x + y, x - y, 2y)$.

We show that f is linear.

Let $u = (x_1, y_1)$, $v = (x_2, y_2)$, and let $\lambda, \mu \in \mathbb{R}$.

Then

$$f(\lambda u + \mu v) = f(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2)$$
$$= (\lambda x_1 + \mu x_2 + \lambda y_1 + \mu y_2, \lambda x_1 + \mu x_2 - \lambda y_1 - \mu y_2, 2\lambda y_1 + 2\mu y_2).$$

Thus,

$f(\lambda u + \mu v) = \lambda f(u) + \mu f(v)$. Hence f is linear.

Example

Let $E = \{f : [a, b] \to \mathbb{R} \mid f \text{ continuous on } [a, b]\}$.

Define $T : E \to \mathbb{R}$ by $T(f) = \int_a^b f(x)\,dx$.

Then

$$T(\alpha f + \beta g) = \int_a^b (\alpha f + \beta g)(x)\,dx$$
$$= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.$$

Therefore,

$T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$, so T is linear.

Example

Consider $g : \mathbb{R}^2 \to \mathbb{R}^2, g(x, y) = (xy, -xy)$.

We take $\lambda = 2, \quad u = (1, -1)$. Then

$g(2u) = g(2, -2) = (-4, 4)$, while $2g(1, -1) = (-2, 2)$.

Hence, $g(2u) \neq 2g(u)$,

therefore g is not linear.

4.2 Identity Map

Definition 4.2.1

The identity map $Id_E : E \to E$ is defined by

$Id_E(u) = u, \quad \forall u \in E.$

It is a linear map.

4.3 Properties

Let $f \in L(E, F)$.

Then:

1) $f(0_E) = 0_F$,

2) $f(-u) = -f(u)$,

3) $f\left(\sum_{i=1}^{p} \lambda_i u_i\right) = \sum_{i=1}^{p} \lambda_i f(u_i)$.

4.4 Special Linear Maps

A linear map $f \in L(E, F)$ is called:

- an **endomorphism** if $F = E$;
- an **isomorphism** if f is bijective;
- an **automorphism** if f is bijective and $F = E$;
- a **linear form** if $F = K$.

4.5 Image and Kernel

Definition 4.5.1

Let $f \in L(E, F)$. The image of f is:
$Im(f) = \{\forall x \in E, \exists y \in F \mid f(x) = y\}$.

Definition 4.5.2

Let $f \in L(E, F)$. The kernel of f is:
$Ker(f) = \{x \in E \mid f(x) = 0_F\}$.

Proposition 4.5.1

- f is injective if and only if $Ker(f) = \{0_E\}$.
- f is surjective if and only if $Im(f) = F$.

4.6 Rank Theorem

Theorem 4.6.1

Let E and F be finite-dimensional vector spaces and let $f \in L(E, F)$.
Then : $\dim(E) = \dim(Ker(f)) + \dim(Im(f))$.

Definition 4.6.1

The rank of f is $rank(f) = \dim(Im(f))$.
Moreover, $rank(f) = \dim(E) - \dim(Ker(f))$.

Examples

Let

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

defined by

$$f(x, y, z) = (x + y + z,\ x - y - z)$$

1. Give the dimension of the kernel of f.

2. Is f injective?

Solution:

The kernel of f is

$$\ker(f) = \{(x, y, z) \in \mathbb{R}^3\ ;\ f(x, y, z) = 0_{\mathbb{R}^2}\}$$

Therefore, $f(x, y, z) = (0, 0)$
gives

$$\begin{cases} x + y + z = 0 \\ x - y - z = 0 \end{cases}$$

Adding the two equations: $2x = 0 \Rightarrow x = 0$
Then: $y + z = 0 \Rightarrow y = -z$
Hence, $(x, y, z) = (0, -z, z)$
so $\ker(f) = \{z(0, -1, 1); z \in \mathbb{R}\}$
Therefore, $\ker(f) = \text{Vect}\{(0, -1, 1)\}$
and $\dim(\ker(f)) = 1$

Since $\ker(f) \neq \{0\}$ we conclude that f is not injective.

Example:

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

defined by $f(x, y, z) = (2x + y + z,\ y - z,\ x - y)$

Find $\ker(f)$, $\dim(\operatorname{Im} f)$. Is f injective? Is f surjective?

Solution:

The kernel of f is $\ker(f) = \{(x, y, z) \in \mathbb{R}^3; f(x, y, z) = 0\}$

Thus,

$$\begin{cases} 2x + y + z = 0 \\ y - z = 0 \\ x - y = 0 \end{cases}$$

From $y - z = 0$ we get $y = z$ and from $x - y = 0$ we get $x = y$

Therefore, $x = y = z$

Substituting in the first equation: $2x + x + x = 0$

so $x = y = z = 0$ Thus, $\ker(f) = \{(0, 0, 0)\}$ and $\dim(\ker f) = 0$, we conclude that f is injective and $\dim(\operatorname{Im} f) = 3$, since f is surjective.

Exercise,

Let f be the map defined by:

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}^3, \qquad (x, y, z) \longmapsto \Big(x,\ x + y - z,\ x\Big).$$

① Determine $\operatorname{Ker}(f)$. Deduce a basis for $\operatorname{Ker}(f)$.

② Determine $Im(f)$. Deduce a basis for $Im(f)$.

Kernel of f

Let $(x, y, z) \in \operatorname{Ker} f$. By definition, $f(x, y, z) = (0, 0, 0)$, so:

$$\begin{cases} x = 0, \\ x + y - z = 0, \\ x = 0. \end{cases}$$

From the first equation, $x = 0$. Substituting into the second equation gives

$$y - z = 0 \quad \Longrightarrow \quad y = z.$$

Therefore, any vector in $\operatorname{Ker} f$ has the form

$$(x, y, z) = (0,\ y,\ y) = y\,(0, 1, 1), \qquad y \in \mathbb{R}.$$

$$\operatorname{Ker} f = Vect\Big\{(0, 1, 1)\Big\}.$$

The family $\{(0, 1, 1)\}$ is a basis of $\operatorname{Ker} f$, so $\dim(\operatorname{Ker} f) = 1$.

Image of f

By definition, $Imf = \{f(x, y, z) \mid (x, y, z) \in \mathbb{R}^3\}$. We expand:

$$\begin{aligned} Im(f) &= \Big\{(x,\ x + y - z,\ x) \mid (x, y, z) \in \mathbb{R}^3\Big\} \\ &= \Big\{x(1, 1, 1) + y(0, 1, 0) + z(0, -1, 0) \mid (x, y, z) \in \mathbb{R}^3\Big\} \\ &= \Big\{x(1, 1, 1) + (y - z)(0, 1, 0) \mid (x, y, z) \in \mathbb{R}^3\Big\} \\ &= Vect\Big\{(1, 1, 1),\ (0, 1, 0)\Big\}. \end{aligned}$$

The vectors $(1, 1, 1)$ and $(0, 1, 0)$ are clearly not proportional, so they are linearly independent.

$$Im(f) = Vect\Big\{(1, 1, 1),\ (0, 1, 0)\Big\}.$$

The family $\{(1, 1, 1), (0, 1, 0)\}$ is a basis of $Im(f)$, so $\dim(Im(f) = rank(f) = 2$.

Verification (Rank-Nullity Theorem):

$$\dim(\operatorname{Ker} f) + \dim(Im(f)) = 1 + 2 = 3 = \dim(\mathbb{R}^3). \quad \checkmark$$

4.7 Composition of Linear Maps

Let

$$f \in L(E, F), \qquad g \in L(F, G).$$

Then $g \circ f \in L(E, G)$.

$$\begin{aligned} g \circ f(\lambda v_1 + \mu v_2) &= g(f(\lambda v_1 + \mu v_2)) \\ &= g(\lambda f(v_1) + \mu f(v_2)) \\ &= \lambda g(f(v_1)) + \mu g(f(v_2)). \end{aligned}$$

Hence, $g \circ f$ is linear.

4.8 Inverse of a Linear Map

Definition 4.8.1

A linear map $f : E \to F$ is invertible if it is bijective.
Its inverse map $f^{-1} : F \to E$
satisfies $f^{-1}(f(x)) = x$ and $f(f^{-1}(y)) = y$.

Example

Consider $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by
$f(x, y) = (x + y, x - y)$.
We solve $(y_1, y_2) = (x_1 + x_2, x_1 - x_2)$. Then $x_1 = \frac{1}{2}(y_1 + y_2), \qquad x_2 = \frac{1}{2}(y_1 - y_2)$.
Hence, $f^{-1}(y_1, y_2) = \left(\frac{1}{2}(y_1 + y_2), \frac{1}{2}(y_1 - y_2)\right)$.

Chapter 5

Matrices, associated matrices, and determinants

5.1 Definitions and Properties

A rectangular array of numbers (elements of $\mathbb{K}$) of the form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is called a *matrix*. The numbers a_{ij} are its *entries*. Horizontal lines are *rows* and vertical lines are *columns*. A matrix with m rows and n columns is of *type* (m, n), denoted (a_{ij}).

Example

1. The *zero matrix* O has all entries equal to zero.
2. A matrix with a single row $(a_1, \ldots, a_n)$ is a *row matrix*.
3. A matrix with a single column is a *column matrix*.
4. A square matrix has the same number of rows and columns (*order* = common value).
5. The *identity matrix* $I = (a_{ij})$ with $a_{ii} = 1$ and $a_{ij} = 0$ for $i \neq j$ satisfies $AI = IA = A$ for every square matrix A of the same order.
6. Two matrices are *equal* iff they have the same type and $a_{ij} = b_{ij}$ for all i, j.

5.2 Operations on Matrices

5.2.1 Addition

The *sum* of two (m, n)-matrices (a_{ij}) and (b_{ij}) is (c_{ij}) with $c_{ij} = a_{ij} + b_{ij}$.

Example $\begin{pmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{pmatrix}$.

Matrix addition satisfies commutativity, associativity, identity (O), and inverses ($-A = (-a_{ij})$).

5.2.2 Scalar Multiplication

For $\lambda \in \mathbb{K}$ and $A = (a_{ij})$: $\lambda A = (\lambda a_{ij})$. This satisfies the four standard properties $\lambda(A + B) = \lambda A + \lambda B$, $(\lambda + \mu)A = \lambda A + \mu A$, $\lambda(\mu A) = (\lambda\mu)A$, and $1 \cdot A = A$.

5.2.3 Matrix Multiplication

Let $A = (a_{ij})$ of type (m, n) and $B = (b_{kl})$ of type (r, p). The *product* AB is defined only when $n = r$; it is the (m, p) matrix (c_{il}) with

$$c_{il} = \sum_{j=1}^{n} a_{ij} b_{jl}.$$

Example $\begin{pmatrix} 3 & 2 & -1 \\ 0 & 4 & 6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 5 & 3 & 1 \\ 6 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 7 & 2 & 6 \\ 56 & 36 & 16 \end{pmatrix}$.

Matrix multiplication is associative and distributive over addition, but **not** commutative in general. Also, $AB = O$ does not imply $A = O$ or $B = O$.

Remark 5.2.1 (Special matrices)**.**

1. A *diagonal matrix*: $a_{ij} = 0$ for $i \neq j$.

2. An *upper-triangular* (resp. *lower-triangular*) matrix: $a_{ij} = 0$ for $i > j$ (resp. $i < j$).

3. Powers: $A^2 = AA$, $A^3 = A^2A$, etc.

4. The *transpose* ${}^tA = (b_{ij})$ with $b_{ij} = a_{ji}$; ${}^t({}^tA) = A$.

5.3 Elementary Matrices

5.3.1 Elementary Row Operations

An *elementary operation* on a matrix A is one of:

(1) $R_j \leftarrow R_j + kR_i$: add a scalar multiple of one row to another.

(2) $R_i \leftarrow kR_i$ $(k \neq 0)$: multiply a row by a non-zero scalar.

(3) $R_i \leftrightarrow R_j$: swap two rows.

The *elementary matrix* $E = e(I)$ corresponding to operation e satisfies $e(A) = EA$.

Example

From I_3: swapping $L_2 \leftrightarrow L_3$ gives $E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; replacing L_2 by $-6L_2$ gives $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; replacing L_3 by $-4L_1 + L_3$ gives $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$.

Transpose of a Matrix

Definition 5.3.1

Let

$$A = (a_{ij})_{1 \leq i \leq m,\, 1 \leq j \leq n} \in M_{m,n}(\mathbb{K}),$$

the transpose of A, denoted by ${}^tA$, is the matrix with n rows and m columns. Thus,

$${}^tA = (a_{ji})_{1 \leq j \leq n,\, 1 \leq i \leq m}$$

Example $A = \begin{pmatrix} 1 & -2 \\ 4 & 1 \\ -4 & 8 \end{pmatrix}$, $A^t = \begin{pmatrix} 1 & 4 & -4 \\ -2 & 1 & 8 \end{pmatrix}$

Proposition 5.3.1

Let A and B be two matrices of $M_{m,n}(\mathbb{K})$ and let $\lambda \in \mathbb{K}$. Then,

1. ${}^t({}^tA) = A$
2. ${}^t(\lambda A) = \lambda\, {}^tA$
3. ${}^t(A+B) = {}^tA + {}^tB$
4. ${}^t(AB) = {}^tB\, {}^tA$

5.3.2 Trace of a Matrix

Definition 5.3.2

The **trace** of a square matrix $A = (a_{ij})$ of order n is the sum of its diagonal entries. It is denoted by $\mathrm{tr}(A)$ and defined by $\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$.

Example Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 5 & 2 & 8 \\ 11 & 0 & -10 \end{pmatrix}$ Then $\mathrm{tr}(A) = 1 + 2 + (-10) = -7$.

Theorem 5.3.1

Let A and B be two square matrices of order n. Then:

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$.
2. For every scalar (k), $\mathrm{tr}(kA) = k\,\mathrm{tr}(A)$.
3. $\mathrm{tr}(A^T) = \mathrm{tr}(A)$.
4. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Definition 5.3.3 Inverse of a Matrix

If A and B are two matrices such that $AB = BA = I$, then each is said to be inverse of the other. The inverse of A is denoted by A^{-1}

Example Let the matrix $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$

Calculate :$A^2 - 3A + 2I$, deduce that A is invertible and then calculate A^{-1}

Solution We calculate:

$$A^2 = \begin{pmatrix} -5 & -6 \\ 9 & 10 \end{pmatrix}$$

Then,

$$A^2 - 3A + 2I = \begin{pmatrix} -5 & -6 \\ 9 & 10 \end{pmatrix} + \begin{pmatrix} 3 & 6 \\ -9 & -12 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Therefore, $A^2 - 3A + 2I = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Hence, $A^2 - 3A + 2I = 0$, so $A(A - 3I) = -2I$

and $A^{-1} = \frac{1}{2}(3I - A)$

Now, $A^{-1} = \begin{pmatrix} 2 & 1 \\ -\frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ Therefore A is invertible.

5.3.3 Computing the Inverse of a Square Matrix

Apply elementary row operations to the augmented matrix $(A \mid I)$ to reduce A to I; the same operations transform I into A^{-1}.

Example $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ 4 & 1 & 8 \end{pmatrix}$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 4 & 1 & 8 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -11 & 2 & 2 \\ 0 & 1 & 0 & -4 & 0 & 1 \\ 0 & 0 & 1 & 6 & -1 & -1 \end{array}\right)$$

Hence $A^{-1} = \begin{pmatrix} -11 & 2 & 2 \\ -4 & 0 & 1 \\ 6 & -1 & -1 \end{pmatrix}$.

5.4 Tutorial 4 – Linear Maps

Exercise 5.4.1. Let : : $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (x + y, x - y, 2y)$

1) Show that f is linear map

2) Give $Ker(f), Rank(f)$.

3) is f injective ? surjective? bijective?

Exercise 5.4.2. Let the following linear maps :

$f : \mathbb{R}^3 \to \mathbb{R}^2$, $g : \mathbb{R}^2 \to \mathbb{R}^3$

$(x, y, z) \to f(x, y, z) = (z, x + y + z)$ $(x, y) \to g(x, y) = (-y, -x + 2y, x)$

1. Determine a basis of $ker(f)$, deduce the rank of f and that f is surjective.
2. a) Show that :g is injective, deduce $rank$ of g.

b) Show that :$\{v_1 = (-1, 2, 0), v_2 = (0, -1, 1)\}$ is a basis of $Im(g)$.

3. Show that : $f \circ g = Id_{\mathbb{R}^2}$.
4. Show that : $ker(g \circ f) = ker(f)$

Exercise 5.4.3. A and B are two matrices so that $A = \begin{pmatrix} 2 & 5 \\ 3 & -1 \end{pmatrix}, B = \begin{pmatrix} 7 & 2 \\ -1 & -3 \end{pmatrix}$

1) Calculate: A+B, A-B, 3A-4B, A×B, B×A, $(A + B)^2$

2) Calculate A^2, B^2, A^T, B^T

3) C is matrix so that $C = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - cb \neq 0$.

-Find the real numbers a, b, c and d such that $A \times C = C \times A = I_2$, then deduce A^{-1}

Exercise 5.4.4. Let the matrix $A = \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix}$

1) Calculate :$A^2 - 3A + 2I$, deduce that A is invertible and then calculate A^{-1}

2) Solve the equation $AX = b$, such that $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Exercise 5.4.5. $M = \begin{pmatrix} 2 & -1 & 2 \\ 5 & -3 & 3 \\ -1 & 0 & -2 \end{pmatrix}$, calculate $(M + I)^3$

- deduce that M is invertible and then calculate M^{-1}

Exercise 5.4.6. Let the matrix $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, we put $B = A - I$

- Calculate B^n, and then calculate A^n, $n \in \mathbb{N}$.

5.5 Determinants

5.6 Determinant of Order 2

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

Example $\begin{vmatrix}1 & 3\\ 2 & 4\end{vmatrix} = -2;\quad \begin{vmatrix}2 & 4\\ 1 & 3\end{vmatrix} = 2;\quad \begin{vmatrix}3 & 1\\ 4 & 2\end{vmatrix} = 2.$

Swapping two rows (or columns) multiplies the determinant by -1. Also $\det A = \det({}^tA)$.

5.7 Determinant of Order 3

$$\det A = a_{11}\underbrace{\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}}_{\text{minor of } a_{11}} - a_{21}\begin{vmatrix}a_{12} & a_{13}\\ a_{32} & a_{33}\end{vmatrix} + a_{31}\begin{vmatrix}a_{12} & a_{13}\\ a_{22} & a_{23}\end{vmatrix}.$$

The *cofactor* of a_{ik} is $C_{ik} = (-1)^{i+k}\cdot(\text{minor of } a_{ik})$. The sign pattern is $\begin{pmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{pmatrix}$.

The determinant can be expanded along *any* row or column: $\det A = \sum_k a_{ik}C_{ik}$.

Example $\begin{vmatrix}1 & 3 & 0\\ 2 & 6 & 4\\ -1 & 0 & 2\end{vmatrix} = 1(12) - 2(6) - 1(12) = -12.$

Remark 5.7.1 (Properties of the 3×3 determinant)**.**

1. Swapping two rows/columns multiplies the det by -1.
2. Multiplying all entries of one row/column by k multiplies det by k.
3. If one entry is a binomial, the det splits into two dets.
4. A zero row/column gives det $= 0$.
5. Proportional rows/columns give det $= 0$.
6. Adding a linear combination of other rows/columns to one row/column leaves det unchanged.

Example

$$\begin{vmatrix}3 & -1 & 2\\ 6 & -2 & 4\\ 1 & 7 & 3\end{vmatrix} \xrightarrow{C_1\leftarrow C_1+C_2-C_3} \begin{vmatrix}0 & -1 & 2\\ 0 & -2 & 4\\ 5 & 7 & 3\end{vmatrix} = 5\begin{vmatrix}-1 & 2\\ -2 & 4\end{vmatrix} = 5(0) = 0.$$

Remark 5.7.2. Warning. Do not multiply the row/column along which you expand, as that would multiply the entire determinant.

5.8 Determinant of Order n

$$\det A = \sum_{k=1}^{n} a_{ik} C_{ik} = \sum_{k=1}^{n} a_{kj} C_{kj}$$

(for any fixed row i or column j), where the order-$(n-1)$ minors are evaluated recursively.

All properties from order 3 carry over. Additional:

1. $\det(A+B) \neq \det A + \det B$ in general.
2. $\det(AB) = \det A \cdot \det B$.
3. $\det(A^{-1}) = (\det A)^{-1}$.

Example
$$\begin{vmatrix} 1 & 1 & 1 & -3 \\ 1 & 1 & -3 & 1 \\ 1 & -3 & 1 & 1 \\ -3 & 1 & 1 & 1 \end{vmatrix} \xrightarrow{C_1+C_2+C_3+C_4} \begin{vmatrix} 0 & 1 & 1 & -3 \\ 0 & 1 & -3 & 1 \\ 0 & -3 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{vmatrix} = 0.$$

5.9 Applications

5.9.1 Inverse of a Square Matrix of Order n

A is invertible $\Leftrightarrow \det A \neq 0$. When invertible:

$$A^{-1} = \frac{1}{\det A} \operatorname{adj}(A), \qquad \operatorname{adj}(A) = {}^t(C_{ij}).$$

Example $A = \begin{pmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{pmatrix}$, $\det A = -46 \neq 0$. After computing the nine cofactors:

$$A^{-1} = -\tfrac{1}{46} \begin{pmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{pmatrix}.$$

5.10 Matrices Associated with Linear Maps

Let E and F be two vector spaces with dimensions p and n respectively.

Let $f : E \longrightarrow F$ be a linear map.

$$\mathcal{B}_E = \{u_1, u_2, \ldots, u_p\} \quad \text{basis of } E, \qquad \mathcal{B}_F = \{v_1, v_2, \ldots, v_n\} \quad \text{basis of } F.$$

Definition 5.10.1 Matrix of a Linear Map

We call the **matrix of the linear map** f with respect to the bases $\mathcal{B}_E$ and $\mathcal{B}_F$ the matrix whose **columns** are the images of the starting basis ($\mathcal{B}_E$) written in the arriving basis ($\mathcal{B}_F$):

$$\begin{cases} f(u_1) = a_{11}\,v_1 + a_{21}\,v_2 + a_{31}\,v_3 + \cdots + a_{n1}\,v_n \\ f(u_2) = a_{12}\,v_1 + a_{22}\,v_2 + a_{32}\,v_3 + \cdots + a_{n2}\,v_n \\ \qquad\qquad \vdots \\ f(u_p) = a_{1p}\,v_1 + a_{2p}\,v_2 + a_{3p}\,v_3 + \cdots + a_{np}\,v_n \end{cases}$$

$$M_{\mathcal{B}_E,B_F} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \begin{matrix} \leftarrow v_1 \\ \leftarrow v_2 \\ \vdots \\ \leftarrow v_n \end{matrix}$$

- The **order** of M is $n \times p$.
- The number of **columns** is the dimension of E.
- The number of **rows** is the dimension of F.

$$f : E \longrightarrow F$$

$$\begin{array}{ccc} \mathcal{B}_E & \xrightarrow{\quad M \quad} & \mathcal{B}_F \\ \Big\downarrow P \text{ (transition)} & & \Big\downarrow Q \text{ (transition)} \\ \mathcal{B}'_E & \xrightarrow{\quad A \quad} & \mathcal{B}'_F \end{array}$$

5.10.1 Example 1 — $f : \mathbb{R}^2 \to \mathbb{R}^3$

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map defined by:

$$f(x, y) = (x + y,\ 2x - y,\ x - 3y).$$

Find the matrix of f with respect to the standard bases.

Solution.

The standard basis of $\mathbb{R}^2$ is $e_1 = (1, 0),\ e_2 = (0, 1)$.
The standard basis of $\mathbb{R}^3$ is $\tilde{e}_1 = (1, 0, 0),\ \tilde{e}_2 = (0, 1, 0),\ \tilde{e}_3 = (0, 0, 1)$.

Images of the basis vectors of $\mathbb{R}^2$:

$$f(1, 0) = (1, 2, 1) = 1 \cdot \tilde{e}_1 + 2 \cdot \tilde{e}_2 + 1 \cdot \tilde{e}_3.$$

$$f(0, 1) = (1, -1, -3) = 1 \cdot \tilde{e}_1 - 1 \cdot \tilde{e}_2 - 3 \cdot \tilde{e}_3.$$

The matrix of the linear map f is:

$$M = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -3 \end{pmatrix}.$$

5.10.2 Example 2 — $f : \mathbb{R}^2 \to \mathbb{R}^3$, non-standard bases

Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ be the linear map defined by:

$$f(x, y) = (x + y,\ 2x,\ -3y).$$

Consider the bases:

$$\mathcal{B}_E :\ u_1 = (1, 1),\ u_2 = (2, 0) \qquad \text{and} \qquad \mathcal{B}_F :\ v_1 = (2, 0, 1),\ v_2 = (1, 1, 0),\ v_3 = (0, 0, -1).$$

Also consider the new bases:

$$\mathcal{B}'_E :\ u'_1 = (2, 4),\ u'_2 = (-1, 1), \qquad \mathcal{B}'_F :\ v'_1 = (1, 1, 1),\ v'_2 = (1, 3, 0),\ v'_3 = (0, 2, -2).$$

1) Find the change-of-basis matrix P from $\mathcal{B}_E$ to $\mathcal{B}'_E$.

2) Find the change-of-basis matrix Q from $\mathcal{B}_F$ to $\mathcal{B}'_F$.

3) Find the matrix M associted f in $\mathcal{B}_E$ and $\mathcal{B}_F$.

4) Find the matrix A associted f in $\mathcal{B}'_E$ and $\mathcal{B}'_F$.

Solution

Step 1 — Compute P (express $\mathcal{B}'_E$ in $\mathcal{B}_E$).

$$(2,4) = \lambda_1(1,1) + \lambda_2(2,0) \implies \begin{cases} \lambda_1 + 2\lambda_2 = 2 \\ \lambda_1 = 4 \end{cases} \implies \lambda_1 = 4,\ \lambda_2 = -1.$$

$$(-1,1) = \lambda'_1(1,1) + \lambda'_2(2,0) \implies \begin{cases} \lambda'_1 + 2\lambda'_2 = -1 \\ \lambda'_1 = 1 \end{cases} \implies \lambda'_1 = 1,\ \lambda'_2 = -1.$$

So

$$P = \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix}.$$

Step 2 — Compute Q (express $\mathcal{B}'_F$ in $\mathcal{B}_F$).

$$(1,1,1) = \lambda_1(2,0,1) + \lambda_2(1,1,0) + \lambda_3(0,0,-1).$$

$$\begin{cases} 2\lambda_1 + \lambda_2 = 1 \\ \lambda_2 = 1 \\ \lambda_1 - \lambda_3 = 1 \end{cases} \implies \lambda_1 = 0,\ \lambda_2 = 1,\ \lambda_3 = -1.$$

$$(1,3,0) = \lambda'_1(2,0,1) + \lambda'_2(1,1,0) + \lambda'_3(0,0,-1).$$

$$\begin{cases} 2\lambda'_1 + \lambda'_2 = 1 \\ \lambda'_2 = 3 \\ \lambda'_1 - \lambda'_3 = 0 \end{cases} \implies \lambda'_1 = -1,\ \lambda'_2 = 3,\ \lambda'_3 = -1. \quad (\lambda'_3 = \lambda'_1 + 2 = -1)$$

$$(0,2,-2) = \lambda''_1(2,0,1) + \lambda''_2(1,1,0) + \lambda''_3(0,0,-1).$$

$$\begin{cases} 2\lambda_1'' + \lambda_2'' = 0 \\ \lambda_2'' = 2 \\ \lambda_1'' - \lambda_3'' = -2 \end{cases} \implies \lambda_1'' = -1,\ \lambda_2'' = 2,\ \lambda_3'' = \lambda_1'' + 2 = 1.$$

Then

$$Q = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 3 & 2 \\ -1 & -1 & 1 \end{pmatrix}.$$

RemarkChange-of-basis formula

If M is the matrix of f in bases $(\mathcal{B}_E, \mathcal{B}_F)$ and M' is the matrix of f in bases $(\mathcal{B}'_E, \mathcal{B}'_F)$, then:

$$A = Q^{-1} \cdot M \cdot P,$$

where P is the transition matrix from $\mathcal{B}_E$ to $\mathcal{B}'_E$, and Q is the transition matrix from $\mathcal{B}_F$ to $\mathcal{B}'_F$.

Step 1 — Compute Q^{-1}

From the previous session we established that $\det(Q) = 1$ and:

$$Adj(Q) = \begin{pmatrix} 5 & 2 & 1 \\ -3 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Since $\det(Q) = 1$, we have $Q^{-1} = Adj(Q)$:

$$Adj(Q) = \begin{pmatrix} +\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} \\ -\begin{vmatrix} -1 & -1 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ -1 & -1 \end{vmatrix} \\ +\begin{vmatrix} -1 & -1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ 1 & 3 \end{vmatrix} \end{pmatrix}^\top = \begin{pmatrix} 5 & -3 & 2 \\ 2 & -1 & 1 \\ 1 & -1 & 1 \end{pmatrix}^\top = \begin{pmatrix} 5 & 2 & 1 \\ -3 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

So

$$Q^{-1} = \begin{pmatrix} 5 & 2 & 1 \\ -3 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix}.$$

Step 2 — Compute M (matrix of f in $\mathcal{B}_E$, $\mathcal{B}_F$)

We express $f(u_k)$ in $\mathcal{B}_F = \{v_1, v_2, v_3\}$, where $v_1 = (2, 0, 1)$, $v_2 = (1, 1, 0)$, $v_3 = (0, 0, -1)$.

Column 1 — image of $u_1 = (1, 1)$:

$$f(1, 1) = (1 + 1,\ 2 \cdot 1,\ -3 \cdot 1) = (2, 2, -3).$$

Write $(2, 2, -3) = \alpha_1(2, 0, 1) + \alpha_2(1, 1, 0) + \alpha_3(0, 0, -1)$:

$$\begin{cases} 2\alpha_1 + \alpha_2 = 2 \\ \alpha_2 = 2 \\ \alpha_1 - \alpha_3 = -3 \end{cases} \Longrightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 2 \\ \alpha_3 = 3 \end{cases}.$$

Column 2 — image of $u_2 = (2, 0)$:

$$f(2, 0) = (2 + 0,\ 2 \cdot 2,\ -3 \cdot 0) = (2, 4, 0).$$

Write $(2, 4, 0) = \alpha'_1(2, 0, 1) + \alpha'_2(1, 1, 0) + \alpha'_3(0, 0, -1)$:

$$\begin{cases} 2\alpha'_1 + \alpha'_2 = 2 \\ \alpha'_2 = 4 \\ \alpha'_1 - \alpha'_3 = 0 \end{cases} \Longrightarrow \begin{cases} \alpha'_1 = -1 \\ \alpha'_2 = 4 \\ \alpha'_3 = -1 \end{cases}.$$

So

$$M = \begin{pmatrix} 0 & -1 \\ 2 & 4 \\ 3 & -1 \end{pmatrix}.$$

Step 3 — Compute $A = Q^{-1}MP$

Recall from the previous session:

$$P = \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix}.$$

First compute MP:

$$MP = \begin{pmatrix} 0 & -1 \\ 2 & 4 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0\cdot 4 + (-1)(-1) & 0\cdot 1 + (-1)(-1) \\ 2\cdot 4 + 4\cdot(-1) & 2\cdot 1 + 4\cdot(-1) \\ 3\cdot 4 + (-1)(-1) & 3\cdot 1 + (-1)(-1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & -2 \\ 13 & 4 \end{pmatrix}.$$

Then compute $A = Q^{-1}(MP)$:

$$A = \begin{pmatrix} 5 & 2 & 1 \\ -3 & -1 & -1 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 4 & -2 \\ 13 & 4 \end{pmatrix}.$$

So

$$A = \begin{pmatrix} 26 & 5 \\ -20 & -5 \\ 19 & 4 \end{pmatrix}.$$

This is the matrix of f with respect to the bases $(\mathcal{B}'_E, \mathcal{B}'_F)$.

Second Method — Direct Computation of A

express $f(u'_k)$ directly in $\mathcal{B}'_F$ The columns of A are the coordinates of $f(u'_1)$ and $f(u'_2)$ expressed in $\mathcal{B}'_F = \{v'_1, v'_2, v'_3\}$, where $v'_1 = (1,1,1)$, $v'_2 = (1,3,0)$, $v'_3 = (0,2,-2)$.

Column 1 — image of $u'_1 = (2,4)$:

$$f(2,4) = (2+4,\ 2\cdot 2,\ -3\cdot 4) = (6,4,-12).$$

Write $(6,4,-12) = \beta_1(1,1,1) + \beta_2(1,3,0) + \beta_3(0,2,-2)$:

$$\begin{cases} \beta_1 + \beta_2 = 6 \\ \beta_1 + 3\beta_2 + 2\beta_3 = 4 \\ \beta_1 \qquad - 2\beta_3 = -12 \end{cases} \Longrightarrow \begin{cases} \beta_1 = 26 \\ \beta_2 = -5 \\ \beta_3 = 19 \end{cases}.$$

Column 2 — image of $u'_2 = (-1,1)$:

$$f(-1,1) = (-1+1,\ 2\cdot(-1),\ -3\cdot 1) = (0,-2,-3).$$

Write $(0,-2,-3) = \beta'_1(1,1,1) + \beta'_2(1,3,0) + \beta'_3(0,2,-2)$:

$$\begin{cases} \beta'_1 + \beta'_2 = 0 \\ \beta'_1 + 3\beta'_2 + 2\beta'_3 = -2 \\ \beta'_1 \qquad - 2\beta'_3 = -3 \end{cases} \Longrightarrow \begin{cases} \beta'_1 = 5 \\ \beta'_2 = -5 \\ \beta'_3 = 4 \end{cases}.$$

So

$$A = \begin{pmatrix} 26 & 5 \\ -20 & -5 \\ 19 & 4 \end{pmatrix}.$$

Both methods give the same result. ✓

$A = Q^{-1}MP$ (matrix of f associated with the new bases)

Interpretation

- P is the transition matrix from $\mathcal{B}_E$ to $\mathcal{B}'_E$ (columns = coordinates of u'_1, u'_2 in $\mathcal{B}_E$).
- Q is the transition matrix from $\mathcal{B}_F$ to $\mathcal{B}'_F$ (columns = coordinates of v'_1, v'_2, v'_3 in $\mathcal{B}_F$).
- M is the matrix of f in the original bases $(\mathcal{B}_E, \mathcal{B}_F)$.
- $A = Q^{-1}MP$ is the matrix of f in the new bases $(\mathcal{B}'_E, \mathcal{B}'_F)$.

5.11 Tutorial 5 – Determinants and Linear Systems

Exercise 5.11.1. Calculate the determinants

$$1)\begin{vmatrix} 3 & 4 \\ -1 & 5 \end{vmatrix}, 2)\begin{vmatrix} 1 & 2 & 3 \\ -1 & 0 & 5 \\ 2 & 4 & 1 \end{vmatrix}, 3)\begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 0 & 1 & 3 \\ 2 & 4 & 5 & 1 \\ 3 & 2 & 1 & -4 \end{vmatrix}, 4)\begin{vmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 1 & 3 \\ 2 & 1 & 0 & 6 \\ 1 & 1 & 1 & 7 \end{vmatrix} 5)\begin{vmatrix} 2 & 2 & -1 & 7 \\ 4 & 3 & -1 & 11 \\ 0 & -1 & 2 & -4 \\ 3 & 3 & -2 & 11 \end{vmatrix}$$

Exercise 5.11.2. Let $A = \begin{pmatrix} 3 & 4 \\ -1 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 5 & -7 & -3 \end{pmatrix}, C = \begin{pmatrix} 2 & -3 & 3 & -4 \\ 2 & 0 & 1 & -1 \\ 1 & 5 & 0 & -1 \end{pmatrix}$

1) Determine the Rank of A, B and C

2) Find $|A|$, adj(A) and A^{-1}.3) Find $|B|$, adj(B) and B^{-1}.

Exercise 5.11.3. Let $A = \begin{pmatrix} 13 & -8 & -12 \\ 12 & -7 & -12 \\ 6 & -4 & -5 \end{pmatrix}$

1) Determine the Rank of A.

2) Show that A is invertible and then calculate A^{-1}

3) Deduce from of the A^n.

Exercise 5.11.4. f is linear map: $\mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x + y - z, x - 2y + 3z)$

Let the following standards basics $\{e_1, e_2, e_3\}$ and $\{\acute{e}_1, \acute{e}_2\}$ of $\mathbb{R}^3$and $\mathbb{R}^2$ respectively

Find the matrix M associated the linear map f in Previous basics

Find Rank of M.

Exercise 5.11.5. f is linear map: $\mathbb{R}^3 \to \mathbb{R}^2$, $f(x, y, z) = (x + y - z, x - 2y + 3z)$

Let $v_1 = (1, 1, 0), v_2 = (1, 0, 1), v_3 = (0, 1, 1)$ and $\Phi_1 = (1, 0), \Phi_2 = (1, 1)$.

New starting basis $B = (v_1, v_2, v_3)$,new arrival basis $\acute{B} = (\acute{v}_1, \acute{v}_2)$

- Find the matrix M associated the linear map f in B and $\acute{B}$.

Exercise 5.11.6. f is linear map: $\mathbb{R}^4 \to \mathbb{R}^3$, A associated f,

$A = \begin{pmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 2 & 1 \\ 1 & -2 & 5 & -11 \end{pmatrix}$

- Give the $\ker(f)$

- Find the $\dim(f)$ and the $rank(f)$.

Homework:4),5)+exercise 3+ exercise 6

Chapter 6

Réduction des matrices

E denotes a finite-dimensional vector space over $\mathbb{K}$ of dimension n, and f an endomorphism of E.

Definition 6.0.1

et A be a square matrix of order n. A **non-zero vector v** is called an **eigenvector** of A if

$$A\mathbf{v} = \lambda\,\mathbf{v}$$

for some scalar λ. The scalar λ is then called an **eigenvalue** of A.

Theorem 6.0.1

The eigenvalues of A are the roots of $p_A(\lambda) = 0$; eigenvectors are the non-trivial solutions of $(A - \lambda I)\mathbf{v} = 0$.

Remark 6.0.1. Characteristic Equation

$$A\mathbf{v} = \lambda\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0} \implies \det(A - \lambda I) = 0.$$

The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Example 1

Given:

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}.$$

Find the eigenvalues and eigenvectors of A.

Step 1 — Characteristic equation.

$$A - \lambda I_2 = \begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix}.$$

$$\det(A - \lambda I_2) = (3-\lambda)(-\lambda) - (2)(-1) = -3\lambda + \lambda^2 + 2 = \lambda^2 - 3\lambda + 2 = 0.$$

Discriminant: $\Delta = 9 - 8 = 1$.

$$\lambda_1 = \frac{3-1}{2} = 1, \qquad \lambda_2 = \frac{3+1}{2} = 2.$$

The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$.

Step 2 — Eigenvector $\mathbf{v}_1$ associated with $\lambda_1 = 1$.

Let $\mathbf{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$. We solve $(A - I_2)\mathbf{v}_1 = \mathbf{0}$:

$$\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} 2x + 2y = 0 \\ -x - y = 0 \end{cases} \implies x = -y.$$

$$\mathbf{v}_1 = (-y,\, y) = y(-1,\, 1), \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Step 3 — Eigenvector $\mathbf{v}_2$ associated with $\lambda_2 = 2$.

Let $\mathbf{v}_2 = \begin{pmatrix} x' \\ y' \end{pmatrix}$. We solve $(A - 2I_2)\mathbf{v}_2 = \mathbf{0}$:

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x' + 2y' = 0 \implies x' = -2y'.$$

$$\mathbf{v}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Example 2

Given:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}.$$

Find: (i) the characteristic equation, (ii) the eigenvalues, (iii) a basis for each

eigenspace.

(i) Characteristic equation.

$$A - \lambda I_3 = \begin{pmatrix} 2-\lambda & 0 & 1 \\ -1 & 4-\lambda & -1 \\ -1 & 2 & -\lambda \end{pmatrix}.$$

Expanding along the first row:

$$\begin{aligned} \det(A - \lambda I_3) &= (2-\lambda)\det\begin{pmatrix} 4-\lambda & -1 \\ 2 & -\lambda \end{pmatrix} + 1 \cdot \det\begin{pmatrix} -1 & 4-\lambda \\ -1 & 2 \end{pmatrix} \\ &= (2-\lambda)\Big[\lambda^2 - 4\lambda + 2\Big] + \Big[-2 + 4 - \lambda\Big] \\ &= (2-\lambda)\Big[\lambda^2 - 4\lambda + 2\Big] + \Big[2 - \lambda\Big] \\ &= (2-\lambda)\Big[\lambda^2 - 4\lambda + 2 + 1\Big] \\ &= (2-\lambda)(\lambda^2 - 4\lambda + 3) \\ &= -(\,\lambda - 2)(\lambda - 1)(\lambda - 3) = 0. \end{aligned}$$

(ii) Eigenvalues:

$$\lambda_1 = 1, \qquad \lambda_2 = 2, \qquad \lambda_3 = 3.$$

(iii) Eigenvectors.

$\underline{\lambda_1 = 1}$: solve $(A - I_3)\mathbf{v} = \mathbf{0}$, with $\mathbf{v} = (x, y, z)^\top$:

$$\begin{pmatrix} 1 & 0 & 1 \\ -1 & 3 & -1 \\ -1 & 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies \begin{cases} x + z = 0 \\ -x + 3y - z = 0 \\ -x + 2y - z = 0 \end{cases} \implies \begin{cases} x = -z \\ y = 0 \end{cases}.$$

$$\mathbf{v}_1 = (-z,\, 0,\, z) = z(-1, 0, 1), \qquad \mathbf{v}_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$\underline{\lambda_2 = 2}$: the eigenvector is $\mathbf{v}_2 = (2, 1, 0)^\top$.

$\underline{\lambda_3 = 3}$: the eigenvector is $\mathbf{v}_3 = (1, 2, 1)^\top$.

Diagonalization

Definition 6.0.2

square matrix A is called **diagonalizable** if there exists an invertible matrix P such that

$$D = P^{-1}AP$$

is a **diagonal matrix**.

Remark 6.0.2. Form the matrix P whose columns are the eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ (linearly independent). Then $D = P^{-1}AP$ is diagonal, with the corresponding eigenvalues on the diagonal:

$$D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n).$$

D and A are **similar matrices**.

Diagonalization of Example 1

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = 2, \quad \mathbf{v}_1 = (-1, 1)^\top,\ \mathbf{v}_2 = (-2, 1)^\top.$$

$$P = \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}, \qquad D = P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Remark: $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis of $\mathbb{R}^2$. The matrix P is the transition matrix from the standard basis to the new basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

Diagonalization of Example 2

$$A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 4 & -1 \\ -1 & 2 & 0 \end{pmatrix}, \quad \lambda_1 = 1,\ \lambda_2 = 2,\ \lambda_3 = 3.$$

$$\mathbf{v}_1 = (-1, 0, 1)^\top, \quad \mathbf{v}_2 = (2, 1, 0)^\top, \quad \mathbf{v}_3 = (1, 2, 1)^\top.$$

$$P = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}, \qquad D = P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Computing P^{-1} — Two small examples

Ex. 01: $\det(P) = 1, \quad P^{-1} = \begin{pmatrix} 1 & 2 \\ -2 & -2 \end{pmatrix}.$

Ex. 02: $\det(P) = 2,$

$$P^{-1} = \frac{1}{\det(P)}\, adj(P) = \frac{1}{2}\begin{pmatrix} 1 & -2 & 3 \\ 4 & -4 & 4 \\ -3 & 6 & -3 \end{pmatrix}.$$

Similar Matrices

Definition 6.0.3

wo matrices A and B are called **similar** if there exists an invertible matrix M such that

$$B = M^{-1}AM.$$

Proposition 6.0.1 Invariants of Similar Matrices

If A and B are similar, then:

$$\operatorname{tr}(A) = \operatorname{tr}(B), \qquad \det(A) = \det(B), \qquad rank(A) = rank(B).$$

ExampleMatrix B with a repeated eigenvalue

$$B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Characteristic equation:

$$(1-\lambda)^2(4-\lambda) = 0 \implies \lambda = 1 \text{ (multiplicity 2)}, \quad \lambda = 4.$$

Eigenvectors for $\lambda_1 = 1$:

$$\mathbf{v}_1 = (-1, 1, 0)^\top, \qquad \mathbf{v}_2 = (-1, 0, 1)^\top.$$

Eigenvector for $\lambda_3 = 4$:

$$\mathbf{v}_3 = (1, 1, 1)^\top.$$

Example $A = \begin{pmatrix} 1 & -2 \\ 4 & -8 \end{pmatrix}$, $A - \lambda I = \begin{pmatrix} 1-\lambda & -2 \\ 4 & -8-\lambda \end{pmatrix}$ $\det(A - \lambda I) = p_A(\lambda) = \lambda^2 + 7\lambda = \lambda(\lambda + 7)$. Eigenvalues $\lambda_1 = 0$ (eigenvectors $\propto (2, 1)^T$) and $\lambda_2 = -7$ (eigenvectors $\propto (1, 4)^T$).

6.1 Properties of Eigenvectors and Eigenvalues

Theorem 6.1.1

For $A \in M_n(\mathbb{K})$: λ is an eigenvalue iff $A - \lambda I$ is not invertible iff $\det(A - \lambda I) = 0$. In particular, 0 is an eigenvalue iff A is singular.

Theorem 6.1.2 Eigenspaces associated with distinct eigenvalues

1. If $\lambda_1 \neq \lambda_2$ are eigenvalues, then $E_{\lambda_1} \cap E_{\lambda_2} = \{0\}$.
2. Eigenvectors for pairwise distinct eigenvalues are linearly independent.

Corollary 6.1.1

1. $A \in M_n(\mathbb{K})$ has at most n distinct eigenvalues.
2. For distinct eigenvalues $\lambda_1, \ldots, \lambda_m$: $E_{\lambda_1} + \cdots + E_{\lambda_m} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_m}$.

Theorem 6.1.3 Necessary and sufficient conditions for diagonalisability

$A \in M_n(\mathbb{K})$ is diagonalisable iff:

(1) The characteristic polynomial p_A splits over $\mathbb{K}$;

(2) For each eigenvalue λ_i of algebraic multiplicity k_i: $\dim E_{\lambda_i} = k_i$.

In particular, if A has n pairwise distinct eigenvalues, A is diagonalisable.

Example $A = \begin{pmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, $p_A(x) = (5 - x)^2(1 - x)$.

For $\lambda = 5$ (mult. 2): $E_5 = \langle (1,-1,0)^T, (0,0,1)^T \rangle$, $\dim E_5 = 2$. For $\lambda = 1$ (mult. 1): $E_1 = \langle (1,1,0)^T \rangle$, $\dim E_1 = 1$.

Both match $\Rightarrow$ A is diagonalisable:

$$P = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P^{-1}AP = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(2) $A = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$, $p_A(\lambda) = (\lambda+1)^2$. $E_{-1} = \langle (1,1)^T \rangle$, so $\dim E_{-1} = 1 < 2$: A is **not** diagonalisable.

(3) Find a basis of $\mathbb{R}^3$ making the matrix of $f(x_1,x_2,x_3) = (3x_1 - 2x_2, -2x_1 + 3x_2, 5x_3)$ diagonal.

$M(f)$ equals the matrix A of (1). The diagonalising matrix is $P = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, and the new basis is $u' = e_1 - e_2$, $v' = e_3$, $u'_3 = e_1 + e_2$.

6.2 Tutorial 6 – Eigenvalues, Eigenvectors and Diagonalisation

Exercise 6.2.1. Let 1) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, 2) $A = \begin{pmatrix} -1 & 18 & 0 \\ 1 & 2 & 0 \\ 5 & -3 & -1 \end{pmatrix}$, 3) $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$,

1) Find the characteristic polynomial of A

2) Determine all eigenvalues of A.

3) Find a basis of eigenvectors for each eigenvalue.

4) Is the matrix A diagonalizable? Justify your answer.

Exercise 6.2.2. Find all solutions to the following system of equations.(Method of: Inverse-Cramer-Gauss)

1) $\begin{cases} -2x + 3y = -5 \\ 4x - 2y = 14 \end{cases}$ 2) $\begin{cases} 3x + 4y + z = 1 \\ 2x + 3y = 0 \\ 4x + 3y - z = -2 \end{cases}$,

Exercise 6.2.3. Solve the following system of equations.(Gaussian Elimination Method)

$$1)\begin{cases} 3x+y-4z=-1 \\ x+10z=5 \\ 4x+y+6z=1 \end{cases} \qquad 2)\begin{pmatrix} -1 & -2 & 1 \\ 2 & 3 & 0 \\ 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Exercise 6.2.4. Solve the following system of equation.

$$1)\begin{cases} 2x_1+x_2+2x_3+x_4=6 \\ x_1-x_2+x_3+2x_4=6 \\ 4x_1+3x_2+3x_3-3x_4=-1 \\ 2x_1+2x_2-x_3+x_4=10 \end{cases} \qquad 2)\begin{cases} x+y+z+t=16 \\ 2x+y-z-3t=-21 \\ 3x-2y+4z-t=10 \\ -2x+3y+z-2t=-2 \end{cases}$$

Chapter 7

Systems of linear equations

7.1 Linear Equations

Definition 7.1.1

A *linear equation in n unknowns* $x_1, x_2, \ldots, x_n$ is any equation of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

where the elements $a_1, a_2, \ldots, a_n$ and b belong to $\mathbb{K}$ ($\mathbb{K}$ denotes $\mathbb{R}$ or $\mathbb{C}$) and are called the *coefficients* of the equation.

The equation is said to be *homogeneous* if $b = 0$.

A *solution* of this equation is an n-tuple $(x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$ that satisfies the equation.

Solving the equation means finding the set S of all its solutions.

Two equations are said to be *equivalent* if they have the same solution set.

Example The equation $y = 2x_1$ represents an affine line in the plane. It is a linear equation in two unknowns.

7.2 Systems of Linear Equations

Definition 7.2.1

A *system of linear equations* consisting of m equations in n unknowns is a system of the form

$$(S_1): \begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1, \\ \quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

where the coefficients a_{ij} and b_j are given, and where the x_i are unknowns in $\mathbb{R}$ or $\mathbb{C}$.

* An important special case is that of *homogeneous systems*, for which $b_1 = b_2 = \cdots = b_n = 0$, i.e. whose right-hand side is zero.

7.2.1 Matrix Form

The system (S_1) can be written in the following matrix form:

$$AX = B,$$

where

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Example

$$\begin{cases} 2x + 4y + 5z = 11 \\ x - 13y - 7z = 5 \\ -x - z = 6 \end{cases}$$

The associated matrix is

$$A = \begin{pmatrix} 2 & 4 & 5 \\ 1 & -13 & -7 \\ -1 & 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad B = \begin{pmatrix} 11 \\ 5 \\ 6 \end{pmatrix}.$$

Then,

$$AX = B \iff \begin{pmatrix} 2 & 4 & 5 \\ 1 & -13 & -7 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ 5 \\ 6 \end{pmatrix}.$$

7.2.2 Equivalent Systems

Definition 7.2.2

Two systems are said to be *equivalent* if they have the same solution set.
To solve a given system, one may try to find simpler equivalent systems.

Example

The two systems

$$(S): \begin{cases} 2x + 2y = 10, \\ 3x - y = 6, \end{cases} \quad \text{and} \quad (S'): \begin{cases} x + y = 5, \\ 3x - y = 6, \end{cases}$$

are equivalent.

7.2.3 Echelon Systems

Definition 7.2.3

A linear system is *triangular* if it has as many unknowns as equations, and if all the coefficients of its matrix lying below the diagonal are zero.

Example The system

$$\begin{cases} x + 2y - z = 5, \\ -3y - z = 9, \\ 2z = -1, \end{cases}$$

is triangular. Its diagonal coefficients 1, -3 and 2 are nonzero. We observe that this system has exactly one solution.

Indeed, the last equation determines z uniquely, then the second equation gives y, and the first gives x.

Definition 7.2.4

A linear system is *in echelon form* if its matrix consists of a triangular matrix with nonzero diagonal coefficients, to which (arbitrary) columns have been appended, followed by rows that are identically zero. The size of the triangular subsystem is called the *rank* of the system.
It is in *reduced echelon form* if, in addition, the first nonzero coefficient of each row equals 1 and is the only nonzero entry in its column.

Example

- The system
$$\begin{cases} x + 2y - 3z = 4, \\ y - 4z = 1, \\ 0 = 1, \end{cases}$$
is in echelon form, of rank 2. It has no solution, since the last equation is never satisfied.

- The system
$$\begin{cases} 2x + 3y + 2z - t = 4 \\ -y - 2z = 6 \\ 3t = 1, \end{cases}$$
is in echelon form (but not reduced).

- The system
$$\begin{cases} 2x + 3y + 2z - t = 4, \\ -2z = 4, \\ z + t = 1, \end{cases}$$
is *not* in echelon form (the last row begins with the same variable as the row above it).

- The system
$$\begin{cases} x + 2z = 25 \\ y - 2z = 16 \\ t = 1 \end{cases}$$
is in echelon form and reduced.

7.3 Methods for Solving a Linear System

7.3.1 Substitution Method

The principle is: choose an equation, solve it for one of the unknowns in terms of the others, substitute this expression into all the other equations, and then repeat.

Example

$$(1): \begin{cases} x + y = 7, \\ 3x - y = 3. \end{cases}$$

$$(1) \iff \begin{cases} x = 7 - y, \\ 3x - y = 3, \end{cases} \iff \begin{cases} x = 7 - y, \\ 3(7 - y) - y = 3, \end{cases} \iff \begin{cases} x = 1, \\ y = 6. \end{cases}$$

The system has a unique solution: $S = \{(1, 6)\}$.

7.3.2 Cramer's Method

Definition 7.3.1

A *Cramer system* is a square system (as many unknowns as equations) that has a unique solution ($\det(A) \neq 0$).

Let (S) be a square system, that is, its matrix A is square, with the matrix interpretation $AX = B$. If the matrix A is invertible, this system can be solved by Cramer's method.

Solving the system by Cramer's method gives

$$x_i = \frac{\det(A_i)}{\det(A)}, \qquad i = 1, \ldots, n.$$

Example Solve, using Cramer's method,

$$\begin{cases} 3x - y = 4 \\ 5x + 2y = -2. \end{cases}$$

We have

$$A = \begin{pmatrix} 3 & -1 \\ 5 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

$$\det A = \begin{vmatrix} 3 & -1 \\ 5 & 2 \end{vmatrix} = 6 - (-5) = 11 \neq 0,$$

and since $\det A \neq 0$, this system can be solved by Cramer's method, which gives:

$$A_1 = \begin{pmatrix} 4 & -1 \\ -2 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 3 & 4 \\ 5 & -2 \end{pmatrix}.$$

Hence:

$$x = \frac{\det(A_1)}{\det A} = \frac{\begin{vmatrix} 4 & -1 \\ -2 & 2 \end{vmatrix}}{1} = \frac{8-2}{1} = 6.$$

$$y = \frac{\det(A_2)}{\det A} = \frac{\begin{vmatrix} 3 & 4 \\ 5 & -2 \end{vmatrix}}{1} = \frac{-6+20}{1} = 14.$$

Therefore, $(x, y) = (6, 14)$ is the unique solution of this system.

7.3.3 Solution by the Inverse Matrix Method

Let (S) be a square system, with the matrix interpretation $AX = B$.

If the matrix A is invertible, this system can be solved by the inverse matrix method as follows:

$$AX = B \iff X = A^{-1}B.$$

Example Solve the following system:

$$\begin{cases} 5x + 7y - 3z = 16, \\ 3x - 2y + 4z = -7, \\ x + y - z = 6. \end{cases}$$

We have

$$A = \begin{pmatrix} 5 & 7 & -3 \\ 3 & -2 & 4 \\ 1 & 1 & -1 \end{pmatrix},$$

$\det A = -24 \neq 0$, so A is invertible and its inverse is

$$A^{-1} = \frac{1}{\det A}\mathrm{Adj}(A).$$

Let us compute A^{-1}.

$$\operatorname{Adj}(A) = \begin{pmatrix} +\begin{vmatrix} -2 & 4 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 7 & -3 \\ 1 & -1 \end{vmatrix} & +\begin{vmatrix} 5 & -3 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 5 & 7 \\ 1 & 1 \end{vmatrix} \\ +\begin{vmatrix} 7 & -3 \\ -2 & 4 \end{vmatrix} & -\begin{vmatrix} 5 & -3 \\ 3 & 4 \end{vmatrix} & +\begin{vmatrix} 5 & 7 \\ 3 & -2 \end{vmatrix} \end{pmatrix}^t = \begin{pmatrix} -2 & 7 & 5 \\ 4 & -2 & 2 \\ 22 & -29 & -31 \end{pmatrix}^t,$$

$$\operatorname{Adj}(A) = \begin{pmatrix} -2 & 4 & 22 \\ 7 & -2 & -29 \\ 5 & 2 & -31 \end{pmatrix}, \qquad \text{so} \qquad A^{-1} = \frac{1}{-24}\begin{pmatrix} -2 & 4 & 22 \\ 7 & -2 & -29 \\ 5 & 2 & -31 \end{pmatrix}.$$

Then,

$$X = A^{-1}B \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{-24}\begin{pmatrix} -2 & 4 & 22 \\ 7 & -2 & -29 \\ 5 & 2 & -31 \end{pmatrix}\begin{pmatrix} 16 \\ -7 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ -5 \end{pmatrix},$$

so $(3, -2, -5)$ is the unique solution of this system.

7.4 Solution of a System of Linear Equations

Definition

We call a **linear system** with unknowns $(x_1, x_2, x_3, \ldots, x_n)$ of the form:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = y_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = y_2 \\ \quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = y_n \end{cases}$$

This can be written in matrix form:

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}_{b}$$

7.5 Method 1: Inverse Matrix Method

If A is an invertible matrix, solving $Ax = b$ gives:

$$A^{-1}Ax = A^{-1}b \implies \boxed{x = A^{-1}b}$$

Example

Solve the system:

$$\begin{cases} 2x + y + z = 7 \\ 3x - 2y + 4z = 11 \\ -x + 4y - 2z = 1 \end{cases}$$

In matrix form $Ax = b$:

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 4 \\ -1 & 4 & -2 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad b = \begin{pmatrix} 7 \\ 11 \\ 1 \end{pmatrix}$$

$$\det(A) = -12 \neq 0 \implies A \text{ is invertible.}$$

$$\text{Adj}(A) = \begin{pmatrix} -12 & 2 & 10 \\ 6 & -3 & -9 \\ 6 & -5 & -7 \end{pmatrix}^t$$

$$A^{-1} = \frac{\text{Adj}(A)}{\det(A)} = \frac{-1}{12} \begin{pmatrix} -12 & 6 & 6 \\ 2 & -3 & -5 \\ 10 & -9 & -7 \end{pmatrix}$$

$$x = A^{-1}b \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{-1}{12} \begin{pmatrix} -12 & 6 & 6 \\ 2 & -3 & -5 \\ 10 & -9 & -7 \end{pmatrix} \begin{pmatrix} 7 \\ 11 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\therefore \quad (x, y, z) = (1, 2, 3)$$

7.6 Method 2: Cramer's Method

General 3-Variable System

Given:

$$\begin{cases} ax + by + cz = d \\ a'x + b'y + c'z = d' \\ a''x + b''y + c''z = d'' \end{cases}$$

The determinant of the system is:

$$\Delta = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

If $\Delta \neq 0$, the system has a unique solution:

$$x = \frac{\Delta_x}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} d & b & c \\ d' & b' & c' \\ d'' & b'' & c'' \end{vmatrix}, \quad y = \frac{\Delta_y}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a & d & c \\ a' & d' & c' \\ a'' & d'' & c'' \end{vmatrix}, \quad z = \frac{\Delta_z}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a & b & d \\ a' & b' & d' \\ a'' & b'' & d'' \end{vmatrix}$$

General 2-Variable System

Given:

$$\begin{cases} ax + by = c \\ a'x + b'y = c' \end{cases}$$

$$\Delta = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = ab' - a'b$$

If $\Delta \neq 0$:

$$x = \frac{\Delta_x}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} c & b \\ c' & b' \end{vmatrix}, \qquad y = \frac{\Delta_y}{\Delta} = \frac{1}{\Delta} \begin{vmatrix} a & c \\ a' & c' \end{vmatrix}$$

Example 1 (2 variables)

$$\begin{cases} 2x + 3y = 5 \\ -x + 4y = -8 \end{cases}$$

$$\Delta = 11, \quad x = \frac{\Delta_x}{\Delta} = \frac{\begin{vmatrix} 5 & 3 \\ -8 & 4 \end{vmatrix}}{11} = 4, \quad y = \frac{\Delta_y}{\Delta} = \frac{\begin{vmatrix} 2 & 5 \\ -1 & -8 \end{vmatrix}}{11} = \frac{-16+5}{11} = -1$$

$$\therefore \quad (x, y) = (4, -1)$$

Example 2 (3 variables)

$$\begin{cases} 2x + y + z = 7 \\ 3x - 2y + 4z = 11 \\ -x + 4y - 2z = 1 \end{cases}$$

$$\Delta = \begin{vmatrix} 2 & 1 & 1 \\ 3 & -2 & 4 \\ -1 & 4 & -2 \end{vmatrix} = -12 \neq 0$$

$$x = \frac{\Delta_x}{\Delta} = \frac{1}{-12} \begin{vmatrix} 7 & 1 & 1 \\ 11 & -2 & 4 \\ 1 & 4 & -2 \end{vmatrix} = \frac{-12}{-12} = 1$$

$$y = \frac{\Delta_y}{\Delta}, \quad \Delta_y = \begin{vmatrix} 2 & 7 & 1 \\ 3 & 11 & 4 \\ -1 & 1 & -2 \end{vmatrix} = -24, \quad y = \frac{-24}{-12} = 2$$

$$z = \frac{\Delta_z}{\Delta}, \quad \Delta_z = \begin{vmatrix} 2 & 1 & 7 \\ 3 & -2 & 11 \\ -1 & 4 & 1 \end{vmatrix} = -36, \quad z = \frac{-36}{-12} = 3$$

$$\therefore \quad (x, y, z) = (1, 2, 3)$$

7.7 Method 3: Gauss (Row Elimination) Method

The idea is to reduce the augmented matrix to **row-echelon form**.

Example 1

$$\begin{cases} 2x + y + z = 7 \\ 3x - 2y + 4z = 11 \\ -x + 4y - 2z = 1 \end{cases}$$

Augmented matrix and row operations:

$$\begin{pmatrix} 2 & 1 & 1 & 7 \\ 3 & -2 & 4 & 11 \\ -1 & 4 & -2 & 1 \end{pmatrix} \xrightarrow{R_2 \leftarrow 3R_1 - 2R_2,\ R_3 \leftarrow R_1 + 2R_3} \begin{pmatrix} 2 & 1 & 1 & 7 \\ 0 & 7 & -5 & -1 \\ 0 & 9 & -3 & 9 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow 9R_2 - 7R_3} \begin{pmatrix} 2 & 1 & 1 & 7 \\ 0 & 7 & -5 & -1 \\ 0 & 0 & -24 & -72 \end{pmatrix}$$

Back substitution:

$$z = \frac{-72}{-24} = 3, \quad y = \frac{-1 + 15}{7} = 2, \quad x = \frac{7 - 2 - 3}{2} = 1$$

$$\therefore \quad (x, y, z) = (1, 2, 3)$$

Example 2 — Infinitely Many Solutions

$$\begin{cases} x + y + z = 0 \\ 2x - 3y + 4z = 0 \\ 4x - 11y + 10z = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & -3 & 4 & 0 \\ 4 & -11 & 10 & 0 \end{pmatrix} \xrightarrow{R_2 \leftarrow 2R_1 - R_2,\ R_3 \leftarrow 4R_1 - R_3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 15 & -6 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow \frac{1}{5}R_3} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 5 & -2 & 0 \\ 0 & 5 & -2 & 0 \end{pmatrix}$$

This gives $5y = 2z$. Setting $z = t$:

$$(x, y, z) = \left(-\tfrac{7}{5}t,\ \tfrac{2}{5}t,\ t\right), \quad t \in \mathbb{R}$$

7.8 Series 3 Solutions — Diagonalization

Exercise 1

(1) $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 5$

Eigenvectors: $v_1 = (-2, 1)$, $v_2 = (1, 1)$

$$P^{-1}AP = D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad P = \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix}$$

(2) $A = \begin{pmatrix} -1 & 13 & 0 \\ 1 & 2 & 0 \\ 5 & -3 & -1 \end{pmatrix}$

Eigenvalues: $\lambda_1 = -4$, $\lambda_2 = -1$, $\lambda_3 = 5$

Eigenvectors: $v_1 = (-6, 1, 11)$, $v_2 = (0, 0, 1)$, $v_3 = (3, 1, 2)$

v_1, v_2, v_3 are linearly independent, so:

$$P^{-1}AP = D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad P = \begin{pmatrix} -6 & 0 & 3 \\ 1 & 0 & 1 \\ 11 & 1 & 2 \end{pmatrix}$$

(3) $A = \begin{pmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$

$\lambda = 4$ is a repeated eigenvalue (algebraic multiplicity 3).

Only one eigenvector: $v_1 = (1, 0, 0)$.

In this case, A cannot be diagonalised.

7.8.1 Series 3 — Exercise 01: System Solutions

System (I): 2 variables

$$\begin{cases} -2x + 3y = -5 \\ 4x - 2y = 14 \end{cases} \implies (x, y) = (4, 1)$$

System (II): 3 variables

$$\begin{cases} 3x + 4y + z = 1 \\ 2x + 3y = 0 \\ 4x + 3y - z = -2 \end{cases}$$

Gauss method:

$$\begin{pmatrix} 3 & 4 & 1 & 1 \\ 2 & 3 & 0 & 0 \\ 4 & 3 & -1 & -2 \end{pmatrix} \xrightarrow{R_2 \leftarrow 2R_1 - 3R_2,\ R_3 \leftarrow 4R_1 - 3R_3} \begin{pmatrix} 3 & 4 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 7 & 7 & 10 \end{pmatrix} \xrightarrow{R_3 \leftarrow 7R_2 + R_3} \begin{pmatrix} 3 & 4 & 1 & 1 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 21 & 24 \end{pmatrix}$$

Back substitution:

$$z = \frac{24}{21} = \frac{8}{7}, \qquad y = \frac{16/7 - 2}{-1} \cdot (-1) = \frac{-2}{7}, \qquad x = \frac{1 - 4(-2/7) - 8/7}{3} = \frac{-3}{7}$$

Cramer's method:

$$\Delta = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 4 & 3 & -1 \end{vmatrix} = -7$$

$$\Delta_x = \begin{vmatrix} 1 & 4 & 1 \\ 0 & 3 & 0 \\ -2 & 3 & -1 \end{vmatrix} = 3 \implies x = \frac{3}{-7} = \frac{-3}{7}$$

$$\Delta_y = \begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 0 \\ 4 & -2 & -1 \end{vmatrix} = -2 \implies y = \frac{-2}{-7} = \frac{2}{7}$$

$$\Delta_z = \begin{vmatrix} 3 & 4 & 1 \\ 2 & 3 & 0 \\ 4 & 3 & -2 \end{vmatrix} = -8 \implies z = \frac{-8}{-7} = \frac{8}{7}$$

Inverse method: $Ax = b \iff x = A^{-1}b$

7.8.2 Series 3 — Exercise 02

System (1): No Solution

$$\begin{cases} 3x + y - 4z = -1 \\ x + 10z = 5 \\ 4x + y + 6z = 1 \end{cases}$$

Augmented matrix:

$$\begin{pmatrix} 3 & 1 & -4 & -1 \\ 1 & 0 & 10 & 5 \\ 4 & 1 & 6 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 0 & 10 & 5 \\ 3 & 1 & -4 & -1 \\ 4 & 1 & 6 & 1 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow 3R_1 - R_2,\ R_3 \leftarrow 4R_1 - R_3} \begin{pmatrix} 1 & 0 & 10 & 5 \\ 0 & -1 & 34 & -16 \\ 0 & -1 & 34 & -19 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{pmatrix} 1 & 0 & 10 & 5 \\ 0 & 1 & -34 & -16 \\ 0 & 0 & 0 & -3 \end{pmatrix}$$

The reduced system gives $0 = -3$, which is a contradiction.

Hence the system has no solution.

System (2): Infinitely Many Solutions

$$\begin{cases} -x + 2y - z = -1 \\ 2x + 3y = 2 \\ y - 2z = 0 \end{cases}$$

$$\begin{pmatrix} -1 & 2 & -1 & -1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix} \xrightarrow{-R_1} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 2 & 3 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix} \xrightarrow{R_3 \leftarrow R_2 + R_3} \begin{pmatrix} 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Back substitution: let $z = a \in \mathbb{R}$:

$$y = 2z = 2a, \qquad x = 1 - 2y + z = 1 - 4a + a = 1 - 3a$$

Hence the system has infinitely many solutions:

$$(x, y, z) = (1 - 3a,\ 2a,\ a), \quad a \in \mathbb{R}$$

7.8.3 Series 3 — Exercise 03

$$(x, y, z, t) = (1, 3, 5, 7)$$

4-variable System

$$\begin{cases} 2x_1 + x_2 + x_3 + 2x_4 = 6 \\ x_1 - x_2 + x_3 + 2x_4 = 6 \\ 4x_1 + 3x_2 + 3x_3 - 3x_4 = -1 \\ 2x_1 + 2x_2 - x_3 + x_4 = 10 \end{cases}$$

Augmented matrix and row reductions:

$$\begin{pmatrix} 2 & 1 & 1 & 2 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow 2R_1 - R_2,\ R_3 \leftarrow 4R_1 - R_3,\ R_4 \leftarrow 2R_1 - R_4} \begin{pmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & -3 & 0 & 3 & 6 \\ 0 & -7 & 1 & 11 & 25 \\ 0 & -4 & 3 & 3 & 2 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow -7R_2 + 3R_3,\ R_4 \leftarrow -4R_2 + 3R_4} \begin{pmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & -3 & 0 & 3 & 6 \\ 0 & 0 & -3 & -19 & -33 \\ 0 & 0 & -9 & 3 & -18 \end{pmatrix}$$

$$\xrightarrow{R_3 \leftarrow \frac{1}{3}R_3,\ R_4 \leftarrow \frac{1}{3}R_4} \begin{pmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & -3 & 0 & 3 & 6 \\ 0 & 0 & -1 & 4 & 11 \\ 0 & 0 & -3 & 1 & 6 \end{pmatrix} \xrightarrow{R_4 \leftarrow 3R_3 + R_4} \begin{pmatrix} 1 & -1 & 1 & 2 & 6 \\ 0 & -3 & 0 & 3 & 6 \\ 0 & 0 & 1 & 4 & 11 \\ 0 & 0 & 0 & 13 & 39 \end{pmatrix}$$

Back substitution:

$$x_4 = \frac{39}{13} = 3, \quad x_3 = 11-4(3) = -1, \quad x_2 = \frac{6-9}{-3} = 1, \quad x_1 = 6+x_2-x_3-2x_4 = 6+1+1-6 = 2$$

$$\therefore \quad (x_1, x_2, x_3, x_4) = (2, 1, -1, 3)$$

7.9 Solved Exercises

Example 7.9.1. Solve the following system in four different ways (by substitution, by Gaussian elimination, by inverting the coefficient matrix, and by Cramer's formulas):

$$(S) : \begin{cases} 2x + y = 1, \\ 3x + 7y = -2. \end{cases}$$

1) By substitution.

The first equation can also be written $y = 1 - 2x$. We now substitute y into the second equation:

$$3x + 7(1 - 2x) = -2 \iff -11x = -9 \iff x = \frac{9}{11}.$$

We deduce y:

$$y = 1 - 2\left(\frac{9}{11}\right) = -\frac{7}{11}.$$

The solution of this system is therefore $\left\{\frac{9}{11}, -\frac{7}{11}\right\}$.

Do not forget to check that your solution works!

2) By the Gaussian pivot method.

$$\begin{matrix} R_1 \\ R_2 \end{matrix} \begin{cases} 2x + y = 1, \\ 3x + 7y = -2. \end{cases} \iff {\scriptstyle 2R_2 - 3R_1 \to R_2} \begin{cases} 2x + y = 1, \\ 11y = -7. \end{cases}$$

We obtain a triangular system, from which we deduce $y = -\frac{7}{11}$.

And from the first row: $x = \frac{9}{11}$.

3) By the inverse matrix.

In matrix form, the system is written: $AX = B$, with

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 7 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

We find the solution of the system by inverting the matrix: $X = A^{-1}B$.

$$\det A = \begin{vmatrix} 2 & 1 \\ 3 & 7 \end{vmatrix} = 14 - 3 = 11 \neq 0.$$

The inverse of A is therefore

$$A^{-1} = \frac{1}{\det A}\, {}^t\mathrm{com}\, A = \frac{1}{11} \begin{pmatrix} 7 & -3 \\ -1 & 2 \end{pmatrix}^t, \qquad \text{so} \qquad A^{-1} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix}.$$

Then

$$X = A^{-1}B \iff \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 7 & -1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 9 \\ -7 \end{pmatrix}.$$

4) By Cramer's formulas.

$\det A = 11 \neq 0$, so Cramer's formulas can be applied.

Cramer's formulas for a system of two equations are:

$$\begin{cases} x = \dfrac{\det A_x}{\det A}, \\ y = \dfrac{\det A_y}{\det A}, \end{cases} \iff \begin{cases} x = \dfrac{\begin{vmatrix} 1 & 1 \\ -2 & 7 \end{vmatrix}}{\det A} = \dfrac{9}{11}, \\ y = \dfrac{\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}}{\det A} = -\dfrac{7}{11}. \end{cases}$$

Exercise 7.9.1. Let the following system be given:

$$\begin{cases} 3x - 2y + 4z = -7, \\ 5x + 7y - 3z = 16, \\ x + y - z = 6. \end{cases}$$

1. Write the matrix A associated with this system.

2. Compute A^2.

3. Compute $\det A$ by expanding the determinant of the matrix A along the first row.

4. Compute $\det A$ using Sarrus's rule.

5. Determine whether Cramer's method is applicable to solve the given system of equations. If so, solve the system.

Solution.

$$\begin{cases} 3x - 2y + 4z = -7, \\ 5x + 7y - 3z = 16, \\ x + y - z = 6. \end{cases}$$

1)

$$A = \begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix}.$$

2)

$$A^2 = \begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 3 & -16 & 14 \\ 47 & 36 & 2 \\ 7 & -4 & 2 \end{pmatrix}.$$

3)

$$\det A = \begin{vmatrix} 3 & -2 & 4 \\ 5 & 7 & -3 \\ 1 & 1 & -1 \end{vmatrix} = 3 \begin{vmatrix} 7 & -3 \\ 1 & -1 \end{vmatrix} + 2 \begin{vmatrix} 5 & -3 \\ 1 & -1 \end{vmatrix} + 4 \begin{vmatrix} 5 & 7 \\ 1 & 1 \end{vmatrix} = -24.$$

Exam 1

Exercise 7.9.2. Let be linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (3x + 2y - 2z, -x + z, x + y)$

1) Give $Ker(f)$, is f injective

2) Give $\dim \mathrm{Im}(f)$, is f surjective

3) Find the matrix M associated the linear map f in the canonical (standards) basics $B = \{e_1, e_2, e_3\}$

4) Let $\acute{e}_1 = (1, -1, 0)$, $\acute{e}_2 = (0, 1, 0)$, $\acute{e}_1 = (2, -1, 1)$

- Show that $\acute{B} = \{\acute{e}_1, \acute{e}_2, \acute{e}_3\}$ is a basis of $\mathbb{R}^3$

- Find the matrix $\acute{M}$ associated the linear map f in the basics $\acute{B}$.

Exercise 7.9.3. Let the matrix $P = \begin{pmatrix} 3 & 0 & -2 \\ 2 & 1 & -2 \\ 4 & 0 & -3 \end{pmatrix}$

1) Calculate P^2. Show that P is invertible and find P^{-1} (without calculation).

2) Determine the characteristic polynomial of (P).

Deduce that 1 and -1 are eigenvalues of (P).

Determine the eigenvectors of (P).

Exercise 7.9.4. Let be the following system (S_m) of linear equations

$$\begin{cases} mx - y + z = m \\ x - my + z = 1 \quad ..... \quad (S_m), \quad m \in \mathbb{R} \\ 2x + y + z = 1 \end{cases}$$

1) Show that the determinent of cramer for (S_m) is : $\Delta = m(1-m)$

2) Solve the system (S_2) by Cramer method for $m = 2$.

Exam 2

Exercise 7.9.5. exercise 1

Let (S) be the following system:

$$(S) \begin{cases} 2x + y = 5, \\ x - y = 1. \end{cases}$$

1. Write the system (S) in matrix form $AX = B$.
2. Compute each of the following for the matrix A: Trace, transpose, rank, and inverse.
3. Solve the system (S) using Cramer's rule.

Exercise 7.9.6. Let F be a subset of $\mathbb{R}^3$ defined by

$$F = \left\{(x, y, z) \in \mathbb{R}^3 \ ; \ 2x + 3y - z = 0\right\}.$$

1. Show that F is a vector subspace of $\mathbb{R}^3$.

2. Determine a basis of F. What is its dimension?

Exercise 7.9.7. Let the linear map $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$f(x, y, z) = (\, 2x - y + z,\ x + y - z\,).$$

1. Determine $\ker(f)$ and $\dim \operatorname{Im}(f)$.

2. Study the injectivity and surjectivity of f.

3. Find the matrix M associeted f with respect to the standard bases of $\mathbb{R}^3$ and $\mathbb{R}^2$.

4. Let

   $$B = \{B_1 = (1,0,0)\,, B_2 = (1,1,0)\,, B_3 = (1,1,1)\} \quad \text{and} \quad \acute{B} = \left\{\acute{B}_1 = (1,1)\,, \acute{B}_2 = (1,-1)\right\}$$

   (a) Prove that B is a basis for $\mathbb{R}^3$, and $\acute{B}$ is a basis for $\mathbb{R}^2$,

   (b) Determine the matrix $\acute{M}$ associeted f with respect to the bases B and $\acute{B}$.

7.10 Bibliographie

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This handout presents the core topics of Algebra taught in the first year of university studies in Science and Technology programmes (Licence and Engineer). It is designed to provide students with a rigorous yet accessible introduction to the fundamental structures of modern mathematics.

Topics Covered

Chapter 1 Sets, Relations and Applications

Chapter 2 Complex Numbers

Chapter 3 Vector Spaces

Chapter 4 Linear Maps

Chapter 5 Matrices, Associated Matrices, and Determinants

Chapter 6 Réduction des matrices

Chapter 7 Systems of Linear Equations

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