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Chaos and Bifurcations in some Zeraoulia-Sprott mappings

Presented by:

Nassima Habhoub.

In front of the jury:

Mr, El-hafsi Boukhalfa	*MCB*	*Larbi Tebessi University*	*President*
Mr, Diab Zouhair	*MCA*	*Larbi Tebessi University*	*Examiner*
Mr, Elhadj Zeraoulia	*Prof*	*Larbi Tebessi University*	*Supervisor*

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بسم الله الرحمن الرحيم

الاهداء

قال تعالى :(قل إعملوا فسيرى الله عملكم و رسوله و المؤمنون).

إلهي لا يطيب الليل إلا بشكرك ولا يطيب النهار إلا بطاعتك ..
ولا تطيب اللحظات إلا بذكرك .. ولا تطيب الاخرة إلابعفوك..
ولا تطيب الجنة إلا برؤيتك.
شارفت رحلتي التعليمية على الختام بعد تعب ومشقة وها أنا أختم بحث تخرجي بكل همة ونشاط
ما كنت لأفعل هذا لولا فضل الله فالحمد له عند البدء والختام , وجعلنا الله ممن قال فيهم ﷺ (من سلك طريقا يلتمس فيه علما سهل الله له به طريقا إلى الجنة).

أهدي هذا العمل المتواضع.....

إلى من منحاني روح الحياة وأضاءا لي عتمة الدنيا وكانا سببا في وجودي بعد الله ,إلى من غرسا في حب العلم والتعلم , إلى روح والدي رحمه الله وأسكنه الفردوس الأعلى وإلى والدتي متعها الله بالصحة والعافية .

إلى اخوتي واخواتي من كان لهم بالغ الأثر في نجاحي و تذليل كثير من العقبات والصعاب في مسيرتي .

إلى زملائي وأساتذتي في مختلف الاطوار التعليمية كل باسمه ومقامه.



شكر وتقدير

الحمد لله كثيرا حتى يبلغ الحمد منتهاه والصلاة والسلام على أشرف مخلوق أناره الله بنوره واصطفاه.

الشكر لله الذي بفضله تتم الصالحات , ما سلكنا البدايات إلا بتيسيره وما بلغنا النهايات إلا بتوفيقه و إليه ينسب الفضل في إتمام بحثي هذا.

ثم أتوجه بجزيل الشكر إلى كل من:
الأستاذ المشرف على هذه المذكرة "د. زراولية الحاج".
أعضاء لجنة المناقشة الكرام

إلى عائلتي :أمي ,إخوتي و أخواتي وأساتذتي وزملائي.

Abstract

The main goal of this thesis is to investigate different kinds of bifurcations in some 2-D Zeraoulia-Sprott chaotic discrete mappings. It exoplores the normal form of these systems, the various kinds of bifurcations, and the conditions under which they occur. the study also inverstigates chaos and bifurcation phenomena in both the modified Lozi and Logistic systems highlighting the main characteristics of each system.

Keywords: Chaos and bifurcations, 2-D discret map, normal form, fixed points.

ملخص

تهتم هذه الأطروحة بدراسة أنواع المفارقات في نماذج **زراولية- سبروت** الفوضوية في الأنظمة الديناميكية المتقطعة ذات البعد الثاني.حيث تعرضنا إلى الشكل النظامي للأنظمة لوزي وأنواع المفارقات وشروطها، ثم درسنا الفوضى و أنواع التشعبات في كل من نظامي المعدل و نظام معدل من الخريطة اللوجستية مع ابراز خصائص كل من هذين النظامين.

الكلمات المفتاحية :فوضى و مفارقات، نظام متقطع ثنائي الأبعاد، شكل نظامي، نقاط ثابتة.

Resumé

Cette recherche se concentre sur l'analyse des formes de paradoxes dans les systèmes dynamiques discrets de dimension deux à travers les modèles chaotiques Zeraoulia-Sprott. Il examine la forme normale de ces systèmes, les différentes types de bifurcations et des conditions qui y sont associées,l'étude exploré le chaos ainsi que les phénomène de bifurcations dans le modèle de Lozi modifié et dans un système modifié dérivé de l'application logistique, en mettant en évidence les principales caractéristiques de chacun de ces systèmes.

Mots-clés :Chaos et bifurcations, système discret 2-D, forme normale, point fixe .

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General Introduction

Bifurcation theory and Chaos focus on the behavior of nonlinear dynamics, where small changes in parameters can lead to drastic and unpredictable shifts. These systems are highly sensitive to initial conditions, making their long term behavior difficult to predict.In this context, our study explores certain types of bifurcations in some Zeraoulia-Sprott maps,which exhibit rich chaotic characteristics and allow for a detailed analysis of transitions between dynamical states. The foundational work of Edward Lorenz and Henri Poincaré laid the groundwork for understanding these complex phenomena and opened new directions in the study of nonlinear dynamics.

We divide this thesis into three chapters as follows:

Chapter 1: Bifurcations of two-dimensional maps.

Chapter 2: Bifurcations and chaos in a piecewise Zeraoulia-Sprott mapping obtained from the Lozi map.

Chapter 3: Bifurcations and chaos in a 2-D modulated smooth map obtained from the logistic map.

Chapter 1

Bifurcations of two-dimensional maps

In this section, it has been established that, upon the occurrence of a parameter change, the resultant eigenvalues of the Jacobian matrix are of unit modulus. Consequently, the aforementioned methodology may be employed in the investigation of the bifurcation of two-dimensional maps.

1.1 Two-dimontional piecewise smooth maps

Let's define a 2-D piecewise smooth system as follows:

$$g\left(\widehat{x},\widehat{y},\rho\right)=\begin{cases} g_1=\begin{pmatrix} f_1\left(\widehat{x},\widehat{y},\rho\right) \\ f_2\left(\widehat{x},\widehat{y},\rho\right) \end{pmatrix},\ \text{if}\ \ \widehat{x}<S\left(\widehat{y},\rho\right). \\ \\ g_1=\begin{pmatrix} f_1\left(\widehat{x},\widehat{y},\rho\right) \\ f_2\left(\widehat{x},\widehat{y},\rho\right) \end{pmatrix},\ \text{if}\ \ \widehat{x}>S\left(\widehat{y},\rho\right). \end{cases} \tag{1.1}$$

Where ρ is the bifurcation parameter. The smooth curve $\widehat{x}=S\left(\widehat{y},\rho\right)$ create two regions in the phase plane R_L and R_R given by:

$$\begin{cases} R_L=\{(\widehat{x},\widehat{y})\in\mathbb{R}^2,\widehat{x}<S\left(\widehat{y},\rho\right)\ \}. \\ \\ R_R=\{(\widehat{x},\widehat{y})\in\mathbb{R}^2,\widehat{x}>S\left(\widehat{y},\rho\right)\ \}. \end{cases}$$

and the line separating them as:

$$\Sigma=\left\{(\widehat{x},\widehat{y})\in\mathbb{R}^2,\widehat{x}=S\left(\widehat{y},\rho\right)\ \right\}.$$

Some characteristics:

- Despite the continuous nature of the map g, its derivation breaks at the borderline $\widehat{x} = S(\widehat{y}, \rho)$.
- Both g_1 and g_2 are continuous functions with continuous derivatives.
- The one-sided partial derivatives close to the border in each subregion R_L and R_R are finite.
- For a value ρ^* of the parameter ρ, the map (1.1) has one fixed point in R_L and one fixed point in R_R.

The normal form of system: The system (1.1) has the normal form (See [2]) given by:

$$N(x,y) = \begin{cases} \begin{pmatrix} \tau_1 & 1 \\ -\delta_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x < 0 \\ \begin{pmatrix} \tau_2 & 1 \\ -\delta_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x > 0, \end{cases} \tag{1.2}$$

Where μ is a parameter, and $\tau_{1,2}$, $\delta_{1,2}$, are the traces and determinants of the corresponding matrices of the linearized map in the two subregion R_L and R_R given by:

$$\begin{cases} R_L = \{(x,y) \in \mathbb{R}^2, x < 0\}. \\ R_R = \{(x,y) \in \mathbb{R}^2, x > 0\}. \end{cases}$$

The fixed points and their types: Let P_L and P_R be the possible fixed points of the system near the border to the right $(\widehat{x} < S(\widehat{y}, \rho))$ and the left $(\widehat{x} > S(\widehat{y}, \rho))$ of the border respectively. Then in the normal form (1.2) we have:

$$\begin{cases} P_L = \left(\frac{\mu}{1-\tau_L+\delta_L}, \frac{-\delta_L \mu}{1-\tau_L+\delta_L}\right) \in R_L, \\ P_R = \left(\frac{\mu}{1-\tau_R+\delta_R}, \frac{-\delta_L \mu}{1-\tau_R+\delta_R}\right) \in R_R. \end{cases}$$

With eigenvalues $\lambda_{L_{1,2}}$ and $\lambda_{R_{1,2}}$ respectively. See Figure(1.1) and see [2].

1.2 Border collision bifurcations

Because our study of these bifurcations in this dimension is limited to the classification of fixed points under the two conditions $|\delta_L| < 1$ and $|\delta_R| < 1$, the border collision bifurcations can be obtained by different combinations of the values τ_L, τ_R, δ_L and δ_R as they are varied through zero. Consequently, the following are the possible fixed point types for the normal form map in Figure (1.2) (For more informations see [2] and [5]):

(1) For positive determinant:

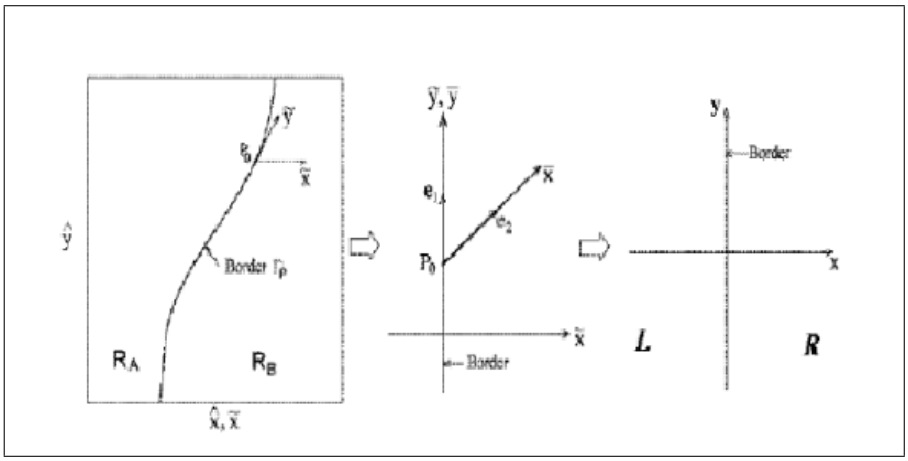


Figure 1.1: The translation of parameters to their normal form from a 2-D piecewise smooth map.

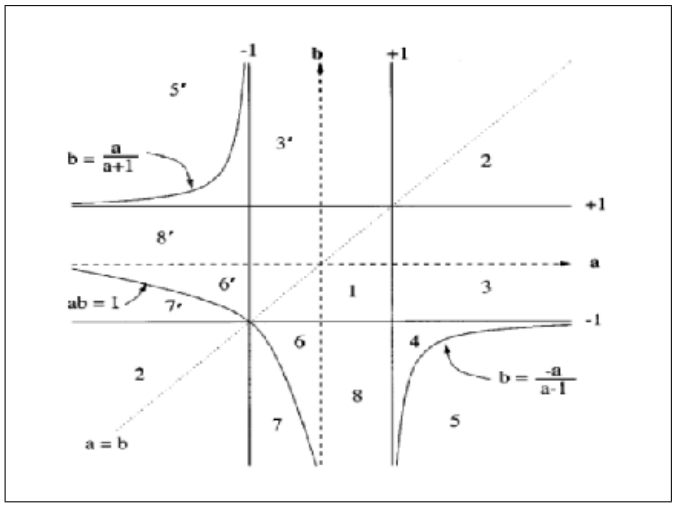


Figure 1.2: The parameter space is divided into areas that exhibit similar qualitative features.

(1.*a*) For $2\sqrt{\delta} < \tau < 1 + \delta$ then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L}, \lambda_{2L} < 1$ and the fixed point is a **regular attractor**.
(1.*b*) For $\tau > 1 + \delta$ then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L} < 1, \lambda_{2L} > 1$ and the fixed point is a **regular saddle.**
(1.*c*) For $-(1 + \delta) < \tau < -2\sqrt{\delta}$ then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L}, \lambda_{2L} < 0$ and the fixed point is a **flip attractor.**
(1.*d*) For $\tau < -(1 + \delta)$ then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L} < 0, \lambda_{2L} < 0$ and the fixed point is a **flip saddle.**
(1.*e*) For $0 < \tau < 2\sqrt{\delta}$ then the Jacobian matrix has two complex eigenvalues $|\lambda_{1L}|, |\lambda_{2L}| < 1$ and the fixed point is a **clockwise spiral.**

(1.*g*) For $-2\sqrt{\delta} < \tau < 0$ then the Jacobian matrix has two complex eigenvalues $|\lambda_{1L}|, |\lambda_{2L}| < 1$ and the fixed point is a **counter-clockwise spiral.**

(2) For negative determinant:

(2.*a*) For $-(1+\delta) < \tau < 1+\delta$ then the Jacobian matrix has two real eigenvalues $-1 < \lambda_{1L} < 0, 0 < \lambda_{2L} < 1$ and the fixed point is a **flip attractor.**

(2.*b*) For $\tau > 1+\delta$ then the Jacobian matrix has two real eigenvalues $\lambda_{1L} > 1, -1 < \lambda_{2L} < 0$ and the fixed point is a **flip saddle.**

(2.*c*) For $\tau < -(1+\delta)$ then the Jacobian matrix has two real eigenvalues $0 < \lambda_{1L} < 1, \lambda_{2L} < -1$ and the fixed point is a **flip saddle.** See Figure (1.3)(see [2]).

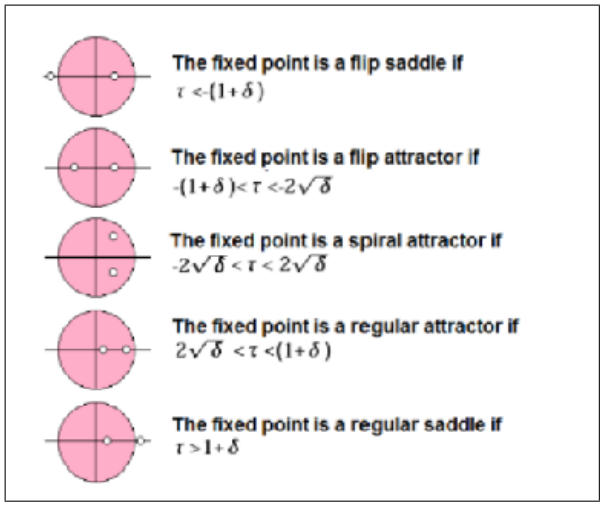


Figure 1.3: The fixed point types of the normal form.

Chapter 2

Bifurcations and chaos in a piecewise Zeraoulia-Sprott mapping

This chapter will cover bifurcations and chaos in a spatial 2-D mapping (a modified Lozi system), also some of its properties characteristics, fixed points are given.

2.1 Existence of fixed points and their stability

The two-dimensional iterative non-reversible map proposed by Lozi is given by:

$$(x, y) \longrightarrow (1 - a|x| + y, bx) \tag{2.1}$$

and its achaotic attractor is shown in Figure (2.1). This type of application is used in secure communications using the concepts of chaos. See [3]. In this section, we will mainly study the modified Lozi map given by:

$$f(x, y) = \left(1 + a\left(|x| - y^2\right) + y, bx\right), a \text{ and } b \in \mathbb{R}.$$

The study in this section is about a new two dimensional discrete quadratic attractor and we describe the map, its fixed points and its bifurcations. We show numerically that the model (2.2) can display two different chaotic attractors, including the classical Lozi type attractor as shown in Figure (2.1) and Figure (2.2).

Comparison with the Lozi system:

a) Both systems are complex.

b) They are both 2-D discrete noninvertible maps.

However, the two models are topologically not equivalent because the Lozi system (2.1) is a piecewise-linear map, but model (2.2) is a nonlinear system. In addition, it can be rigorously

proved that a non-singular homeomorphism that transforms each system to other does not exist. The proof needed some straightforward but tedious algebra which leads to a system of algebraic equations without solutions (See [5]).

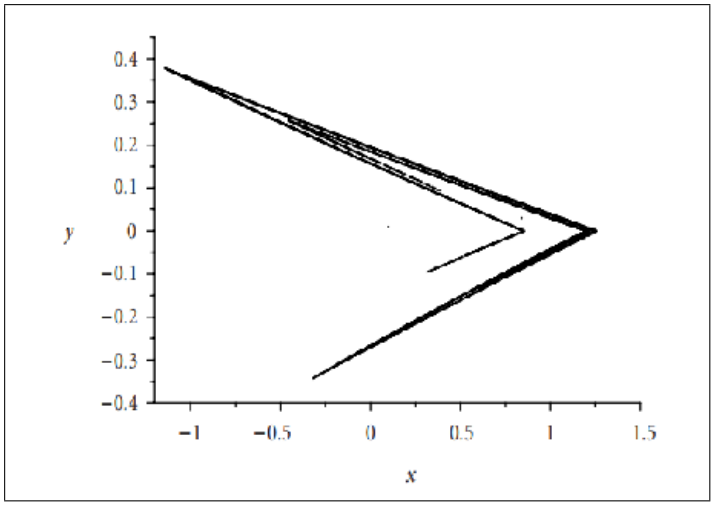


Figure 2.1: System (2.1) gives a Lozi-type chaotic attractor for $a = -1.8, b = 0.3$.

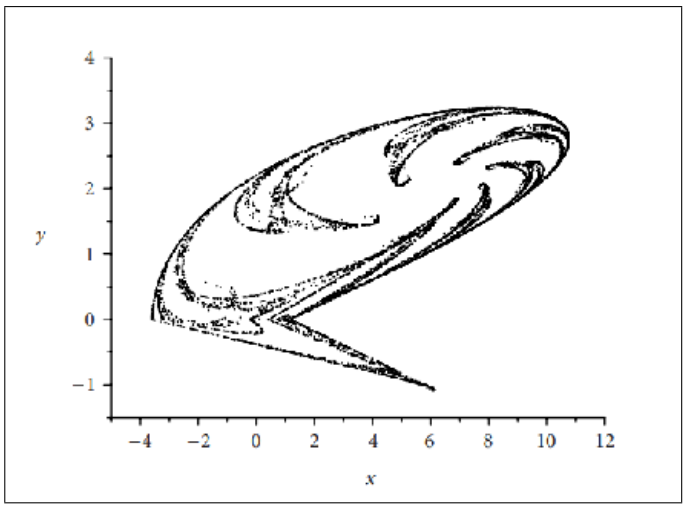


Figure 2.2: The Lozi chaotic attractor after modification obtained for $a = 1.4, b = 0.3$.

The fixed points: Two regions were constructed on the map: D_1 and D_2 given by:

$$\begin{cases} D_1 = \{(x, y) \in \mathbb{R}^2 : x > 0\}. \\ D_2 = \{(x, y) \in \mathbb{R}^2 : x < 0\}. \end{cases}$$

So that the map can be rewritten as follow:

$$f(x,y)=\begin{pmatrix} 1+a\left(x-y^2\right)+y, \text{ if } (x,y)\in D_1 \\ 1-a\left(x+y^2\right)+y, \text{ if } (x,y)\in D_2 \\ bx \end{pmatrix} \tag{2.2}$$

The fixed point are given by:

$$f(x,y)=(x,y)\Leftrightarrow f(x,y)=\begin{pmatrix} 1+a\left(x-y^2\right)+y, \text{if } (x,y)\in D_1 \\ 1-a\left(x+y^2\right)+y, \text{if } (x,y)\in D_2 \\ bx \end{pmatrix}=\begin{pmatrix} x_1 \\ x_2 \\ y \end{pmatrix} \tag{2.3}$$

By substitution for $x\in D_1$; we get

$$\begin{cases} 1+a\left(x-y^2\right)+y=x \\ bx=y \end{cases} \Rightarrow \begin{cases} 1+a\left(x-b^2x^2\right)+bx=x \\ bx=y \end{cases}$$
$$\Leftrightarrow \quad -ab^2x^2+(a+b-1)x+1=0$$

We have $\Delta=(a+b-1)^2+4ab^2$. If $\Delta<0$ then the map (2.2) has not fixed points (See[7]). If $\Delta\geq 0\Leftrightarrow (a+b-1)^2+4ab^2>0$ and $(a+b-1)^2>0\ \Rightarrow 4ab^2>0\Rightarrow a>0$, and $b>0$.

$$\begin{cases} x_1=\frac{-(a+b-1)-\sqrt[2]{(a+b-1)^2+4ab^2}}{-2ab^2} \\ \\ x_2=\frac{-(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{-2ab^2} \end{cases} \Leftrightarrow \begin{cases} x_1=\frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab^2}>0 \\ \\ x_2=\frac{(a+b-1)-\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab^2}<0 \end{cases}$$

So $x_1=\frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab^2}$ and

$$\begin{aligned} p_1(x,y) &= p_1(x_1,bx_1) \\ &= \left(\frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab^2},\frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab}\right) \end{aligned}$$

We have on other hand,

$$\begin{cases} 1-a\left(x+y^2\right)+y=x \\ bx=y \end{cases} \Rightarrow \begin{cases} 1-a\left(x+b^2x^2\right)+bx=x \\ bx=y \end{cases}$$
$$\Leftrightarrow \quad -ab^2x^2+(b-a-1)x+1=0$$

We have $\Delta=(b-a-1)^2+4ab^2$. If $\Delta<0$ then, no solutions exists. If $\Delta\geq 0$ and $(b-a-1)^2>0\ \Rightarrow 4ab^2>0\Rightarrow a,b>0$.

$$\begin{cases} x_1=\frac{-(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{-2ab^2} \\ \\ x_2=\frac{-(b-a-1)+\sqrt[2]{(b-a-1)^2+4ab^2}}{-2ab^2} \end{cases} \Leftrightarrow \begin{cases} x_1=\frac{(b-a-1)+\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab^2}>0 \\ \\ x_2=\frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab^2}<0 \end{cases}$$

Let $x \in D_2$ so $x < 0$. So we find $x_2 = \frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab^2}$ and

$$\begin{aligned} p_2(x,y) &= p_2(x_2, bx_2) \\ &= \left(\frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab^2}, \frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab} \right). \end{aligned}$$

Finally, the piesewise smooth map (2.2) has two simultaneous fixed points defined for $a > 0$ and $b > 0$ as $P_1 = (x_1, bx_1) \in D_1$ and $P_2 = (x_2, bx_2) \in D_2$, i.e,

$$\begin{cases} P_1 = \left(\frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab^2}, \frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{2ab} \right) \\ P_2 = \left(\frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab^2}, \frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{2ab} \right) \end{cases} \tag{2.4}$$

with eigenvalues $\lambda_{1,2}$ and $\omega_{1,2}$ respectively. The Jacobian matrix of the map (2.2) evaluated at a point (x, y) in the region D_1 is given by:

$$J_1(x,y) = \begin{pmatrix} a & -2ay+1 \\ b & 0 \end{pmatrix} \tag{2.5}$$

and at a point (x, y) in the region D_2 the Jacobian matrix is given by:

$$J_2(x,y) = \begin{pmatrix} -a & -2ay+1 \\ b & 0 \end{pmatrix} \tag{2.6}$$

Thus, at P_1 one has:

$$J_1(P_1) = J_1(x_1, bx_1) = \begin{pmatrix} a & \frac{(a+b-1)+\sqrt[2]{(a+b-1)^2+4ab^2}}{b} + 1 \\ b & 0 \end{pmatrix}$$

The eigenvalues of $J_1(P_1)$ are:

$$\begin{cases} \lambda_1 = \frac{a+\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} \\ \lambda_2 = \frac{a-\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} \end{cases} \tag{2.7}$$

with $a, b > 0$ and $a + 2b - 1 > 0 \Rightarrow a + 2b > 1 \Rightarrow a > 1 - 2b > 0 \Rightarrow b < \frac{1}{2}$, and at P_2 one has:

$$J_2(P_2) = \begin{pmatrix} -a & \frac{(b-a-1)-\sqrt[2]{(b-a-1)^2+4ab^2}}{b} + 1 \\ b & 0 \end{pmatrix} \tag{2.8}$$

The eigenvalues of $J_2\left(P_2\right)$ are:

$$\begin{cases} \omega_1 = \frac{-a-\sqrt[2]{a^2+4\left(2b-a-1-\sqrt[2]{(b-a-1)^2+4ab^2}\right)}}{2} \\ \\ \omega_2 = \frac{-a+\sqrt[2]{a^2+4\left(2b-a-1-\sqrt[2]{(b-a-1)^2+4ab^2}\right)}}{2} \end{cases} \tag{2.9}$$

with $a, b > 0$ and $b > \frac{1}{2}$.

The normal form: In the case of two-dimensional piecewise smooth maps, it is possible to choose an appropriate coordinate transformation, so that the choice of axis is independent of the parameter. In so doing, the normal form of map (2.2) is given by:

$$N\left(x,y\right) = \begin{cases} \begin{pmatrix} \tau_1 & 1 \\ -\delta_1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x < 0, \\ \begin{pmatrix} \tau_2 & 1 \\ -\delta_2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x > 0. \end{cases} \tag{2.10}$$

Where μ is a parameter, and $\tau_{1,2}, \delta_{1,2}$ are the traces and determinants of the corresponding matrices of the linearized map in the two subregion D_1 and D_2 evaluated at P_1 and P_2 respectively. By substitution we find:

$$N\left(x,y\right) = \begin{cases} \begin{pmatrix} a & 1 \\ b-2aby & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{if } x < 0 \\ \begin{pmatrix} -a & 1 \\ b-2aby & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{if } x > 0, \end{cases} \tag{2.11}$$

where, $J_{21} = \lim\limits_{x\to 0, y\to 0} \frac{\partial}{\partial x} f_2\left(x,y,0\right) = b = \delta$, so:

$$\begin{cases} j_{11} = \tau_1 = a \\ \\ j_{11} = \tau_2 = -a \\ \\ j_{21} = \delta_1 = \delta_2 = b \end{cases} \tag{2.12}$$

Then, the normal form of map (2.2) is given as follow:

$$N\left(x,y\right) = \begin{cases} \begin{pmatrix} a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x < 0 \\ \begin{pmatrix} -a & 1 \\ b & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x > 0, \end{cases} \tag{2.13}$$

2.2 The types of fixed points of the normal form

(1) For positive determinant:

We have that from (2.15): $\tau_1 = a, \tau_2 = -a, \delta_1 = \delta_2 = b$ and from (2.10):

$$\begin{cases} \lambda_1 = \frac{a+\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} \\ \lambda_2 = \frac{a-\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} \end{cases}$$

and from (2.12):

$$\begin{cases} \omega_1 = \frac{-a-\sqrt[2]{a^2+4\left(2b-a-1-\sqrt[2]{(b-a-1)^2+4ab^2}\right)}}{2} \\ \omega_2 = \frac{-a+\sqrt[2]{a^2+4\left(2b-a-1-\sqrt[2]{(b-a-1)^2+4ab^2}\right)}}{2} \end{cases}$$

All forms of bifurcation are possible if: $\lambda_{1,2}$ existe if $a, b > 0$ and $b < \frac{1}{2}$ that is from (2.10) and $\omega_{1,2}$ existe if $a, b > 0$ and $b > \frac{1}{2}$ that is from (2.12).

(1.a) For $2\sqrt[2]{\delta} < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_1, \lambda_2 < 1$ and the fixed point is a **regular attractor**.

For P_1: Let us consider the following conditions:
1). $2\sqrt[2]{\delta} < \tau < (1+\delta)$.

2). $\lambda_1, \lambda_2 \in \mathbb{R}$ if $2b-1 > a$ and $0 < \lambda_1, \lambda_2 < 1$.
$2\sqrt[2]{\delta_1} < \tau_1 < (1+\delta_1)$:

If $a > b$, then $\tau_1 = a > b$, and $a \geq b+1$, so P_1 is not a **regular attractor**.
If $a < b$, then $\tau_1 = a < b+1$ and $a < 2\sqrt{b}$, so P_1 is a not **regular attractor**.
For P_2 : Let us consider the following conditions:
1). $2\sqrt[2]{\delta} < \tau < (1+\delta)$.

2). $\omega_1, \omega_2 \in \mathbb{R}$ if $2b-1 > a$ and $0 < \omega_1, \omega_2 < 1$.
$2\sqrt[2]{\delta_2} < \tau_2 < (1+\delta_2)$:

$\tau_2 = -a < b+1$ and since $a, b > 0$ then $-a < 2\sqrt[2]{\delta_2}$. So P_2 is not a **regular attractor** over the previous conditions.

(1,b) For $\tau > 1+\delta$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_1 < 1, \lambda_2 > 1$ and the fixed point is a **regular saddle**.

For P_1: Let us consider the following conditions:
3). $\tau > 1+\delta$.
4). $\lambda_{1,2} \in \mathbb{R}$ and $0 < \lambda_1 < 1, \lambda_2 > 1$.
$\tau_1 > 1+\delta_1$:
If $a < b$,then $\tau_1 = a < b+1$,so P_1 is not a **regular saddle**.
If $a > b$ then $\tau_1 = a \geq b+1$ equivalent to $\tau_1 \geq 1+\delta_1$ then P_1 is not a **regular saddle**.
For P_2: Let us consider the following conditions:
3). $\tau > 1+\delta$.
4). $\omega_{1,2} \in \mathbb{R}$ and $0 < \omega_1 < 1, \omega_2 > 1$.
Since $a, b > 0$ and $\tau_2 = -a < 1+b < 1+\delta$, then P_2 is not a **regular saddle**.

(1.c) For $-(1+\delta) < \tau < -2\sqrt[2]{\delta}$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_1, \lambda_2 < 0$ and the fixed point is a **flip attractor**.

For P_1: Let us consider the following conditions:
5). $-(1+\delta) < \tau < -2\sqrt[2]{\delta}$
6). $\lambda_{1,2} \in \mathbb{R}$ and $-1 < \lambda_1, \lambda_2 < 0$.
$-(1+\delta_1) < \tau_1 < -2\sqrt[2]{\delta_1}$.
$\tau_1 = a$
If $a < b$ implies $\tau_1 > -(1+\delta)$ but $a > -2\sqrt{\delta}$ so P_1 is not a **flip attractor**.
If $a > b$ implies $\tau_1 > -(1+\delta)$ but $a > -2\sqrt{\delta}$ but $a > -2\sqrt{\delta}$ so P_1 is not a **flip attractor**.
For P_2: Let us consider the following conditions:
5). $-(1+\delta) < \tau < -2\sqrt[2]{\delta}$.
6). $\omega_{1,2} \in \mathbb{R}$ and $-1 < \omega_1, \omega_2 < 0$.
$\tau_2 = -a < b$ then $-a < b+1$ so $\tau_2 > -(1+\delta)$, but $\tau \geq -2\sqrt[2]{\delta}$. So P_2 is not a **flip attractor**.

(1,d) For $\tau < -(1+\delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_1 < 0, \lambda_2 < -1$ and the fixed point is a **flip saddle**.

For P_1: Let us consider the following conditions:
7). $\tau < -(1+\delta)$.
8). $\lambda_{1,2} \in \mathbb{R}$ and $-1 < \lambda_1 < 0, \lambda_2 < -1$.
$\tau_1 < -(1+\delta_1)$:
If $a > b$ implies that $a > -(1+b)$ so P_1 is a not **flip saddle.**

If $a < b$ implies $\tau_1 \geq -(1+\delta)$ so P_1 is a not **flip saddle.**
For P_2: Let us consider the following conditions:
7). $\tau < -(1+\delta)$.
8). $\omega_{1,2} \in \mathbb{R}$ and $-1 < \omega_1 < 0, \omega_2 < -1$.
$\tau_2 = -a$ we have $a, b > 0$ then $-a < b$, $\tau_2 \geq -(1+\delta)$. So P_2 is not a **flip saddle** under the previous conditions.

(1,e) For $0 < \tau < 2\sqrt[2]{\delta}$, then the Jacobian matrix has two complex eigenvalues $|\lambda_1|, |\lambda_2| < 1$ and the fixed point is a **clockwise spiral**.

For P_1: Let us consider the following conditions:
9). $0 < \tau < 2\sqrt[2]{\delta}$ if $a < \sqrt[2]{\delta}$.
10). $\lambda_{1,2} \in \mathbb{C}$ if $a^2 + 4a + 8b - 4 > 0$. and $|\lambda_1|, |\lambda_2| < 1$.
$|\lambda_1| < 1$ if:

- $2 - a < 0$,$2 - 2a - 2b < 0$ and $-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 < 0$.
- $2 + a < 0$, $2b - 2 < 0$ and $a^2 - 3b^2 + 4ab^2 + 2ab + 6b - 2a - 3 > 0$.

$|\lambda_2| < 1$ if:

- $a - 2 < 0$, $a + b < 0$, $-3a^2 - 3b^2 + 4ab^2 - 6ab - 2a - 2b + 1 > 0$.
- $2 + a < 0$, $2 - 2b < 0$ and $a^2 - 3b^2 + 4ab^2 + 2ab - 2a + 6b - 3 < 0$.

Lemma 2.1 *If (9),(10) holds true, then P_1 is a **clockwise spiral**.*

Proof. 9) $0 < \tau_1 < 2\sqrt[2]{\delta_1}$.
$\tau_1 = a$ if $a > 0$ and $a < \sqrt[2]{\delta}$ then $a < 2\sqrt[2]{\delta}$ and $\tau_2 < 2\sqrt[2]{\delta}$.
10) $\lambda_{1,2} \in \mathbb{C}$ if $a^2 + 4a + 8b - 4 > 0$.
a) $\lambda_{1,2} \in \mathbb{C}$. We have:

$$\lambda_1 = \frac{a + \sqrt{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)}}{2},$$

$$\lambda_2 = \frac{a - \sqrt{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)}}{2}.$$

$$\sqrt{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} < 0.$$

If

$$\begin{aligned} a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right) &< 0. \\ \sqrt[2]{(a+b-1)^2+4ab^2} &< -\frac{a\,(a+4)}{4}-2b+1. \\ (a+b-1)^2+4ab^2 &< \left(-\frac{a\,(a+4)}{4}-2b+1\right)^2 \leq 0. \end{aligned}$$

Then

$$\left(-\frac{a\,(a+4)}{4}-2b+1\right)^2=-\left(\frac{a\,(a+4)}{4}+2b-1\right)^2=i^2\left(\frac{a\,(a+4)}{4}+2b-1\right)^2.$$

and

$$\begin{aligned} \frac{a\,(a+4)}{4}+2b-1 &> 0. \\ \text{then } a^2+4a+8b-4 &> 0. \end{aligned}$$

b) $|\lambda_1|, |\lambda_2|<1$:

$|\lambda_1|<1$ if: $2+a<0$, $2b-2<0$ and $a^2-3b^2+4ab^2+2ab+6b-2a-3>0$.

Then we have:

$$\begin{cases} \lambda_1=\frac{a+\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2}, \\ \lambda_2=\frac{a-\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} \end{cases},$$

$|\lambda_1|<1$ if $-1<\lambda_1<1$:

$\lambda_1<1$ if

$$\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}<2-a$$

If $a<2$ then $2-a<0$ thus we get:

$$\begin{aligned} a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right) &< (2-a)^2. \\ \sqrt[2]{(a+b-1)^2+4ab^2} &< 2-2b-2a. \end{aligned}$$

Then $(a+b-1)^2+4ab^2-(2-2b-2a)^2<0$, i.e.,

$$\begin{aligned} & (a+b-1)^2+4ab^2-(2-2b-2a)^2 \\ = \;& (a+b-1-2+2b+2a)(a+b-1+2-2b-2a)+4ab^2. \\ = \;& (3a+3b-3)(1-a-b)+4ab^2. \\ = \;& 3a-3a^2-3ab+3b-3ab-3b^2-3+3a+3b+4ab^2. \\ = \;& -3a^2-3b^2+4ab^2-6ab+6a+6b-3. \end{aligned}$$

that is,

$$-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 < 0.$$

We have:

$$\lambda_1 = \frac{a + \sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)}}{2} > -1.$$

Then

$$\begin{aligned}
\sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} &> -2 - a. \\
a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right) - (2+a)^2 &> 0. \\
2b - 2 + \sqrt[2]{(a+b-1)^2 + 4ab^2} &> 0. \\
(a+b-1)^2 + 4ab^2 - (2b-2)^2 &> 0.
\end{aligned}$$

Hence,

$$\begin{aligned}
& (a+b-1)^2 + 4ab^2 - (2b-2)^2 \\
= \; & (a+b-1+2b-2)(a+b-1-2b+2) + 4ab^2. \\
= \; & (a+3b-3)(a-b+1) + 4ab^2. \\
= \; & a^2 - ab + a + 3ab - 3b^2 + 3b - 3a + 3b - 3 + 4ab^2. \\
= \; & a^2 - 3b^2 + 4ab^2 + 2ab + 6b - 2a - 3.
\end{aligned}$$

That is,

$$a^2 - 3b^2 + 4ab^2 + 2ab + 6b - 2a - 3 > 0.$$

$|\lambda_2| < 1$ if: $a - 2 < 0$, $a + b < 0$ and $-3a^2 - 3b^2 + 4ab^2 - 6ab - 2a - 2b + 1 > 0$, $2 + a < 0$, $2 - 2b < 0$ and $a^2 - 3b^2 + 4ab^2 + 2ab - 2a + 6b - 3 < 0$.

$|\lambda_2| < 1 : -1 < \lambda_2 < 1 :$

$\lambda_2 < 1$ if

$$\sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} > a - 2.$$

$\lambda_2 \in \mathbb{C}$ if $a-2<0$, that is, $\sqrt[2]{(a+b-1)^2+4ab^2} > -2a-2b$ and $a+b<0$ and $(a+b-1)^2+4ab^2-(2a+2b)^2>0$.

$$\begin{aligned}
&(a+b-1)^2+4ab^2-(2a+2b)^2\\
=\;&(a+b-1+2a+2b)(a+b-1-2a-2b)+4ab^2.\\
=\;&(3a+3b-1)(-a-b-1)+4ab^2.\\
=\;&-3a^2-3ab-3a-3ab-3b^2-3b.+a+b+1+4ab^2.\\
=\;&-3a^2-3b^2+4ab^2-6ab-2a-2b+1.
\end{aligned}$$

Hence,

$$-3a^2-3b^2+4ab^2-6ab-2a-2b+1>0.$$

$\lambda_2 > -1$ if

$$\begin{aligned}
\frac{a-\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)}}{2} &> -1.\\
\sqrt[2]{a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)} &< 2+a.\\
a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)-(2+a)^2 &< 0.
\end{aligned}$$

$$a^2+4\left(a+2b-1+\sqrt[2]{(a+b-1)^2+4ab^2}\right)-(2+a)^2 = 8b-8+4\sqrt[2]{(a+b-1)^2+4ab^2}$$

We find

$$\sqrt[2]{(a+b-1)^2+4ab^2} < 2-2b.$$

$$\begin{aligned}
(a+b-1)^2+4ab^2-(2-2b)^2 &= (a+b-1+2-2b)(a+b-1-2+2b)+4ab^2.\\
&= (a-b+1)(a+3b-3)+4ab^2.\\
&= a^2+3ab-3a-ab-3b^2+3b+a+3b-3+4ab^2.\\
&= a^2-3b^2+4ab^2+2ab-2a+6b-3.
\end{aligned}$$

That is,

$$a^2-3b^2+4ab^2+2ab-2a+6b-3<0.$$

■

$\tau_1 = a$:

If $a>0$ and if $a > \sqrt[2]{b}$ then $a \geq 2\sqrt[2]{\delta}$ and $\tau_1 \geq 2\sqrt[2]{\delta}$,so P_1 is not a **clockwise spiral.**

For P_2: Let us consider the following conditions:
9). $0 < \tau < 2\sqrt[2]{\delta}$.
10). $\omega_{1,2} \in \mathbb{C}$ and $|\omega_1|, |\omega_2| < 1$.
We have: $-a < 0$ and $\tau_2 = -a < 2\sqrt[2]{\delta}$.
P_2 is not a **clockwise spiral** under the previous conditions.

(1,g) For $-2\sqrt[2]{\delta} < \tau < 0$, then the Jacobian matrix has two complex eigenvalues $|\lambda_1|, |\lambda_2| < 1$ and the fixed point is a **counter-clockwise spiral**.

For P_1: Let us consider the following conditions:
11). $-2\sqrt[2]{\delta} < \tau < 0$.
12). $\lambda_{1,2} \in \mathbb{C}$ and $|\lambda_1|, |\lambda_2| < 1$.
$-2\sqrt[2]{\delta_1} < \tau_1 < 0$.
$\tau_1 = a > 0$ and $a > -2\sqrt[2]{\delta}$. So P_1 is not a **counter-clockwise spiral.**

For P_2: Let us consider the following conditions:
11). $-2\sqrt[2]{\delta} < \tau < 0$.
12). $\omega_{1,2} \in \mathbb{C}$ and $|\omega_1|, |\omega_2| < 1$.
$-2\sqrt[2]{\delta} < \tau < 0$.
$\tau_2 = -a$:
If $a > b$ then $-a \leq -2\sqrt[2]{\delta}$ so P_2 is not a **counter-clockwise spiral.**
If $a < b$ then $-a \geq -2\sqrt[2]{\delta}$ so P_2 is not a **counter-clockwise spiral.**
So, P_2 is not a **counter-clockwise spiral** under the previous conditions.
(2) For negative determinant:

(2,a) For $-(1+\delta) < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_1 < 0, 0 < \lambda_2 < 1$ and the fixed point is a **flip attractor.**

For P_1: Let us consider the following conditions:
13). $-(1+\delta_1) < \tau_1 < (1+\delta_1)$ if $a < b$.
14). $\lambda_{1,2} \in \mathbb{R}$ and $-1 < \lambda_1 < 0, 0 < \lambda_2 < 1$.
$-1 < \lambda_1 < 0$:

- $\lambda_1 > -1$ if $2 + a > 0$ and $2 - 2b > 0$ and $a^2 - 3b^2 + 2ab - 2a + 6b + 4ab^2 - 3 < 0$.
- $\lambda_1 < 0$ if $1 - a - 2b > 0$ and $2b - 3b^2 - 2ab + 4ab^2 < 0$.

$0 < \lambda_2 < 1$:

- $0 < \lambda_2$ if $1 - a - 2b > 0$ and $2b - 2ab - 3b^2 + 4ab^2 < 0$.
- $\lambda_2 < 1$ if $a - 2 > 0$ and $2 - 2a - 2b > 0$ and $-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 > 0$.

Lemma 2.2 *If (13),(14) holds true, then P_1 is a **flip attractor**.*

Proof. 13) $-(1+\delta_1) < \tau_1 < (1+\delta_1)$:
$\tau_1 = a$, if $a < b$, implies $a < b + 1$ and $a > -(1+b)$ equivalent to $-(1+\delta_2) < \tau_1 < (1+\delta_1)$.
14)
a) $\lambda_{1,2} \in \mathbb{R}$:
b) $-1 < \lambda_1 < 0, 0 < \lambda_2 < 1$.
$-1 < \lambda_1 < 0$:
$\lambda_1 > -1$ if $2 + a > 0$ and $2 - 2b > 0$ and $a^2 - 3b^2 + 2ab - 2a + 6b + 4ab^2 - 3 < 0$
$\lambda_1 < 0$ if $1 - a - 2b > 0$ and $2b - 3b^2 - 2ab + 4ab^2 < 0$.
$\lambda_1 < 0$ if

$$\begin{aligned} a + \sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} &< 0. \\ \sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} &< -a. \\ \sqrt[2]{(a+b-1)^2 + 4ab^2} &< 1 - a - 2b. \end{aligned}$$

If $1 - a - 2b < 0$ then $b > \frac{1-a}{2}$ and $\lambda_1 \geq 0$. Hence, if $1 - a - 2b > 0$ then $\lambda_1 < 0$ and we get:

$$(a+b-1)^2 + 4ab^2 - (1-a-2b)^2 < 0.$$

$$\begin{aligned} & (a+b-1)^2 + 4ab^2 - (1-a-2b)^2 \\ = \; & (a+b-1-1+a+2b)(a+b-1+1-a-2b) + 4ab^2. \\ = \; & 2b - 3b^2 - 2ab + 4ab^2. \end{aligned}$$

That is,

$$2b - 3b^2 - 2ab + 4ab^2 < 0.$$

$\lambda_1 > -1$ if

$$\sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} < 2 + a.$$

If $2 < -a$ then $2 + a < 0$ and $\lambda_1 \leq 1$. Hence, if $2 > -a$ then $2 + a > 0$ and thus, we get:

$$\begin{aligned} a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right) - (2+a)^2 &< 0. \\ 4a + 8b - 4 + 4\sqrt[2]{(a+b-1)^2 + 4ab^2} - 4 - 4a &< 0. \end{aligned}$$

i,e,

$$\sqrt[2]{(a+b-1)^2 + 4ab^2} < 2 - 2b.$$

If $2 - 2b < 0$ then $1 < b$ and $\lambda_1 \leq 1$. Hence,if $1 > b$ then $2 - 2b > 0$ and thus, we get:

$$(a+b-1)^2 + 4ab^2 - (2-2b)^2 < 0.$$

$$\begin{aligned} (a+b-1)^2 + 4ab^2 - (2-2b)^2 &= (a+b-1+2-2b)(a+b-1-2+2b) + 4ab^2. \\ &= (1+a-b)(a+3b-3) + 4ab^2. \\ &= a + 3b - 3 + a^2 + 3ab - 3a - ab - 3b^2 + 3b + 4ab^2. \\ &= a^2 - 3b^2 + 2ab - 2a + 6b + 4ab^2 - 3. \end{aligned}$$

That is,

$$a^2 - 3b^2 + 2ab - 2a + 6b + 4ab^2 - 3 < 0.$$

$0 < \lambda_2 < 1$:
$0 < \lambda_2$ if $1 - a - 2b > 0$ and $2b - 2ab - 3b^2 + 4ab^2 < 0$. Then

$$\sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2}\right)} < a.$$

If $a > 0$ then

$$\begin{aligned} a + 2b - 1 + \sqrt[2]{(a+b-1)^2 + 4ab^2} &< 0. \\ \sqrt[2]{(a+b-1)^2 + 4ab^2} &< 1 - a - 2b. \end{aligned}$$

If $1 - a - 2b < 0$ then $b > \frac{1-a}{2}$ and $\lambda_2 \leq 0$. Hence, if $b < \frac{1-a}{2}$ then $1 - a - 2b > 0$ and thus, we get:

$$(a+b-1)^2 + 4ab^2 - (1-a-2b)^2 < 0.$$

i.e,

$$\begin{aligned} (a+b-1)^2 + 4ab^2 - (1-a-2b)^2 &= (a+b-1+1-a-2b)(a+b-1-1+a+2b) + 4ab^2. \\ &= 2b - 2ab - 3b^2 + 4ab^2. \end{aligned}$$

that is,

$$2b - 2ab - 3b^2 + 4ab^2 < 0.$$

Now, $\lambda_2 < 1$ if $a - 2 > 0$ and $2 - 2a - 2b > 0$ and $-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 > 0$. Then

$$\sqrt[2]{a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a + b - 1)^2 + 4ab^2}\right)} > a - 2.$$

If $a < 2$ then $a - 2 < 0$ and $\lambda_2 \geq 1$. Hence, if $a > 2$ then $a - 2 > 0$ and thus, we get:

$$a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a + b - 1)^2 + 4ab^2}\right) - (a - 2)^2 > 0.$$

$$a^2 + 4\left(a + 2b - 1 + \sqrt[2]{(a + b - 1)^2 + 4ab^2}\right) - (a - 2)^2 = 8a + 8b - 8 + 4\sqrt[2]{(a + b - 1)^2 + 4ab^2}.$$

i.e,

$$\sqrt[2]{(a + b - 1)^2 + 4ab^2} > 2 - 2a - 2b.$$

If $2 - 2a - 2b < 0$ then $b > 1 - a$ and $\lambda_2 \geq 1$.Hence, if $b > 1 - a$ then $2 - 2a - 2b > 0$ and thus, we get:

$$(a + b - 1)^2 + 4ab^2 - (2 - 2a - 2b)^2 > 0.$$

$$\begin{aligned}(a + b - 1)^2 + 4ab^2 - (2 - 2a - 2b)^2 &= (a + b - 1 + 2 - 2a - 2b)(a + b - 1 - 2 + 2a + 2b) + 4ab^2. \\ &= (1 - a - b)(3a + 3b - 3) + 4ab^2. \\ &= 3a + 3b - 3 - 3a^2 - 3ab + 3a - 3ab - 3b^2 + 3b + 4ab^2. \\ &= -3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3.\end{aligned}$$

that is,

$$-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 > 0.$$

■

$\tau_1 = a$, if $a > b$, implies $\tau_1 \geq (1 + \delta_1)$ and $a \leq -(1 + b)$ so P_1 is not a **flip attractor** under the previous conditions.

For P_2: Let us consider the following conditions:

13). $-(1 + \delta) < \tau < (1 + \delta)$.

14). $\omega_{1,2} \in \mathbb{R}$ and $-1 < \omega_1 < 0, 0 < \omega_2 < 1$.

- $-1 < \omega_1 < 0$:

$\omega_1 < 0$ if $2b - a - 1 > 0$ and $-3b^2 + 2ab + 2b < 0$.

$\omega_1 > -1$ if $2 - a > 0$ and $2b - 2 > 0$ and $a^2 - 3b^2 - 2ab + 6b + 2a + 4ab^2 - 3 > 0$.

- $0 < \omega_2 < 1$:

$\omega_2 > 0$ if $2b - a - 1 > 0$ and $-3b^2 + 2ab + 2b + 4ab^2 < 0$.
$\omega_2 < 1$ if $2 + a > 0$ and $2b - 2a - 2 > 0$ and $-3a^2 - 3b^2 + 6ab - 6a + 6b + 4ab^2 - 3 > 0$.

Lemma 2.3 *If (13),(14) holds true, then P_2 is a **flip attractor**.*

Proof. 13) $-(1 + \delta_2) < \tau_2 < (1 + \delta_2)$.
$\tau_2 = -a$, $-a < b + 1$ and $-a > -(b + 1)$, equivalent to $-(1 + \delta) < \tau_2 < (1 + \delta)$.
14)
a) $\omega_{1,2} \in \mathbb{R}$.
b) $-1 < \omega_1 < 0, 0 < \omega_2 < 1$.
$-1 < \omega_1 < 0$:
$\omega_1 < 0$ if $2b - a - 1 > 0$ and $-3b^2 + 2ab + 2b < 0$.
$\omega_1 < 0$ if

$$\begin{aligned} -a - \sqrt[2]{a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}\right)} &< 0. \\ \sqrt[2]{a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}\right)} &> -a. \\ 2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2} &> 0. \\ \sqrt[2]{(b - a - 1)^2 + 4ab^2} &< 2b - a - 1. \end{aligned}$$

If $2b - a - 1 < 0$ then $b < \frac{a+1}{2}$ and $\omega_1 \geq 0$. Hence, if $b > \frac{a+1}{2}$ then $2b - a - 1 > 0$ and thus, we get:

$$(b - a - 1)^2 + 4ab^2 - (2b - a - 1)^2 < 0.$$

i.e,

$$\begin{aligned} (b - a - 1)^2 + 4ab^2 - (2b - a - 1)^2 &= (b - a - 1 + 2b - a - 1)(b - a - 1 - 2b + a + 1) + 4ab^2. \\ &= -3b^2 + 2ab + 2b. \end{aligned}$$

that is,

$$-3b^2 + 2ab + 2b < 0.$$

$\omega_1 > -1$ if $2 - a > 0$ and $2b - 2 > 0$ and $a^2 - 3b^2 - 2ab + 6b + 2a + 4ab^2 - 3 > 0$.Then

$$\sqrt[2]{a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}\right)} < 2 - a.$$

If $2 < a$ then $2 - a < 0$ and $\omega_1 \leq -1$. Hence, if $2 > a$ then $2 - a > 0$ and thus, we get:

$$a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b-a-1)^2 + 4ab^2}\right) - (2-a)^2 < 0.$$

$$a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b-a-1)^2 + 4ab^2}\right) - (2-a)^2 = 8b - 8 - 4\sqrt[2]{(b-a-1)^2 + 4ab^2}.$$

$$8b - 8 - 4\sqrt[2]{(b-a-1)^2 + 4ab^2} < 0.$$

i.e,

$$\sqrt[2]{(b-a-1)^2 + 4ab^2} > 2b - 2.$$

If $b < 1$ then $2b - 2 < 0$ and $\omega_1 \leq -1$. Hence, if $b > 1$ then $2b - 2 > 0$ and thus, we get:

$$(b-a-1)^2 + 4ab^2 - (2b-2)^2 > 0.$$

$$\begin{aligned}(b-a-1)^2 + 4ab^2 - (2b-2)^2 &= (b-a-1-2b+2)(b-a-1+2b-2) + 4ab^2. \\ &= (1-a-b)(3b-a-3) + 4ab^2 \ . \\ &= 3b - a - 3 - 3ab + a^2 + 3a - 3b^2 + ab + 3b + 4ab^2. \\ &= a^2 - 3b^2 - 2ab + 6b + 2a + 4ab^2 - 3.\end{aligned}$$

that is,

$$a^2 - 3b^2 - 2ab + 6b + 2a + 4ab^2 - 3 > 0.$$

$0 < \omega_2 < 1 : \omega_2 > 0$ if $2b - a - 1 > 0$ and $-3b^2 + 2ab + 2b + 4ab^2 < 0$. Then

$$\begin{aligned}\sqrt[2]{a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b-a-1)^2 + 4ab^2}\right)} &> a. \\ 2b - a - 1 - \sqrt[2]{(b-a-1)^2 + 4ab^2} &> 0. \\ \sqrt[2]{(b-a-1)^2 + 4ab^2} &< 2b - a - 1.\end{aligned}$$

If $b < \frac{a+1}{2}$ then $2b - a - 1 < 0$ and $\omega_2 \leq 0$. Hence, if $b > \frac{a+1}{2}$ then $2b - a - 1 > 0$ and thus we get:

$$(b-a-1)^2 + 4ab^2 - (2b-a-1)^2 < 0.$$

$$\begin{aligned}(b-a-1)^2 + 4ab^2 - (2b-a-1)^2 &= (b-a-1+2b-a-1)(b-a-1-2b+a+1) + 4ab^2. \\ &= -3b^2 + 2ab + 2b + 4ab^2.\end{aligned}$$

that is,

$$-3b^2 + 2ab + 2b + 4ab^2 < 0.$$

$\omega_2 < 1$ if $2 + a > 0$ and $2b - 2a - 2 > 0$ and $-3a^2 - 3b^2 + 6ab - 6a + 6b + 4ab^2 - 3 > 0$.
$\omega_2 < 1$ if

$$\sqrt[2]{a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}\right)} < 2 + a.$$

If $2 < -a$ then $2 + a < 0$ and $\omega_2 \geq 1$. Hence, if $2 > -a$ then $2 - a > 0$ and thus we get:

$$\begin{aligned} & a^2 + 4\left(2b - a - 1 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}\right) - (2 + a)^2 \\ = \; & 8b - 8a - 8 - \sqrt[2]{(b - a - 1)^2 + 4ab^2}. \end{aligned}$$

i.e,

$$\sqrt[2]{(b - a - 1)^2 + 4ab^2} > 2b - 2a - 2.$$

If $b < 1 + a$ then $2b - 2a - 2 < 0$ and $\omega_2 \geq 1$. Hence, if $b < 1 + a$ then $2b - 2a - 2 > 0$ and thus we get:

$$(b - a - 1)^2 + 4ab^2 - (2b - 2a - 2)^2 > 0.$$

i.e,

$$\begin{aligned} & (b - a - 1)^2 + 4ab^2 - (2b - 2a - 2)^2 \\ = \; & (b - a - 1 - 2b + 2a + 2)(b - a - 1 + 2b - 2a - 2) + 4ab^2. \\ = \; & (a - b + 1)(3b - 3a - 3) + 4ab^2. \\ = \; & 3ab - 3a^2 - 3a - 3b^2 + 3ab + 3b + 3b - 3a - 3 + 4ab^2. \\ = \; & -3a^2 - 3b^2 + 6ab - 6a + 6b + 4ab^2 - 3. \end{aligned}$$

that is,

$$-3a^2 - 3b^2 + 6ab - 6a + 6b + 4ab^2 - 3 > 0.$$

■

(2,b) For $\tau > (1 + \delta)$, then the Jacobian matrix has two real eigenvalues $1 < \lambda_1, -1 < \lambda_2 < 0$ and the fixed point is a **flip saddle**.

For P_1: Let us consider the following conditions:
15). $\tau > (1 + \delta)$.
16). $\lambda_{1,2} \in \mathbb{R}$ and $1 < \lambda_1, -1 < \lambda_2 < 0$.

- $-1 < \lambda_2 < 0$:

$\lambda_2 < 0$ if $1 - a - 2b > 0$ and $2b - 2ab - 3b^2 + 4ab^2 > 0$.
$\lambda_2 > -1$ if $2 + a > 0$ and $2 - 2b > 0$ and $a^2 - 3b^2 + 4ab^2 + 2ab - 2a + 6b - 3 < 0$.

- $\lambda_1 > 1$ if $2 - a > 0$ and $2 - 2a - 2b > 0$ and $-3a^2 - 3b^2 + 4ab^2 - 6ab + 6a + 6b - 3 > 0$.

If $a > b$, then $\tau \geq (1 + \delta)$.
If $a < b$,then $a < b + 1$. So P_1 is not a **flip saddle** under the previous conditions.
For P_2: Let us consider the following conditions:
15). $\tau > (1 + \delta)$.
16). $\omega_{1,2} \in \mathbb{R}$ and $1 < \omega_1, -1 < \omega_2 < 0$.

$\tau_2 = -a$, $-a < b$ then $\tau_2 < 1 + \delta$.

P_2 is not a **flip saddle**.

(2,c) For $\tau < -(1 + \delta)$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_1 < 1, \lambda_2 < -1$ and the fixed point is a **flip saddle**.

For P_1: Let us consider the following conditions:
17). $\tau < -(1 + \delta)$.
18). $\lambda_{1,2} \in \mathbb{R}$ and $0 < \lambda_1 < 1, \lambda_2 < -1$.
We have $\tau_1 = a$ and:

- If $a < b$ then $\tau_1 > -(1 + \delta)$.

  So P_1 is not a **flip saddle** under the previous conditions.

- If $a > b$ then $a > -(1 + \delta)$.

So P_1 is not a **flip saddle** under the previous conditions.
For P_2: Let us consider the following conditions:
17). $\tau < -(1 + \delta)$.
18). $\omega_{1,2} \in \mathbb{R}$ and $0 < \omega_1 < 1, \omega_2 < -1$.
$\tau_2 = -a$ then $-a < b$ and $-a < b + 1$:
If $a < b$ then $\tau_2 > -(1 + \delta)$ so P_2 is not a **flip saddle**.
If $a > b$ then $\tau_2 < b + 1$ and $\tau_2 \leq -(1 + \delta)$, so P_2 is not a **flip saddle**.

Chapter 3

Bifurcations of a modulated 2-D smooth map

In this chapter,we shall consider a 2-D discrete time system (See [6]) given by:

$$\begin{cases} x_{n+1} = ax_n\left(1-x_n\right). \\ y_{n+1} = \left(b+c\left|x_n\right|\right)y_n\left(1-y_n\right). \end{cases}$$

Let us consider the following 2-D piecewise smooth system given by:

$$f\left(x,y\right) = \left(ax\left(1-x\right), \left(b+c\left|x\right|\right)\times y\left(1-y\right)\right).$$

The map create two regions D_1' and D_2' given by:

$$\begin{cases} D_1' = \left\{(x,y)\in\mathbb{R}^2 : x>0\right\}. \\ D_2' = \left\{(x,y)\in\mathbb{R}^2 : x<0\right\}. \end{cases}$$

So the map can be rewritten as follow:

$$f\left(x,y\right) = \begin{pmatrix} ax\left(1-x\right) \\ y\left(b+cx\right)\left(1-y\right), \text{ if } (x,y)\in D_1' \\ y\left(b-cx\right)\left(1-y\right), \text{ if } (x,y)\in D_2' \end{pmatrix}. \tag{3.1}$$

3.1 Existence of fixed points and their stability

The fixed point are given by:

$$f\left(x,y\right) = (x,y) \Leftrightarrow f\left(x,y\right) = \begin{pmatrix} ax\left(1-x\right) \\ \left(b+cx\right)y\left(1-y\right), \text{ if } (x,y)\in D_1 \\ \left(b-cx\right)y\left(1-y\right), \text{ if } (x,y)\in D_2 \end{pmatrix} = \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix} \tag{3.2}$$

- For $y \in D_1'$:

$$\begin{cases} ax\,(1-x) = x \\ (b+cx)\,y\,(1-y) = y \end{cases} \Rightarrow \begin{cases} x = 1 - \frac{1}{a} \\ (b+cx)\,(1-y) = 1 \end{cases}$$

and by substitution we find:

$$\begin{cases} x = 1 - \frac{1}{a},\ a \neq 0. \\ y = 1 - \frac{a}{ab+ac-c}, ab + ac - c \neq 0. \end{cases}$$

So $y_1 = 1 - \frac{a}{ab+ac-c}$ under the two condetions $a \neq 0$ and $a \neq \frac{c}{b+c}$ and $p_3\,(x,y) = p_3\,(x,y_1) = \left(1 - \frac{1}{a}, 1 - \frac{a}{ab+ac-c}\right)$ with laste two conditions.

We have on other hand:

$$\begin{cases} ax\,(1-x) = x \\ (b-cx)\,y\,(1-y) = y \end{cases} \Rightarrow \begin{cases} x = 1 - \frac{1}{a} \\ (b-cx)\,(1-y) = 1 \end{cases}$$

and by substitution we find:

$$\begin{cases} x = 1 - \frac{1}{a},\ a \neq 0. \\ y = 1 - \frac{a}{ab-ac-c}, ab - ac - c \neq 0. \end{cases}$$

So $y_2 = 1 - \frac{a}{ab-ac-c}$ under the two condetions $a \neq 0$ and $a \neq \frac{c}{b-c}$. So we find $y_2 = 1 - \frac{a}{ab-ac-c}$ and $p_4\,(x,y) = p_4\,(x,y_2) = \left(1 - \frac{1}{a}, 1 - \frac{a}{ab-ac-c}\right)$ with the last two conditions. Then the map (3.1) has two fixed points given by $p_3\,(x_1,y_1) \in D_1'$ and $p_4\,(x_1,y_2) \in D_2'$, where:

$$\begin{cases} y_1 = 1 - \frac{a}{ab+ac-c}. \\ \\ y_2 = 1 - \frac{a}{ab-ac-c}. \end{cases} \tag{3.3}$$

Finally, the piecewise smooth map (3.1) has two simultaneous fixed points defined for $a > 0$ and $b > 0$ as $P_3 = (x_1,y_1) \in D_1'$ and $P_4 = (x_1,y_2) \in D_2'$:

$$\begin{cases} P_3 = \left(1 - \frac{1}{a}, 1 - \frac{a}{ab+ac-c}\right). \\ \\ P_4 = \left(1 - \frac{1}{a}, 1 - \frac{a}{ab-ac-c}\right). \end{cases} \tag{3.4}$$

With eigenvalues $\lambda_{3,4}$ and $\omega_{3,4}$ respectively. The Jacobian matrix of the map (3.1) evaluated at a point (x,y) in the region D_1' (See [4]) is given by:

$$J_3\,(x,y) = \begin{pmatrix} -2ax + a & 0 \\ cy\,(1-y) & (1-2y)\,(b+cx) \end{pmatrix}. \tag{3.5}$$

and at a point (x, y) in the region D_2' the Jacobian matrix is given by:

$$J_4(x,y) = \begin{pmatrix} -2ax + a & 0 \\ -cy(1-y) & (1-2y)(b-cx) \end{pmatrix}. \tag{3.6}$$

Thus, at P_3 one has:

$$J_3(P_3) = \begin{pmatrix} 2-a & 0 \\ \frac{ac(ab-ac-c-a)}{(ab+ac-c)^2} & \frac{2a-ab-ac+c}{a} \end{pmatrix}. \tag{3.7}$$

The eigenvalues of $J_3(P_3)$ are:

$$\begin{cases} \lambda_3 = 2-a. \\ \\ \lambda_4 = \frac{2a-ab-ac+c}{a}. \end{cases} \tag{3.8}$$

With $a \neq 0$ and $a \neq \frac{c}{b+c}$ and at P_4 one has:

$$J_4(P_4) = \begin{pmatrix} 2-a & 0 \\ \frac{a\left(ac-abc+ac^2+c^2\right)}{(ab-ac-c)^2} & \frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \end{pmatrix}. \tag{3.9}$$

The eigenvalues of $J_4(P_4)$ are:

$$\begin{cases} \omega_3 = 2-a \\ \\ \omega_4 = \frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \end{cases} \tag{3.10}$$

With $a \neq 0$ and $ab - ac - c \neq 0$, then $a \neq 0$ and $a \neq \frac{c}{b-c}$.

The normal form: The normal form of map (3.1) is given by:

$$N(x,y) = \begin{cases} \begin{pmatrix} \tau_3 & 1 \\ -\delta_3 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{if } x < 0 \\ \\ \begin{pmatrix} \tau_4 & 1 \\ -\delta_4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{if } x > 0, \end{cases} \tag{3.11}$$

Where μ is a parameter, and $\tau_{3,4}, \delta_{3,4}$, are the traces and determinants of the corresponding matrices of the linearized map in the two subregion D_1' and D_2' evaluated at P_3 and P_4 respectively, and they are given by:

$$N(x,y) = \begin{cases} \begin{pmatrix} 2-a+\frac{2a-ab-ac+c}{a} & 1 \\ (2-a)\frac{(2a-ab-ac+c)}{a} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x > 0. \\ \\ \begin{pmatrix} 2-a+\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} & 1 \\ (2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x < 0. \end{cases} \tag{3.12}$$

We have

$$\begin{cases} j_{11} = \tau_3 = 2 - a + \frac{(2a-ab-ac+c)}{a}. \\ \\ j_{11} = \tau_4 = 2 - a + \frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)}. \\ \\ j_{21} = \delta_3 = (2-a)\frac{(2a-ab-ac+c)}{a}. \\ \\ j_{21} = \delta_4 = (2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)}. \end{cases} \tag{3.13}$$

Then, the normal form of map (3.1) is:

$$N(x,y) = \begin{cases} \begin{pmatrix} 2-a+\frac{(2a-ab-ac+c)}{a} & 1 \\ (2-a)\frac{(2a-ab-ac+c)}{a} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x < 0, \\ \\ \begin{pmatrix} 2-a+\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} & 1 \\ (2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mu, \text{ if } x > 0. \end{cases} \tag{3.14}$$

3.2 Types of fixed points of the normal form

(1) For positive determinant: We have that from (3.14):

$$\begin{cases} \tau_3 = 2 - a + \frac{2a-ab-ac+c}{a}, \tau_4 = 2 - a + \frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \\ \\ \delta_3 = (2-a)\frac{(2a-ab-ac+c)}{a}, \delta_4 = (2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \end{cases}$$

and from (3.9):

$$\begin{cases} \lambda_3 = 2 - a \\ \\ \lambda_4 = \frac{2a-ab-ac+c}{a} \end{cases}$$

and from (3.11):

$$\begin{cases} \omega_3 = \frac{2a-ab-ac+c}{a} \\ \\ \omega_4 = \frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \end{cases}$$

We have $\lambda_{3,4}, \omega_{3,4} \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$. All forms of bifurcation is possible if: $\lambda_{1,2}$ exist if $a \neq 0$ and $a \neq \frac{c}{b+c}$ from (3.9). $\omega_{1,2}$ exist if $a \neq 0$ and $a \neq \frac{c}{b-c}$from (3.11).

(1.a) For $2\sqrt{\delta} < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalnues $0 < \lambda_3, \lambda_4 < 1$ and the fixed point is a **regular attractor**.

For P_3: Let us consider the following conditions:
19). $2\sqrt[2]{\delta_3} < \tau_3 < (1+\delta_3)$, if $a \in]a_1, a_2[$ where

$$\begin{cases} a_1 = \frac{1-b-2c+\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \\ a_2 = \frac{1-b-2c-\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \end{cases}$$

and $-a^2 - ab - ac + c > 0$.
20). $\lambda_3, \lambda_4 \in \mathbb{R}$ and $0 < \lambda_3, \lambda_4 < 1$.

- $0 < \lambda_3 < 1$ if $a \in]1, 2[$.
- $0 < \lambda_4 < 1$ if $2a - ab - ac + c > 0$ and $a - ab - ac + c < 0$.

Lemma 3.1 *If (19),(20) holds true, then P_3 is a **regular attractor**.*

Proof. 19) $2\sqrt[2]{\delta_3} < \tau_3 < (1+\delta_3)$:

$$\tau_3 < 1 + \delta_3 \Leftrightarrow \frac{(2-a)\,a + 2a - ab - ac + c}{a} < \frac{(2-a)\,(2a - ab - ac + c) + a}{a}.$$

$a \neq 0$ then

$$\begin{aligned} (2-a)\,a + 2a - ab - ac + c &< (2-a)\,(2a - ab - ac + c) + a. \\ -a^2 + 4a - ab - ac + c &< -2a^2 + a^2 b + a^2 c + 4a - 2ab - 3ac + 2c + a. \\ a^2 + ab + 2ac - c - a - a^2 b - a^2 c &< 0. \\ a^2\,(1-b-c) - a\,(1-b-2c) - c &< 0. \end{aligned}$$

We have $\Delta = (1-b-2c)^2 + 4c\,(1-b-c)$. If $\Delta < 0$ we have no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ then we have $c > 0$ and $1-b-c > 0$ or $1-b-c < 0$ and $c < 0$ so we get a_1, a_2: Since $a \neq 0$ then $\tau_3 < 1 + \delta_3$if $a \in]a_1, a_2[$.
$\tau_3 > 2\sqrt{\delta_3}$: $\tau_3 = 2 - a + \frac{2a-ab-ac+c}{a}$ and $\delta_3 = \frac{(2-a)a+2a-ab-ac+c}{a}$, then

$$\tau_3 > 2\sqrt{\delta_3} \Leftrightarrow \frac{(2-a)\,a + 2a - ab - ac + c}{a} > 2\sqrt{(2-a)\,\frac{(2a - ab - ac + c)}{a}}$$

Thus,

$$\begin{aligned} ((2-a)\,a + 2a - ab - ac + c)^2 &> 4a\,((2-a)\,a + 2a - ab - ac + c)\,. \\ (2-a)\,a + 2a - ab - ac + c - 4a &> 0. \end{aligned}$$

$2\sqrt{\delta_3} < \tau_3 < 1 + \delta_3$ if $a \in]a_1, a_2[$ and $-a^2 - ab - ac + c > 0$.

20)

a) $\lambda_{3,4} \in \mathbb{R}$ if $a, b, c \in \mathbb{R}$.

b) $0 < \lambda_{3,4} < 1$.

$0 < \lambda_3 < 1$if $a \in]1, 2[$.

$\lambda_3 > 0$:

$\lambda_3 = 2 - a$ then $2 - a > 0$ then $a < 2$ and $\lambda_3 < 1$ then $2 - a < 1$ so $a > 1$.

So $0 < \lambda_3 < 1$ if $a \in]1, 2[$.

$0 < \lambda_4 < 1$ if :

$\lambda_4 > 0$:

$\lambda_4 = \frac{2a-ab-ac+c}{a}, \frac{2a-ab-ac+c}{a} > 0$ then $a > 0$ and $2a - ab - ac + c > 0$.

$\lambda_4 < 1$ then $2a - ab - ac + c < a$ then $a - ab - ac + c < 0$.

So $0 < \lambda_4 < 1$ if $2a - ab - ac + c > 0$ and $a - ab - ac + c < 0$. ■

For P_4: Let us consider the following conditions:

19). $2\sqrt[2]{\delta_4} < \tau_4 < (1 + \delta_4)$.

20). $\omega_3, \omega_4 \in \mathbb{R}$ and $0 < \omega_3, \omega_4 < 1$.

- $0 < \omega_3 < 1$ if:

$\omega_3 > 0$ if $a < 0$ and $2a - ab - ac + c < 0$ or $a > 0$ and $2a - ab - ac + c > 0$.

$\omega_3 < 1$ if $a - ab - ac + c < 0$.

- $0 < \omega_4 < 1$ if:

$\omega_4 > 0$ if $2a - ab + ac + c > 0$ and $ab - ac + c > 0$ or $2a - ab + ac + c < 0$ and $ab - ac + c < 0$.

$\omega_4 < 1$ if $(2a - ab + ac + c)(ab - ac + c) - a < 0$.

Lemma 3.2 *If (19),(20) holds true, then P_4 is a **regular attractor**.*

Proof. 19) $2\sqrt{\delta_4} < \tau_4 < \delta_4 + 1$ if $\tau_4 < \delta_4 + 1$ if

$$\begin{aligned} & 2 - a + \frac{(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)} \\ < \; & 1 + (2 - a)\frac{(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)}. \end{aligned}$$

If $ab - ac - c \neq 0$ then

$$\begin{aligned} & a(ab - ac - c)(2 - a) + (2a - ab + ac + c)(ab - ac + c) - a(ab - ac - c) \\ & - (2 - a)(2a - ab + ac + c)(ab - ac + c) \\ < \; & 0. \end{aligned}$$

Equivalent to:

$$(ab-ac-c)\left(a-a^2\right)+(ab-ac+c)(2a-ab+ac+c)(a-1)<0.$$

$2\sqrt{\delta_4}<\tau_4$ if

$$\begin{cases} 2\sqrt{(2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)}} \\ <2-a+\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \\ 4(2-a)\frac{(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)} \\ <\left(\frac{a(2-a)(ab-ac-c)+(2a-ab+ac+c)(ab-ac+c)}{a(ab-ac-c)}\right) \\ (8-4a)(2a-ab+ac+c)(ab-ac+c) \\ <\frac{(a(2-a)(ab-ac-c)+(2a-ab+ac+c)(ab-ac+c))^2}{a(ab-ac-c)} \\ a(ab-ac-c)(8-4a)(2a-ab+ac+c)(ab-ac+c) \\ <(a(2-a)(ab-ac-c)+(2a-ab+ac+c)(ab-ac+c))^2 \end{cases}.$$

Then $2\sqrt{\delta_4}<\tau_4<\delta_4+1$ if

$$\begin{aligned} & (8-4a)(2a-ab+ac+c)(ab-ac+c) \\ < & \left(a(2-a)(ab-ac-c)+(2a-ab+ac+c)(ab-ac+c)\right)^2. \end{aligned}$$

20)

a) $\omega_3,\omega_4\in\mathbb{R}$.

b) $0<\omega_3,\omega_4<1$.

$0<\omega_3<1$ if:

$\omega_3=\frac{2a-ab-ac+c}{a}$.

$\omega_3>0$ if

$\frac{2a-ab-ac+c}{a}>0$ if $a<0$ and $2a-ab-ac+c<0$ or $a>0$ and $2a-ab-ac+c>0$.

$\omega_3<1$ if $a-ab-ac+c<0$.

If $a<0$ and $2a-ab-ac+c<0$ or $a>0$ and $2a-ab-ac+c>0$.

$\omega_3<1$ if

$\frac{2a-ab-ac+c}{a}<1$ if $a-ab-ac+c<0$.

$0<\omega_4<1$ if:

$\omega_4>0$ if $2a-ab+ac+c>0$ and $ab-ac+c>0$ or $2a-ab+ac+c<0$ and $ab-ac+c<0$.

$\omega_4<1$ if $(2a-ab+ac+c)(ab-ac+c)-a<0$. $\omega_4>0$:

If $a>0$ and $(2a-ab+ac+c)(ab-ac+c)>0$ that is, $2a-ab+ac+c>0$ and $ab-ac+c>0$ or $2a-ab+ac+c<0$ and $ab-ac+c<0$.

$\omega_4<1$ if

$$(2a-ab+ac+c)(ab-ac+c)<aif\,(2a-ab+ac+c)(ab-ac+c)-a<0.$$

■

(1,b) For $\tau > 1+\delta$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_3 < 1, \lambda_4 > 1$ and the fixed point is a **regular saddle**.

For P_3: Let us consider the following conditions:
21). $\tau_3 > 1+\delta_3$ if $a \in]-\infty, a_1[\cup]a_2, +\infty[$.
22). $\lambda_{3,4} \in \mathbb{R}$ and $0 < \lambda_3 < 1, \lambda_4 > 1$.

- $0 < \lambda_3 < 1$ if $a \in]1, 2[$.
- $\lambda_4 > 1$ if $a - ab - ac + c > 0$.

Lemma 3.3 *If (21),(22) holds true, then P_3 is a **regular saddle**.*

Proof. 21) $\tau_3 > 1+\delta_3$ if

$$\tau_3 > 1+\delta_3 \Leftrightarrow \frac{(2-a)\,a + 2a - ab - ac + c}{a} > \frac{(2-a)\,(2a - ab - ac + c) + a}{a}.$$

$a \neq 0$ then

$$\begin{aligned}
(2-a)\,a + 2a - ab - ac + c &> (2-a)\,(2a - ab - ac + c) + a. \\
-a^2 + 4a - ab - ac + c &> -2a^2 + a^2 b + a^2 c + 4a - 2ab - 3ac + 2c + a. \\
a^2 + ab + 2ac - c - a - a^2 b - a^2 c &> 0. \\
a^2\,(1-b-c) - a\,(1-b-2c) - c &> 0.
\end{aligned}$$

$\Delta = (1-b-c)^2 + 4c\,(1-b-2c)$. If $\Delta < 0$ then no solutions exists. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ then we have $c > 0$ and $1-b-2c > 0$ or $1-b-2c < 0$ and $c < 0$ so we get $a_1 = \frac{1-b-2c+\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)}$ and $a_2 = \frac{1-b-2c-\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)}$. Since $a \neq 0$ and $a^2\,(1-b-c) - a\,(1-b-2c) - c > 0$ then $\tau_1 > 1+\delta_1$ if $a \in]-\infty, a_1[\cup]a_2, +\infty[$.
22)
a) $\lambda_{3,4} \in \mathbb{R}$.
b) $0 < \lambda_3 < 1, \lambda_4 > 1$.
$0 < \lambda_3 < 1$ if:
$\lambda_3 > 0$:
$\lambda_3 = 2 - a$ then $2 - a > 0$ then $a < 2$.
$\lambda_3 < 1$ then $2 - a < 1$ then $a > 1$.
So $0 < \lambda_3 < 1$ if $a \in]1, 2[$.

$\lambda_4 > 1$ if $a > 0$ and $2a - ab - ac + c > 0$.

$\lambda_4 > 1$ if $a - ab - ac + c > 0$. ■

For P_4: Let us consider the following conditions:

21). $\tau_4 > 1 + \delta_4$ if

$$\begin{aligned} & a\,(ab - ac - c)\,(8 - 4a)\,(2a - ab + ac + c)\,(ab - ac + c) \\ < \; & \left(\left(2a - a^2\right)(ab - ac - c) + (2a - ab + ac + c)\,(ab - ac + c)\right)^2 \end{aligned}$$

22). $\omega_{3,4} \in \mathbb{R}$ and $0 < \omega_3 < 1, \omega_4 > 1$.

- $0 < \omega_3 < 1$:

$\omega_3 > 0$ if $a < 0$ and $2a - ab - ac + c < 0$ or $a > 0$ and $2a - ab - ac + c > 0$.

$\omega_3 < 1$ if $a - ab - ac + c < 0$.

- $\omega_4 > 1$ if $(2a - ab + ac + c)\,(ab - ac + c) - a > 0$.

Lemma 3.4 *If (21),(22) holds true, then P_4 is a **regular saddle**.*

Proof. 21) $\tau_4 > 1 + \delta_4$: If $ab - ac - c \neq 0$ then

$$(a - 1) \times [(ab - ac + c)\,(a - ab + ac + c)] < 0.$$

If $a < 1$ and $(ab - ac + c)\,(a - ab + ac + c) > 0$ or $a > 1$ and $(ab - ac + c)\,(a - ab + ac + c) < 0$.

$2\sqrt{\delta_2} < \tau_2$ if

$$\begin{aligned} & 2\sqrt{(2 - a)\,\frac{(2a - ab + ac + c)\,(ab - ac + c)}{a\,(ab - ac - c)}} \\ < \; & 2 - a + \frac{(2a - ab + ac + c)\,(ab - ac + c)}{a\,(ab - ac - c)}. \end{aligned}$$

$$\begin{aligned} & 4\,(2 - a)\,\frac{(2a - ab + ac + c)\,(ab - ac + c)}{a\,(ab - ac - c)} \\ < \; & \left(\frac{a\,(2 - a)\,(ab - ac - c) + (2a - ab + ac + c)\,(ab - ac + c)}{a\,(ab - ac - c)}\right)^2 \end{aligned}$$

$$\begin{aligned} & a\,(ab - ac - c)\,(8 - 4a)\,(2a - ab + ac + c)\,(ab - ac + c) \\ < \; & \left(\left(2a - a^2\right)(ab - ac - c) + (2a - ab + ac + c)\,(ab - ac + c)\right)^2 \end{aligned}$$

Then $\tau_4 < \delta_4 + 1$ if

$$\begin{aligned} & a\,(ab - ac - c)\,(8 - 4a)\,(2a - ab + ac + c)\,(ab - ac + c) \\ < \; & \left(\left(2a - a^2\right)(ab - ac - c) + (2a - ab + ac + c)\,(ab - ac + c)\right)^2 \end{aligned}$$

22)
a) $\omega_{3,4} \in \mathbb{R}$.
b) $0 < \omega_3 < 1, \omega_4 > 1$.
$0 < \omega_3 < 1$:
$\omega_3 > 0$ if $a < 0$ and $2a - ab - ac + c < 0$ or $a > 0$ and $2a - ab - ac + c > 0$.
$\omega_3 > 0$:
If $a > 0$ and $2a - ab - ac + c > 0$ or $a < 0$ and $2a - ab - ac + c < 0$.
If $a < 0$ and $2a - ab - ac + c < 0$ or $a > 0$ and $2a - ab - ac + c > 0$.
$\omega_3 < 1$ if $a - ab - ac + c < 0$.
$\omega_4 > 1$ if $(2a - ab + ac + c)(ab - ac + c) - a > 0$. ■

(1.c) For $-(1+\delta) < \tau < -2\sqrt[2]{\delta}$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_3, \lambda_4 < 0$ and the fixed point is a **flip attractor**.

For P_3: Let us consider the following conditions:
23). $-(1+\delta_3) < \tau_3 < -2\sqrt[2]{\delta_3}$ if $a \neq 0$ and $a^2(b+c-3) - a(3b+3c-9) > 0$ then $\tau_3 > 1 + \delta_3$if $a \in]-\infty, \widetilde{a_1}[\cup]\widetilde{a_2}, +\infty[$.
24). $\lambda_{3,4} \in \mathbb{R}$ and $-1 < \lambda_3, \lambda_4 < 0$.

- $-1 < \lambda_3 < 0$ if $a \in]2, 3[$.
- $-1 < \lambda_4 < 0$ if $3a - ab - ac + c > 0$ and $2a - ab - ac + c < 0$.

Lemma 3.5 *If (23),(24) holds true, then P_3 is a **flip attractor**.*

Proof. 23) $-(1+\delta_3) < \tau_3 < -2\delta_3$:

$$\tau_3 > -(1+\delta_3) \Leftrightarrow \frac{(2-a)a + 2a - ab - ac + c}{a} > -\frac{(2-a)(2a - ab - ac + c) + a}{a}$$

$a \neq 0$ then

$$\begin{aligned}
(2-a)a + 2a - ab - ac + c &> -[(2-a)(2a - ab - ac + c) + a] \\
-a^2 + 4a - ab - ac + c &> 2a^2 - a^2b - a^2c - 4a + 2ab + 2ac - 2c - a \\
-3a^2 + a^2b + a^2c - 3ab - 3ac + 3c + 9a &> 0 \\
a^2(b+c-3) - a(3b+3c-9) &> 0
\end{aligned}$$

We have $\Delta = (3b+3c-9)^2$. If $\Delta < 0$ then no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ since

$(3b+3c-9)^2 \geq 0$ so we get: $\widetilde{a_1} = \frac{3b+3c-9}{b+c-3}, \widetilde{a_2} = 0$. Since $a \neq 0$ and $a^2(b+c-3) - a(3b+3c-9) > 0$ then $\tau_1 > 1+\delta_1$if $a \in]-\infty, \widetilde{a_1}[\cup]\widetilde{a_2}, +\infty[$.

24)

a) $\lambda_{3,4} \in \mathbb{R}$.

b) $-1 < \lambda_3, \lambda_4 < 0$.

$-1 < \lambda_3 < 0$ if $a < 3$ and $a > 2$.

$-1 < \lambda_4 < 0$ if $3a - ab - ac + c > 0$ and $2a - ab - ac + c < 0$. ■

For P_4: Let us consider the following conditions:

23). $-(1+\delta_4) < \tau_4 < -2\sqrt[2]{\delta_4}$ if $-(a-3)(-a^2b^2 + 2a^2bc + 3a^2b - a^2c^2 - 3a^2c + ac + c^2) > 0$.

24). $\omega_{3,4} \in \mathbb{R}$ and $-1 < \omega_3, \omega_4 < 0$:

- $-1 < \omega_3 < 0$ if $3a - ab - ac + c > 0$, $2a - ab - ac + c < 0$ and $a > 0$ or $2a - ab - ac + c > 0$ and $a < 0$.
- $-1 < \omega_4 < 0$ if $(2a - ab + ac + c)(ab - ac + c) > 0$ and $a(ab - ac - c) > 0$, and $2a - ab + ac + c > 0$ and $ab - ac + c < 0$ or $2a - ab + ac + c < 0$ and $ab - ac + c > 0$.

Lemma 3.6 *If (23),(24) holds true, then P_4 is a **flip attractor**.*

Proof. 23) $-(1+\delta_4) < \tau_4 < -2\sqrt[2]{\delta_4}$.

$\tau_4 > -(1+\delta_4)$ if

$$\begin{aligned} & -\frac{(2-a)(2a-ab+ac+c)(ab-ac+c) + a(ab-ac-c)}{a(ab-ac-c)} \\ < & \frac{a(2-a)(ab-ac-c) + (ab-ac+c)(2a-ab+ac+c)}{a(ab-ac-c)}. \end{aligned}$$

If $ab - ac - c \neq 0$ then

$$\begin{aligned} & -[(2-a)(2a-ab+ac+c)(ab-ac+c) + a(ab-ac-c)] \\ < & a(2-a)(ab-ac-c) + (ab-ac+c)(2a-ab+ac+c). \\ & a(2-a)(ab-ac-c) + (ab-ac+c)(2a-ab+ac+c) + (2-a) \\ & (2a-ab+ac+c)(ab-ac+c) + a(ab-ac-c) \\ > & 0. \end{aligned}$$

24)

a) $\omega_{3,4} \in \mathbb{R}$.

b) $-1 < \omega_3, \omega_4 < 0$.

$-1 < \omega_3 < 0$ if $3a - ab - ac + c > 0$ and $2a - ab - ac + c < 0$ and $a > 0$ or $2a - ab - ac + c > 0$ and $a < 0$.
$-1 < \omega_4 < 0$ if $(2a - ab + ac + c)(ab - ac + c) > 0$ and $a(ab - ac - c) > 0$ and $ab - ac - c \neq 0$ then $(2a - ab + ac + c)(ab - ac + c) < 0$. ■

(1,d) For $\tau < -(1 + \delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_3 < 0, \lambda_4 < -1$ and the fixed point is a **flip saddle**.

For P_3: Let us consider the following conditions:
25). $\tau_3 < -(1 + \delta_3)$ if $a \in]\widetilde{a_1}, \widetilde{a_2}[$.
26). $\lambda_{3,4} \in \mathbb{R}$ and $-1 < \lambda_3 < 0, \lambda_4 < -1$:

- $-1 < \lambda_3 < 0$ if $a \in]2, 3[$ and $-1 < \lambda_4 < 0$ if $3a - ab - ac + c > 0$ and $2a - ab - ac + c < 0$.
- $\lambda_4 < -1$ if $3a - ab - ac + c < 0$.

Lemma 3.7 *If (25),(26) holds true, then P_3 is a **flip saddle**.*

Proof. 25) $\tau_3 < -(1 + \delta_3)$

$$\frac{(2-a)a + 2a - ab - ac + c}{a} < -\frac{(2-a)(2a - ab - ac + c) + a}{a}$$

$a \neq 0$ then

$$\begin{aligned}
(2-a)a + 2a - ab - ac + c &< -\left[(2-a)(2a - ab - ac + c) + a\right]. \\
-a^2 + 4a - ab - ac + c &< 2a^2 - a^2b - a^2c - 4a + 2ab + 2ac - 2c - a. \\
-3a^2 + a^2b + a^2c - 3ab - 3ac + 3c + 9a &< 0. \\
a^2(b + c - 3) - a(3b + 3c - 9) &< 0.
\end{aligned}$$

$\Delta = (3b + 3c - 9)^2$. If $\Delta < 0$ then no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ since $(3b + 3c - 9)^2 \geq 0$ so

we get:
$\widetilde{a_1} = \frac{3b+3c-9}{b+c-3}, \widetilde{a_2} = 0$. Since $a \neq 0$ and $a^2(b + c - 3) - a(3b + 3c - 9) > 0$ then $\tau_1 < -(1 + \delta_1)$ if $a \in]\widetilde{a_1}, \widetilde{a_2}[$.
26)
a) $\lambda_{3,4} \in \mathbb{R}$.
b) $-1 < \lambda_3 < 0, \lambda_4 < -1$:
$-1 < \lambda_3 < 0$:

$\lambda_3 > -1$ if $2 - a > -1$ then $a < 3$.
$\lambda_3 < 0$ if $2 - a < 0$ then $a > 2$.
$\lambda_4 < -1$if $3a - ab - ac + c < 0$. ■
For P_4: Let us consider the following conditions:
25). $\tau_4 < -(1 + \delta_4)$ if $-(a-3)(-a^2b^2 + 2a^2bc + 3a^2b - a^2c^2 - 3a^2c + ac + c^2) < 0$.
26). $\omega_{3,4} \in \mathbb{R}$ and $-1 < \omega_3 < 0, \omega_4 < -1$:

- $-1 < \omega_3 < 0$ if $3a - ab - ac + c > 0$,$2a - ab - ac + c < 0$ and $a > 0$ or $2a - ab - ac + c > 0$ and $a < 0$.
- $\omega_4 < -1$ if $(2a - ab + ac + c)(ab - ac + c) < 0$ and $a(ab - ac - c) < 0$.

Lemma 3.8 *If (25),(26) holds true, then P_4 is a **flip saddle**.*

Proof. 25) $\tau_4 < -(1 + \delta_4)$.

$$\begin{aligned} & -\frac{(2-a)(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c)}{a(ab - ac - c)} \\ > & \frac{a(2-a)(ab - ac - c) + (ab - ac + c)(2a - ab + ac + c)}{a(ab - ac - c)}. \end{aligned}$$

If $ab - ac - c \neq 0$ then

$$\begin{aligned} & -[(2-a)(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c)] \\ > & a(2-a)(ab - ac - c) + (ab - ac + c)(2a - ab + ac + c). \end{aligned}$$

Then $\tau_4 < -(1 + \delta_4)$ if

$$\begin{aligned} & a(2-a)(ab - ac - c) + (ab - ac + c)(2a - ab + ac + c) \\ & + (2-a)(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c) \\ < & 0. \end{aligned}$$

26)
a) $\omega_{3,4} \in \mathbb{R}$.
b) $-1 < \omega_3 < 0, \omega_4 < -1$.
$-1 < \omega_3 < 0$:
$-1 < \omega_3$ if $2a - ab - ac + c > -a$ then $3a - ab - ac + c > 0$.
$\omega_3 < 0$ if $2a - ab - ac + c < 0$ and $a > 0$ or $2a - ab - ac + c > 0$ and $a < 0$.
$-1 > \omega_4$ if $(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c) < 0$ then $(2a - ab + ac + c)(ab - ac + c) < 0$ and $a(ab - ac - c) < 0$. ■

(1,e) For $0 < \tau < 2\sqrt[2]{\delta}$, then the Jacobian matrix has two complex eigenvalues $|\lambda_3|, |\lambda_4| < 1$ and the fixed point is a **clockwise spiral**.

For P_3: Let us consider the following conditions:
27). $0 < \tau_3 < 2\sqrt[2]{\delta_3}$.
28). $\lambda_{3,4} \in \mathbb{C}$ and $|\lambda_3|, |\lambda_4| < 1$.
we have from (3.9):

$\lambda_3 = 2 - a$ and $\lambda_4 = \frac{2a-ab-ac+c}{a}$ and comme $a, b, c \in \mathbb{R}$ so $\lambda_3, \lambda_4 \notin \mathbb{C}$.
P_3 is not a **clockwise spiral**.
For P_4: Let us consider the following conditions:
27). $0 < \tau_4 < 2\sqrt[2]{\delta_4}$.
28). $\omega_{3,4} \in \mathbb{C}$ and $|\omega_3|, |\omega_4| < 1$.
Since $a, b, c \in \mathbb{R}$ then $\omega_{3,4} \notin \mathbb{C}$.
P_4 is not a **clockwise spiral.**

(1,g) For $-2\sqrt[2]{\delta} < \tau < 0$, then the Jacobian matrix has two complex eigenvalues $|\lambda_3|, |\lambda_4| < 1$ and the fixed point is a **counter-clockwise spiral**.

For P_3: Let us consider the following conditions:

29). $-2\sqrt[2]{\delta_3} < \tau_3 < 0$.
30). $\lambda_{3,4} \in \mathbb{C}$ and $|\lambda_3|, |\lambda_4| < 1$.

Since $a, b, c \in \mathbb{R}$ so $\lambda_3, \lambda_4 \notin \mathbb{C}$. So P_3 is not a **counter-clockwise spiral**

For P_4: Let us consider the following conditions:
29). $-2\sqrt[2]{\delta_4} < \tau_4 < 0$.
30). $\omega_{3,4} \in \mathbb{C}$ and $|\omega_3|, |\omega_4| < 1$.
Since $a, b, c \in \mathbb{R}$ then $\omega_{3,4} \notin \mathbb{C}$. So P_4 is not a **counter-clockwise spiral** under the previous conditions.
(2) For negative determinant:

(2,a) For $-(1+\delta) < \tau < (1+\delta)$, then the Jacobian matrix has two real eigenvalues $-1 < \lambda_3 < 0, 0 < \lambda_4 < 1$ and the fixed point is a **flip attractor**.

For P_3: Let us consider the following conditions:
31). $-(1+\delta_3) < \tau_3 < (1+\delta_3)$ if $a^2 - 3ab + -4ac - c + 9a + a^2b + a^2c > 0$.

32). $\lambda_{3,4} \in \mathbb{R}$ and $-1 < \lambda_3 < 0, 0 < \lambda_4 < 1$.
$0 < \lambda_3 < 1$ if $a \in]1, 2[$.
$\lambda_4 > 1$ if $a - ab - ac + c > 0$.

Lemma 3.9 *If (31),(32) holds true, then P_3 is a **flip attractor**.*

Proof. 31) $-(1 + \delta_3) < \tau_3 < (1 + \delta_3)$: We have:

$$\tau_3 < 1 + \delta_3 \Leftrightarrow \frac{(2-a)a + 2a - ab - ac + c}{a} < \frac{(2-a)(2a - ab - ac + c) + a}{a}.$$

If $a \neq 0$ then

$$\begin{aligned}
(2-a)a + 2a - ab - ac + c &< (2-a)(2a - ab - ac + c) + a. \\
-a^2 + 4a - ab - ac + c &< -2a^2 + a^2b + a^2c + 4a - 2ab - 3ac + 2c + a. \\
a^2 + ab + 2ac - c - a - a^2b - a^2c &< 0. \\
a^2(1 - b - c) - a(1 - b - 2c) - c &< 0.
\end{aligned}$$

We have $\Delta = (1 - b - c)^2 + 4c(1 - b - 2c)$. If $\Delta < 0$ then no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ then we have $c > 0$ and $1 - b - 2c > 0$ or $1 - b - 2c < 0$ and $c < 0$ so we get:

$$\begin{cases} a_1 = \frac{1-b-2c+\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \\ a_2 = \frac{1-b-2c-\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \end{cases}$$

Since $a \neq 0$ and $a^2(1 - b - c) - a(1 - b - 2c) - c > 0$ then $\tau_3 > 1 + \delta_3$ if $a \in]a_1, a_2[$. We have:

$$\tau_3 > -(1 + \delta_3) \Leftrightarrow \frac{(2-a)a + 2a - ab - ac + c}{a} > -\frac{(2-a)(2a - ab - ac + c) + a}{a}.$$

If $a \neq 0$ then $a^2 - 3ab + -4ac - c + 9a + a^2b + a^2c > 0$.
32)
a) $\lambda_{3,4} \in \mathbb{R}$.
b) $-1 < \lambda_3 < 0, 0 < \lambda_4 < 1$.
$-1 < \lambda_3 < 0 : \lambda_3 > 0$ if $a < 2$ and $\lambda_3 < 1$ if $a > 1$. So $0 < \lambda_3 < 1$ if $a \in]1, 2[$.
$0 < \lambda_4 < 1 : \lambda_4 > 1$ if $a > 0$ and $2a - ab - ac + c > 0$ and $\lambda_4 > 1$ if $2a - ab - ac + c > a$ then $a - ab - ac + c > 0$. ■
For P_4: Let us consider the following conditions:
31). $-(1 + \delta_4) < \tau_4 < (1 + \delta_4)$.
32). $\omega_{3,4} \in \mathbb{R}$ and $-1 < \omega_3 < 0, 0 < \omega_4 < 1$.

- $0 < \omega_3 < 1$:

$\omega_3 < 0$ if $a < 0$ and $2a - ab - ac + c < 0$ or $a > 0$ and $2a - ab - ac + c > 0$.
$\omega_3 > -1$ if $a - ab - ac + c < 0$.

- $0 < \omega_4 < 1$:

$\omega_4 < 1$ if $(2a - ab + ac + c)(ab - ac + c) - a < 0$.
$\omega_4 > 0$ if $(2a - ab + ac + c)(ab - ac + c) > 0$ and $a > 0$ or $(2a - ab + ac + c)(ab - ac + c) < 0$ and $a < 0$.

Lemma 3.10 *If (31),(32) holds true, then P_4 is a **flip attractor**.*

Proof. 31) $-(1 + \delta_4) < \tau_4 < (1 + \delta_4)$
$\tau_4 < (1 + \delta_4)$:

$$\frac{a(ab - ac - c) + (2 - a)(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)} > \frac{(2a - a^2)(ab - ac - c) + (2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)}.$$

Since $a \neq 0$ and $ab - ac - c \neq 0$ then:

$$(a - 1)\left(-a^2b + a^2c + a^2b^2 + a^2c^2 - 3ac - c^2 - 2a^2bc\right) > 0.$$

$-(1 + \delta_4) < \tau_4$ if

$$-\frac{a(ab - ac - c) + (2 - a)(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)} < \frac{(2a - a^2)(ab - ac - c) + (2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)}.$$

Since $a \neq 0$ and $ab - ac - c \neq 0$ then:

$$(a - 3)\left(-3a^2b + 3a^2c + a^2b^2 + a^2c^2 - ac - c^2 - 2a^2bc\right) > 0.$$

32)
a) $\omega_{3,4} \in \mathbb{R}$.
b) $-1 < \omega_3 < 0, 0 < \omega_4 < 1$.
$-1 < \omega_3 < 0$:
$\omega_3 < 0$ if $a < 0$ and $2a - ab - ac + c > 0$ or $a > 0$ and $2a - ab - ac + c < 0$.

$\omega_3 > -1$if $3a - ab - ac + c > 0$.
$0 < \omega_4 < 1$:
$\omega_4 < 1$if $(2a - ab + ac + c)(ab - ac + c) - a < 0$.
$\omega_4 > 0$ if $(2a - ab + ac + c)(ab - ac + c) > 0$ and $a > 0$ or $(2a - ab + ac + c)(ab - ac + c) < 0$ and $a < 0$. ■

(2,b) For $\tau > (1 + \delta)$, then the Jacobian matrix has two real eigenvalues $1 < \lambda_3, -1 < \lambda_4 < 0$ and the fixed point is a **flip saddle**.

For P_3: Let us consider the following conditions:
33). $\tau_3 > (1 + \delta_3)$ if $a \in]-\infty, a_1[\cup]a_2, +\infty[$.
34). $\lambda_{3,4} \in \mathbb{R}$ and $1 < \lambda_3, -1 < \lambda_4 < 0$:

- $1 < \lambda_3$ if $a < 1$.
- $-1 < \lambda_4 < 0$ if $2a - ab - ac + c < 0$ and $a > 0$ or $a < 0$ and $2a - ab - ac + c > 0$.

Lemma 3.11 *If (33),(34) holds true, then P_3 is a **flip saddle**.*

Proof. 33) $\tau_3 > (1 + \delta_3)$:

$$\tau_3 > 1 + \delta_3 \Leftrightarrow \frac{(2 - a)a + 2a - ab - ac + c}{a} > \frac{(2 - a)(2a - ab - ac + c) + a}{a}.$$

Since $a \neq 0$ then

$$\begin{aligned}
(2 - a)a + 2a - ab - ac + c &> (2 - a)(2a - ab - ac + c) + a. \\
-a^2 + 4a - ab - ac + c &> -2a^2 + a^2b + a^2c + 4a - 2ab - 3ac + 2c + a. \\
a^2 + ab + 2ac - c - a - a^2b - a^2c &> 0. \\
a^2(1 - b - c) - a(1 - b - 2c) - c &> 0.
\end{aligned}$$

We have $\Delta = (1 - b - c)^2 + 4c(1 - b - 2c)$. If$\Delta < 0$, then no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ then we have $c > 0$ and $1 - b - 2c > 0$ or $1 - b - 2c < 0$ and $c < 0$ so we get:

$$\begin{cases} a_1 = \frac{1-b-2c+\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \\ a_2 = \frac{1-b-2c-\sqrt{(1-b-2c)^2+4c(1-b-c)}}{2(1-b-c)} \end{cases} \tag{3.15}$$

Since $a \neq 0$ and $a^2(1 - b - c) - a(1 - b - 2c) - c > 0$ then $\tau_3 > 1 + \delta_3$ if $a \in]-\infty, a_1[\cup]a_2, +\infty[$.
34)
a) $\lambda_{3,4} \in \mathbb{R}$.

b) $1 < \lambda_3, -1 < \lambda_4 < 0$.
$\lambda_3 > 1$ if $a < 1$.
$-1 < \lambda_4 < 0$:
$-1 < \lambda_4$ if $3a - ab - ac + c > 0$.
$\lambda_4 < 0$ if $2a - ab - ac + c < 0$ and $a > 0$ or $a < 0$ and $2a - ab - ac + c > 0$. ■
For P_4: Let us consider the following conditions:
33). $\tau_4 > (1 + \delta_4)$ if $(1 - a)\left[(2a - ab + ac + c)(ab - ac + c) - (a^2b - a^2c - ac)\right] > 0$.
34). $\omega_{3,4} \in \mathbb{R}$ and $1 < \omega_3, -1 < \omega_4 < 0.1 < \lambda_3, -1 < \lambda_4 < 0$.

- $1 < \omega_3$: If $a - ab - ac + c > 0$.

- $-1 < \omega_4 < 0$: If $(2a - ab + ac + c)(ab - ac + c) + a > 0$ and $(2a - ab + ac + c)(ab - ac + c) < 0$

Lemma 3.12 *If (33),(34) holds true, then P_4 is a **flip saddle**.*

Proof. 33)$\tau_4 > (1 + \delta_4)$:

$$\begin{aligned} & \frac{a(ab - ac - c) + (2 - a)(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)} \\ > & \frac{(2a - a^2)(ab - ac - c) + (2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)} \end{aligned}$$

Since $a \neq 0$ and $ab - ac - c \neq 0$ then:

$$\begin{aligned} (2a - ab + ac + c)(ab - ac + c)(1 - a) - a(ab - ac - c)(1 - a) &> 0. \\ (1 - a)\left[(2a - ab + ac + c)(ab - ac + c) - \left(a^2b - a^2c - ac\right)\right] &> 0. \end{aligned}$$

34)
a) $\omega_{3,4} \in \mathbb{R}$.
b) $1 < \omega_3, -1 < \omega_4 < 0$.
$\omega_3 > 1$ if $a - ab - ac + c > 0$.
$-1 < \omega_4 < 0$:
$\omega_4 > -1$ if $(2a - ab + ac + c)(ab - ac + c) > -a$ if $(2a - ab + ac + c)(ab - ac + c) + a > 0$.
$\omega_4 < 0$ if $(2a - ab + ac + c)(ab - ac + c) < 0$. ■

(2,c) For $\tau < -(1 + \delta)$, then the Jacobian matrix has two real eigenvalues $0 < \lambda_3 < 1, \lambda_4 < -1$ and the fixed point is a **flip saddle**.

For P_3: Let us consider the following conditions:
35). $\tau_3 < -(1+\delta_3)$ if $a \in]-\infty, \widehat{a_1}[\cup]\widehat{a_2}, +\infty[$.
36. $\lambda_{3,4} \in \mathbb{R}$ and $0 < \lambda_3 < 1, \lambda_4 < -1$:

- $0 < \lambda_3 < 1$ if $a \subset]1, 2[$.
- $\lambda_4 < -1$ if $3a - ab - ac + c < 0$.

Lemma 3.13 *if (35),(36) holds true, then P_3 is a **flip saddle**.*

Proof. 35) $\tau_3 < -(1+\delta_3)$:

$$\tau_3 > -(1+\delta_3) \Leftrightarrow \frac{(2-a)a + 2a - ab - ac + c}{a} > -\frac{(2-a)(2a - ab - ac + c) + a}{a}.$$

Since $a \neq 0$ then

$$\begin{aligned}
(2-a)a + 2a - ab - ac + c &> -[(2-a)(2a - ab - ac + c) + a]. \\
-a^2 + 4a - ab - ac + c &> 2a^2 - a^2b - a^2c - 4a + 2ab + 3ac - 2c - a. \\
-3a^2 - 3ab - 4ac + 3c + a + a^2b + a^2c &> 0. \\
a^2(-3 + b + c) + a(1 - 3b - 4c) + 3c &> 0.
\end{aligned}$$

We have $\Delta = (1 - 3b - 4c)^2 - 12c(-3 + b + c)$. If $\Delta < 0$ then no solutions. If $a, b, c \in \mathbb{R}$ and $\Delta \geq 0$ then we have $c > 0$ and $-3 + b + c < 0$ or $-3 + b + c > 0$ and $c < 0$ so we get:

$$\begin{cases} \widehat{a_1} = \frac{3b+4c-1-\sqrt{(1-3b-4c)^2 - 12c(-3+b+c)}}{2(-3+b+c)} \\ \widehat{a_2} = \frac{3b+4c-1+\sqrt{(1-3b-4c)^2 - 12c(-3+b+c)}}{2(-3+b+c)} \end{cases} \tag{3.16}$$

Since $a \neq 0$ and $a^2(-3 + b + c) + a(1 - 3b - 4c) + 3c > 0$ then $\tau_1 > 1 + \delta_1$ if $a \in]-\infty, \widehat{a_1}[\cup]\widehat{a_2}, +\infty[$.
36)
a) $\lambda_{3,4} \in \mathbb{R}$.
b) $0 < \lambda_3 < 1, \lambda_4 < -1$.
$0 < \lambda_3 < 1$ if $a \in (1, 2)$
$\lambda_4 < -1$ if $3a - ab - ac + c < 0$. ■
For P_4: Let us consider the following conditions:
35). $\tau_4 < -(1+\delta_4)$ if $(3-a)[(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c)] > 0$.
36). $\omega_{3,4} \in \mathbb{R}$ and $0 < \omega_3 < 1, \omega_4 < -1$:

- $0 < \omega_3 < 1$:if $a - ab - ac + c < 0$, and if $2a - ab - ac + c > 0$ and $a > 0$ or $2a - ab - ac + c < 0$ and $a < 0$.

- $\omega_4 < -1$: if $(2a - ab + ac + c)(ab - ac + c) + a < 0$.

Lemma 3.14 *If (35),(36) holds true, then P_4 is a **flip saddle**.*

Proof. 35)$\tau_4 < -(1 + \delta_4)$

$$\frac{(2a - a^2)(ab - ac - c) + (2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)}$$
$$> -\frac{a(ab - ac - c) + (2 - a)(2a - ab + ac + c)(ab - ac + c)}{a(ab - ac - c)}.$$

Since $a \neq 0$ and $ab - ac - c \neq 0$ then:

$$\begin{aligned}(2a - ab + ac + c)(ab - ac + c)(3 - a) + a(ab - ac - c)(3 - a) &> 0.\\ (3 - a)[(2a - ab + ac + c)(ab - ac + c) + a(ab - ac - c)] &> 0.\end{aligned}$$

36)

a) $\omega_{3,4} \in \mathbb{R}$.

b) $0 < \omega_3 < 1, \omega_4 < -1$:

$0 < \omega_3 < 1$ if $a - ab - ac + c < 0$.

$\omega_3 > 0$ if $2a - ab - ac + c > 0$ and $a > 0$ or $2a - ab - ac + c < 0$ and $a < 0$.

$\omega_4 < -1$ if $(2a - ab + ac + c)(ab - ac + c) + a < 0$. ■

Conclusion

Chaos and bifurcations are fundamental concepts in nonlinear dynamical systems. Zeraoulia-Sprott's works has provided important insights into how simple, yet mathematically rich, systems can exhibit complex behaviors such as chaotic attractors and bifurcations. This study consist to understand clearly chaos and bifurcations in some Zeraoulia sprott maps and discover their bifurcations. These systems helps illustrating the core principles of chaos theory in an accessible and computationally manageable form, making them an important tool in both theoretical and applied studies of nonlinear dynamics.

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