PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

ECHAHID CHEIKH LARBI TEBESSI UNIVERSITY

FACULTY OF SCIENCES AND TECHNOLOGY

FIELD OF STUDY: SCIENCES AND TECHNOLOGY (ST)

Course Handout

Course Title : Steel Construction
Specialization : Civil Engineering
Level : 3rd Year

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Academic Year: 2024/2025

Foreword

This course manual provides a comprehensive overview of calculation methods for steel constructions, contributing to the expanding body of resources on EUROCODE 3-compliant calculation techniques for steel structures. Explicitly designed for third-year civil engineering students, it aligns with the national curriculum guidelines of the L.M.D. Academic Program, National Program for 2021–2022 (second update). As an introduction and a reference, it equips students with the terminology, foundational principles, and calculation techniques needed in steel construction analysis and design.

The manual is organised into four detailed chapters, each focusing on critical aspects of structural analysis and stability for steel members; the first Chapter, "Phenomena of Elastic Instabilities," introduces the concept of elastic instability, examining how slender structural elements under certain conditions can experience sudden and large deformations due to instability. Students are introduced to different forms of elastic instability, including lateral-torsional, flexural, and global instability. The Chapter explores the theoretical underpinnings of these phenomena, guiding students to understand when and why elastic instability may occur and providing insights into designing structures that prevent such behaviour.

The second Chapter, "Calculations for Members Subjected to Simple Compression," Focuses on compression members and delves into the principles and formulas for analysing steel members under axial compressive loads. Topics include calculating compressive resistance, factors influencing steel members' compressive capacity, and applying EUROCODE 3 rules in practical cases. Students learn to determine the safety and stability of columns and other structural members experiencing direct compression through examples and exercises.

The third Chapter, "Calculations for Members Under Combined Buckling," addresses the complexities of members subjected to combined bending and compression, where the potential for instability increases due to the interaction of forces. Concepts such as effective length, critical stress, and buckling curves are covered in depth. The Chapter guides students through applying buckling calculations to structures subject to complex load scenarios, emphasising the methods for predicting and mitigating risks associated with combined buckling.

The final Chapter, "Lateral Torsional Buckling in Steel Members," examines lateral-torsional buckling, which occurs when a steel member bends about a weak axis while

simultaneously twisting. This Chapter covers the causes and consequences of lateral-torsional buckling, emphasising factors such as cross-section shape, loading type, and length. Students are introduced to calculation methods and design strategies to counteract lateral-torsional buckling, using guidelines from EUROCODE 3 to inform safe and effective structural designs.

Each Chapter contains practical examples, step-by-step calculations, and exercises designed to reinforce students' understanding and skills in applying these methods to real-world structural analysis challenges. By the end of the manual, students will have a solid foundation in assessing the stability and safety of steel structures, preparing them for more advanced courses and professional practice in civil engineering.

Boursas Farid

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CHAPTER 01: Phenomena of Elastic Instabilities

1.1 Introduction:

Structural stability forms the foundation of safe engineering design, yet it is a complex property to define and analyse. The structural integrity of a building or framework relies on its capacity to withstand applied loads, ensuring that it meets the essential conditions of equilibrium, compatibility, and force-displacement consistency within the material. In steel structures, elastic instability is critical, presenting as a vulnerability that can compromise load-carrying ability. This phenomenon often emerges through local or global buckling, where a minimal increase in load or minor geometric shift can lead to dramatic changes in deformation and stability. Understanding elastic instabilities is vital for engineers, as these effects influence design codes and safety assessments. Elastic instabilities typically occur well before the material reaches its ultimate strength, resulting in a complex behaviour profile with implications for the structure's performance under stress. Stability analysis provides the basis for predicting critical loads and modes of failure, drawing on various methods such as equilibrium states, energy considerations, and dynamic responses. These approaches allow engineers to assess different instability types, from flexural and torsional buckling to snap-through scenarios.

Structural stability calculations are rooted in theoretical assumptions often deviate from real-world scenarios. Factors such as material imperfections, manufacturing tolerances, assembly inaccuracies, and operational changes introduce inherent uncertainties. Engineering safety assessment addresses these deviations by adopting a probabilistic approach or permissible stress methods to ensure that the final design can withstand unanticipated stresses.

1.2 Definition of Elastic Instabilities:

An instability can be defined as a loss of equilibrium or a disruption of the internal force system within a structure. Instabilities can be triggered at relatively low load levels compared to the structure's ultimate load capacity [1].

A phenomenon is said to be elastic instability if the equilibrium is lost at load levels that are considerably below the load at which material strength is exceeded and excessive deflection occurs. Elastic instabilities are of significant concern to designers of steel structures because a large proportion of design codes are based on the concept of elastic behaviour. When elastic equilibrium is lost, the remaining load-carrying capacity of the structure can be challenging to determine, and the post-buckling strength can be lower than the strength of the structure in the

linear elastic range. This concept has important implications in that, for specific structures, it may not be cost-effective to design for a load-carrying capacity beyond the buckling load.

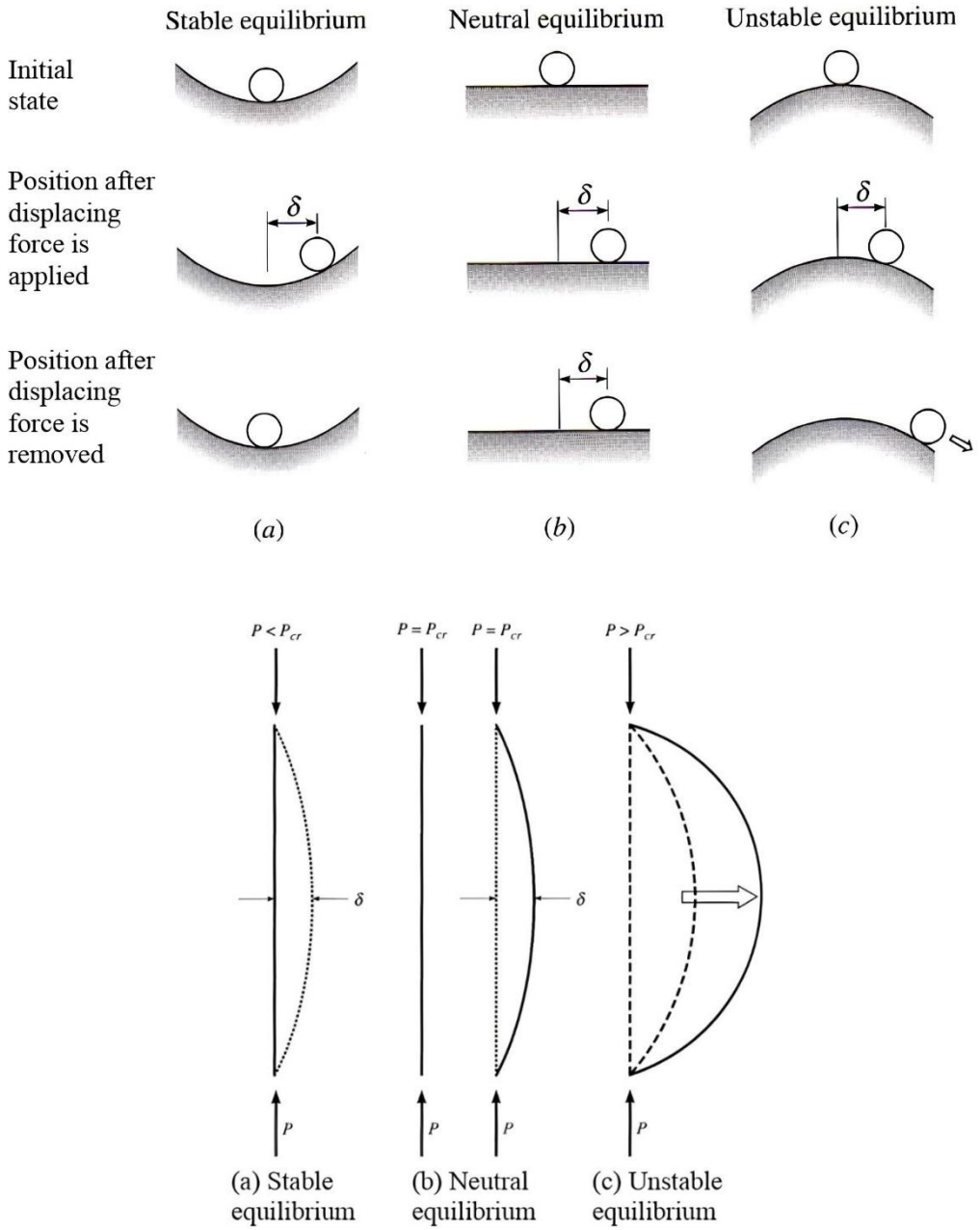


Figure 1. 1: Concept of Stability[1]

1.3 Safety Concepts:

The process of structural calculation relies heavily on a multitude of theoretical, mathematical, and physical assumptions, which, unfortunately, often diverge from real-world conditions. This

disparity arises from the sheer magnitude of inaccuracies, imperfections, and errors inherent in every stage of construction, spanning from initial design to final utilization of the structure.

These discrepancies exhibit a highly variable and entirely stochastic nature. They manifest predominantly in the following aspects:

1.3.1 Design Phase:

- Underestimation of both permanent and variable expenses.
- Inaccurate determination of nodal or support conditions, such as articulation or embedding.
- Suboptimal assembly design, where concepts of rigidity and rotational capacity are inadequately grasped.
- Neglect of dilation effects and their impact.
- The tendency towards excessive deformations is due to flawed design assumptions.

1.3.2 Manufacturing of Elements:

- Lack of precise determination of the elastic limit (f_y) of steel.
- Influence of poorly understood residual rolling stresses, which distort the calculation of resulting stresses.
- Non-constant modulus of elasticity (E) across a steel section.
- Steel's inherent lack of perfect elasticity, homogeneity, and isotropy.
- Significance of rolling tolerances, which can disrupt inertia and stress calculations by up to 15% quite readily[2].

1.3.3 Parts Transformation:

- Errors induced during part fabrication within the factory environment due to discrepancies in execution plans or workshop practices.
- Occurrence of omissions concerning secondary components (e.g., stiffeners).
- Incidence of erroneous executions, such as oversized drilling diameters.
- The presence of defective welds contributes to structural weaknesses and potential failures.
- These inherent inaccuracies and imperfections necessitate a nuanced approach to structural engineering, emphasizing thorough validation and verification processes to mitigate risks and ensure the safety and reliability of constructed systems[2].

1.3.4 Assembly on Site:

- Structural calculation methods often prioritize considerations for structures in their final installed phase, sometimes neglecting the assembly phase. This oversight can lead to various on-site challenges, such as the improper discharge of beams during lifts or collapses due to overlooked temporary braces.
- Instances of incorrect bolt tightening, including those with RH (Right-Handed) threads, discrepancies in bolt diameter and steel grade, or insufficient plate friction coefficients, can compromise the integrity of the assembled structure.

1.3.5 Operation and Maintenance by the Contracting Authority:

- Changes in the usage of premises may result in significantly higher loads on floors than initially anticipated.
- Additional loads, such as hoists that were not initially accounted for, may be introduced over time[3].
- Inadequate maintenance practices, including neglecting corrosion prevention measures for steel components, can lead to structural degradation and loss of section resistance.
- Considering real-world deviations from idealized assumptions, such as non-centred loads, non-vertical poles, non-straight beams, and stresses occurring outside the principal planes of inertia, it's inevitable that structures will experience stresses exceeding those initially calculated during their lifespan.
- To ensure construction safety, two approaches are viable:
- The first method involves deterministic calculations using "permissible stresses," wherein in-service stresses must be verified to be within a fraction of the material's ultimate stress. This method assumes known calculation parameters and is non-random.
- The second approach entails limit state calculations, where the increased or weighted in-service stresses are verified to remain below the material's ultimate stress. This method is probabilistic, introducing variable weighting coefficients and randomness[2].

1.4 Elastic Instabilities Modes in Steel Structures:

Instability in compression members is a critical condition where the member can no longer effectively withstand increasing loads, leading to a decrease in its load-carrying capacity. This phenomenon is particularly significant in structural engineering, where the stability of components is paramount for the safety and integrity of the overall structure[4].

The types of instability in a structure or its members can be categorized in several ways:

- **Buckling concerning the principal minor axis:** This occurs when a compression member, such as a column or strut, fails due to buckling along its minor axis, which is the axis with the smaller moment of inertia. The member can buckle outward in a direction perpendicular to this axis.

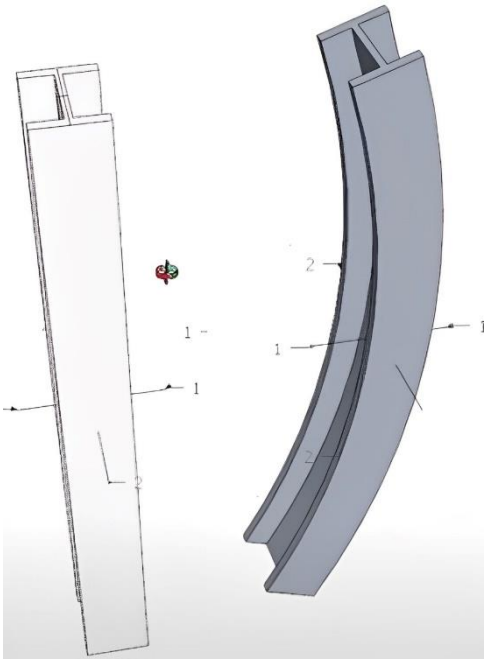


Figure 1. 2 : Buckling along its minor axis [4]

- **Buckling with respect to principal major axis:** In this case, the member fails due to buckling along its central axis, the axis with the more significant moment of inertia. Similar to minor axis buckling, the member can buckle outward along a different axis.

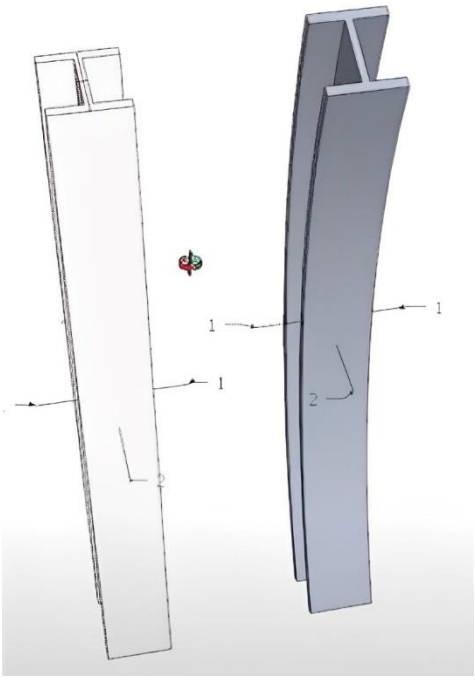


Figure 1. 3 : Buckling along its major axis[4]

- **Pure torsional instability:** Torsional instability happens when a compression member fails under torsional loading, typically due to insufficient torsional stiffness or inadequate cross-sectional properties to resist twisting forces.

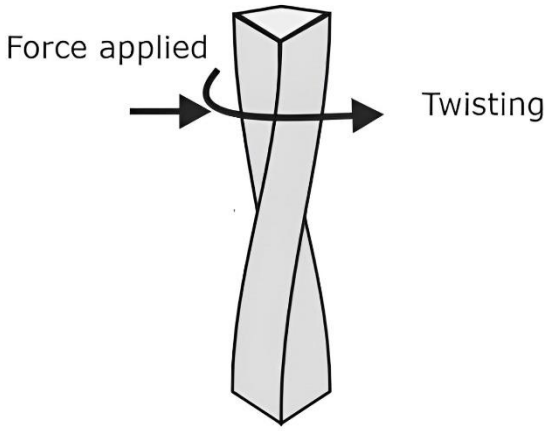


Figure 1. 4 : Pure torsion[4]

Classification of instability can also be provided based on the specific modes of buckling:

- **Flexural buckling (Euler Buckling)** is the most common form of instability. It occurs when a compression member fails due to bending under axial compression. The failure typically involves lateral deflection or buckling of the member[5].

- **Torsional buckling** refers explicitly to the instability caused by torsional loads. In this case, the member may twist or deform laterally due to the applied torsional moment.
- **Torsional-flexural buckling:** This type of instability involves a combination of flexural and torsional effects, where the member may fail due to a combination of bending and twisting.
- **Snap-through buckling:** Snap-through buckling occurs when a compression member suddenly transitions from one stable state to another unstable state, often accompanied by a rapid change in shape or configuration.

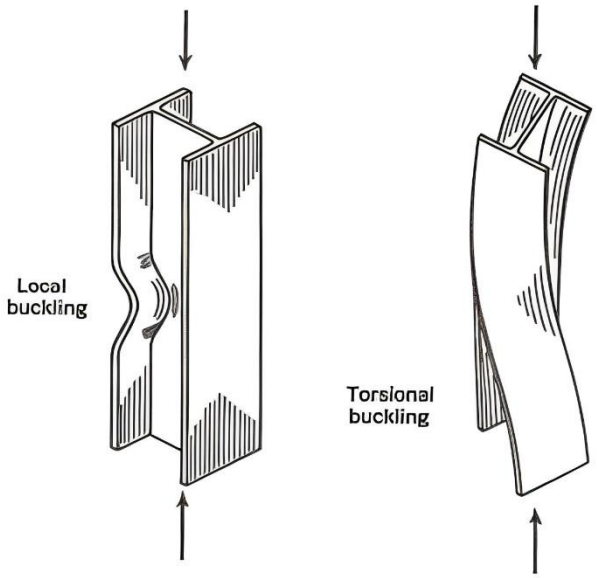


Figure 1. 5 : Unstable state of compression member[5]

1.5 Methods of analysis:

Stability analysis consists of determining the mode of structural stability loss and the corresponding load, called critical load. Four classical methods are definitely available for solving buckling problems.

1.5.1 Non-trivial Equilibrium State Approach:

This method involves analysing the equilibrium states of the structure under consideration, focusing on states other than the trivial stable equilibrium. Trivial stable equilibrium refers to the state where the structure is stable without any external loads applied.

In the non-trivial equilibrium state approach, the analysis typically involves studying the behaviour of the structure under various loading conditions, with particular attention to critical points where the structure's stability may be compromised. By identifying and analysing these

essential equilibrium states, engineers can determine the mode of loss of strength (buckling) and the corresponding critical load at which this occurs[6].

1.5.2 Work Approach:

The work approach, also known as the virtual work method or the principle of virtual displacements, is based on the principle of virtual work in mechanics. In this method, the work done by external forces and internal forces (due to deformation) is equated, considering virtual displacements of the structure subjected to small perturbations. By setting the virtual work done by external and internal forces equal to zero for equilibrium, engineers can derive the governing equations that describe the behaviour of the structure under different loading conditions, including buckling.

1.5.3 Energy Approach:

The energy approach utilizes the principle of energy conservation to analyse structures' stability. In this method, the total potential energy of the structure, including both elastic strain energy and potential energy due to applied loads, is considered. By minimizing the total potential energy concerning the degrees of freedom of the structure, engineers can obtain the critical load and corresponding mode of instability (buckling)[6].

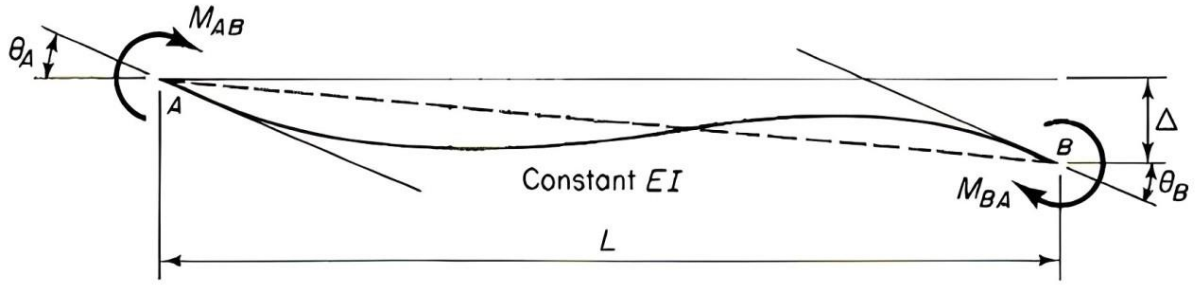
1.5.4 Kinetic or Dynamical Approach:

The kinetic or dynamical approach involves analysing the structure's dynamic behaviour, particularly its response to dynamic loading or perturbations. This method considers the structure's dynamic equilibrium and consists of solving equations of motion to predict the dynamic response, including any instabilities such as buckling. By studying the structure's dynamic characteristics, engineers can determine critical loads and modes of instability that may arise under dynamic loading conditions[6].

1.6 Critical load analysis using stability functions:

The slope deflection equation of a member related moments and slopes can be written as follows:

$$\mathbf{M_{AB}} = \frac{\mathbf{EI}}{\mathbf{L}}\left(4\boldsymbol{\theta}_{\mathbf{A}} + 2\boldsymbol{\theta}_{\mathbf{B}} - \frac{6\Delta}{\mathbf{L}}\right) \qquad \mathbf{1.1}$$



Where;

M_{AB} = moment at A

EI = flexural stiffness

L = member length;

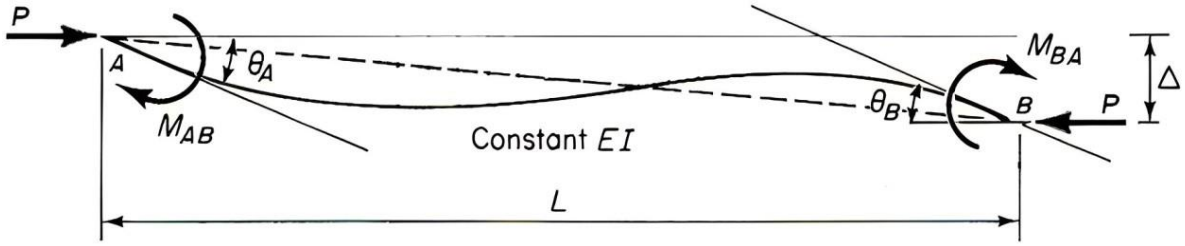
θ_A = slope at A

θ_B = slope at B

Δ = member deformation

The equation of a member subjected to axial load can be written as follows:

$$\mathbf{M_{AB} = \frac{EI}{L}\left(s\theta_A + sc\theta_B - s(1+c)\frac{\Delta}{L}\right)} \qquad \mathbf{1.2}$$



Where;

s = member stiffness

c = carry-over factor

- ✓ The parameters s and c are called" stability functions."
- ✓ The values of parameters s and c are functions of the ratio of axial load, P, and Euler's load, Pe.

1.7 Classification of cross sections:

Determining the strength of a structural steel component necessitates the designer's thorough consideration of both its **cross-sectional behaviour** and its overall member behaviour[7].

The classification system assesses the extent to which cross sections' resistance and rotational capacity are constrained by their susceptibility to **local buckling**.

Clauses 5.5.1 and 6.2 detail this classification process and specifically address the cross-sectional aspects of the design process[8].

- Under the Eurocode EC3, cross-sections are categorized into **four distinct behavioural classes**.
- This categorization hinges on various factors, including the material yield strength, the width-to-thickness ratios **(b/t_f or d/t_w)** of the individual compression parts (such as the web and flanges) within the cross-section, and the loading arrangement applied to the component.
- Local buckling phenomena are critical considerations in this context. Local buckling manifests as localized deformation, particularly noticeable in components like the flanges of I beams. Flange buckling occurs when the ratio of the flange outstands to its thickness **(b/t_f)** is relatively high, indicating a susceptibility to deformation under compressive loads.

Moreover, the web, another integral part of many structural steel components, is also prone to compressive stresses from bending. There exists a limiting ratio of width to thickness **(d/t_w)** beyond which the web may buckle, even if the component's axis remains ostensibly straight. This type of buckling is called web buckling and underscores the importance of understanding and managing local buckling phenomena in the design and assessment of structural steel components[8].

1.7.1 Definition of classes:

Classes of sections refer to categorizations established within the Eurocode 3 (EC3) standards, which aim to define the structural behaviour and performance of various cross-sectional shapes used in engineering and construction. Eurocode 3 delineates four distinct classes to characterize these sections based on their response to loading conditions and material properties[9].

In hot-rolled structural design, which commonly employs standard cross-sections, most of these profiles are typically categorized under classes 1, 2, or 3.

The four behavioural classes delineated by Eurocode 3 serve as a framework for assessing and selecting appropriate cross-sectional shapes for structural applications. Each class signifies a specific level of performance and suitability for particular design requirements, ranging from standard to more specialized applications.

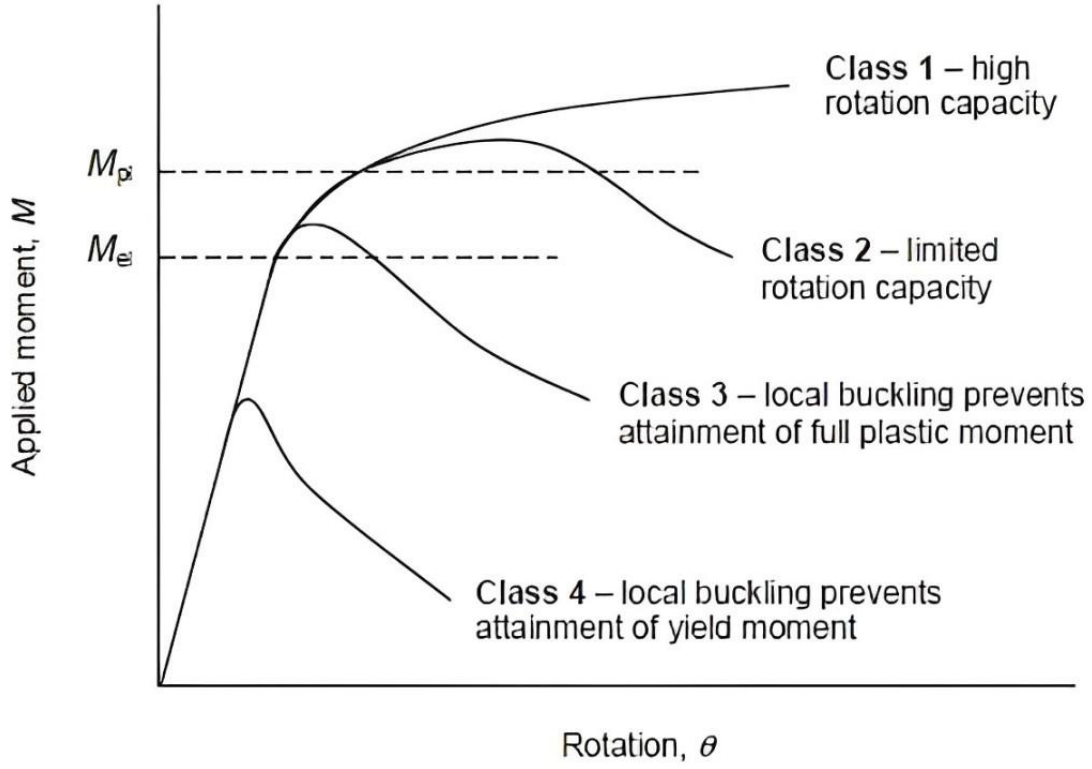


Figure 1. 6 : Moment-rotation curves for the four cross-section classifications [8].

1.7.2 Assessment of Individual parts:

The classification of structural elements is a crucial aspect of assessing their performance and Suitability for engineering applications. By EN 1993-1-1 standards, each compressed or partially compressed element within a structure undergoes individual evaluation against specific criteria outlined in Table 5.2 [8]. This table delineates the limiting width-to-thickness ratios applicable to elements categorized under Classes 1, 2, and 3.

If an element's width-to-thickness ratio surpasses the prescribed limits for Class 3, indicating a potential compromise in its structural integrity, it is automatically designated as Class 4. This

reclassification underscores the significance of adhering to stringent dimensional constraints to ensure the structural robustness and safety of the overall system.

Moreover, the prescribed width-to-thickness ratios are subject to modification by introducing a factor ε, which is contingent upon the material's nominal yield strength (f_y). This adjustment factor accounts for variations in material properties, allowing for a more tailored assessment of structural elements based on their specific material characteristics.

It's important to note that the section classification is ultimately determined by the element with the lower performance level, emphasizing the need for comprehensive evaluation and consideration of all contributing factors to ascertain the most accurate classification for structural elements within a given system[9].

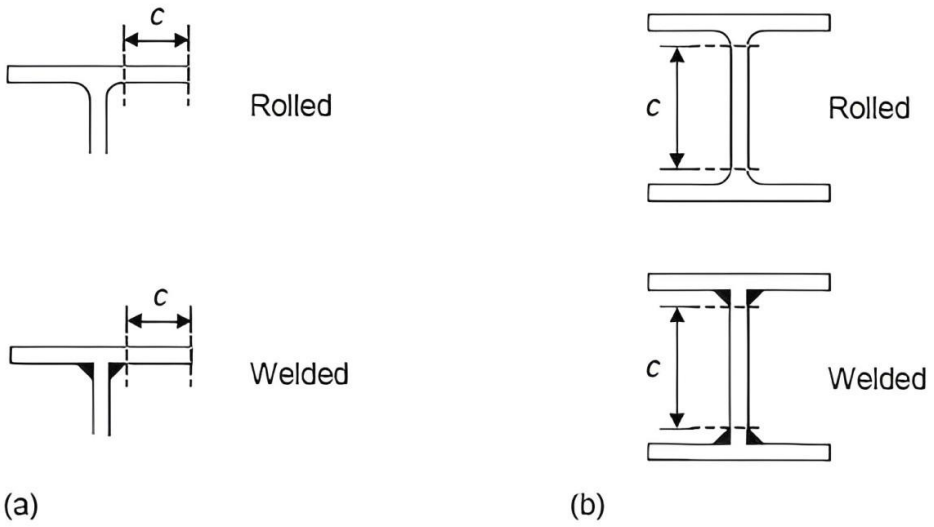


Figure 1. 7 : Definition of compression width c for common cases a) outstand flanges b) internal compression parts

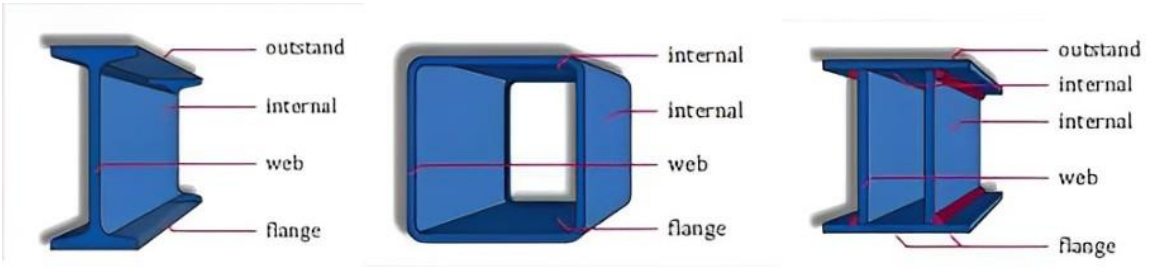
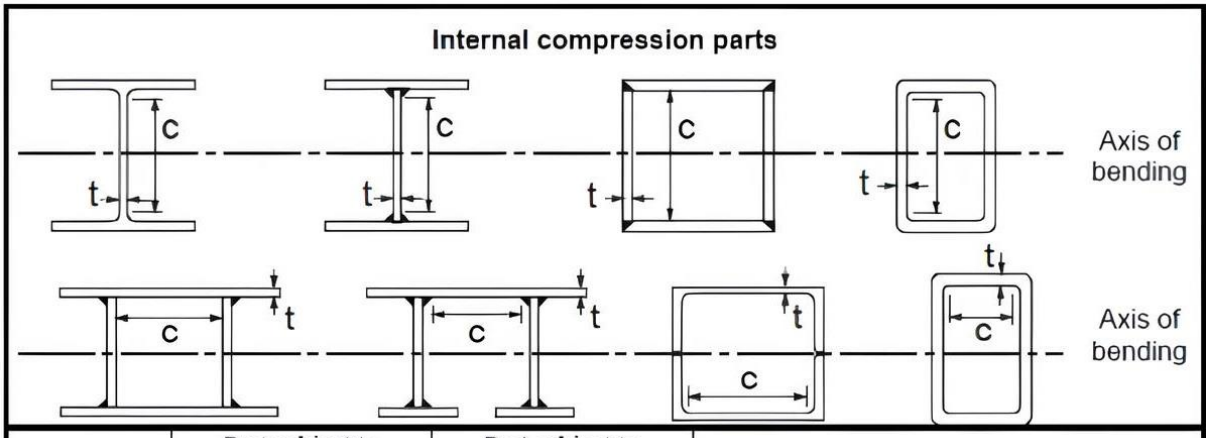
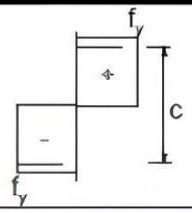
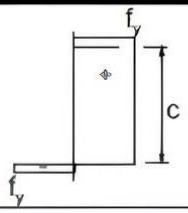
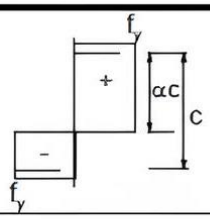
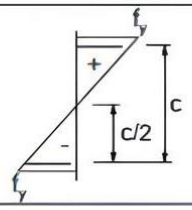
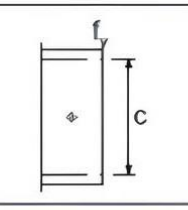
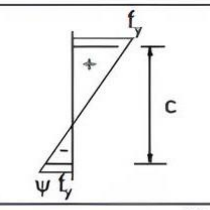


Figure 1. 8 : Compression parts [9]

Table 1. 1 : (Sheet 1 of 3) Maximum width-to-thickness ratios for compression parts [8]



Class	Part subject to bending	Part subject to compression	Part subject to bending and compression
Stress distribution in parts (compression positive)			
1	$c/t \le 72\varepsilon$	$c/t \le 33\varepsilon$	when $\alpha > 0{,}5$: $c/t \le \frac{396\varepsilon}{13\alpha - 1}$ when $\alpha \le 0{,}5$: $c/t \le \frac{36\varepsilon}{\alpha}$
2	$c/t \le 83\varepsilon$	$c/t \le 38\varepsilon$	when $\alpha > 0{,}5$: $c/t \le \frac{456\varepsilon}{13\alpha - 1}$ when $\alpha \le 0{,}5$: $c/t \le \frac{41{,}5\varepsilon}{\alpha}$
Stress distribution in parts (compression positive)			
3	$c/t \le 124\varepsilon$	$c/t \le 42\varepsilon$	when $\psi > -1$: $c/t \le \frac{42\varepsilon}{0{,}67 + 0{,}33\psi}$ when $\psi \le -1^{*)}$: $c/t \le 62\varepsilon(1-\psi)\sqrt{(-\psi)}$

$\varepsilon = \sqrt{235/f_y}$	f_y	235	275	355	420	460
	ε	1,00	0,92	0,81	0,75	0,71

*) $\psi \le -1$ applies where either the compression stress $\sigma \le f_y$ or the tensile strain $\varepsilon_y > f_y/E$

Table 1. 2: (Sheet 2 of 3) Maximum width-to-thickness ratios for compression parts [8]

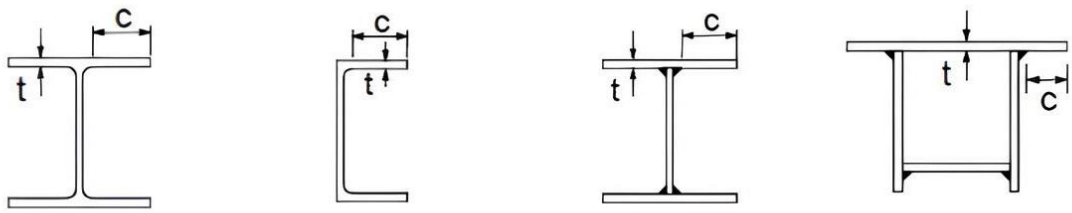
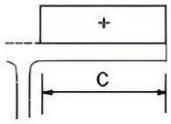
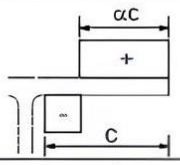
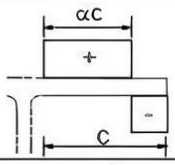
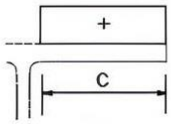
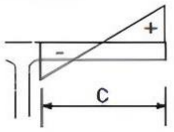
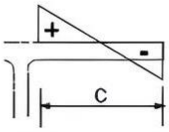
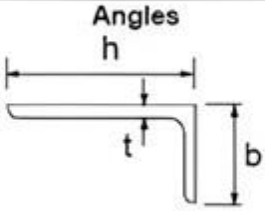
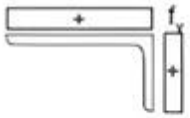
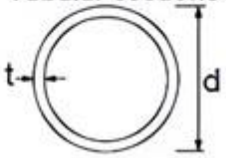
<table>
<tr><td colspan="7">Outstand flanges

Rolled sections Welded sections</td></tr>
<tr><td rowspan="2">Class</td><td rowspan="2">Part subject to compression</td><td colspan="5">Part subject to bending and compression</td></tr>
<tr><td colspan="2">Tip in compression</td><td colspan="3">Tip in tension</td></tr>
<tr><td>Stress distribution in parts (compression positive)</td><td></td><td colspan="2"></td><td colspan="3"></td></tr>
<tr><td>1</td><td>$c/t \le 9\varepsilon$</td><td colspan="2">$c/t \le \frac{9\varepsilon}{\alpha}$</td><td colspan="3">$c/t \le \frac{9\varepsilon}{\alpha\sqrt{\alpha}}$</td></tr>
<tr><td>2</td><td>$c/t \le 10\varepsilon$</td><td colspan="2">$c/t \le \frac{10\varepsilon}{\alpha}$</td><td colspan="3">$c/t \le \frac{10\varepsilon}{\alpha\sqrt{\alpha}}$</td></tr>
<tr><td>Stress distribution in parts (compression positive)</td><td></td><td colspan="2"></td><td colspan="3"></td></tr>
<tr><td>3</td><td>$c/t \le 14\varepsilon$</td><td colspan="5">$c/t \le 21\varepsilon\sqrt{k_\sigma}$
For k_σ see EN 1993-1-5</td></tr>
<tr><td rowspan="2">$\varepsilon = \sqrt{235/f_y}$</td><td>$f_y$</td><td>235</td><td>275</td><td>355</td><td>420</td><td>460</td></tr>
<tr><td>ε</td><td>1,00</td><td>0,92</td><td>0,81</td><td>0,75</td><td>0,71</td></tr>
</table>

Table 1. 3 : (Sheet 3 of 3) Maximum width-to-thickness ratios for compression parts [8]

<table>
<tr><td colspan="7">Angles


Refer also to "Outstand flanges" (see sheet 2 of 3)
Does not apply to angles in continuous contact with other components</td></tr>
<tr><td>Class</td><td colspan="6">Section in compression</td></tr>
<tr><td>Stress distribution across section (compression positive)</td><td colspan="6">
</td></tr>
<tr><td>3</td><td colspan="6">[(1) $h/t \le 15\varepsilon$ and $\frac{b+h}{2t} \le 11.5\varepsilon$ (1)]</td></tr>
<tr><td colspan="7">Tubular sections

</td></tr>
<tr><td>Class</td><td colspan="6">Section in bending and/or compression</td></tr>
<tr><td>1</td><td colspan="6">$d/t \le 50\varepsilon^2$</td></tr>
<tr><td>2</td><td colspan="6">$d/t \le 70\varepsilon^2$</td></tr>
<tr><td>3</td><td colspan="6">$d/t \le 90\varepsilon^2$
NOTE For $d/t > 90\varepsilon^2$ see EN 1993-1-6.</td></tr>
<tr><td rowspan="3">$\varepsilon = \sqrt{235/f_y}$</td><td>$f_y$</td><td>235</td><td>275</td><td>355</td><td>420</td><td>460</td></tr>
<tr><td>ε</td><td>1.00</td><td>0.92</td><td>0.81</td><td>0.75</td><td>0.71</td></tr>
<tr><td>ε^2</td><td>1.00</td><td>0.85</td><td>0.66</td><td>0.56</td><td>0.51</td></tr>
</table>

1.7.3 Overall Cross-Section Classification:

EC3 provides two methods for defining the overall cross-section classification:

- The overall classification is determined by the highest (least favourable) class of its parts, with the following exceptions (Clause 5.5.2):
  - Cross-sections with Class 3 webs and Class 1 or 2 flanges may be classified as Class 2 cross-sections with an effective web by 6.2.2.4 of EC3-1-1.
  - If the web is assumed to carry only shear force (and does not contribute to the bending or axial resistance of the cross-section), the classification may be based on that of the flanges (Class 1 is not allowed in this case).

- The overall classification can also be defined by quoting the flange and web classifications separately [8].

1.7.4 Class 4 Cross-Section:

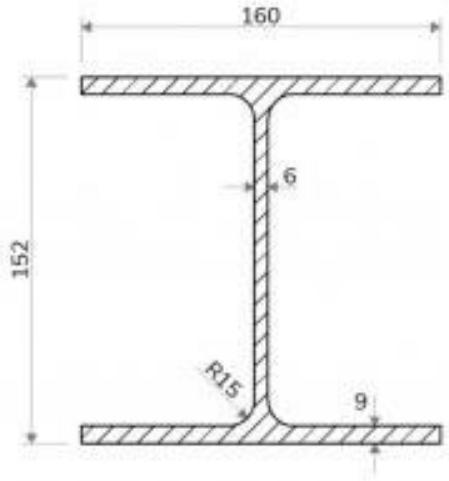
- **Susceptibility to Local Buckling:** Class 4 cross-sections are characterized by slender elements that are prone to local buckling even within the elastic material range. Under compressive loads, these slender parts of the section can buckle before the material yields, reducing the overall load-carrying capacity of the cross-section [9].
- **Reduction in Resistance Due to Local Buckling:** To account for the reduction in resistance caused by local buckling in Class 4 cross-sections, an "effective width" concept is applied. The effective width is a reduced width of the compression elements, calculated to represent the part of the section that effectively resists the applied loads after accounting for the effects of local buckling. This ensures a more accurate and conservative assessment of the section's load-bearing capacity [10].
- **Reference to Relevant Eurocode Parts:** The detailed calculation procedures for determining the effective width and assessing the resistance of Class 4 cross-sections are not provided in Eurocode 3 Part 1-1. Instead, the user is referred to other parts of the Eurocode for specific types of sections:
  - **Part 1.3:** For cold-formed sections, the thin-walled nature of the sections increases susceptibility to local buckling.
  - **Part 1.5:** This is for hot-rolled and fabricated sections, which may include slender plate elements that require special consideration for local buckling.
  - **Part 1.6:** For circular hollow sections (CHS), which, due to their geometry, also require specific local buckling considerations.

1.7.5 Examples of cross section classification:

- **Example 01: Cross-section classification under pure compression**

Determine the classifications and resistance $N_{c,Rd}$ for a HEA160 in pure compression, assuming grade S235 steel.

$h = 152$ mm
$b = 160$ mm
$t_w = 6$ mm
$t_f = 9$ mm
$r = 15$ mm
$A = 3877$ mm^2



$$\varepsilon = \sqrt{235/f_y} = \sqrt{235/235} = 1$$

- **Web-internal compression part (Table 5.2 sheet 1)**

C = (h-2tf-2r) =(152-2.9-2.15) = 104 mm

C /t_w = 104/6 = 17.33

Limit for Class 1 web, 33ε = 33 > 17.33

Hence web are Class 1

- **Outstand flanges (Table 5.2, sheet 2)**

C = (b-t_w-2r)/2 = (160-6-2.15)/2 = 62 mm

C / t_f = 62/9 = 6.88

Limit for class 1 flange 9ε = 9 x 1 = 9 > 6.88

Hence flanges are Class 1

Overall cross-section classification is therefore **Class 1**

- **Example 02: Cross-section classification under bending**

A welded I section is to be designed in bending. Determine the classification for a welded section with 200 x 20 mm flanges and a 600 x 6 mm web. Assuming grade S275 steel.

$$\varepsilon = \sqrt{235/f_y} = \sqrt{235/275} = 0.92$$

- **Web-internal compression part (Table 5.2 sheet 1)**

C = (h-2tf) =(600-2.20) = 560 mm

C /t_w = 560/6 = 93.33

Limit for Class 3 web, 124ε = 114.08 > 93.33

Hence web are Class 3

- **Outstand flanges (Table 5.2, sheet 2)**

C = (b-t_w)/2 = (200-6)/2 = 97 mm

C / t_f = 97/20 = 4.85

Limit for class 1 flange 9ε = 9 x 0.92 = 8.28 > 4.85

Hence flanges are Class 1

Overall cross-section classification is therefore **Class 3**

- **Example 03: Cross-section classification under combined bending and compression**

A member must be designed to carry combined bending and axial load. In the presence of a major axis (y-y) bending moment and an axial force of 100kN, determine the cross-section classification of a HEB220 in grade S355 steel.

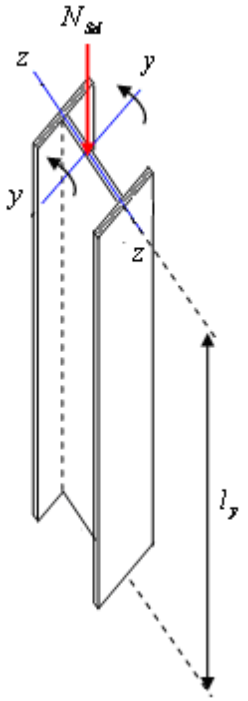
Classification under combined bending and axial force

Should be classified based on the actual stress distribution of the combined loadings

- For simplicity, the initial check was carried out under the severe loading condition of pure axial compression.
- If the section is classified as Class 1 or 2, there is nothing to be gained by conducting additional calculations with the actual pattern of stress
- If the classification is Class 3 or 4, it is advisable for the economy to conduct a more precise classification under combined loading

- **Cross-section classification (Clause 5.5.2)**

$$\varepsilon = \sqrt{235/f_y} = \sqrt{235/355} = 0.81$$



- **Web-internal compression part (Table 5.2 sheet 1)**

C = (h-2tf-2r) =(220-2.16-2.18) = 152 mm

C /t_w = 152/9.5 = 18.66

Limit for Class 1 web, 33ε = 33 x 0.81 = 26.73 > 18.66

For Class 1:

$$\frac{d}{t} \leq \frac{396\varepsilon}{(13\alpha - 1)} \quad pour \quad \alpha > 0{,}5$$

$$\frac{d}{t} \leq \frac{36\varepsilon}{\alpha} \quad pour \quad \alpha \leq 0{,}5$$

$$\alpha = \frac{1}{d}\left(\frac{h}{2} + \frac{1}{2}\frac{N_{sd}}{2t_w f_y} - (t_f - r)\right) \leq 1$$

$$\alpha = \frac{1}{152}\left(\frac{220}{2} + \frac{1}{2}\frac{100000}{2.9{,}5.355} - (16 - 18)\right) = 0{,}78$$

$\alpha > 0{,}5$ so $\frac{152}{9{,}5} = 16 \leq \frac{396}{(13.0{,}81-1)} = 41{,}55$

Hence web is Class 1

- **Outstand flanges (Table 5.2, sheet 2)**

C = (b-t_w-2r)/2 = (220-9.5-2.18)/2 = 87.25 mm

C / t_f = 87.25 /16 = 5.45

Limit for class 1 flange 9ε = 9 x 0.81 = 7.29 > 5.45

Hence flanges are Class 1

Overall cross section classification under the combined loading is therefore **Class 1**.

1.8 Characteristics of Cross-Sections:

1.8.1 Gross Section:

The characteristics of the gross section are determined using the nominal dimensions without deducting the areas of any holes.

1.8.2 Net Area:

The net area A_{net} of a cross-section is equal to its gross area minus the areas of the holes.

1.9 Partial Safety Factors:

The design resistances are multiplied by a partial safety factor γ_M, with the following values:

1.9.1 Design of Cross-Sections:

- Gross sections of Class 1, 2, or 3:

γ_{M0} = 1 (or γ_{M0} = 1.1 if non-certified steel is used)

- Gross sections of Class 4:

γ_{M1} =1,1

- Net sections at holes:

γ_{M2} = 1,25

1.9.2 Design of Members for Elastic Instability:

- Buckling
- Lateral-Torsional Buckling γ_{M1} =1,1
- Local Buckling

1.9.3 Design of Connections:

- Connections with Non-Preloaded Bolts:

Subjected to Shear: γ_{MB} =1,25

Subjected to Tension: γ_{MB} =1,50

- Connections with Preloaded Bolts:

- At Ultimate Limit State (ULS):

- Normal Tolerance Holes: γ_{MS} =1,25

- Slotted Holes: γ_{MS} =1,40

- At Serviceability Limit State (SLS):

- Normal Tolerance Holes: γ_{MS} =1,10

- Welded Connections:

- For S235 Steel: γ_{MW} =1,25

- For S275 Steel: γ_{MW} =1, 30

- For S355 Steel: γ_{MW} =1, 35

CHAPTER 02: Calculations of Members Subjected to Simple Compression

2.1 Introduction:

Steel columns, a fundamental component in structural engineering, are widely used for their strength and versatility in various types of steel buildings. These columns are commonly produced with cross-sectional shapes such as circular, square, or rectangular profiles. This chapter focuses on these particular shapes' structural and functional characteristics, especially square, rectangular, and circular sections. These offer efficient material use and prevent failure under load compared to other sections like HEB, IPE, or HEA. With advancements in high-strength steel and the expanded application of cold-formed sections, newer forms, such as diamond or thin-walled sections, are emerging for applications beyond lightweight structures. These new forms demand precise technical specifications and strict quality controls to ensure safety and performance. In designing steel columns, it is critical to consider both their load-bearing capacity and their resistance to buckling, which becomes more complex as columns become longer and slenderer. Historical contributions, such as those from Euler in the 18th century, laid the foundation for the modern understanding of column behaviour under compression. Euler's formula introduced critical load, slenderness, and effective length, which are crucial for predicting when a column might buckle. This chapter examines these principles and extends them with modern considerations, including end restraints, effective length factors, and the role of bracing in enhancing stability. Design codes, specifically Eurocode 3, offer standardized methods for evaluating the buckling resistance of columns under compression. The code incorporates ultimate and serviceability limit states to ensure safety and usability, using factors like slenderness ratios, reduction factors, and effective lengths based on real-world column behaviour. Additionally, the chapter explores buckling curves, imperfection factors, and cold-formed hollow sections to address the stability needs of various steel profiles under different loading conditions.

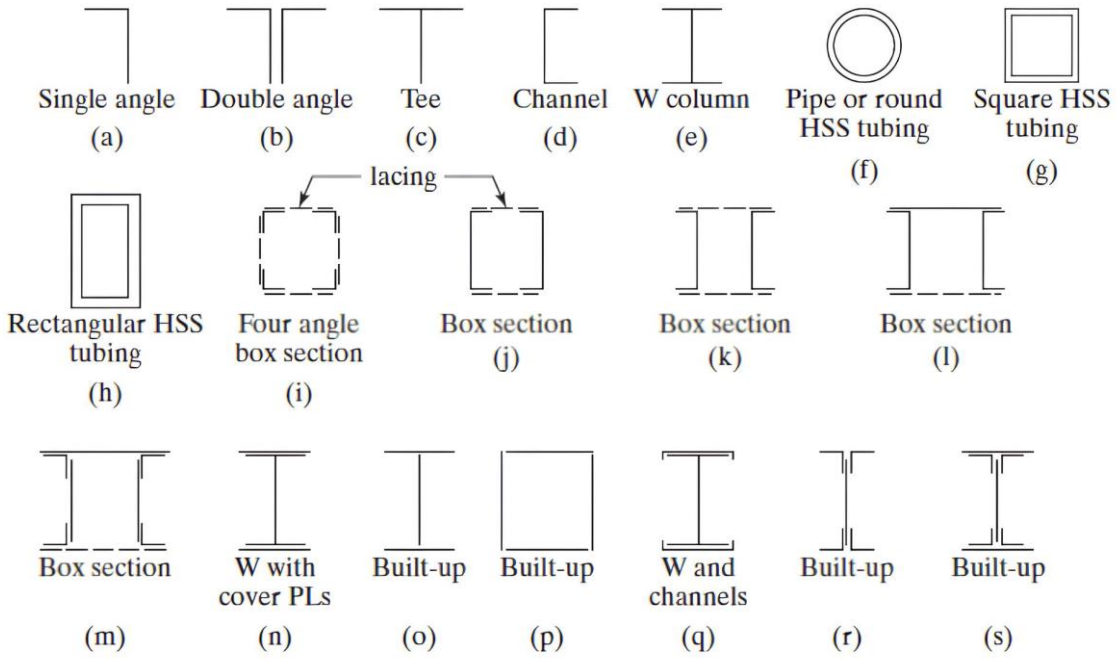


Figure 2. 1 : Types of compression members[11]

2.2 Development of column formulas

In 1729, Dutch mathematician Pieter van Musschenbroek introduced one of the first empirical formulas to estimate the strength of rectangular columns, pioneering an understanding of column behaviour under axial compression. Although limited in scope, his work highlighted the susceptibility of columns to instability, sparking further interest in their structural behaviour. In 1757, Swiss mathematician Leonhard Euler built upon this foundational work by deriving a theoretical formula that determined the critical load at which a slender, ideally straight column would buckle. Euler's formula, based on material properties, column length, and support conditions, marked a significant advancement by shifting from empirical observations to rigorous mathematical analysis, enabling precise predictions of buckling load and laying the groundwork for modern column design. Euler's insights into slenderness, effective length, and stability became essential in structural engineering, opening the door to systematic studies and reliable design methods for compression members[12].

2.3 The Euler formula:

The Euler formula reveals that as a column's length increases, its buckling stress decreases, making long, slender columns particularly vulnerable to elastic buckling. Beyond a certain length, known as the critical slenderness limit, the column buckles at stress levels below the

steel's proportional limit, ensuring that the buckling response remains within the material's elastic range[13].

Either:

A straight post-bi-articulated length $\mathbf{l_0}$

A normal compression effort **N-centred**

The transverse dimensions of the beam are small in front of the length

The maximum inertia of the beam is along the y-axis, and the minimum inertia is along the z-axis. Under compressive force N, the pole flexes around the weak axis ZZ (in the xy plane), producing a deformation y(x).

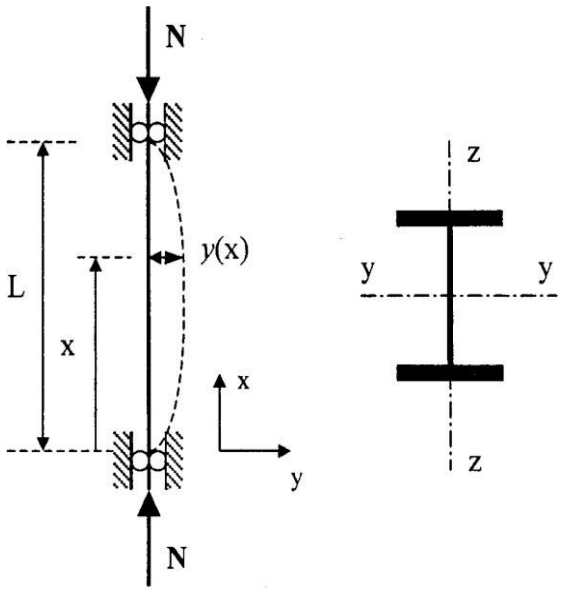


Figure 2. 2 : Column Theory [14]

When N believes, the initial rectilinear state evolves into a curvilinear state.

$$\boldsymbol{M = -EI_z \frac{d^2y}{dx}} \qquad \textbf{2. 1} \qquad \boldsymbol{M=Ny} \qquad \textbf{2. 2}$$

Therefore $\boldsymbol{Ny + EI_z \frac{d^2y}{dx} = 0}$ **2. 3**

Either $\boldsymbol{\frac{N}{EI_z} = \omega^2}$ **2. 4**

$\boldsymbol{y'' + \omega^2 y = 0}$ **2. 5** Solution of equation

$$\boldsymbol{y= A\sin \omega x + B \cos \omega x} \qquad \textbf{2. 6}$$

The conditions in limits $y(0)=0$ so $B=0$ and $y(l_0)=0$ $\sin \omega l_0 = 0$

Either $\omega l_0 = k\pi$ therefore $\omega = k\pi / l_0$

$$\boldsymbol{N = (\pi^2\, EI_z)/l_0^2} \qquad \textbf{2. 7}$$

The minimum value of N will be for **k=1** and is:

$$\boldsymbol{Ncr = \frac{\pi^2 EI_z}{l_0^2}} \qquad \textbf{2. 8}$$

N_{cr} is the critical force of Euler.

This critical force is matched by a critical stress [13]:

$$\boldsymbol{\sigma_{cr} = \frac{\pi^2 EI_z}{l_0^2 A} = \frac{\pi^2 E}{l_0^2} \cdot \frac{I_z}{A} = \frac{\pi^2 E}{l_0^2} \cdot i^2} \qquad \textbf{2. 9}$$

With ***i*** radius of gyration of the cross-section of the column $i = \sqrt{\frac{I_Z}{A}}$

- **radius of gyration i** is a geometric section data which characterizes the ability of a section to resist buckling (normally the smallest ***i*** is used when the effective lengths of the column about both principal axes are the same.).
- **Slenderness Ratio λ** : A column has been described as relatively long, slender member loaded in compression. This description is stated in relative terms and is not very useful for analysis. The measure of the slenderness of a column must consider the length, the cross-sectional shape and dimensions of the column, and the manner of attaching the ends of the column to the structures that supply loads and reactions to the column. The commonly used measure of slenderness is the slenderness ratio, defined as:

$$\boldsymbol{\lambda = \frac{l_0}{i_z}}$$

The critical buckling stress $\boldsymbol{\sigma_{cr}}$ is the average stress over the cross-sectional area A of a column at the critical load $\mathbf{P_{cr}}$.

$$\boldsymbol{\sigma_{cr} = \frac{\pi^2 E}{\lambda^2}}$$

$\boldsymbol{\lambda_{cr}}$ is the slenderness for which the critical stress $\boldsymbol{\sigma_{cr}}$ is equal to $\mathbf{f_y}$

Example for S235 steel: $\lambda_{cr} = \pi\sqrt{\frac{E}{f_y}}\lambda_{cr} = \pi\sqrt{\frac{2.1\cdot 10^5}{235}} = 93.9$

For any part, it is $\bar{\lambda} = \frac{\lambda}{\lambda_{cr}}$, $\boldsymbol{\bar{\lambda}}$ called **reduced slenderness**

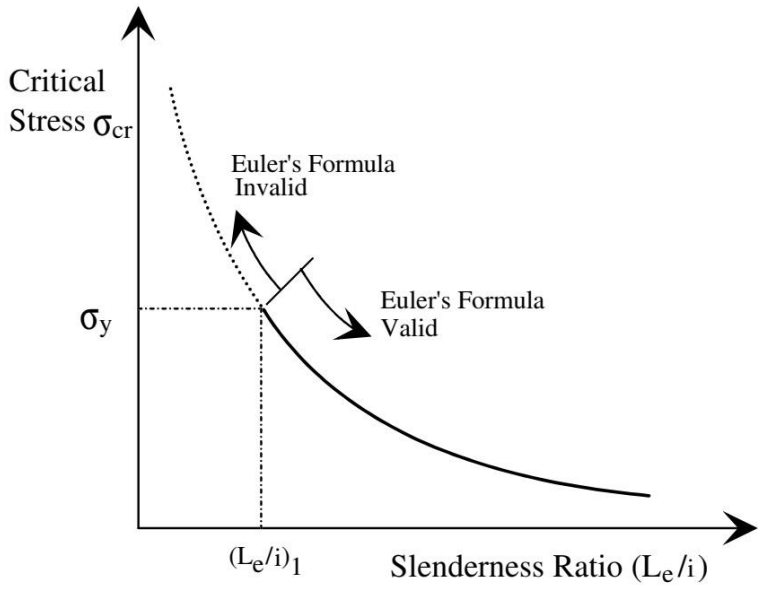


Figure 2. 3 : Critical Buckling Stress for Columns [15]

- When $\bar{\lambda} < 1$ there is no risk of buckling, the critical stress is greater than the yield strength of the steel, the column is not sufficiently slanted. The force N is limited by the strength of the post to compression.
- When $\bar{\lambda} \geq 1$ collapse by buckling. The critical stress is greater than the yield strength of steel. The force **N** is limited by the resistance of the column to buckling.

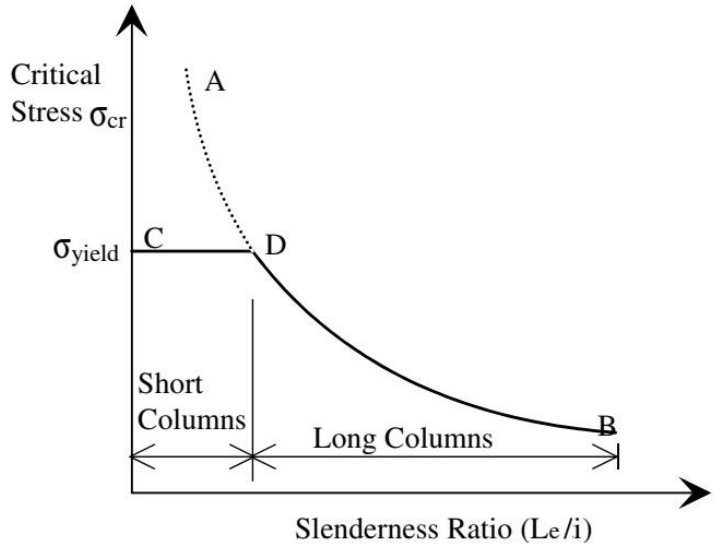


Figure 2. 4 : Short and long columns [15]

2.3.1 End restraint and effective lengths of columns:

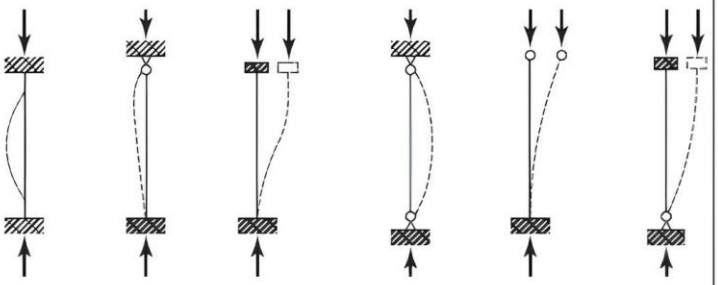
The effective length of a column is the distance between points of zero moment in the column, that is, the distance between its inflection points. In steel specifications, the effective length of a column is referred to as **KL**, where **K** is the influential length factor. The magnitude of **K**

depends on the rotational restraint supplied at the ends of the column and upon the resistance to lateral movement provided.

Columns with different end conditions have entirely different effective lengths. The smaller the effective length of a particular column, the smaller its danger of lateral buckling and the greater its load-carrying capacity will be.

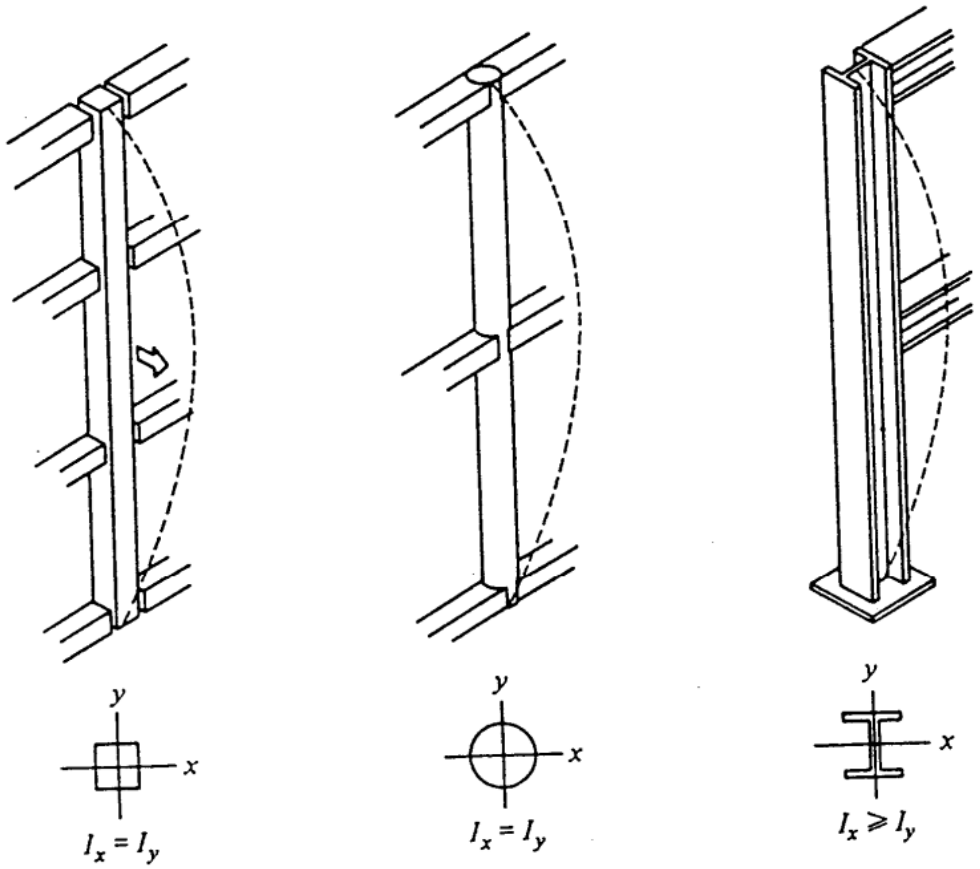
- Two sets of **K** values are provided in the Table-2.1, the theoretical values and the recommended design values based on perfectly pinned and fixed conditions being impossible.
- If the ends of the column were not entirely fixed, the column would be more accessible to bend laterally, and its inflection points would be farther apart. The recommended design **K** given in Table 2.1 is 0.65, while the theoretical value is 0.5. As no column ends are perfectly fixed or hinged, the designer may wish to interpolate between the values in the table, basing the interpolation on his or her judgment of the actual restraint conditions[16].

Table 2. 1 : Effective length (KL) for columns [16]

Approximate Values of Effective Length Factor, *K*						
Buckled shape of column is shown by dashed line						
	(a)	*(b)*	*(c)*	*(d)*	*(e)*	*(f)*
Theoretical *K* value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value when ideal conditions are approximated	0.65	0.80	1.2	1.0	2.10	2.0
End condition code		*Rotation fixed and translation fixed* *Rotation free and translation fixed* *Rotation fixed and translation free* *Rotation free and translation free*				

2.3.2 Bracing:

Bracing is strategically added to columns to increase their load-carrying capacity by reducing their effective length, which directly enhances stability and delays buckling. Effective bracing divides a long column into shorter, more stable segments, thus raising its critical buckling load. However, improper bracing can reduce effectiveness or even compromise structural stability, as poorly placed or weak braces may introduce unintended forces, create lateral instability, or act as points of weakness rather than support. Therefore, careful design and positioning of braces are essential to ensure they effectively enhance the column's performance under compression without inadvertently weakening the structure[17].

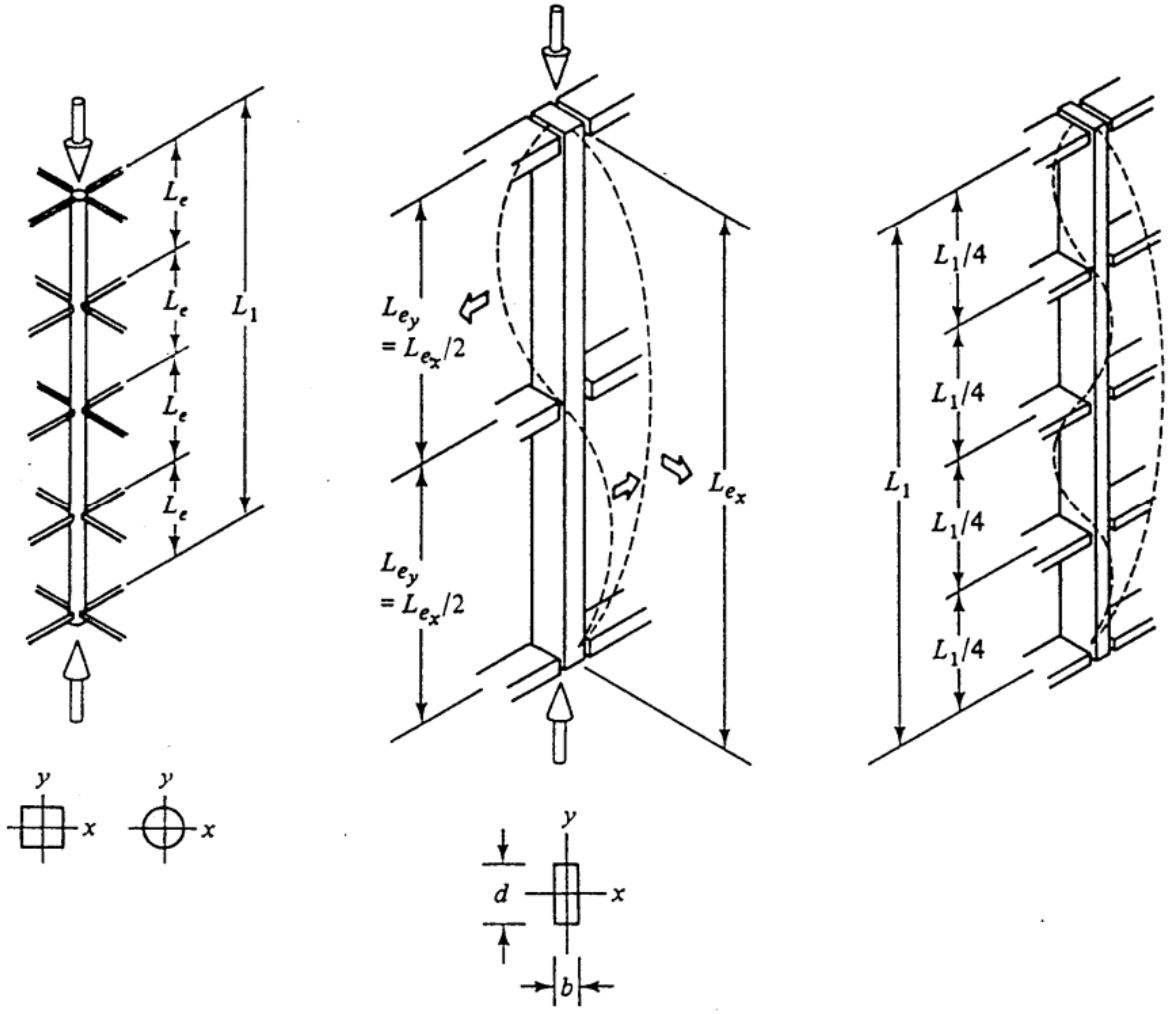


(a) (b) (c)

Figure 2. 5 : Bracing[17]

In (a) and (b), I_{xx} and I_{yy} are the same. Bracing is added in one direction only; the column will buckle in the other unrestrained direction as it has a higher slenderness ratio in this direction.

Although bracing is added in (c), it is ineffective as it restrains the more decisive direction. The column will buckle in the weaker direction as it has a higher slenderness ratio in this direction.



(a) (b) (c)

Figure 2. 6 : Effective use of bracing[17]

- For symmetrical sections in (a), bracing should be added in both directions in order to reduce the effective length of the column.
- For column sections with different I_{xx} and I_{yy} (see (b) and (c)), bracing should be added in the weaker direction to reduce the effective length in the weaker direction.

The column in (b) is braced in the weaker direction. The column will tend to buckle in the direction associated with the highest slenderness ratio.

$$r_x = \sqrt{I_x/A}$$

$$r_y = \sqrt{I_y/A}$$

$$P_{cr_x} = \pi^2 E I_x / L_{e_x}^2$$

$$P_{cr_y} = \pi^2 E I_y / L_{e_y}^2$$

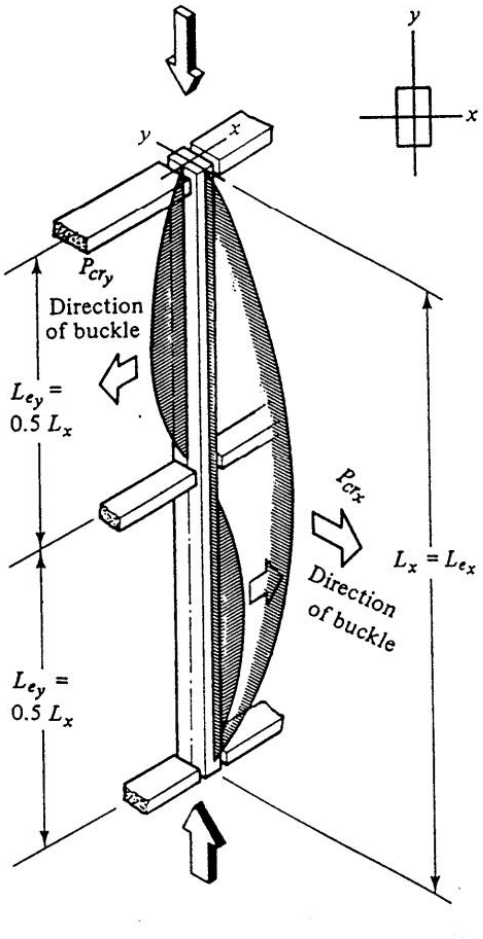
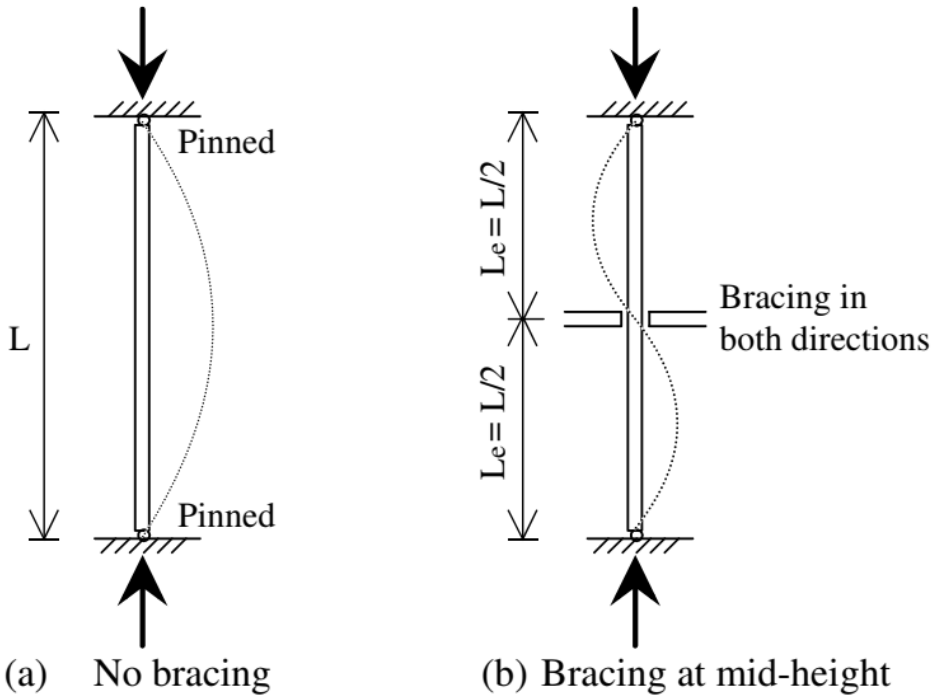


Figure 2. 7 : Buckling load for column with bracing in two directions [17]

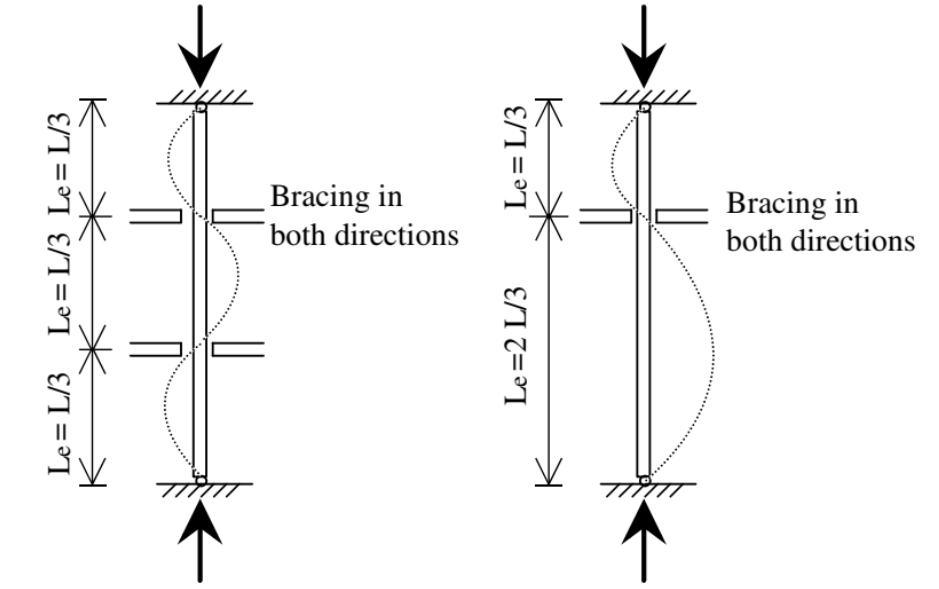
In order to determine the buckling load of this column, it is necessary to calculate the buckling load about x-x axis, $\mathbf{P_{crx}}$, and the buckling load about y-y axis, $\mathbf{P_{cry}}$. The buckling load of the column is the smaller of $\mathbf{P_{crx}}$ and $\mathbf{P_{cry}}$.

Bracing a column changes its effective length and consequently its buckling mode. The more a column is braced, the shorter its effectively length becomes and the greater the load that is required to cause buckling. If bracing is used, it is usually more effective when placed symmetrically.



$$P_1 = \frac{\pi^2 EI}{L^2} \qquad \textbf{2. 10}$$

$$P_2 = \frac{\pi^2 EI}{\left(\frac{L}{2}\right)^2} = \frac{4\pi^2 EI}{L^2} = 4P_1 \qquad \textbf{2. 11}$$



$$P_3 = \frac{\pi^2 EI}{\left(\frac{L}{3}\right)^2} = 9P_1 \qquad \textbf{2. 12}$$

$$P_4 = \frac{\pi^2 EI}{\left(\frac{2L}{3}\right)^2} = \frac{9\pi^2 EI}{4L^2} = \frac{9}{4}P_1 \qquad \textbf{2. 13}$$

Figure 2. 8 : Effects of lateral bracing on column buckling [17]

2.3.3 Allowable load:

The allowable load is calculated by dividing the critical buckling load by a Factor of Safety, F_S. That is**,**

$$P_a = \frac{P_{cr}}{F_s} \qquad \textbf{2. 14}$$

where:

P_a = allowable load

P_{cr} = critical buckling load

F_S= Factor of Safety

Similarly, the allowable stress is computed by dividing the crushing stress or buckling stress by a **F**S depending on whether the column is a short or slender column.

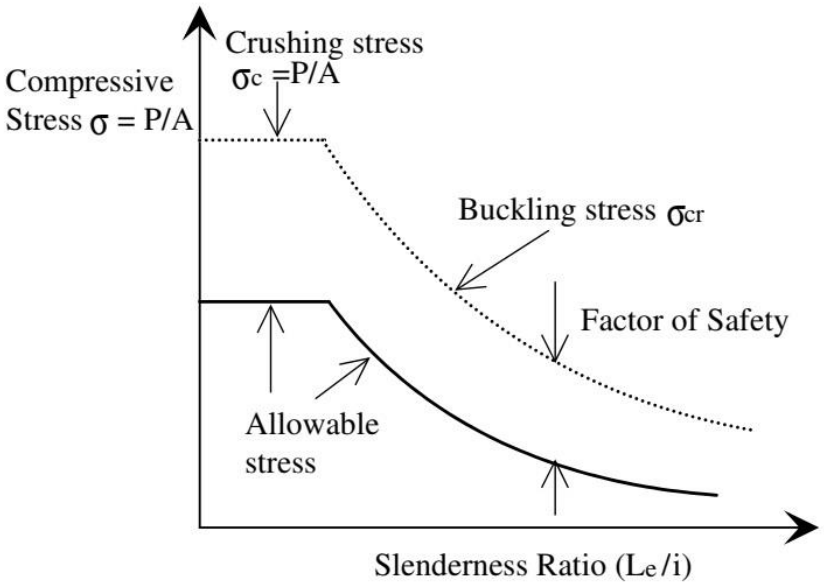


Figure 2. 9 : Allowable stresses in compression [11]

2.4 Design According to the Eurocode:

2.4.1 Introduction:

In the following notes, we will limit our focus to verification against simple buckling (under pure compression). All Eurocodes, including Eurocode 3 (specifically for steel structures), employ the concept of a limit state to define conditions beyond which the structure no longer meets the required performance standards. There are two categories:

- **Ultimate Limit States (ULS):** correspond to structural safety. These are associated with the collapse of the structure or other forms of structural failure that may endanger the safety of individuals.

- **Serviceability Limit States (SLS):** relate to serviceability. These are the conditions beyond which specified operational criteria are no longer met:
  - deformations or deflections affecting the appearance or effective operation of the structure,
  - vibrations that disturb occupants, damage the building, or limit its functional effectiveness.

2.4.2 ULS Verification:

Here, we will use Eurocode notations:

A Cross-sectional area of the bar

f_y Elastic limit of steel (i.e., yield strength = R_e or R_p 0.2)

L_{cr} Critical buckling length (i.e., lfb)

$N_{b,Rd}$ Weighted allowable force in the compressed bar at buckling (i.e., N_{adm} S)

N_{ed} Design axial force (i.e., N)

χ_y, χ_z Reduction factor for the appropriate buckling curve

γ_{M1} Partial safety factor for bar instability resistance (i.e., S)

λ_y, λ_z Slenderness ratio of the bar (i.e., λ_{col})

λ_1 Euler limit slenderness (i.e., λ_{lim} Euler)

i_y, i_z Radius of gyration along the y and z axes (i.e., ig y or ig z)

β_a Coefficient based on the section class, which must be identified per **EC3-DAN, Section 5.3**.

Factors like slight deviations and residual stresses must be integrated into design considerations. Eurocode 3 (EC3) addresses this by introducing an imperfection factor, denoted as "**α**", to account for these discrepancies. The bending resistance of a column under pure compression is detailed in **EC3, section 5.5.1 [8]**, which uses standardized European buckling curves derived from extensive testing by the European Convention for Constructional Steelwork (ECCS).

For any column subjected to a calculated compressive load **N_{Sd}**, the following must be satisfied:

$$N_{csd} \leq N_{crd} = \chi \beta_a A f_y / \gamma_M \qquad \textbf{2. 15}$$

- For sections classified as **Class 1, 2, or 3**:

Standard formulas apply to calculate buckling resistance directly from the gross section area.

$$\beta_a = 1$$

- For **Class 4** sections:

Local buckling risk must be considered. Here, compressive strength is calculated based on the effective area $\mathbf{A_{eff}}$, as **Section 5.3.5 of EC3-DAN** outlines [8].

$$\beta_a = A_{eff} / A$$

If the section is mono-symmetric (e.g., a welded I-beam with unequal flanges), the effective area centroid shifts from the gross section centroid, necessitating additional moment calculations. Applying an axial force-bending moment interaction formula from **EC3-DAN, Section 5.5.4**, verified buckling resistance.

- **Reduction Factor χ and Slenderness**

The reduction factor χ , corresponds to the relevant buckling direction derived from the reduced slenderness λ, and the appropriate buckling curve. When buckling can occur about both principal axes, the lowest of the two χ values is used.

$$\chi = \mathbf{Min}(\chi_y, \chi_z)$$

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}} \text{ mais } \chi \leq 1$$

$$\phi = 0,5\left[1 + \alpha(\bar{\lambda} - 0,2) + \bar{\lambda}^2\right]$$

Reduced slenderness is defined as:

$$\bar{\lambda} = \sqrt{\frac{\beta_A A f_y}{N_{cr}}}$$

Where $\mathbf{N_{cr}}$ is the critical elastic axial force in the relevant buckling direction, calculated based on the gross section properties and effective buckling length. The effective buckling length depends on whether a fixed or flexible node configuration is assumed, as per **EC3-DAN Annex**

E [8]. This configuration selection is also closely related to the analysis method for determining bar forces (**EC3-DAN Section 5.2**)[8].

$$\bar{\lambda} = \frac{\lambda}{\lambda_1}$$
$$\lambda_1 = \pi\left(E/f_y\right)^{0.5}$$
$$\lambda = \frac{L_e}{i}$$

- **Buckling Risk Assessment and Selection of Buckling Curve**

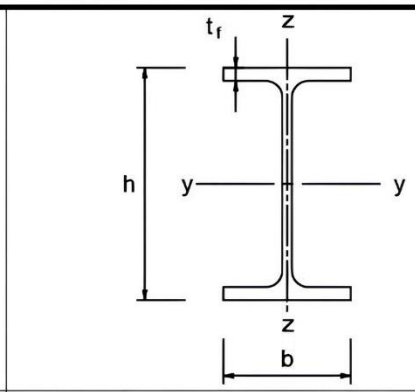
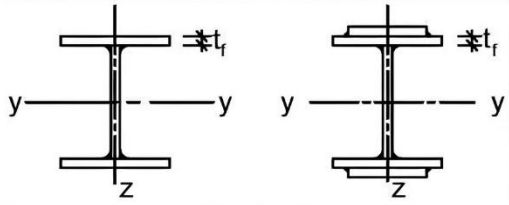
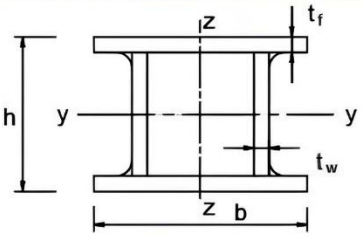
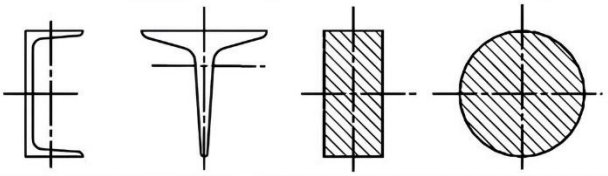
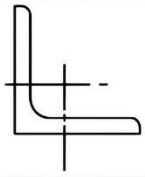
Practical experience shows that the risk of buckling must be considered when $\bar{\lambda} \geq 0,2$ **(**slenderness ratio exceeds **18.8** for steel grade **S235**). Beyond this threshold, the structure's resistance to buckling becomes critical, requiring detailed design consideration.

The appropriate buckling curve is selected based on the following factors:

- **Profile Type**: This includes the specific type of steel section, such as I-beams (both rolled and welded), tubes, U-sections, angles, and solid sections.
- **Dimensional Characteristics**: The geometric properties of the section, such as its height, width, and thickness, which impact stability and susceptibility to buckling.
- **Buckling Axis (Bending Direction)**: The principal axis along which buckling is likely to occur, depending on the orientation of the load and section geometry.

The Table 2.2 below (often found in the Eurocode 3 standards) provides guidance on selecting the appropriate buckling curve based on these parameters. This selection ensures that the design aligns with the specific stability characteristics of each profile type under given load and slenderness conditions, ensuring a safe and efficient structure.

Table 2. 2 : Selection of buckling curve [8]

Cross section		Limits		Buckling about axis	Buckling curve S 235 S 275 S 355 S 420	Buckling curve S 460
Rolled sections		h/b > 1,2	$t_f \leq 40$ mm	y – y z – z	a b	a_0 a_0
			40 mm < $t_f \leq 100$	y – y z – z	b c	a a
		h/b ≤ 1,2	$t_f \leq 100$ mm	y – y z – z	b c	a a
			$t_f > 100$ mm	y – y z – z	d d	c c
Welded I-sections		$t_f \leq 40$ mm		y – y z – z	b c	b c
		$t_f > 40$ mm		y – y z – z	c d	c d
Hollow sections		hot finished		any	a	a_0
		cold formed		any	c	c
Welded box sections		generally (except as below)		any	b	b
		thick welds: a > 0,5t_f $b/t_f < 30$ $h/t_w < 30$		any	c	c
U-, T- and solid sections				any	c	c
L-sections				any	b	b

In Eurocode 3, typical buckling curves **(a, b, c, and d)** are assigned based on these parameters, each corresponding to specific imperfection factors and used to calculate the reduction factor **χ** for the design verification of elements under compression.

- **Imperfection Factor and Buckling Length**

The imperfection factor **α,** which varies based on the selected buckling curve, is given in EC3-DAN **(Table 5.5.1. Annex D) [8].** For elements that may buckle in more than one direction, χ is calculated for each direction, and the lower value is selected.

Table 2. 3 : Imperfection factor for buckling curve

Buckling Curve	Imperfection Factor (α)
a_0	0.13
a	0.21
b	0.34
c	0.49
d	0.76

- **Special Considerations for Cold-Formed Hollow Sections**

For cold-formed hollow sections, buckling resistance verification follows **EC3-DAN, Section 5.5.1.4 [8]**, by considering either:

- the base yield strength ****$\mathbf{f_{yb}}$**** of the parent material with curve **b**, or
- the average yield strength ****$\mathbf{f_{ya}}$**** post-cold forming with curve **c** (**see EC3-DAN Figure 5.5.2**) [8].

The appropriate buckling curve depends on section geometry, buckling direction, steel grade, and manufacturing process, as provided in **EC3-DAN (Table 5.5.3 and Annex D)[8]**.

- **Determining Effective Buckling Length**

Determining a component's effective buckling length is one of the most challenging steps in design. This length is influenced by the translational and rotational conditions at each end of the component. End conditions are categorized based on whether both nodes are fixed in translation. For fixed-translation nodes, the buckling length will be less than or equal to the component's physical length. For movable nodes, buckling modes allow relative displacement between ends, leading to an effective length greater than or equal to the component's physical length [18].

- **European Buckling Curves**

Research on buckling phenomena during the 1960s, alongside extensive experimental testing, led the European Convention for Constructional Steelwork (ECCS) [19] to develop three non-dimensional buckling curves labelled **a, b,** and **c**. These curves were created based on various profile types and account for geometric and structural variations within bars, making them suitable for design purposes across different structural profiles.

The development process began with a comprehensive testing program. Seven countries (France, Italy, the UK, Belgium, Germany, the Netherlands, and Yugoslavia) conducted over **1,000 buckling tests**. These tests provided critical insights but were time-intensive and costly. By the end of 1966, the first European buckling curve—based solely on experimental data—was officially adopted. This curve was based on the **IPE 160 rolled section** in **steel grade Ac 37**.

The standardized European buckling curves are illustrated in Figure 2.10, including the Euler curve. The **Euler curve** represents the theoretical elastic buckling of an ideally straight and perfect bar, providing a reference for understanding how real sections deviate due to imperfections. These buckling curves are now integral to structural design, helping engineers accurately assess the stability of compression elements across various profiles and loading conditions.

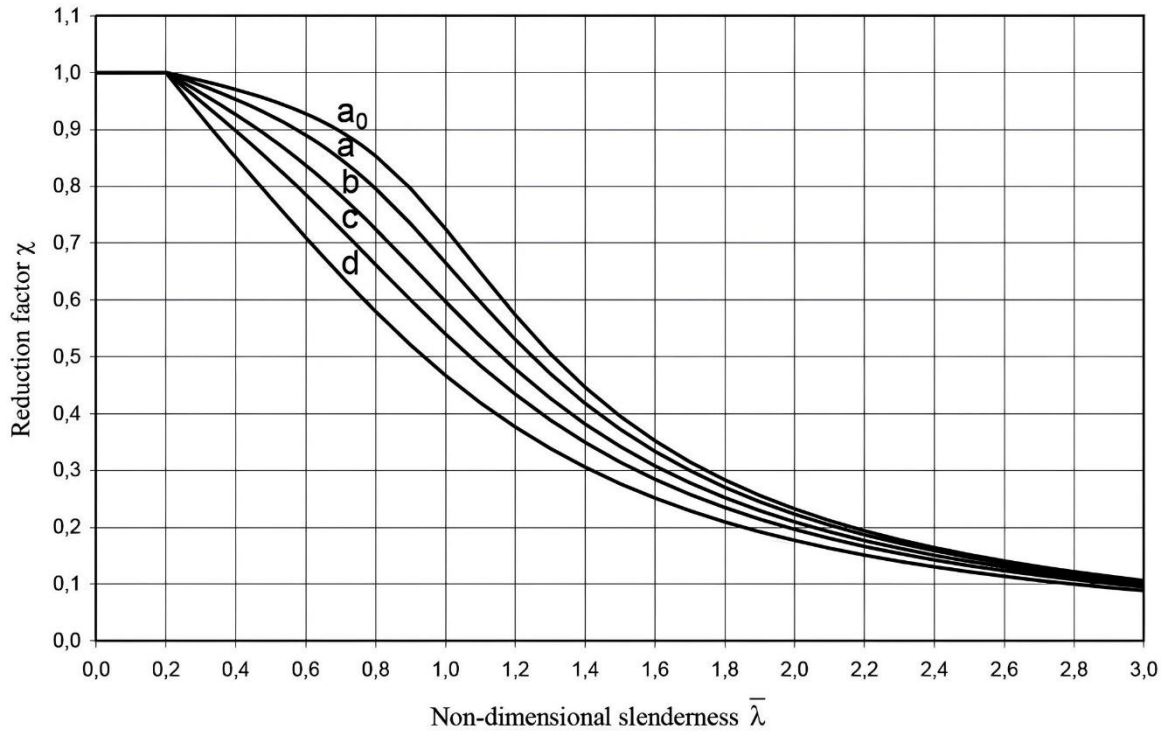


Figure 2. 10 : European buckling curves[19]

The values for the reduction factor, χ, are listed in the Table 2.4 below. This factor is essential for calculating the buckling resistance of structural members under compression. For cases where exact values fall between those provided, intermediate values of χ can be determined using linear interpolation.

Table 2. 4 : Determination of reduction factor, χ [8]

$\bar{\lambda}$	a_0	a	b	c	d
0.2	1.0000	1.0000	1.0000	1.0000	1.0000
0.3	0.9859	0.9775	0.9641	0.9491	0.9235
0.4	0.9701	0.9528	0.9261	0.8973	0.8504
0.5	0.9513	0.9243	0.8842	0.8430	0.7793
0.6	0.9276	0.8900	0.8371	0.7854	0.7100
0.7	0.8961	0.8477	0.7837	0.7247	0.6431
0.8	0.8533	0.7957	0.7245	0.6622	0.5797
0.9	0.7961	0.7339	0.6612	0.5998	0.5208
1.0	0.7253	0.6656	0.5970	0.5399	0.4671
1.1	0.6482	0.5960	0.5352	0.4842	0.4189
1.2	0.5732	0.5300	0.4781	0.4338	0.3762
1.3	0.5053	0.4703	0.4269	0.3888	0.3385
1.4	0.4461	0.4179	0.3817	0.3492	0.3055
1.5	0.3953	0.3724	0.3422	0.3145	0.2766
1.6	0.3520	0.3332	0.3079	0.2842	0.2512
1.7	0.3150	0.2994	0.2781	0.2577	0.2289
1.8	0.2833	0.2702	0.2521	0.2345	0.2093
1.9	0.2559	0.2449	0.2294	0.2141	0.1920
2.0	0.2323	0.2229	0.2095	0.1962	0.1766
2.1	0.2117	0.2036	0.1920	0.1803	0.1630
2.2	0.1937	0.1867	0.1765	0.1662	0.1508
2.3	0.1779	0.1717	0.1628	0.1537	0.1399
2.4	0.1639	0.1585	0.1506	0.1425	0.1302
2.5	0.1515	0.1 467	0.1397	0.1315	0.1214
2.6	0.1404	0.1362	0.1299	0.1234	0.1134
2.7	0.1305	0.1267	0.1211	0.1153	0.1062
2.8	0.1216	0.1182	0.1132	0.1079	0.0997
2.9	0.1136	0.1105	0.1060	0.1012	0.0937
3.0	0.1063	0.1036	0.0994	0.0951	0.0882

2.5 Examples:

- **Example 01: Checking the Stability of the Welded built-up Section Column**

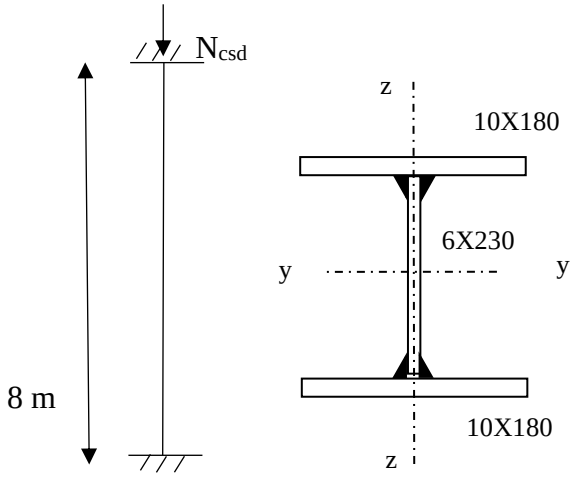
Verify the stability of the column with

Welded built-up section in the figure.

The column is subjected to

N_{csd}=350 kN

The steel grade is S235.



Solution

- **Section class:**
  - **Web-internal compression part (Table 5.2 sheet 1)**

C/t= 230/6=38,33 < 42ε =42

Hence web are Class 03

- **Outstand flanges (Table 5.2, sheet 2)**

C/t= 87/10=8,7 < 10ε =10

Hence flanges are Class 01

Overall cross-section classification is therefore **Class 03**

Cross-sectional area $A = (2.(180.10) + 230.6) = 4980 mm^2$

- **Moments of inertia:**

$Iy = 6.230^3/12 + 2.(180.10^3/12 + 180.10.120^2)$

$Iy = 5{,}79 .10^7\ mm^4$

$Iz = 230.6^3/12 + 2.(10.180^3/12)$

$Iz = 9{,}72 .10^6\ mm^4$

- **Effective length and radius of gyration:**

L_f=0,5×8 =4 m

$i_y = 107,87$ mm

$i_z = 44,17$ mm

- **Slenderness ratio:**

$$\lambda_y = 4000/34,09 = 37,08$$

$$\lambda_z = 4000/44,17 = 90,54$$

- **Reduced slenderness:**

$$\overline{\lambda}_y = 37,08/93,9\sqrt{1} = 0,39$$

$$\overline{\lambda}_z = 90,54/93,9\sqrt{1} = 0,97$$

$\overline{\lambda} \geq 0,2$ Buckling Risk

- **Verification:**

$N_{csd} \leq N_{crd} = \chi A f_y / \gamma m_1$

t_f= 10 mm $<$ 40 mm buckling curve for yy and c for zz

$$\overline{\lambda}_y = \mathbf{0,39}$$

0,3 ⟶ 0,9641 0,1 ↗ ⟶ ↘ 0,038

0,4 ⟶ 0,9261 0,09 ⟶ x=0,0342

χ= 0,9299

N_{crdy} = 0,9299×4980×235/1,1=989,32 KN

$$\overline{\lambda}_z = \mathbf{0,97}$$

0,9 ⟶ 0,5998 0,1 ↗ ⟶ ↘ 0,0599

1,0 ⟶ 0,5399 0,07 ⟶ x=0,0419

χ= 0,5578

N_{crdz} = 0,5578×4980×235/1,1=593,44 KN

N_{csd} = 350 KN ≤ N_{crd} = (989,32 KN; 593,44 KN) The column is stable.

- **Example 02: Analysis of an HE 450 B Column under Axial Compression**

the member was supported with hinge supports about

its both principal axis at its both ends. The type of the

member is HE 450 B Length of the column is 9 m and

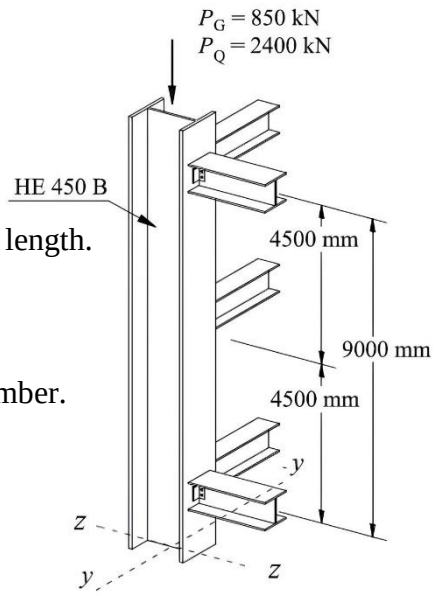
the column is subjected to axial compressive load of

P_G = 850 kN and P_Q = 2400 kN The column is supported

against to displacement and buckling in the direction which

is perpendicular to the y-axis (weak axis) at the midpoint of its length.

The steel grade is S235.



- ✓ Determine the nominal compressive strength of the member.
- ✓ Check the ULS design strength of the member.

Solution

- **Section class:**
  - ➢ **Web-internal compression part (Table 5.2 sheet 1)**

C/t= 344/14=24,57 < 33ε =26,73

Hence web are Class 01

- ➢ **Outstand flanges (Table 5.2, sheet 2)**

C/t= 150/26=5,77 < 9ε =7,29

Hence flanges are Class 01

Overall cross-section classification is therefore **Class 01**

Cross-sectional area $A = 21800 mm^2$

- **Effective length and radius of gyration:**

L_{fy}=1×9 =9 m and L_{fz}=0,5×9 =4,5 m

$i_y = 191,40$ mm

$i_z = 73,30$ mm

- **Slenderness ratio:**

$\lambda_y = 9000/191,40 = 47,02$

$\lambda_z = 4500/73,30 = 61,39$

- **Reduced slenderness:**

$$\overline{\lambda}_y = 47{,}02/93{,}9\sqrt{1} = 0{,}50$$

$$\overline{\lambda}_z = 61{,}39/93{,}9\sqrt{1} = 0{,}65$$

$\overline{\lambda} \geq 0{,}2$ Buckling Risk

- **Verification:**

  $N_{csd} \leq N_{crd} = \chi A f_y / \gamma m_1$

  h/b= 1,5 and t_f= 26 mm $<$ 40 mm buckling curve a for yy and b for zz

  $$\overline{\boldsymbol{\lambda}}_{\mathbf{y}} = \mathbf{0,50}$$

  χ= 0,9243

  $$\overline{\boldsymbol{\lambda}}_{\mathbf{z}} = \mathbf{0,65}$$

  χ= 0,8104

  Nominal compressive strength of the column

  N_{crdy} = 0,9243×21800×355/1,1=6502,87 KN

  N_{crdz} = 0,8104×21800×355/1,1=5701,53 KN

  Checking the ULS design strength of the member

  N_{csd} = 1,35P_G+1,5P_Q=1,35x850+1,5x2400= 4747,5 KN

  N_{csd} = 4747,5 KN ≤ N_{crd} = (6502,87 KN; 5701,53 KN) The **column is stable**.

CHAPTER 03: Calculations of Members Under Combined Flexural Buckling

3.1 Introduction:

This chapter explores the principles of beam-column behaviour, focusing on structural members under combined axial compression and bending moments. Structural engineering often uses the term "compressed member" to refer to elements subjected solely to axial compressive forces, as seen in components like truss bars and bracing members. However, many structural members also experience bending moments due to load eccentricities or lateral forces, transforming them into **"beam-columns."** Unlike purely compressed members analysed in previous discussions, beam-columns require a detailed understanding of stability under combined loading conditions, where axial forces and bending interact, amplifying deflections and influencing the risk of buckling. This complex loading scenario is commonly found across structural systems roof trusses, portal frames, and column-supported frameworks. For instance, in roof trusses, top chords face bending from purlin loads and compression due to truss action. At the same time, portal frames experience significant bending and compression from lateral and gravity loads. Columns in simple framing also endure combined loading due to eccentric beam reactions and gravity-induced axial compression. Each case requires specific stability criteria to prevent buckling and ensure structural integrity[20].

The chapter will delve into the interaction formulas and analytical approaches within Eurocode 3, which offer practical methods for designing members under combined loads. This includes accounting for material and geometric imperfections, secondary moments, and local buckling considerations, all influencing the load-bearing capacity and failure modes. By examining elastic and plastic cross-section classifications and the associated resistance criteria, this chapter provides a comprehensive framework for evaluating and ensuring beam-column stability and safety under real-world loading conditions.

3.2 Combined moment and axial compression load:

3.2.1 Occurrence of Combined Loading in Structural Members:

Combined loading occurs frequently in structural elements due to the simultaneous application of bending moments and axial forces. This loading condition is typical in several structural systems and is influenced by factors such as load direction, structural geometry, and support conditions. Below are some common scenarios where combined bending and compression occur in structural systems:

- **Roof Truss - Top Chord Members Subject to Combined Loading:** In roof truss systems, the top chord members are subjected to both bending and axial compression due to the combined effects of purlin loads and the overall truss bending behaviour. Here are how these forces interact:

    - Bending from Purlin Loads: Purlins, horizontal members supporting roofing material, transfer their load to the truss top chord. These loads introduce bending moments in the top chord due to the distributed weight across the span between truss connections.

    - Compression from Overall Truss Action: The truss structure is under compression in the top chord, acting as a compression member in response to the truss's load path. This compression results from the overall structural behaviour where the top chord is a compressive element under typical gravity loads.

Together, these forces lead to a combination of bending and axial compression in the top chord, which can increase the risk of buckling, especially in slender sections. Engineers must account for this by considering local and global stability criteria and, in some cases, may provide lateral bracing to resist buckling.

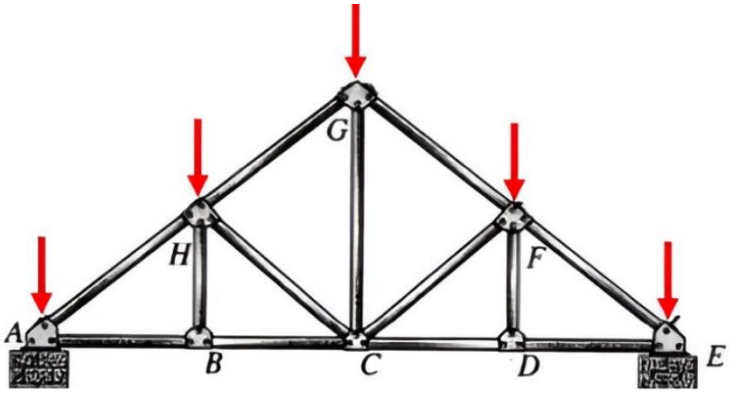


Figure 3. 1 : Roof truss - top chord members [7]

- **Simple Framing – Columns Subject to Combined Loading:** In simple framing, columns often face combined loading conditions due to eccentric beam reactions and the axial compression from gravity loads:

    - Bending from Eccentric Beam Reactions: When beams are connected eccentrically to columns, this eccentricity introduces bending moments in the column. For instance, in cases where the beam's line of action does not align

with the column's centroid, a moment is created at the connection, leading to bending in the column.

- Compression Due to Gravity Loads: The column also experiences axial compression from gravity loads (e.g., self-weight, live loads from floors above) acting down the member's length. Combined with bending, this compression creates a scenario where the column must resist both axial and bending stresses.

In design, this combined loading requires careful analysis of the member's buckling capacity, considering factors like slenderness, lateral-torsional buckling (if unbraced), and the influence of second-order effects (P-Δ effects) due to bending under axial loads.

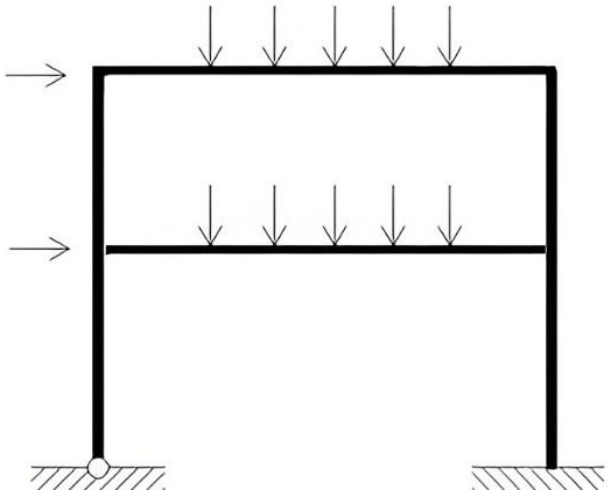


Figure 3. 2 : Simple framing columns subject to combined loading [7]

- **Portal Frame – Rafters and Columns Under Combined Loading:** In portal frames, rafters, and columns are subject to significant bending and compression due to frame action. These structures are designed to provide open spans with minimal internal supports, commonly found in warehouses, industrial buildings, and agricultural structures.

  - Bending in Rafters and Columns: Rafters in portal frames experience substantial bending moments, especially near connections to columns, due to lateral loads (e.g., wind pressure) and gravity loads from the roof. Columns also undergo bending, particularly at the base, due to the lateral resistance provided by the frame action.

- Compression Due to Frame Action: The rafters and columns experience axial compression due to the overall portal frame geometry and load path. For rafters, this compression combines with bending along the span; for columns, it adds to the bending moments at the frame connections and may create significant compressive stresses.

In portal frame design, combined loading is a primary consideration, with engineers focusing on stability against both local buckling of thin-walled sections and lateral-torsional buckling. The base connections are often critical for columns, as they must resist moments and compressive forces simultaneously. In addition, portal frames typically require robust bracing or stiffening strategies to prevent instability under lateral loads, with second-order (P-Δ) effects commonly considered to ensure stability.

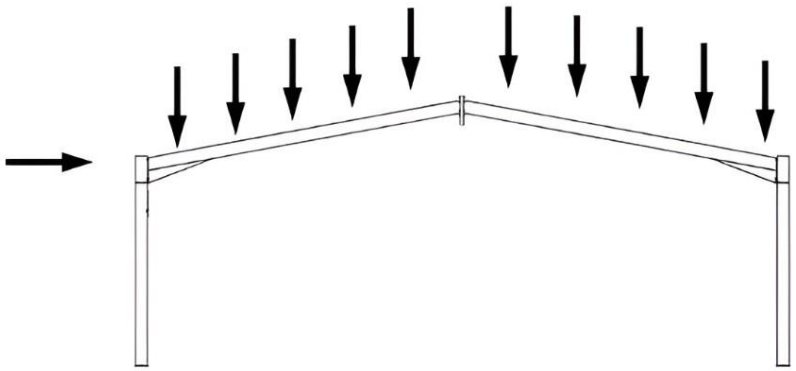


Figure 3. 3 : Portal frame under combined loading [7]

3.2.2 Interaction formulae:

The behaviour of a structural member subjected to simultaneous bending and axial forces arises from a complex interplay between instability (often in the form of buckling) and plasticity. This interaction is significantly influenced by both geometrical and material imperfections present within the member. These imperfections can stem from slight misalignments, deviations in straightness, and inconsistencies in material properties, all of which can reduce the load-bearing capacity and influence the failure mode.

In engineering codes, interaction formulas are applied when load cases can be broken down into simpler, individual components. These formulas consist of multiple terms, each representing the relationship between the effect of a specific load and the carrying capacity of the associated sub-case. For the member to be considered safe, the sum of these terms should be less than a specified limit.

The interaction formula essentially represents the limit of bearing capacity: if the calculated value is less than one, the cross-section can support the combined loads. In practical design, equality signs are typically replaced with inequality signs (≤), and a partial safety factor, γ_M, is used to adjust the bearing capacity [20].

For interaction formulas involving two sub-cases, diagrams such as those in Figure 3.4 can help visualize the relationships. Here, the horizontal axis represents the ratio of load effect to carrying capacity for one case, while the vertical axis shows the same ratio for the other. A value of 1 on either axis represents a pure load case, where only one part of the load is acting. Thus, the relationship line between these two points illustrates the combined effect, shaped by how the two load cases interact and other influencing phenomena.

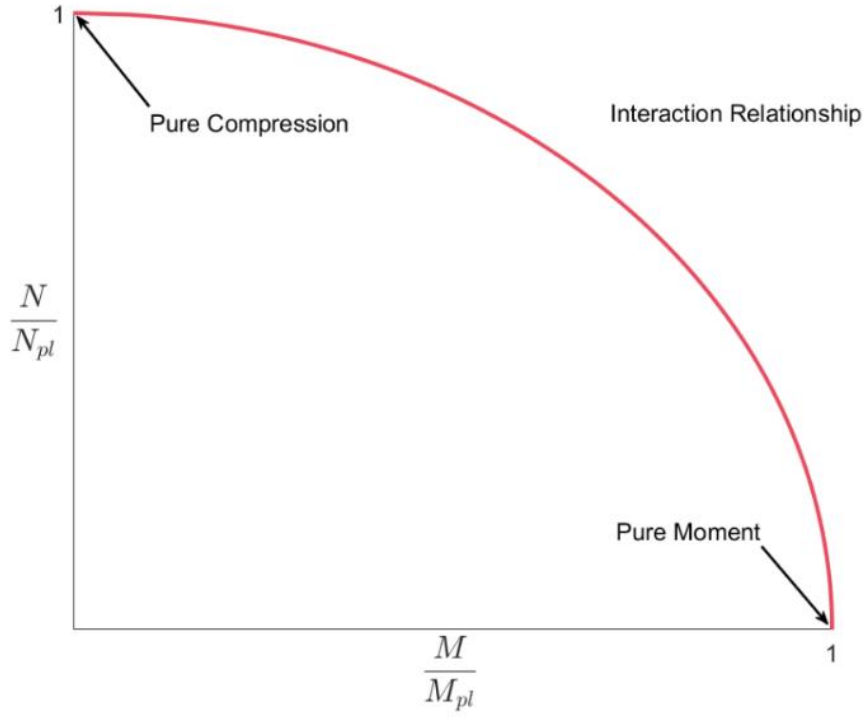


Figure 3. 4 : Interaction relationship between axial force and moment ratios[20]

However, before diving into EC3 methods in detail in the following sections, it is essential to highlight that before these codes, many authors have proposed direct relationships to express interaction domains M – N based on an interpretation of experimental or numerical simulation results in terms of column slenderness, cross-sectional shape, the presence of imperfections and bending moment distribution. Because of the complexity of such formulations and the fact that

their application is frequently limited to specific cases, it has become preferable – practically for specifications – to use simplified criteria.

The simplest method for assuming an interaction domain in the case of a constant bending moment is to consider the linear relationship (Figure 3.4):

$$\frac{N}{N_{Rd}} + \frac{M}{M_{Rd}} \leq 1 \quad 3.1$$

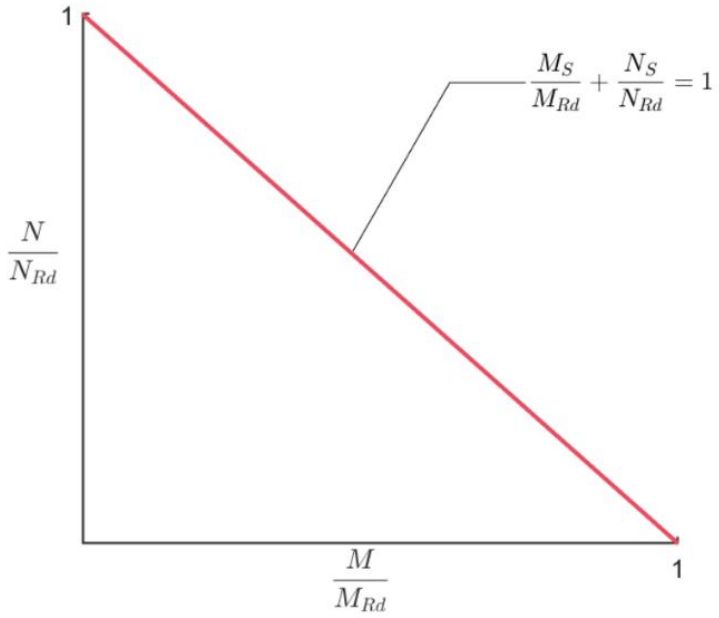


Figure 3. 5 : Linear relationship between axial and moment ratios [20]

N_{Rd} represents the maximum axial load capacity of a column under pure compression, while M_{Rd} is the ultimate bending moment capacity of a simply loaded section. However, this approach is not fully conservative, as the actual failure domains tend to exhibit a convex shape toward the origin (as shown in Figure 3.5). For beam-columns, it is crucial to account for additional "secondary moments" generated by axial forces acting through lateral displacements of the member, known as the P-Δ effect. These secondary moments combine with primary moments caused by end loads or transverse forces, increasing stresses and deflections, which must be considered in beam-column design[21].

the theoretical and experimental results often fall below the straight-line interaction graph, suggesting that the straight-line formula (Equation 3.1) overestimates beam-column capacity by neglecting the secondary moment caused by axial force and lateral displacements. To account for this, the formula was modified by incorporating an amplification factor that adjusts

the primary bending moment M to include the secondary moment (Equation 3.2). This adjustment brings theoretical and experimental values closer to the curved failure domain.

$$\frac{N}{N_c} + \frac{M}{M_u\left(1-\frac{N}{N_{cr}}\right)} = 1 \qquad \textbf{3. 2}$$

Here, N_{cr} represents the Euler critical load in the bending plane. This interaction curve is convex toward the origin for any slenderness ratio, providing a conservative design approach. Many design codes base their interaction formulas on this principle, though different codes implement it with slight variations.

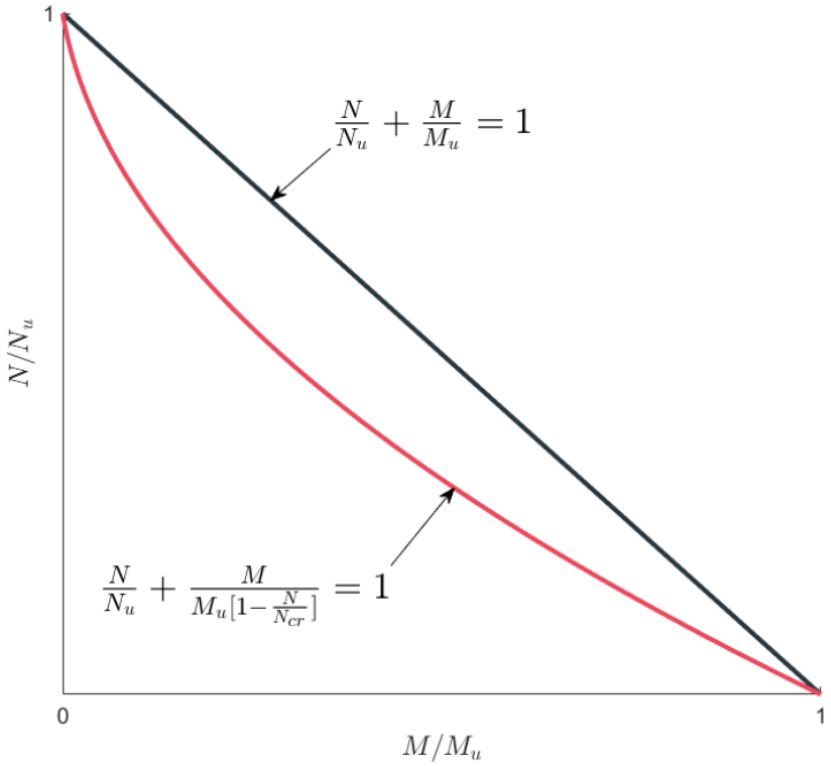


Figure 3. 6 : Interaction curve for a beam-column [20]

Eurocode 3 [8] takes the following approach (assuming bending along the y-axis):

$$\frac{N_{Ed}}{\chi_y A f_y} + k_{yy}\frac{M_{Ed}}{W_{pl.y} f_y} \leq 1 \qquad \textbf{3. 3}$$

And around the z-axis:

$$\frac{N_{Ed}}{\chi_z A f_y} + k_{zz}\frac{M_{Ed}}{W_{pl.z} f_y} \leq 1 \qquad \textbf{3. 4}$$

Where the value of k_y depends in a relatively complicated way on:

- The level of axial compression load is estimated by the ratio.

- The member slenderness λ_y.
- The ratio between the cross-section's elastic and plastic section moduli (W_{el} and W_{pl}).
- The pattern of primary moments.

3.3 Cross-section resistance under combined axial force and bending:

3.3.1 Plastic cross-section resistance:

The simultaneous presence of the moment and axial force is common for beams and columns. In addition, the carrying capacity is often determined by global instability, e.g., flexural buckling. Bending moments can be caused by transverse loads along the column, connecting beams at the column ends, or the known eccentricity of the axial force at one or both ends. However, the primary focus is on isolated columns subjected to known axial forces and moments. according to the model presented below. The axial force is assumed to be carried by a part of the cross-section on both sides of the neutral axis (N.A.) for a cross-section in class 1 or 2 in Figure 3.7. (Where the contribution to the section modulus is minimum). Outer parts support the bending moment (the flanges and perhaps part of the web)[22].

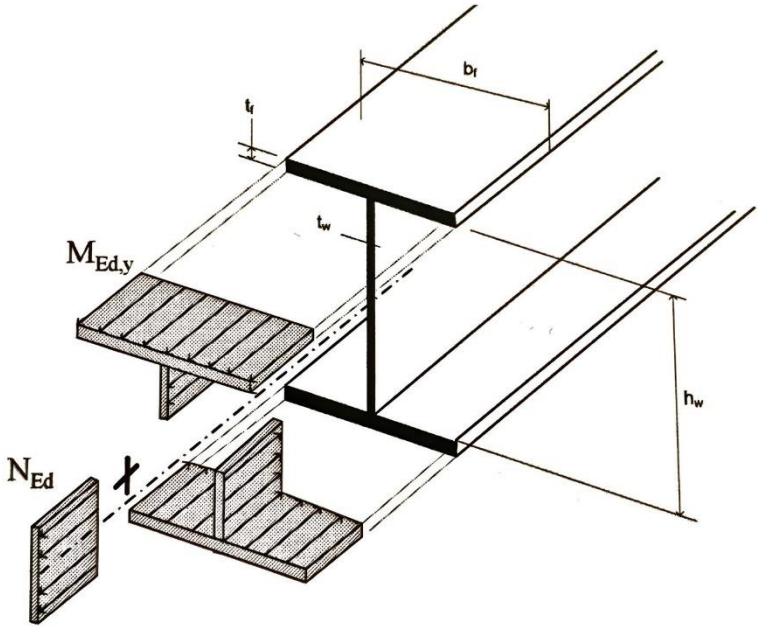


Figure 3. 7 : Model for a double-symmetric I-section loaded in bending and axial load[23]

In the most general case, dependent on the position of the NA (i.e., the ratio of axial force to bending moment), two situations exist:

- In the web, the neutral axis (NA) is located.
- The NA can be found on the flange.

Equilibrium of stresses over the cross-section can be used in both cases to identify the parts of the cross-section carrying the moment and the part carrying the axial force. If full Plasticity is allowed to occur, the failure condition is shown in Figure 3.8 and Figure 3.9 for the strong y-y axis and Figure 3.10 and Figure 3.11 for the weak axis (z-z).

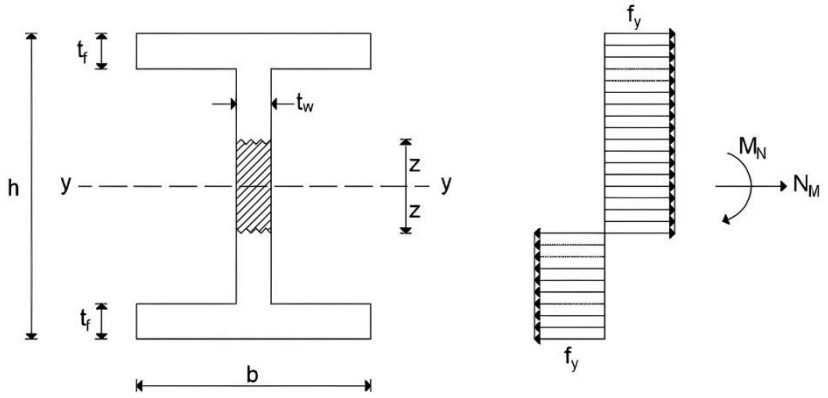


Figure 3. 8 : The neutral axis in the web[23]

For

$$z \leq t_w$$
$$N_M = 2f_y t_w z$$
$$M_N = f_y b t_f (h - t_f) + f_y t_w \left[\left(\frac{h-2t_f}{2} \right)^2 - z^2 \right] \qquad \textbf{3. 5}$$

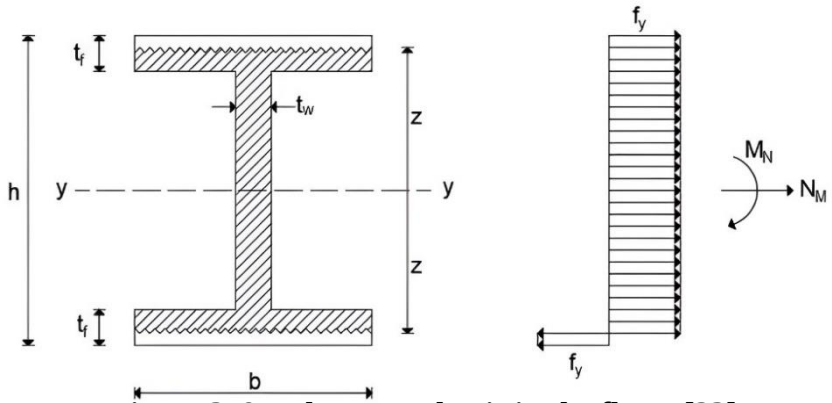


Figure 3. 9 : The neutral axis in the flange[23]

For

$$z_n \geq \frac{h-2t_f}{2}$$
$$N_M = f_y \left[t_w (h - 2t_f) + 2b \left(t_f - \frac{h}{2} + z_n \right) \right] \qquad \textbf{3. 6}$$
$$M_N = f_y \left[2b \left(\frac{h}{2} - z_n \right) \left(\frac{1}{2} \left(\frac{h}{2} - z_n \right) + z_n \right) \right]$$

And around the weak axis (z-z):

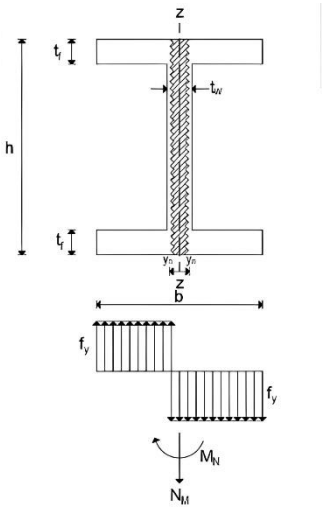


Figure 3. 10 : The neutral axis in the web[23]

$$y_n \leq t_w$$

For

$$N_M = 2f_y h y_n \qquad \textbf{3. 7}$$

$$M_N = f_y\left[t_f\frac{b^2}{2} + (h - 2t_f)\frac{t_w^2}{4} - {y_n}^2 h\right]$$

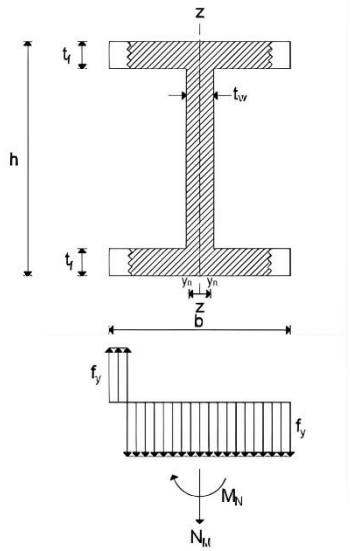


Figure 3. 11 : The neutral axis in the flange[23]

$$y_n \geq t_w$$

For

$$N_M = f_y\left[2t_f b + (h - 2t_f) - 4f_y\left(\frac{b}{2} - y_n\right)\right] \qquad \textbf{3. 8}$$

$$M_N = 4f_y t_f\left(\frac{b}{2} - y_n\right)\left[\frac{1}{2}\left(\frac{b}{2} - y_n\right) + y_n\right]$$

3.3.2 Elastic cross-section design:

The load-carrying capacity is assumed to be reached when the stress in the cross-outermost section yields loaded fibres. The principle of superposition can be used to add two stress distributions for elastic behaviour, as shown in Figure 3.12. As a result, the first yield, or the point at which the material begins to deform plastically, will occur at the location where the maximum compressive bending stress is present. This yield point corresponds to the condition where the material reaches its maximum capacity to resist deformation corresponding to the following condition[24]:

$$\boldsymbol{f_y = \sigma_c + \sigma_b} \qquad \textbf{3. 9}$$

Where, f_y Is the material yield stress

the stress because of the compressive load N $\sigma_c = \frac{N}{A}$

the maximum compressive stress caused by the moment M, h is the overall section depth, and I is the second area moment of the y-axis $\sigma_b = \frac{M\frac{h}{2}}{I}$

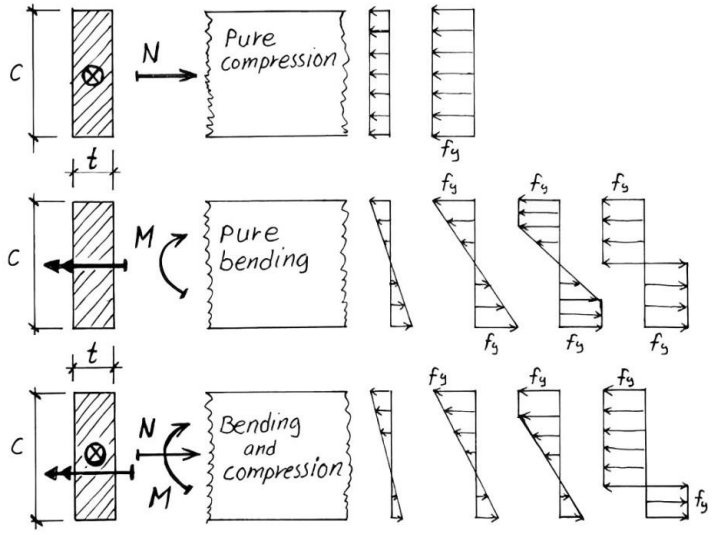


Figure 3. 12 : Cross-section elastic behaviour in compression and bending[23]

However, this assumption corresponds to an elastic cross-section resistance. In this case, the design criterion is

$$\boldsymbol{\sigma = \frac{N_{Ed}}{A} + \frac{M_{Ed}}{W} \leq f_y} \qquad \textbf{3. 10}$$

$$\boldsymbol{\sigma = \frac{N_{Ed}}{A} + \frac{M_{Ed}}{W_{eff}} \leq f_y}$$

The design condition in equation (3.9) above applies to cross-section classes in which the stress in the extreme fibres is limited to the yield stress due to local buckling that prevents the formation of a full plastic hinge. While equation (3.10) applies for cross-sections in which yield in the extreme fibres is not possible due to local buckling.

3.4 Calculation and verification according to Eurocode 3:

Design codes at limit states, such as Eurocode 3, typically consider the ultimate load as the calculation criterion when analysing resistance under static loading. Therefore, these equations must be reformulated in terms of actions. During this process, it is also necessary to account for specific effects present in structures that have yet to be explicitly considered, such as initial straightness deviations, residual stresses...etc. For consistency in the calculations, the interaction equation for the load combination must be reduced to the design procedures for beams and columns, while the moment and axial load are each reduced to zero. The verification of the safety of members subject to bending and axial force is made in two steps[25]:

- verification of the member buckling resistance (in general governed by flexural or lateral-torsional buckling).
- verification of the resistance of cross sections;

3.4.1 Verification of the member buckling resistance:

3.4.1.1 Combined Axial Compression and Uniaxial Bending:

- **Class 1 and 2 Cross-Sections:**

The approach adopted in Eurocode 3 [8] (assuming bending about the y-axis) allows for safe use without risk of yielding:

$$\frac{N_{sd}}{\chi_y . A f_y} + \frac{K_y . M_y}{W_{ply} . f_y} \leq 1 \qquad \textbf{3. 11}$$

χ_y is the reduction factor for column buckling.

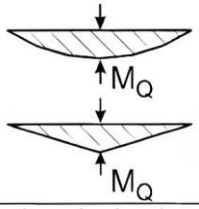
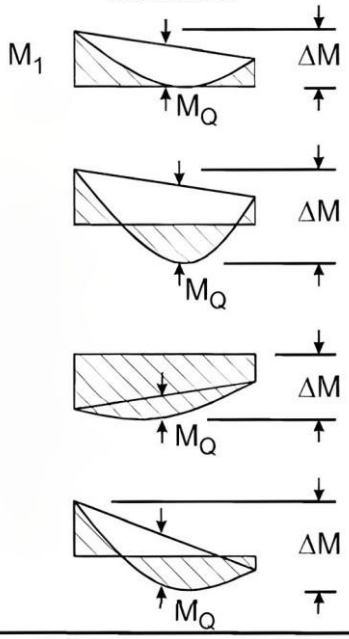
$$k_y = 1 - \frac{\mu_y N_{sd}}{\chi_{zy} A f_y} \qquad \text{but} \qquad k_y \leq 1{,}5$$

k_y is an adjustment factor

$$\mu_y = \overline{\lambda}_y (2\beta_{M.y} - 4) + \frac{W_{pl.y} - W_{el.\,y}}{W_{el.\,y}} \quad \text{but} \quad \mu_y \leq 0{,}9$$

Where $\beta_{M.y}$ represents an equivalent uniform moment factor that accounts for the non-uniformity of the moment diagram; see Table 3.1 (moment diagram about the y-axis and fixities in the z direction)

Table 3. 1 : Equivalent uniform moment factor[8]

Moment diagram	Equivalent uniform moment factor β_M
End moments  $-1 \le \psi \le 1$	$\beta_{M,\psi} = 1{,}8 - 0{,}7\,\psi$
Moments due to in-plane lateral loads 	$\beta_{M,Q} = 1{,}3$ $\beta_{M,Q} = 1{,}4$
Moments due to in-plane lateral loads plus end moments 	$\beta_M = \beta_{M,\psi} + \frac{M_Q}{\Delta M}(\beta_{M,Q} - \beta_{M,\psi})$ $M_Q = \lvert \max M \rvert$ due to lateral load only ΔM { $\lvert \max M \rvert$ for moment diagram without change of sign; $\lvert \max M \rvert + \lvert \min M \rvert$ for moment diagram with change of sign }

- **Class 3 Cross-Sections:**

Elements with Class 3 cross-sections subjected to bending and axial load must satisfy the following expression:

$$\frac{N_{sd}}{\chi_y . A f_y} + \frac{K_y . M_y}{W_{ely} . f_y} \le 1 \qquad \textbf{3. 12}$$

Where k_y and χ_y are defined as previously and:

$\mu_y = \overline{\lambda}_y(2\beta_{M.y} - 4)$ but $\mu_y \leq 0{,}9$

- **Class 4 Cross-Sections:**

When checking a Class 4 cross-section under combined axial force and bending according to Eurocode 3 (EN 1993-1-1) [8], the approach is more involved than with Class 1, 2, or 3 cross-sections due to the local buckling issues inherent in slender elements. For a Class 4 cross-section the formula can be interpreted as:

$$\frac{N_{sd}}{\chi_{min} \cdot A_{eff} \cdot f_y} + \frac{K_y(M_{ysd+} N_{sd} \cdot eN_z)}{W_{eff.y} \cdot f_y} \leq 1 \qquad \textbf{3. 13}$$

Where:

- N_{sd}: Design axial force (factored axial load applied on the cross-section).
- $M_{,y\,sd}$: Design bending moments about the y.
- $eN_{,z}$: Eccentricities of the axial load relative to the centroidal axes (z direction).
- A_{eff}: Effective cross-sectional area of the Class 4 section, which accounts for reductions due to local buckling.
- $W_{eff,y}$: Effective section moduli about the y, taking into account the reduced section due to local buckling.
- χ_{min} : Minimum buckling reduction factor for the section, derived from relevant buckling curves in Eurocode 3.
- K_y: Interaction factor, which is typically defined by the code.

3.4.1.2 Combined Axial Compression and Biaxial Bending:

When a structural member is subjected to both axial compression and bending around two principal axes, it experiences combined axial compression and biaxial bending. This complex loading scenario is common in steel structures, where columns or beams must withstand axial loads alongside bending moments in both directions. Proper design and verification of these members are essential to ensure safety, accounting for possible interactions between axial and bending stresses. Eurocode 3 provides guidelines for assessing such combined effects, which are significant for slender or non-compact cross-sections sensitive to buckling and local instabilities[26].

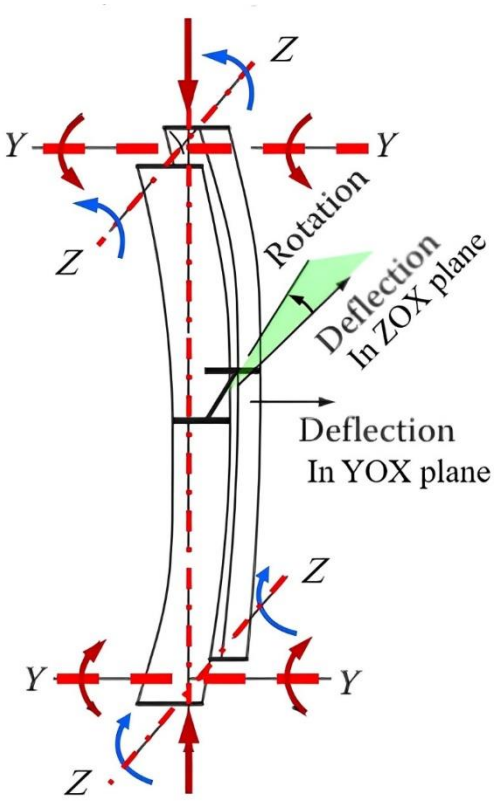


Figure 3. 13 : Combined Axial Compression and Biaxial Bending[26]

- **Class 1 and 2 Cross-Sections:**

$$\frac{N_{sd}}{\chi_{min}.Af_y}+\frac{K_y.M_y}{W_{ply}.f_y}+\frac{K_z.M_{zsd}}{W_{plz}.f_y}\leq 1 \qquad \textbf{3. 14}$$

χ_{min} is the minimum reduction factor for column buckling.

$$k_y = 1-\frac{\mu_y N_{sd}}{\chi_y A f_y} \quad \text{but} \quad k_y \leq 1{,}5$$

$$k_z = 1-\frac{\mu_z N_{sd}}{\chi_z A f_y} \quad \text{but} \quad k_z \leq 1{,}5$$

$$\mu_y = \overline{\lambda}_y(2\beta_{M.y}-4)+\frac{W_{pl.y}-W_{el.\,y}}{W_{el.\,y}} \quad \text{but} \quad \mu_y \leq 0{,}9$$

$$\mu_z = \overline{\lambda}_z(2\beta_{M.z}-4)+\frac{W_{pl.z}-W_{el.\,z}}{W_{el.\,z}} \quad \text{but} \quad \mu_z \leq 0{,}9$$

- **Class 3 Cross-Sections:**

Elements with Class 3 cross-sections subjected to bending and axial load must satisfy the following expression:

$$\frac{N_{sd}}{\chi_{min}.Af_y} + \frac{K_y.M_{ysd}}{W_{ely}.f_y} + \frac{K_z.M_{zsd}}{W_{elz}.f_y} \leq 1 \qquad \textbf{3. 15}$$

Where k_y k_z and χ_{min} are defined as previously and:

$\mu_z = \overline{\lambda}_z(2\beta_{M.z} - 4)$ but $\mu_z \leq 0{,}9$

- **Class 4 Cross-Sections:**

for a Class 4 cross-section the formula can be interpreted as:

$$\frac{N_{sd}}{\chi_{min}.A_{eff}.f_y} + \frac{K_y(M_{ysd+}N_{sd}.eN_Z)}{W_{eff.y}.f_y} + \frac{K_z(M_{zsd+}N_{sd}.eN_{yZ})}{W_{eff.z}.f_y} \leq 1 \qquad \textbf{3. 16}$$

- N_{sd}: Design axial force (factored axial load applied on the cross-section).
- $M_{,y\,sd}$, $M_{,z\,sd}$: Design bending moments about the yy an zz respectively.
- $eN_{,z,}$ $eN_{,y}$: Eccentricities of the axial load relative to the centroidal axes (z and y directions).
- A_{eff}: Effective cross-sectional area of the Class 4 section, which accounts for reductions due to local buckling.
- $W_{eff,y}$, $W_{eff,z}$: Effective section moduli about the y and z respectively , taking into account the reduced section due to local buckling.
- χ_{min} : Minimum buckling reduction factor for the section, derived from relevant buckling curves in Eurocode 3.
- K_y, K_z: Interaction factors, which are typically defined by the code.

3.4.2 Verification of the cross-section's resistance:

In designing steel structures, it is crucial to verify that each cross-section can withstand the applied loads, mainly when dealing with complex loading scenarios like combined axial compression and bending. Eurocode 3 (EN 1993-1-1) [8] provides guidelines to assess the resistance of cross-sections under these combined effects, whether they involve uniaxial or biaxial bending. If the moment distribution across the member is not uniform (for example, when moments are not simply the result of uniform curvature), adjusting the k factors to reflect the actual moment gradient along the member is essential. This adjustment, often achieved through the modification factor $\boldsymbol{\beta_M}$, reduces the severity of bending effects when the moment distribution is favourable (e.g., non-uniform moment diagrams with lower peak moments). However, even with these adjustments, a localized verification of cross-sectional resistance is

required to ensure that each section along the length can resist the combined effects of axial compression and principal moments[27].

Eurocode 3 [8] provides different formulas depending on the class and type of the section for cross-sections subjected to uniaxial bending combined with axial compression. These expressions are generally more straightforward but must be carefully applied based on cross-sectional classification. However, for cross-sections under biaxial bending with axial compression, Eurocode 3 requires a more comprehensive approach. The following interaction formula must be used to verify the resistance:

$$\left(\frac{M_{y.Sd}}{M_{Ny.Rd}}\right)^{\alpha} + \left(\frac{M_{z.Sd}}{M_{NzRd}}\right)^{\beta} \leq 1 \qquad \textbf{3. 17}$$

where:

- $M_{y.Sd}$ and $M_{z.Sd}$ are the design bending moments about the principal **y** and **z** axes, respectively.
- $M_{Ny.Rd}$ and M_{NzRd} are the corresponding design bending resistances about these axes.
- **α** and **β** are interaction exponents that vary based on the cross-section class and loading conditions.

These exponents **α** and **β** can be found in **Table 3.2**, where different values are provided based on whether the cross-section is compact (Class 1 or 2) or non-compact (Class 3 or 4).

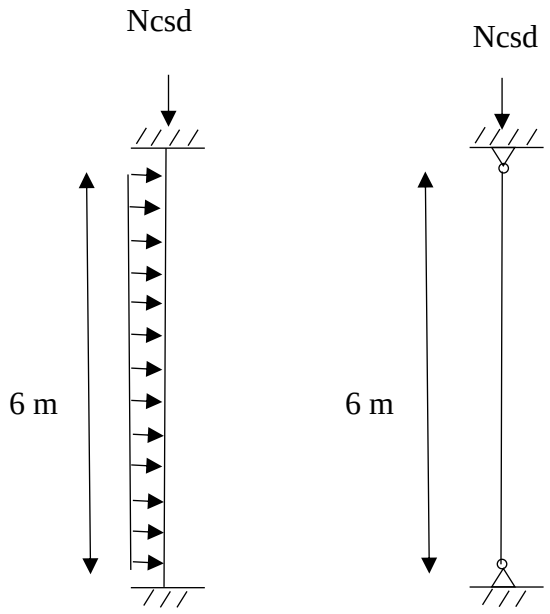
Table 3. 2 : Interaction exponents α and β[8]

Cross-Section Type	α	β
I and H sections	2	$5n$ but ≥ 1
Circular hollow sections	2	2
Rectangular hollow sections	$\frac{1,66}{1-1,33n^2}$ but ≤ 6	$\frac{1,66}{1-1,33n^2}$ but ≤ 6
Flat and solid rectangular bars	$1,73+1,8n^3$	$1,73+1,8n^3$

3.5 Examples:

- **Example 01: Checking the Stability of a Column Under Axial Compression and Bending**

Verify the stability and resistance of an HEB220 column under a compression force and uniformly distributed load.



Solution:

- ***Section Classification:***

➢ *web:*

d/t= 152/9,5=17,33 < 33ε =33

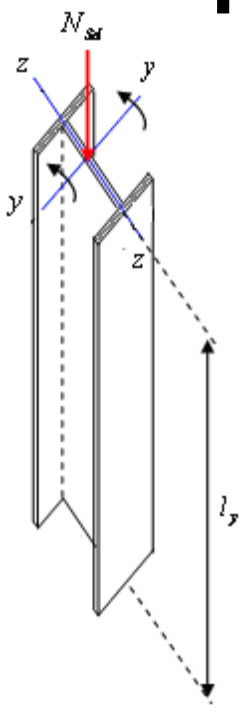
For Class 1 sections:

$$\frac{d}{t} \leq \frac{396\varepsilon}{(13\alpha-1)} \quad for \quad \alpha > 0{,}5$$

$$\frac{d}{t} \leq \frac{36\varepsilon}{\alpha} \quad for \quad \alpha \leq 0{,}5$$

$$\alpha = \frac{1}{d}\left(\frac{h}{2} + \frac{1}{2}\frac{N_{sd}}{2t_w f_y} - \left(t_f - r\right)\right) \leq 1$$

$$\alpha = \frac{1}{152}\left(\frac{220}{2} + \frac{1}{2}\frac{100000}{2.9{,}5.235} - (16-18)\right) = 0{,}81$$



α> 0,5 so $\frac{152}{9,5} = 16 \leq \frac{396}{(13.0,81-1)} = 41,55$

The web is classified as Class 1

- *Flange:*

C/t= 105,25/16=8,55 < 9ε =9 the flange is also classified as Class 1.

Consequently, this section is classified as Class 1.

- ***Buckling Length:***

L_{fy}*=1×6 =6 m and* L_{fz}*=0,5×6 =3 m*

$i_y = 94,3\ mm$

$i_z = 55,9\ mm$

- ***Effective slenderness:***

$\lambda_y = 6000/94,3 = 64$

$\lambda_z = 3000/55,9 = 54$

- ***Reduced slenderness:***

$\bar{\lambda}_y = 64/93,9\sqrt{1} = 0,69$

$\bar{\lambda}_z = 54/93,9\sqrt{1} = 0,58$

Since $\bar{\lambda} \geq 0,2$*, buckling risk is present.*

- ***Stability Verification:***

The interaction formula used for verification is:

$$\frac{N_{sd}}{\chi_{min}.Af_y} + \frac{K_y.M_y}{W_{ply}.f_y} + \frac{K_z.M_{zsd}}{W_{plz}.f_y} \leq 1$$

- *Calculate* χ_{min}*:*

Since h/b<1,2 and t_f*= 16 mm, use buckling curves* ***b*** *for y-axis and* ***c*** *for z-axis*

$\bar{\lambda}_y = 0,69$

0,6 ⟶ 0,8371 *0,1 ⟶ 0,0534*

0,7 ⟶ 0,7837 *0,09 ⟶ x=0,0480*

χ_y= *0,7890*

$\bar{\lambda}_z = 0{,}58$

0,5 ⟶ 0,8430 *0,1 ⟶ 0,0504*

0,6 ⟶ 0,7854 *0,08 ⟶ x=0,0403*

χ_z= *0,8028*

χ_{min}= *0,7890*

- *Calculate K_y and K_z :*

K=1- (μ.N_{sd})/ (χ.Af_y)

$\mu_z = \bar{\lambda}_z (2\beta_M - 4) + (W_{plz} - W_{elz})/W_{elz} = 0{,}58.(2.1{,}23-4)+(393{,}88-258{,}48)/258{,}48 = -0{,}35$

$K_z = 1 - \frac{(-0{,}35.100000)}{(0{,}4438.9104.235)} = 1{,}02$

$$\frac{100000}{0{,}7890.9104.235} + \frac{0}{827050.235} + \frac{1{,}03.6000000}{393880.235} = 0{,}120 \leq 1$$

The column is stable

- ***Moment resistance check:***

$M_{sd} \leq M_N$

- *Axial force interaction check:*

$$\boldsymbol{N_{sd} = 100\ KN < min\begin{cases} 0,25N_{pl} = 534,86\ KN \\ 0,5A_w f_y = 169\ KN \end{cases}}$$

There is no interaction with the axial force N_{sd}

M_{zsd} =9 *KN.m* ≤ M_N=Mplz= $W_{ply}.f_y/\gamma_{m0}$=92,56 *KN.m*

verified as safe.

- ***Cross-Section Resistance Verification:***

$$\left(\frac{M_{y.Sd}}{M_{Ny.Rd}}\right)^{\alpha} + \left(\frac{M_{z.Sd}}{M_{NzRd}}\right)^{\beta} \leq 1$$

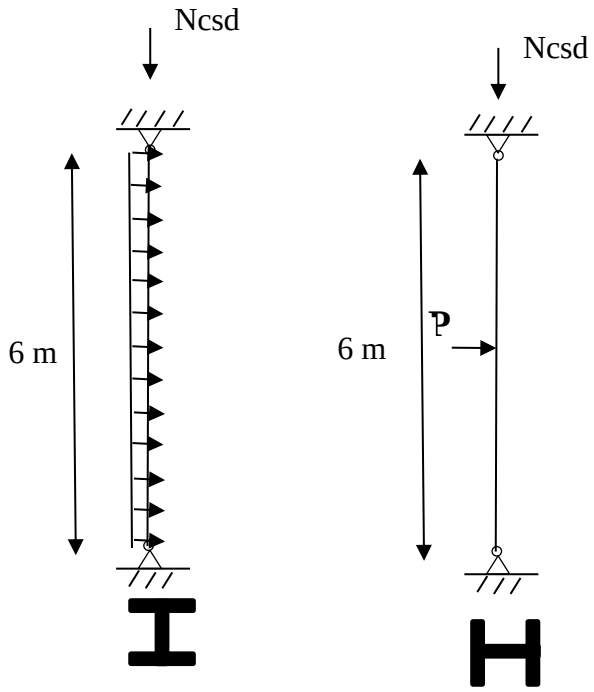
$$\left(\frac{6}{92{,}56}\right)^{2} = 0{,}0042 \leq 1$$

The section is verified.

- **Example 02: Checking the Stability of a Column Under Axial Compression and biaxial Bending**

Verify the HEM600 column in steel S235 as shown in the figure, subjected to the following loads:

- Axial force Nsd=700 kN
- Concentrated load P=50 kN
- Distributed load q=20 kN



Solution:

- ***Section Classification:***

➢*web:*

$$\frac{d}{t} = \frac{486}{21} = 23.14 < 33\varepsilon = 33.$$

For Class 1 section*:*

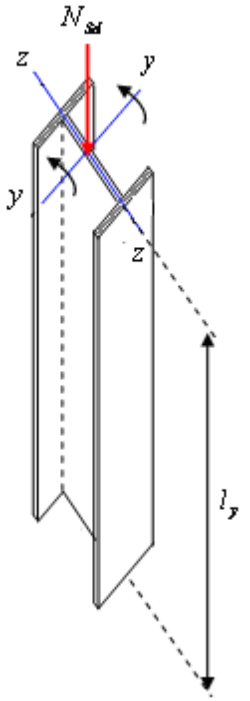
$$\frac{d}{t} \leq \frac{396\varepsilon}{13\alpha - 1} \quad ; \quad pour: \alpha > 0.5.$$

$$\frac{d}{t} \leq \frac{36\varepsilon}{\alpha} \quad ; \quad pour: \alpha \leq 0.5.$$

$$\alpha = \frac{1}{d}\left[\frac{h}{2} + \frac{1}{2}\frac{N_{sd}}{2t_w f_y} - \left(t_f - r\right)\right] \leq 1.$$

$$\alpha = \frac{1}{486}\left[\frac{620}{2} + \frac{1}{2}\frac{700000}{2 \times 21 \times 235} - (40 - 27)\right] = 0.68.$$

$$\alpha > 0.5 \quad so \quad \frac{486}{21} = 23.14 \leq \frac{396}{13 \times 0.68 - 1} = 50.51.$$



Web is Class 1.

- *Flange:*

$$\frac{c}{t} = \frac{142}{40} = 3.55 \leq 10\varepsilon = 10.$$

Flange is Class 1.

Therefore, the section is classified as Class 1.

.

- ***Buckling Length:***

$$\left\{\begin{matrix} L_{fy} = 1 \times 6 = 6m. \\ L_{fz} = 6 \times 1 = 6m. \end{matrix}\right\} et$$

With:

$i_y = 255.5mm.$

$i_z = 72.2mm.$

- ***Slenderness:***

$$\lambda_y = \frac{l_0}{i_y} = \frac{6000}{72.2} = 83.1 = 83.$$

$$\lambda_z = \frac{l_0}{i_z} = \frac{6000}{255.5} = 23.5 = 24.$$

- ***Reduced Slenderness:***

$$\overline{\lambda_y} = \frac{\lambda_y}{\lambda_1}\sqrt{B_A} = \frac{24}{93.9}\sqrt{1} = 0.25.$$

$$\overline{\lambda_z} = \frac{\lambda_z}{\lambda_1}\sqrt{B_A} = \frac{83}{93.9}\sqrt{1} = 0.88.$$

$\bar{\lambda} \geq 0.2 \quad \rightarrow \quad there\ is\ a\ risk\ of\ buckling.$

- ***Stability Verification:***

$$\frac{N_{sd}}{\chi_{min} A f_y} + \frac{K_y M_y}{W_{ply} f_y} + \frac{K_z M_{z.sd}}{W_{plz} f_y} \leq 1.$$

➢ *Calculate* χ_{min} *:*

$\frac{h}{b} = \frac{620}{305} = 2.03 > 1.2 \qquad and \qquad t_f = 40mm.$

*Buckling curve "**a**" for yy and "**b**" for zz.*

$\overline{\lambda_y} = 0.25$

$0.2 \rightarrow 1.0000.$ $\therefore 0.1 \rightarrow 0.0225 \because$

$0.3 \rightarrow 0.9775.$ $0.05 \rightarrow x = 0.01125.$

$$\boldsymbol{x_y = 0.9888}.$$

$\overline{\lambda_z} = 0.88$

$0.8 \rightarrow 0.7245.$ $\therefore 0.1 \rightarrow 0.0633 \because$

$0.9 \rightarrow 0.6612.$ $0.08 \rightarrow x = 0.05064.$

$$\boldsymbol{x_y = 0.6738}.$$

$$\boldsymbol{\chi_{min} = min(x_y; x_z) = 0.6738.}$$

- *Calculate k_y and k_z:*

$$k_z = 1 - (\mu_z . N_{sd})/(\chi_z . Af_y)$$

$$\mu_z = \overline{\lambda_z}(2\beta_{MQ} - 4) + (W_{plz} - W_{elz})/W_{elz}$$

$$\mu_z = 0.88(2 \times 1.4 - 4) + \frac{1930.38 - 1244.29}{1244.29} = \mathbf{-0.504}.$$

$$k_z = 1 - \frac{-0.504 \times 700000}{0.9888 \times 363.66 \times 235} = \mathbf{1.061}.$$

$$k_y = 1 - (\mu_y . N_{sd})/(\chi_y . Af_y)$$

$$\mu_y = \overline{\lambda_y}(2\beta_{My} - 4) + \frac{(W_{ply} - W_{ely})}{W_{ely}} = \mathbf{-0.154}.$$

$$k_y = \mathbf{1.012}.$$

$$\frac{700000}{0.6738 \times 36366 \times 235} + \frac{1.012 \times 75 \times 10^6}{5350390 \times 235} + \frac{1.06 \times 90 \times 10^6}{1930380 \times 235} = 0.28 \leq 1.$$

The column is stable.

- ***Moment resistance check:***

$$M_{sd} \leq M_N$$

- *Axial force N_{sd} interaction check:*

$$N_{sd} = 700\ KN\ > \min\begin{cases} 0.25N_{pl} = 0.25\left(\frac{Af_y}{\gamma_{m0}}\right) = 2136.5KN. \\ 0.5A_w f_y = 1332.45KN. \end{cases}$$

$$N_{sd} = 700KN < 1332.45KN.$$

There is no interaction with the axial force N_{sd}

:

$$M_{ysd} = 75\ KN < M_{ply} = 2061{,}41\ KN \quad ; \quad M_{zsd} = 90\ KN < M_{PLZ} = 453{,}63\ KN$$

verified as safe.

- ***Cross-Section Resistance Verification:***

$$\left(\frac{M_{y.Sd}}{M_{Ny.Rd}}\right)^{\alpha} + \left(\frac{M_{z.Sd}}{M_{NzRd}}\right)^{\beta} \leq 1$$

$$\left(\frac{75}{2061{,}41}\right)^{2} + \left(\frac{90}{453{,}63}\right)^{2} = 0{,}04 \leq 1$$

The section is verified.

CHAPTER 04: Lateral Torsional Buckling of Steel Members

4.1 Introduction:

Beams under bending stress are generally constructed from I-shaped profiles. Due to their significantly lower moment of inertia about the weak z-axis compared to the strong y-axis, these beams are susceptible to a stability issue known as lateral-torsional buckling. To illustrate this phenomenon, consider the example of a cantilever beam subjected to a concentrated vertical load at its free end (see Figure 4.1). Under a relatively low load, the beam primarily deforms perpendicular to its strong axis (in this case, vertically). However, as the load increases, the compressed portion of the profile may shift laterally, attempting to relieve the compressive stress. This lateral deformation, known as buckling, causes each section of the beam to undergo vertical deflection and experience horizontal displacement coupled with rotation about its shear centre. This combined motion reflects the complex interplay of forces that drive lateral-torsional instability in flexural members.

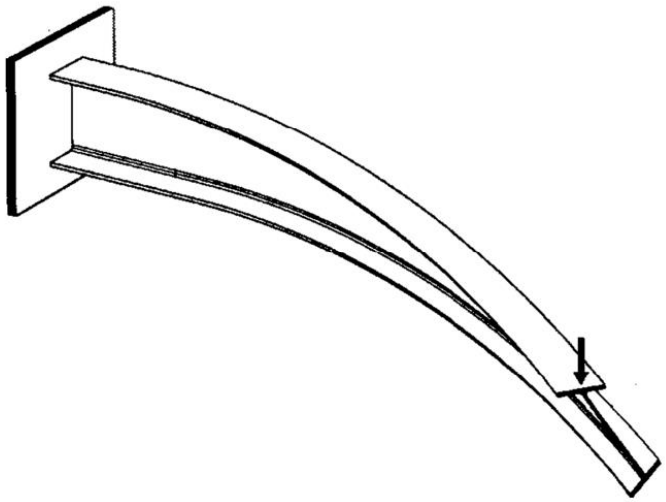


Figure 4. 1 : Lateral torsional buckling [10]

4.2 Definitions and description of concepts:

Lateral-torsional buckling is closely related to bending, involving the buckling of the compressed flange and sharing several characteristics with general buckling phenomena. When subjected to a critical moment, the beam bends and twists along its weakest plane of inertia. This phenomenon typically occurs in beams with low transverse bending and torsional rigidity. Lateral-torsional buckling is the second primary form of elastic instability in flexural members, alongside global buckling, to which it bears a clear analogy.

4.2.1 Mechanisms behind lateral torsional buckling:

Consider a beam in major axis bending subjected to increasing loading. If the beam is slender, it may buckle before the sectional capacity is reached. This kind of buckling involves both lateral deflection and twisting, and is called lateral torsional buckling. See Figure 4.2.

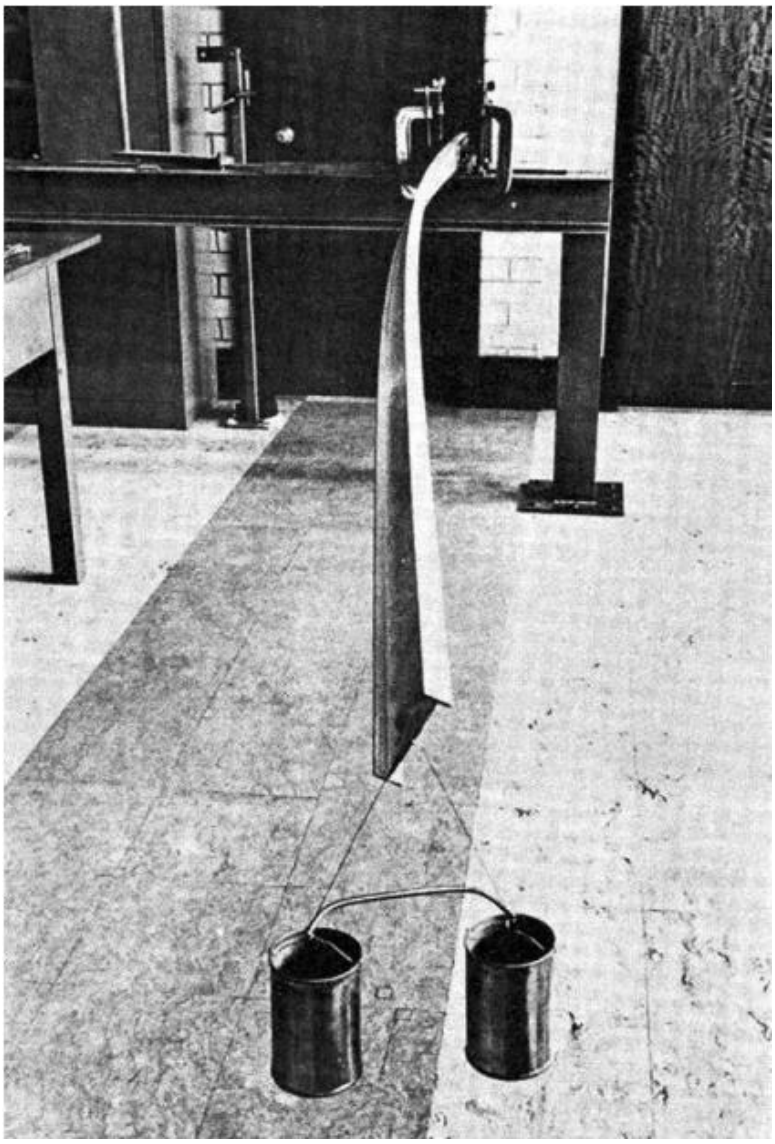


Figure 4. 2 : Lateral torsional buckling of a cantilever beam [28]

Longer beams become unstable at smaller loads compared to shorter beams. Also, if beams of the same length but with different cross-sections are tested, those with slender cross-sections buckle at smaller loads than those with stubby sections. However, the length of the beam and the cross-section shape are important. Many other factors also affect beams' sensitivity to lateral torsional buckling, see Table 4.1.

Table 4. 1 : Factors that affect lateral torsional buckling

Material properties	Shear modulus (G) Youngs's modulus (E)
Cross-section properties	Torsion constant (I_t) Warping constant (I_w) Second moment of inertia about weak axis (I_z)
Geometric properties	Length of the beam (L)
Boundary conditions	Bending about major axis Bending about minor axis Warping
Load	Type of loading (distributed, concentrated etc.) Point of load application (top flange, in shear centre, bottom flange etc.)

By considering the factors in Table 4.1 and choosing a beam with the right properties, lateral torsional buckling can be avoided. The risk of lateral torsional buckling is high for beams with the below listed properties [28]:

- Low flexural stiffness about the weak axis (EI_z)
- Low torsional stiffness (GI_t)
- Low warping stiffness (EI_w)
- High point of load application
- Long unrestrained spans (L)

4.2.2 Measurements against lateral torsional buckling:

To avoid lateral torsional buckling the affecting factors presented in Table 4.1 can be manipulated in different ways. Choosing a stable cross-section with high flexural stiffness about

the weak axis is one method. If the geometric and material properties are already determined, a low point of load application or restraints can be used.

- **Influence of the cross-section:**

Lateral torsional buckling is only possible in major axis bending. If the flexural stiffness is high enough about the weak axis or if the stiffness is equal about both axes, lateral torsional buckling will not occur. Figure 4.3 show sections that are safe with regard to lateral torsional buckling.

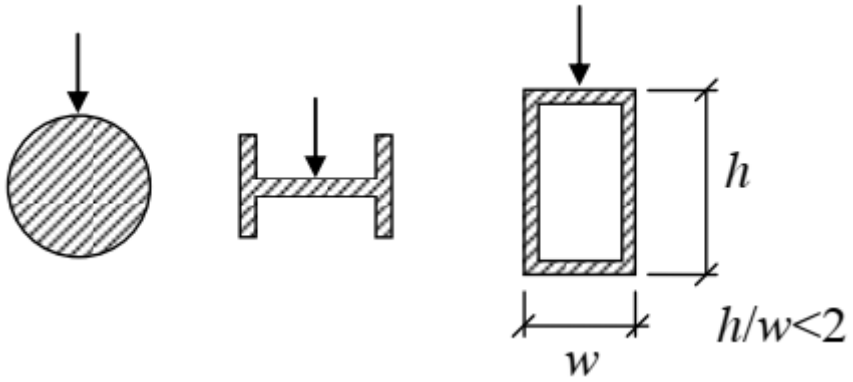


Figure 4. 3 : Sections with little or no risk of lateral torsional buckling [28].

- **Influence of the point of load application:**

A low point of load application reduces the risk of lateral torsional buckling. Hence, a load placed on the bottom flange makes a beam considerably more stable than a load on the top flange. This is provided that the load does not contribute to any restraining effects [28].

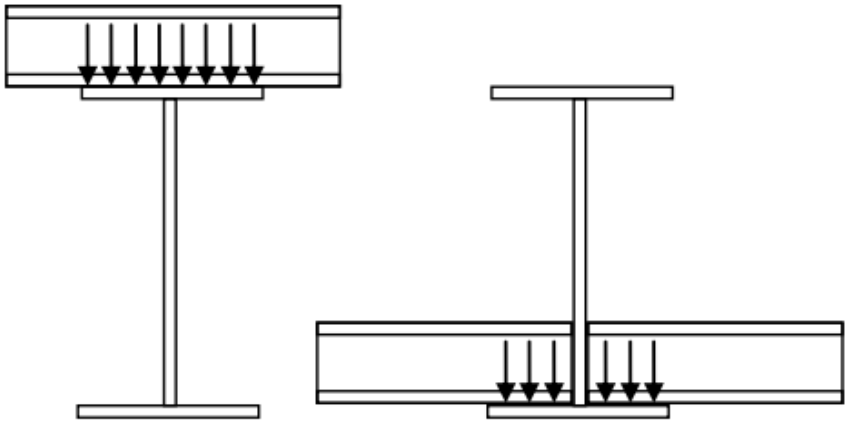


Figure 4. 4 : Beam loaded by a concentrated load from a secondary girder.

If a low point of load application is used the load helps to stabilize the beam. A high point of load application contributes to twisting moments and makes the beam less stable.

When a beam deflects due to lateral torsional buckling, loads above the centre of the twist will contribute to a twisting moment. Loads below the centre of the twist counteract rotations of the sections and stabilize the beam[28].

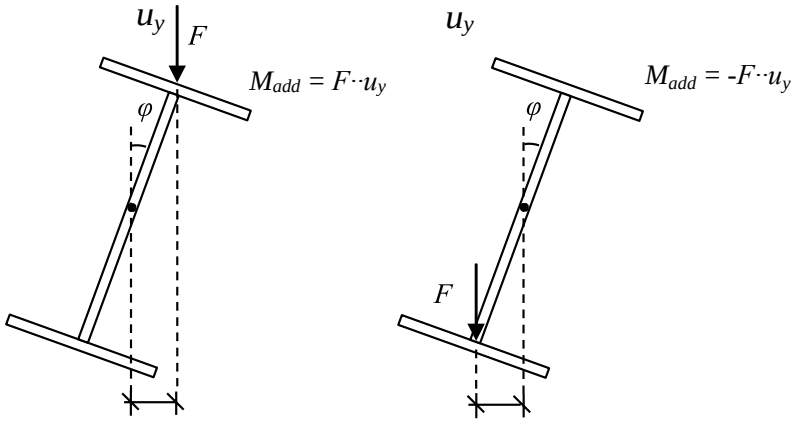


Figure 4. 5 : Additional moment due to load application point

A load, F, acting above the centre of twist contributes to the rotation of the section with an additional moment. When placed under the centre of twist, the load stabilizes the section.

- **Influence of lateral restraints**

If the flange in compression is restrained from lateral bending, lateral torsional buckling can be fully prevented. The spacing of the restraints must be small enough so buckling cannot occur between them.

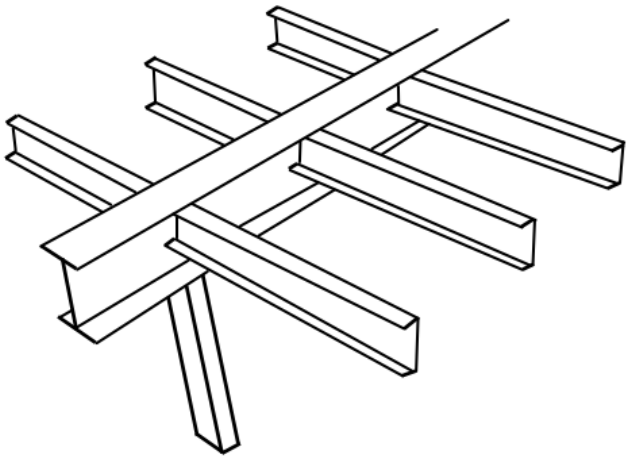


Figure 4. 6 : Restrained beam from lateral torsional buckling by secondary girders [29]

As in Figure 4.6, restraints can often be provided from secondary members in a construction. If a simply supported beam is loaded directly by an evenly distributed load from a slab, the beam can be fully prevented from lateral torsional buckling provided that the slab is part of a stable system. The upper flange of the beam is in compression in the whole span, and restrained at all

points. A twist of such a beam will be counteracted by the slab. See Figure 4.7 [28]. Even if the slab is part of a stable system in the finished structure, other conditions may apply during the erection of the building. Hence, commonly the beam is attached to the slab by some kind of connection to assure that it is restrained also during assembly.

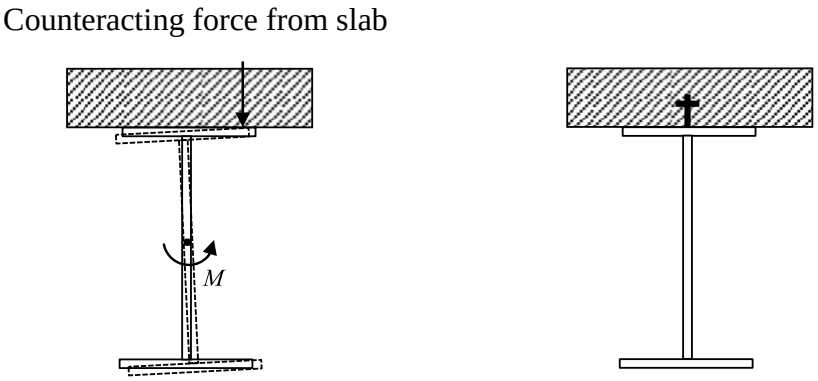


Figure 4. 7 : Beam subjected to a twisting moment is restrained by a floor slab.

A counteracting moment is created from the reaction force from the slab, holding the beam in place (left). Often the beam is attached to the slab to make sure that system is stable (right).

If the beam is continuous instead, the top flange will be in tension close to the mid-supports. Restraining the tension flange will also increase the stability of the beam, but in some cases, this is not enough to prevent the compression flange from buckling. This type of buckling is called distortional buckling and can be prevented by restraining the compression flange, i.e., restraining the bottom flange close to the mid-supports [28].

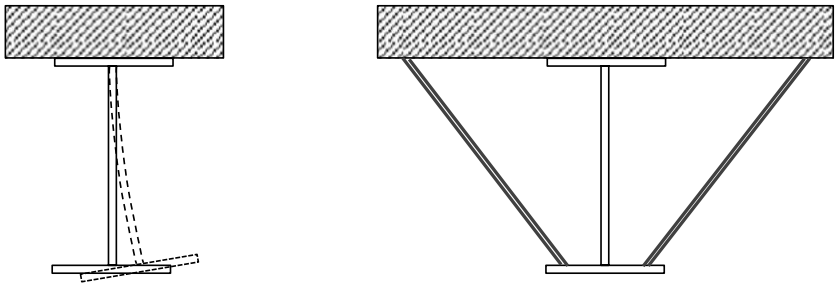


Figure 4. 8 : Restrained and unrestrained beam on tension flange

If a beam is restrained on the tension flange, the unrestrained compression flange may buckle laterally due to distortional buckling (to the left). This can be prevented by placing restraints onto the compression flange (to the right).

Unrestrained beams sometimes buckle between lateral torsional buckling and distortional buckling. This buckling mode is possible for beams with slender flexible webs and rigid flanges and is called lateral distortional buckling, see Figure 4.9 [30].

u_t

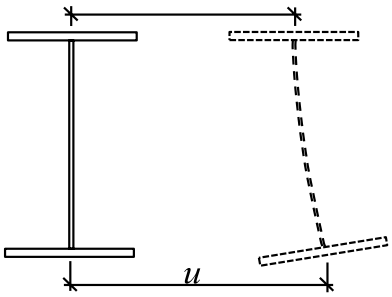
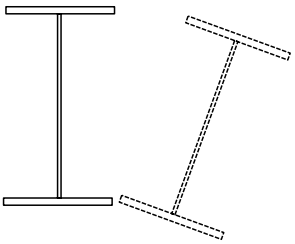
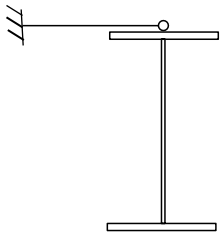
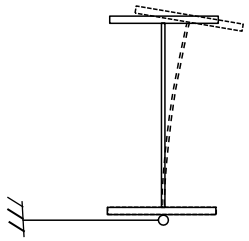
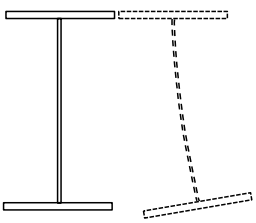


Figure 4. 9 : Lateral distortional buckling is possible for beams with slender webs and rigid flanges.

Table 4. 2 : Buckling modes and corresponding restraints [30]

No restraints - Free to buckle due to lateral torsional buckling	
Compression flange restrained - No lateral torsional buckling is possible	
Flange in tension restrained - Distortional buckling is possible	
No restraints - Lateral distortional buckling is possible for beams with slender flexible webs and rigid flanges	

4.3 Behaviour of real beams:

The theory of lateral-torsional buckling applies to beams under ideal elastic conditions, assuming perfect geometry with no initial imperfections[18]. In reality, however, no beam is flawless, which causes real beams to behave differently than their ideal counterparts (see Figure 4.10).

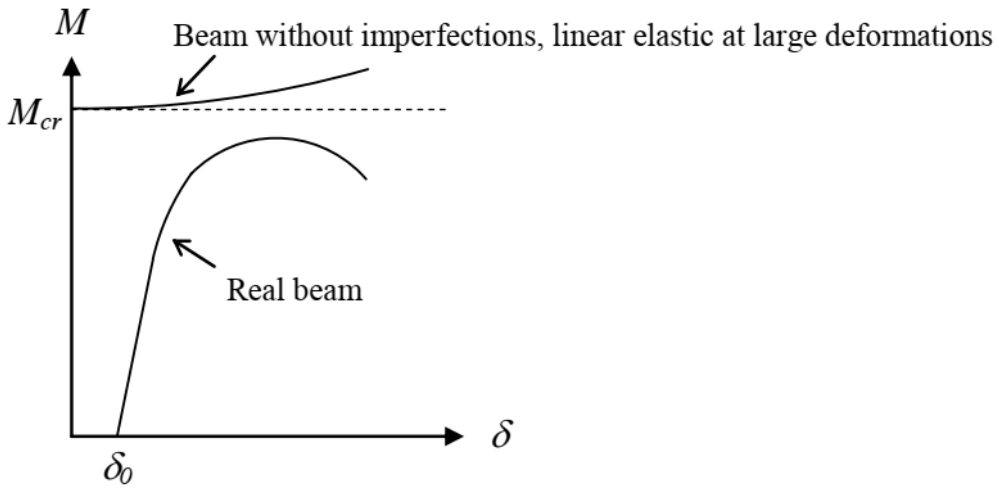


Figure 4. 10 : Elastic critical moment[31]

Increasing moment plotted against lateral deflection. An ideal beam shows no deflection until the elastic critical moment is reached, where a large instantaneous deflection occurs. A real beam has imperfections and residual stresses, and deflects as the load is increased. The failure of the real beam is governed by its imperfections and non-linear response [28].

An ideal beam remains laterally undeformed until it reaches the elastic critical moment. At this point, it enters an indifferent state of instability, resulting in a significant lateral deflection. Due to the material's ideal elastic nature, theoretically infinite deformations can occur, allowing the beam to establish a new equilibrium in its deflected position. Any small additional load beyond this point will cause even larger deflections (see Figure 4.10) [28].

In contrast, a real beam has reduced capacity compared to an ideal beam because of imperfections, residual stresses, and other factors. When a load is applied to a real beam, lateral imperfections are already present. These initial deformations increase as the load rises, and near the critical load, deflection increases rapidly. However, the theoretical, critical moment $\mathbf{M_{cr}}$ is never fully reached, as failure occurs due to the plastic behaviour of the material, geometric non-linearity, and potential local buckling.

Factors That Reduce the Load-Bearing Capacity of Real Beams are:

- Non-linear material behaviour, including plastic deformation.
- Initial lateral imperfections, such as minor curvature in the beam.
- Residual stresses introduced during manufacturing.
- Local buckling in section class 4 beam segments.
- Openings, asymmetries, and other defects.

These factors are considered in design using buckling curves that model real beam behaviour.

4.4 Design Considerations for Lateral Torsional Buckling:

In Eurocode 3, the standard approach for designing structural members against buckling involves applying a buckling factor, denoted as $\chi \leq 1$, to reduce the member's load-bearing capacity. Specifically, for lateral torsional buckling, this factor is represented as χ_{LT} and incorporates all the influences that lower the member's capacity due to buckling effects. Accordingly, the moment capacity, as defined by Eurocode 3[8], is expressed as follows:

$$M_{b.Rd} = \chi_{LT} \frac{W_y f_y}{\gamma_{M1}} \qquad \textbf{4. 1}$$

Where

W_y: Bending resistance corresponding to the cross-section classification of the member

f_y = Yield strength of the steel

γ_{M1} = Partial safety factor

The buckling factor χ_{LT} is defined as:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \bar{\lambda}_{LT}^2}}, \chi_{LT} \leq 1.0$$

$$\Phi_{LT} = 05\left(1 + \alpha_{LT}\left(\bar{\lambda}_{LT} - 0.2\right) + \bar{\lambda}_{LT}^2\right)$$

Where α_{LT} =Imperfection factor

Table 4. 3 : Imperfection factor α_{LT} for lateral-torsional buckling curves[8]

Buckling curve	a	b	c	d
Imperfection factor α_{LT}	0,21	0,34	0,49	0,76

From equation (4.2) and (4.3) it is clear that the buckling behaviour is strongly dependent on the slenderness parameter, $\bar{\lambda}_{LT}$.

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y \cdot f_y}{M_{cr}}} \qquad \textbf{4. 2}$$

M_{cr} in equation (4.3) is the elastic critical moment in classical buckling theory, i.e. the theoretical critical moment without regard to initial imperfections or residual stresses. Eurocode 3 does not provide any information on how to determine M_{cr}, but simply states that it should be based on the gross cross-section, the loading conditions, the real moment distribution and the lateral restraints.

4.4.1 Buckling curves:

Slender members are susceptible to buckling before reaching their full bending capacity, even for idealised, imperfection-free beams. Based on the concept of slenderness, this capacity reduction is represented by a theoretical buckling factor, $\chi_{LT} = 1/\bar{\lambda}_{LT}^{\,2}$, as shown in Equation 4.3.

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}} \leftrightarrow \bar{\lambda}_{LT}^{\,2} = \frac{W_y f_y}{M_{cr}} \leftrightarrow M_{cr} = \frac{1}{\bar{\lambda}_{LT}^{\,2}} W_y f_y \qquad \textbf{4. 3}$$

By plotting the theoretical buckling factor $1/\bar{\lambda}_{LT}^{\,2}$ against varying values of slenderness $\bar{\lambda}_{LT}$, an idealized buckling curve can be generated. This curve, representing a perfect beam with no imperfections, appears as the dashed line in Figure 4.11.

However, imperfections and residual stresses reduce the moment capacity in practical applications. Eurocode 3 incorporates design buckling curves derived from extensive testing to account for these real-world factors. These curves reflect different levels of residual stresses and imperfections in structural members, as outlined in Table 4.4 and illustrated in Figure 4.11.

Table 4. 4 : Selection of Buckling Curves Based on Cross-Section Geometry, Steel Quality, and Fabrication Method (EN-1993-1-1:2005, Table 6.2) [8].

Cross-section	Limits	Buckling curve
Rolled I-sections	h/b ≤ 2 h/b > 2	**b** **c**
Welded I-sections	h/b ≤ 2 h/b > 2	**c** **d**
Other cross-sections	-	**d**

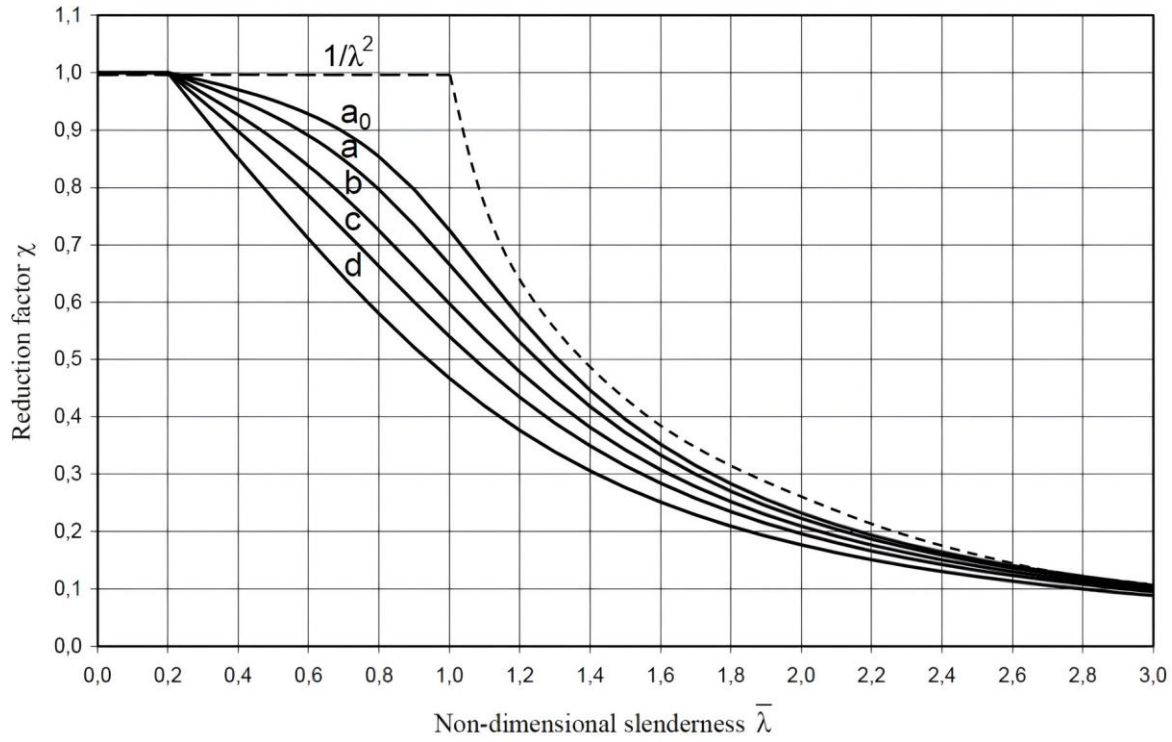


Figure 4. 11 : Buckling curves (adapted from EN-1993-1-1:2005)[8].

By examining Figure 4.11, several key insights emerge regarding the buckling behaviour of structural members. For members with low slenderness, most of the section's full capacity can be utilized, as indicated by a buckling factor close to or equal to 1.0. At high slenderness values, the member's resistance aligns closely with that of an ideal, imperfection-free beam, with minimal effects from imperfections and residual stresses. However, at intermediate slenderness levels, the impacts of initial imperfections and residual stresses become pronounced, resulting in significantly lower design buckling factors compared to the idealized buckling factor.

- **Comment:**

The procedures and buckling curves outlined in this chapter are broadly applicable. In specific scenarios, alternative methods and different buckling curves may be more appropriate. For comprehensive guidance, refer to EN-1993-1-1:2005 [8].

4.4.2 Critical Elastic Lateral-Torsional Buckling Moment M_{cr}:

At the critical moment M_{cr}, the beam will change from a straight state of equilibrium to a deflected state of equilibrium. To determine the critical load analytically, it is necessary to investigate whether such a change of equilibrium state is possible. This can be done in two ways: either by an equilibrium study of a small element of the deflected member or by a study

of the system's potential energy. The system's potential can be used because the theorem states that the potential will have a minimum at the equilibrium [32].

In complex situations where several types of buckling are present (e.g. lateral torsional buckling), it is common to derive the theoretical critical load based on energy methods.

Early editions of Eurocode 3 (ENV versions) provided an expression for M_{cr}, which was later removed in subsequent versions without a direct replacement. This expression, widely known as the **"3-factor formula,"** remains a popular analytical method for estimating the elastic critical moment[28].

The 3-factor formula builds upon a standard reference load case and introduces correction factors to adapt the Equation for various design situations. The reference case involves a doubly symmetric beam supported by two fork supports and subjected to a constant moment distribution. The critical moment associated with this reference scenario, denoted as M_{cr0}, is derived in Appendix C and is represented by Equation (4.4) below.

$$\boldsymbol{M_{cr,0} = \frac{\pi^2 \cdot EI_z}{L^2}\sqrt{\frac{I_w}{I_z} + \frac{L^2 \cdot GI_t}{\pi^2 \cdot EI_z}}} \qquad \textbf{4. 4}$$

The **3-factor formula** expands the expression of $M_{cr,0}$ to be valid for single-symmetric cross sections with arbitrary moment distributions. This is done by introducing three correction factors, C_1, C_2 and C_3 that account for these conditions. The factors can either be found in tables and figures, or estimated from approximate closed-form expressions. To account for warping degrees of freedom and lateral rotation at the supports, the expression is also expanded with the factors, k_z and k_w. The **3-factor formula** is presented below in equation (4.5)

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(k_z \cdot L)^2}\left(\left(\sqrt{\left(\frac{k_z}{k_w}\right)\frac{I_w}{I_z} + \frac{(k_z \cdot L)^2 GI_t}{\pi^2 \cdot EI_z} + \left(C_2 \cdot z_g\right)^2}\right) - \left(C_2 \cdot z_g - C_3 \cdot z_j\right)\right)} \qquad \textbf{4. 5}$$

Where

E = Young's modulus

G = Shear modulus

I_z= Second moment of inertia about weak axis

I_t= Torsion constant

I_w= Warping constant

L = Length between lateral restraints

k_z= Effective length factor which is related to the restrain against lateral bending at the boundaries

k_w= Effective length factor which is related to the restrain against warping at the boundaries

z_g = Distance between the point of load application and the shear centre

z_j = Distance related to the effects of asymmetry about *y*-axis

C_1 = Factor that account for the shape of the moment diagram

C_2 = Factor that account for the point of load application in relation to the shear centre

C_3 = Factor that account for asymmetry about *y*-axis

- **Comment:**

The 3-factor formula is only valid under the following conditions.

- Uniform cross-section with symmetry about weak axis
- Bending about major axis

- **The Equivalent Uniform Moment Factor C_1:**

In structural literature, the C_1 factor is commonly known as the *equivalent uniform moment factor* or the *moment gradient factor*. As discussed in Section 4.4, the critical buckling load for various loading cases can be derived using potential energy methods. Examples are given below for a double symmetric beam on fork supports, subjected to a concentrated load and a uniformly distributed load.

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(L)^2}\left(\left(\sqrt{\frac{I_w}{I_z} + \frac{(L)^2 GI_t}{\pi^2 \cdot EI_z}}\right)\right)} \qquad \textbf{4. 6}$$

Where C_1 take the following values

- A concentrated load in mid-span, fork supports, C_1 = 1.36
- A uniformly distributed load, fork supports, C_1 = 1.13
- Reference load case, $M_{cr,0}$, C_1 = 1.0

Since the boundary conditions affect the moment diagram, it is expected that they also will influence the values of C_1 and M_{cr}

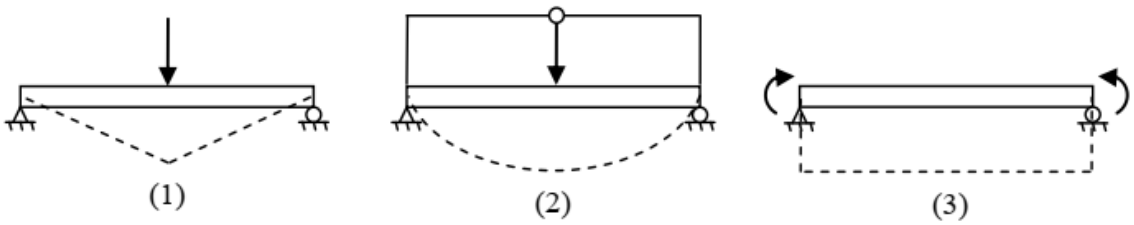


Figure 4. 12 : The moment diagrams in the different load cases.

- **Correction Factor for Point of Load Application C_2:**

Equation (4.6), including the C_1 factor, is accurate only when the load is applied directly at the beam's shear centre. In practice, however, loads are frequently applied at the top or bottom flange rather than at the shear centre, which necessitates the introduction of an additional correction factor C_2 to account for the *point of load application* (PLA).

When the PLA is offset from the shear centre (SC), it introduces an additional twisting moment, which affects the system's potential energy. If the load is applied below the SC, it stabilizes the beam, while a load applied above the SC has a destabilizing effect. This concept is illustrated in Figure 4.13.

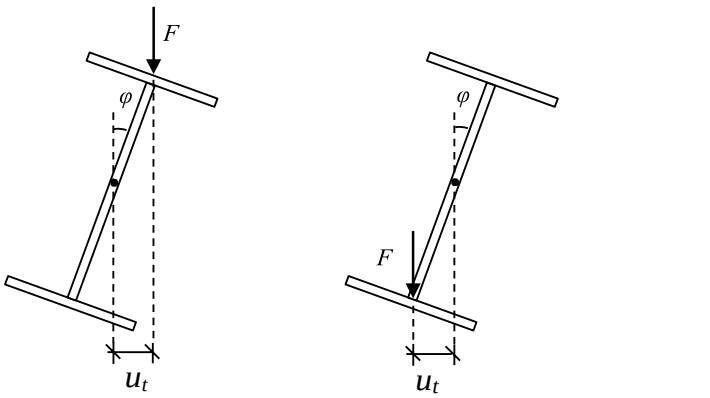


Figure 4. 13 : Additional moment due to the point of load application

With the *C2*-factor added to the formula, equation (4.7) below is obtained.

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(L)^2} \left(\left(\sqrt{\frac{I_w}{I_z} + \frac{(L)^2 GI_t}{\pi^2 \cdot EI_z} + \left(C_2 \cdot z_g\right)^2} \right) - \left(C_2 \cdot z_g\right) \right)} \qquad \textbf{4. 7}$$

Where:

C_2 = Correction factor for the point of load application

z_g = Distance between the point of load application and the shear centre, measured positive above *SC*. See Figure 4.14

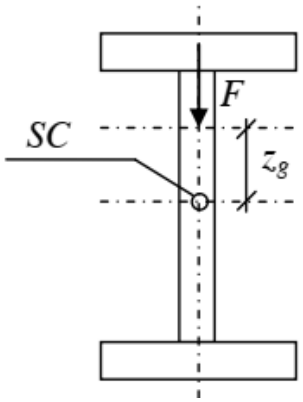


Figure 4. 14 : The distance z_g,

- **Correction factor for cross-section asymmetry C_3:**

Incorporating the $C_1$and C_2 factors enables the use of the 3-factor formula to calculate the critical moment M_{cr} for doubly symmetric beams under various loading conditions and points of load application. However, this formula is limited to sections with symmetry about the major and minor axes, meaning the section must be doubly symmetric.

To analytically estimate M_{cr} for sections that are symmetric only about the minor axis, a third correction factor C_3 must be introduced. This additional factor allows for accurate calculations of the critical moment in sections lacking full symmetry, broadening the applicability of the 3-factor formula to a wider range of structural configurations.

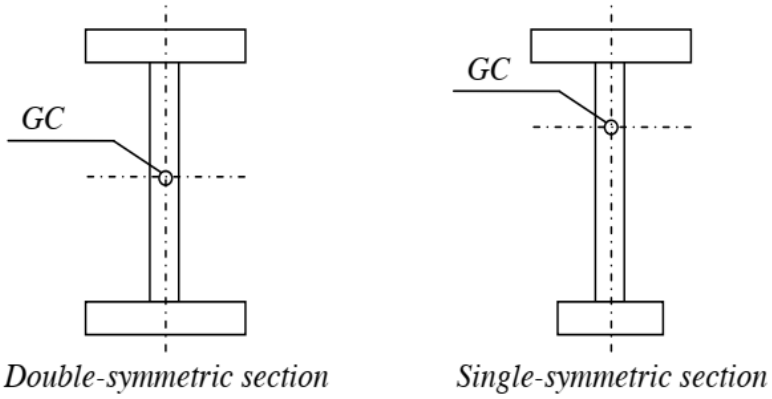


Figure 4. 15 : Examples of a double- and a single-symmetric cross-section

- **Effective Length Factors k_z and k_w:**

With the inclusion of the three *C* factors, the 3-factor formula allows for calculating the critical moment M_{cr} across various loading cases, cross-sections, and points of load application. However, to account for different boundary conditions, further modification is required. Research by Salvadori, Lee, and Vlasov in the 1950s and 60s [33] demonstrated that this extension could be achieved by introducing ***effective length factors***, similar to those used in column buckling analysis.

These effective length factors relate the distance between lateral restraints L to the buckling length L_b, which depends on the specific boundary conditions. This relationship is expressed as: $L_b = kL$

This approach allows the formula to accommodate a variety of boundary conditions, thereby enhancing its versatility in practical design situations.

Table 4. 5 : Effective length factors for lateral torsional buckling

Boundary conditions	k_z
Free at both sides	1.0
One end fixed, one end free	0.7
Fixed at both sides	0.5

4.5 Specific Cases of I-Beam Verification for Lateral-Torsional Buckling according to Eurocode 3:

4.5.1 Pure Bending:

Lateral-torsional buckling resistance (clause 6.3.2 of EC3-1-1) [8]

$$\frac{M_{e.Rd}}{M_{b.Rd}} \leq 1 \qquad \textbf{4. 8}$$

with

$$M_{b.Rd} = \chi_{LT} \frac{W_y f_y}{\gamma_{M1}} \qquad \textbf{4. 9}$$

$W_y = W_{pl.y}$ Class 1 and 2;

$Wy = W_{el.y}$ Class 3;

$Wy = W_{eff.y}$ Class 4.

χ_{LT} is the reduction factor for lateral-torsional buckling, which can be calculated by one of two methods, depending of member cross section.

For rolled sections or equivalent welded sections:

$$\chi_{LT} = \frac{1}{\Phi_{LT} + \sqrt{\Phi_{LT}^2 - \beta\bar{\lambda}_{LT}^2}}, \qquad \chi_{LT} \le 1.0 \qquad \chi_{LT} \le \frac{1}{\bar{\lambda}_{LT}^2}$$

$$\Phi_{LT} = 05\left(1 + \alpha_{LT}\left(\bar{\lambda}_{LT} - 0.2\right) + \beta\bar{\lambda}_{LT}^2\right)$$

$$\bar{\lambda}_{LT} \le 0{,}4 \qquad and \qquad \beta = 0{,}75$$

$$\bar{\lambda}_{LT} = \sqrt{\frac{W_y f_y}{M_{cr}}}$$

For the case of a doubly symmetric beam, the elastic critical lateral-torsional buckling moment of a constant cross-section doubly symmetric beam, where z_j=0, is given by:

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(k_z \cdot L)^2}\left(\left(\sqrt{\left(\frac{k_z}{k_w}\right)\frac{I_w}{I_z} + \frac{(k_z \cdot L)^2 GI_t}{\pi^2 \cdot EI_z} + \left(C_2 \cdot z_g\right)^2}\right) - \left(C_2 \cdot z_g\right)\right)} \qquad \textbf{4. 10}$$

In the case of loading by end moments and transverse loads applied at the shear center C_2, where $C_2.z_g$=0 For these cases:

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(k_z \cdot L)^2}\left(\left(\sqrt{\left(\frac{k_z}{k_w}\right)\frac{I_w}{I_z} + \frac{(k_z \cdot L)^2 GI_t}{\pi^2 \cdot EI_z} + }\right)\right)} \qquad \textbf{4. 11}$$

For simply support conditions at each end, $k_z = k_w = 1.0$.

$$\boldsymbol{M_{cr} = C_1 \frac{\pi^2 \cdot EI_z}{(L)^2}\left(\left(\sqrt{\frac{I_w}{I_z} + \frac{(L)^2 GI_t}{\pi^2 \cdot EI_z} + }\right)\right)} \qquad \textbf{4. 12}$$

4.5.2 Combined Compression + Bending (beam-columns):

$$\boldsymbol{\frac{N_{sd}}{\chi_z . Af_y} + \frac{K_{LT} . M_{ysd}}{\chi_{LT} . \beta_w . W_y . f_y} + \frac{K_z . M_{zsd}}{\beta_w . W_z . f_y} \le 1{,}1} \qquad \textbf{4. 13}$$

$W_y = W_{pl.y}$ Class 1 and 2;

$W_y = W_{el.y}$ Class 3;

and for Class 4 $$\frac{N_{sd}}{\chi_z . A f_y} + \frac{K_{LT}.(M_{ysd}+N_{sd}.e_{Ny})}{\chi_{LT}.W_{eff.y}.f_y} + \frac{K_z .(M_{zsd}+N_{sd}.e_{Nz})}{\beta_w .W_{eff.z}.f_y} \leq 1,1 \qquad \textbf{4. 14}$$

$$k_{LT} = 1 - \frac{\mu_{LT} N_{sd}}{\chi_z A f_y} \qquad \text{but} \qquad k_{LT} \leq 1{,}0$$

$$\mu_{LT} = 0{,}15.\overline{\lambda}_y.\beta_{M.LT} - 0{,}15 \qquad \text{but} \qquad \mu_{LT} \leq 0{,}9$$

$\beta_{M.LT}$: The calculation is identical to the equivalent uniform moment factor for buckling by bending β_{My}.

$$k_z = 1 - \frac{\mu_z N_{sd}}{\chi_z A f_y} \qquad \text{but} \qquad k_z \leq 1{,}5$$

$$\mu_z = \overline{\lambda}_z(2\beta_{M.z} - 4) + \frac{W_{pl.z} - W_{el.\,z}}{W_{el.\,z}} \quad \text{but} \quad \mu_z \leq 0{,}9$$

$\beta_{M.z}$: Calculation identical to the equivalent uniform moment factor for flexural buckling, β_{Mz}.

e_{Ny} and e_{Nz} : represent the neutral axis eccentricities, assuming the cross-section is subjected to compression only.

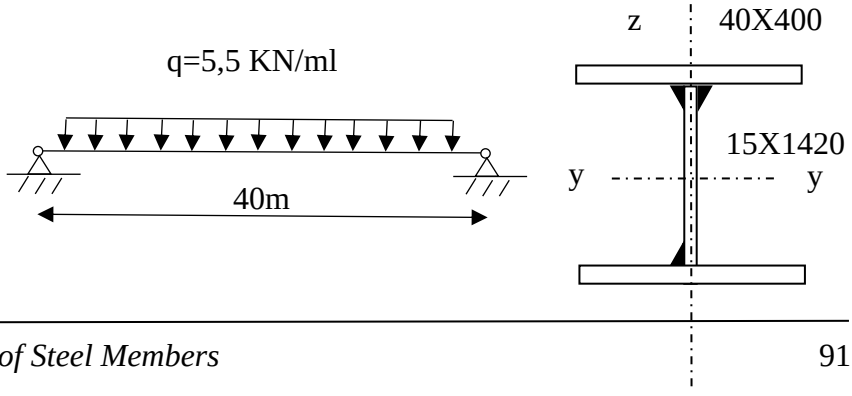
4.6 Examples:

- **Example 01: Lateral Torsional Buckling Checking of a Steel Beam**

Consider the Welded built-up beam made of S235 steel shown in the figure.

Verify the beam against lateral-torsional buckling, given that:

- *The load is applied at the centroid of the cross-section.*
- *The beam is unsupported at the supports.*
- *The section is classified as Class 3.*



Solution:

40X400

z

$$Mcr = C1\frac{\pi^2 EIz}{(KzL)^2}\sqrt{\left(\frac{Kz}{Kw}\right)^2\frac{Iw}{Iz} + \frac{(KzL)^2 GIt}{\pi^2 EIz} + (C2Zg)^2} - \ (C2Zg)$$

1. ***Geometrical and mechanical properties of the section:***

K_z=1 et K_w=1 → boundary conditions

C1=1,13 , C2=0,46 , Zg=0 , E=210000 MPa , G=80000 MPa .

$$M_{cr} = C1\ \frac{\pi^2 EI_z}{(L)^2}\frac{\sqrt{I_w}}{Iz} + (L)^2 GI_t/\pi^2 I_z$$

$$A = 1420 * 15 + 2[400 * 40] = 53300\ mm^2$$

$$I_z = 1420 * \frac{15^3}{12} + \ 40 * \frac{400^3}{12} = 4{,}27 * 10^8\ mm^4$$

$$I_t = \frac{1}{3}(2 * 400 * 40^3 + 1420 * 15^3) = 1{,}86 * 10^7 mm^6$$

$$I_w = 4{,}27 * 10^8\frac{(1500-40)^2}{4} = 2{,}27 * 10^{14} mm^6$$

$$W_{ely} = \frac{I_y}{y} = 2.06 * \frac{10^{10}}{750} = 2.75 * 10^7\ mm^3$$

$$I_y = 1420^3 * \frac{15}{12} + 2 * \left(400 * \frac{40^3}{12} + 400 * 40 * 730^2\right) = 2{,}06 * 10^{10}\ mm^4$$

$$M_{cr} = \frac{1{,}13 * \ \pi^2 * 2{,}1 * 10^5 * 4{,}27 * 10^8}{(40000)^2}\sqrt{\frac{2{,}27 * 10^{14}}{4{,}27 * 10^8}} + \frac{(40000)^2 * 0{,}8 * 10^3 * 1{,}86 * 10^7}{\pi^2 * 2{,}1 * 10^5 * 4{,}27 * 10^8}$$

$$M_{cr} = 1025\ KN.m$$

2. ***Checking:***

$$\frac{M_{yed}}{M_{brd}} \leq 1$$

$$M_{brd} = \chi_{LT} \times \ W_{ely} \times f_y/\gamma_{m1}$$

$$M_{yed} = \frac{qL^2}{8} = 5{,}5 * \frac{40^2}{8} = 1100KN.m$$

$$\lambda_{LT} = \sqrt{W_{ely} * \frac{f_y}{M_{cr}}} = \sqrt{2{,}75 * 10^7 * \frac{235}{1{,}03} * 10^9}$$

$\lambda_{LT} = 2{,}51 > 0{,}4 \rightarrow$ *Lateral Torsional Buckling Risk*

$\frac{h}{b} = \frac{1500}{400} = 3{,}75 > 2$

Welded I: curve "d"

$\lambda_{LT} = 2{,}51 \rightarrow \chi_{LT} = 0{,}1205$

$$M_{brd} = 0{,}1205 * 2{,}75 * 10^7 * \frac{235}{1{,}1} = 708KN.m$$

$$\frac{M_{yed}}{M_{brd}} = \frac{1100}{713} = 1{,}54 > 1$$

*The beam **is not stable** against lateral-torsional buckling.*

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APPENDICES

Appendix A: Cross Sections Properties

A.1 Sections Other Than Hollow Sections

A.2.1 Second Moment of Area

The second moment of area has been calculated considering all tapers, radii, and fillets of the sections. Values are given about both the yy and zz axes.

A.2.2 Radius of Gyration (i)

The radius of gyration is a parameter used in the calculation of buckling resistance and is derived as follows**: $i = \sqrt{(I/A)}$**

Where:

- I is the second moment of area about the relevant axis

- A is the area of the cross section

A.2.3 Elastic Section Modulus (W_{el})

The elastic section modulus is used to calculate the elastic design resistance for bending or to calculate the stress at the extreme fibre of the section due to a moment. It is derived as follows:

$W_{el,y} = I_y/z$

$W_{el,z} = I_z/y$

Where:

- y, z are the distances to the extreme fibres of the section from the elastic yy and zz axes respectively

For parallel flange channels, the elastic section modulus about the minor (zz) axis is given for the extreme fibre at the toe of the section only.

For angles, the elastic section moduli about both axes are given for the extreme fibres at the toes of the section only. For elastic section moduli about the principal axes uu and vv see AD[2].

For asymmetric beams, the elastic section moduli about the yy axis are given for both top and bottom extreme fibres and about the zz axis for the extreme fibre.

A.2.4 Plastic Section Modulus (W_{pl})

The plastic section moduli about both yy and zz axes are tabulated for all sections except angle sections.

A.2.5 Warping Constant (Iw) and Torsional Constant (It)

- **Rolled I-sections**

The warping constant and St Venant torsional constant for rolled I-sections have been calculated using the formulae given in the SCI publication P385 Design of members subject to combined bending and torsion [4]. In Eurocode terminology, these formulae are as follows:

$$\mathbf{I_w = (I_z \times h_s^2)/4}$$

Where:

- I_z is the second moment of area about the minor axis

- h_s is the distance between shear centres of flanges (i.e., h - tf)

$$\mathbf{I_t = (1/3) \times [bf \times tf^3 + (h - tf) \times tw^3 + \alpha 1 \times D - tf^3]}$$

Where:

$\alpha 1 = [2 \times (r + tw/2)]/tf - 1$

$D1 = [(r + tw/2) \times (2r + tw)] \times [(r + tw/2) + (r + tf)]$

Where:

- b is the width of the section

- h is the depth of the section

- t_f is the flange thickness

- t_w is the web thickness

- r is the root radius

- **Tee sections**

For Tee sections cut from UB and UC sections, the warping constant (I_w) and torsional constant (I_t) have been derived as given below:

$$\mathbf{I_w = (t_w^3 \times bf/36) \times [(h - t_f)^2 + (3 \times bf^2/4)]}$$

$$\mathbf{I_t = (1/3) \times [bf \times t_f^3 + (h - t_f) \times t_w^3 + \alpha 1 \times D - tf^3 - tw^3]}$$

Where:

$\alpha 1 = [2 \times (r + tw/2)]/tf - 1$

$D1 = [(r + tw/2) \times (2r + tw)] \times [(r + tw/2) + (r + tf)]$

Note: These formulae do not apply to Tee sections cut from joists which have tapered flanges. For such sections, expressions are given in SCI Publication[4].

- **Parallel Flange Channels**

For parallel flange channels, the warping constant (Iw) and torsional constant (It) have been calculated as follows:

$$\mathbf{I_w = [(Iz \times (h - tf)^2)/4] \times [1 - (2 \times cz \times (h - tf))/(Iz \times A)]}$$

Where:

- cz is the distance from the back of the web to the centroidal axis

$$\mathbf{I_t = (1/3) \times [bf \times tf^3 + (h - tf) \times tw^3 + \alpha 1 \times D - 2 \times tf^3]}$$

Where:

$\alpha 1 = [(2 \times (r + tw/2))/tf] - 1$

$D1 = [(r + tw/2) \times (2r + tw)] \times [(r + tw/2) + (r + tf)] - [(tw/2)^2]$

Note: The formula for the torsional constant (It) is applicable to parallel flange channels only and does not apply to tapered flange channels.

- **Angles**

For angles, the torsional constant (It) is calculated as follows:

$$\mathbf{I_t = (1/3) \times [b \times t^3 + (h - t) \times t^3 + \alpha 1 \times D - t^3]}$$

Where:

$\alpha 1 = t/r$

$D1 = [(r + t) \times (2r + t)] - [(r + t)^2]$

- **ASB Sections**

For ASB sections, the warping constant (Iw) and torsional constant (It) are as given in Tata brochure Advanced sections [5].

A.2 Hollow Sections

Section properties are given for both hot-finished and cold-formed hollow sections (but not for cold-formed elliptical hollow sections). For the same overall dimensions and wall thickness, the section properties for square and rectangular hot-finished and cold-formed sections are different because the corner radii are different.

A.2.1 Common Properties

For general comment on second moment of area, radius of gyration, elastic and plastic modulus, see Sections 1-4.

For hot-finished square and rectangular hollow sections, the section properties have been calculated using corner radii of 2t externally and t internally as specified by BS EN 10210[6].

For cold-formed square and rectangular hollow sections, the section properties have been calculated using the external corner radii of:

- 2t if $t \leq 6mm$

- 2.5t if $6mm < t \leq 10mm$

- 3t if $t > 10mm$

The internal corner radii used are:

- t if $t \leq 6mm$

- 1.6t if $6mm < t \leq 10mm$

- 2t if $t > 10mm$

As specified by BS EN 10219[7].

A.2.2 Plastic Section Modulus of Hollow Sections (W_{pl})

The plastic section moduli (W_{pl}) about both principal axes are given in the tables.

A.2.3 Torsional Constant (It)

For circular hollow sections: $\mathbf{I_t = I}$

For square, rectangular and elliptical hollow sections:

$$\mathbf{I_t = [(2 \times A \times t)/p] \times [1 - (p^2 \times t)/(12 \times Ap)]}$$

Where:

- I is the second moment of area of a CHS

- t is the thickness of the section

- p is the mean perimeter length:

  - For square and rectangular hollow sections: $p = [2(b-t) + 2(h-t)] - 4Rc \times \pi$

  - For elliptical hollow sections: $p = \pi \times [(h + b - 2t) - \sqrt{((h - b)^2/2)}]$

- Ap is the area enclosed by the mean perimeter:

  - For square and rectangular hollow sections: $Ap = (b-t)(h-t) - Rc^2\pi$

  - For elliptical hollow sections: $Ap = (\pi/4) \times (h - t)(b - t)$

- Rc is the average of the internal and external corner radii

A.2.4 Torsional Section Modulus (Wt)

For circular hollow sections: $\mathbf{W_t = W_{el}}$

For square, rectangular and elliptical hollow sections:

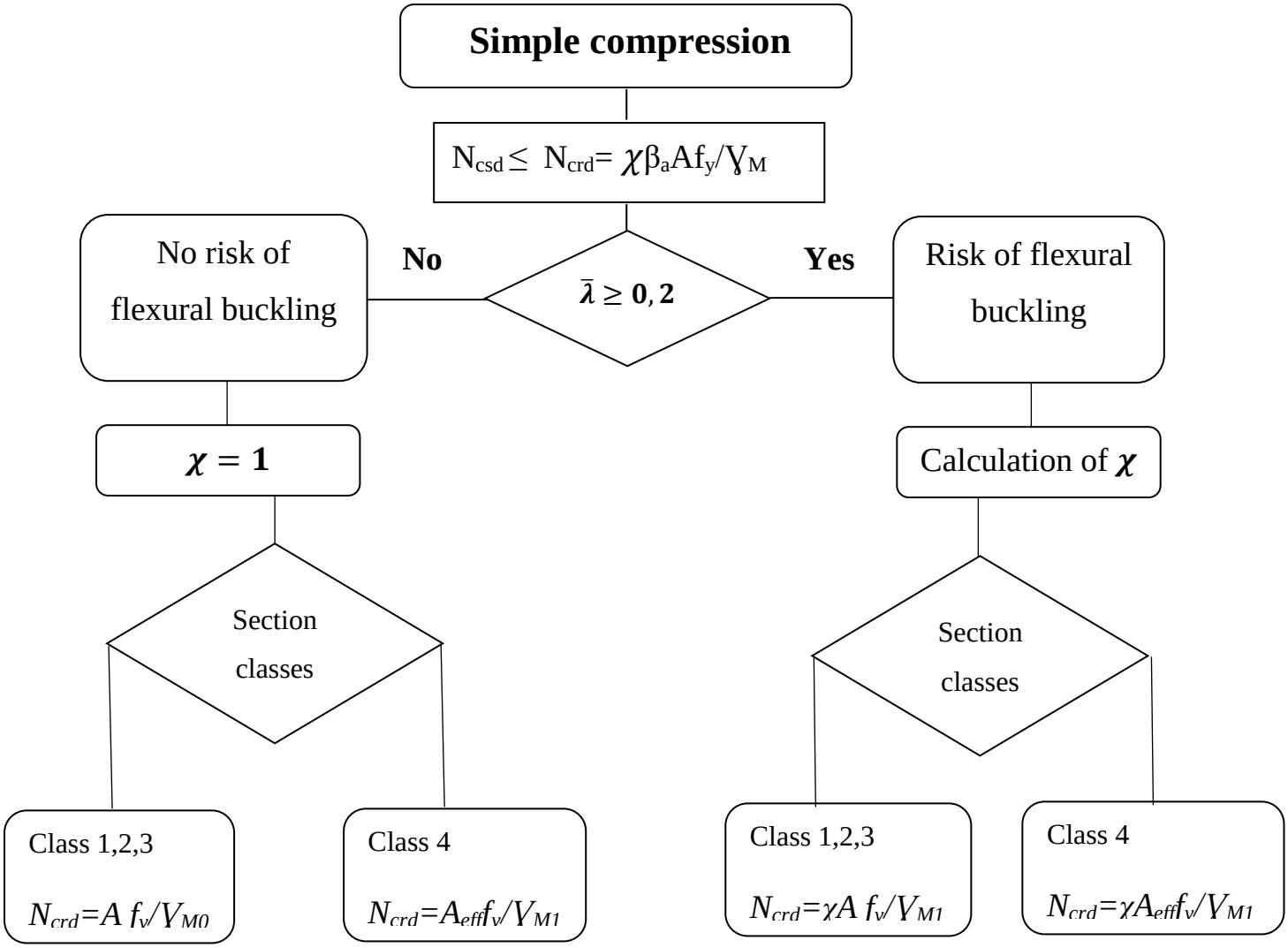
$$\mathbf{W_t = [(2 \times It \times A)/p] \times [1 + (p \times t)/(2 \times Ap)]}$$

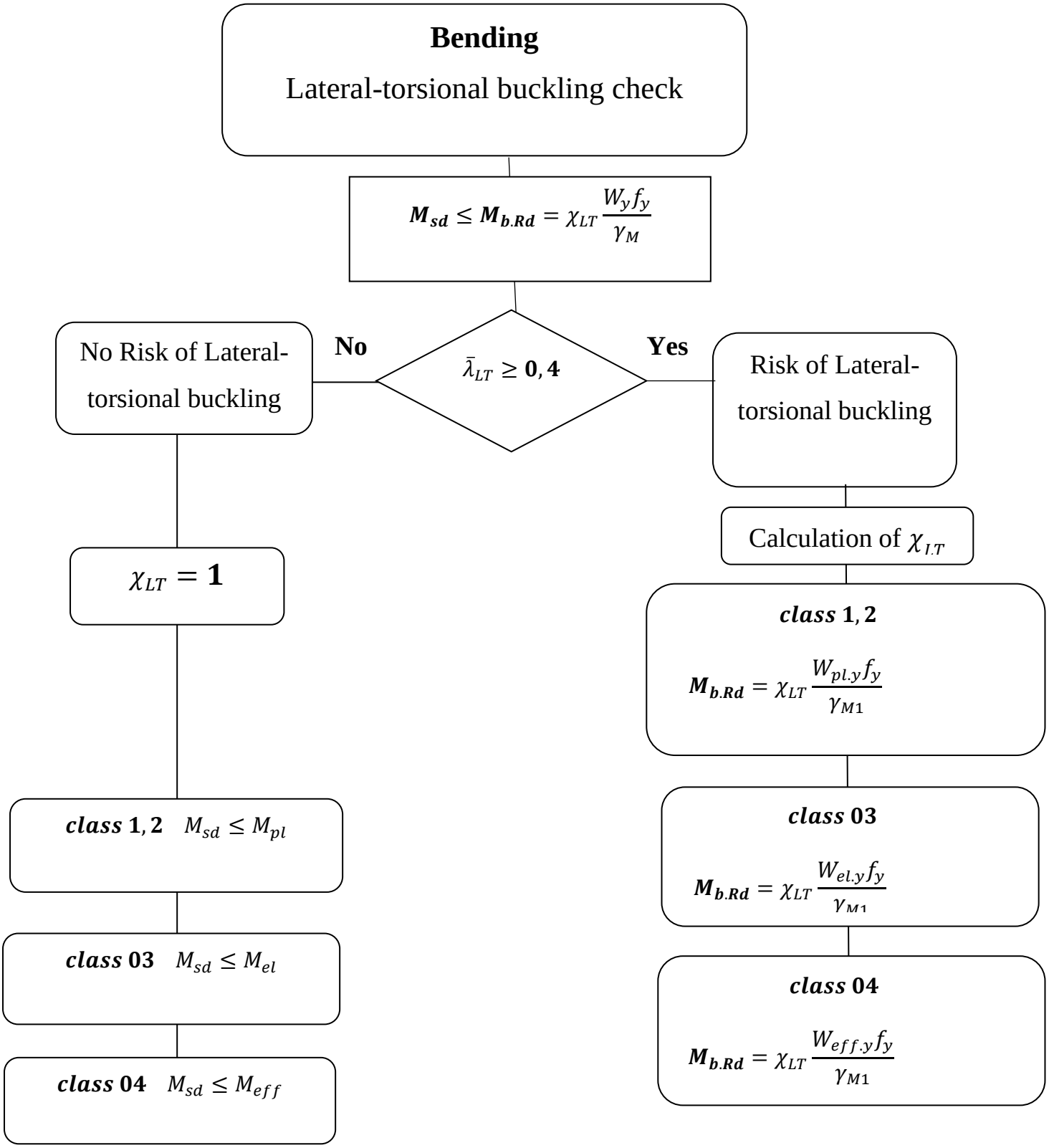
Where:

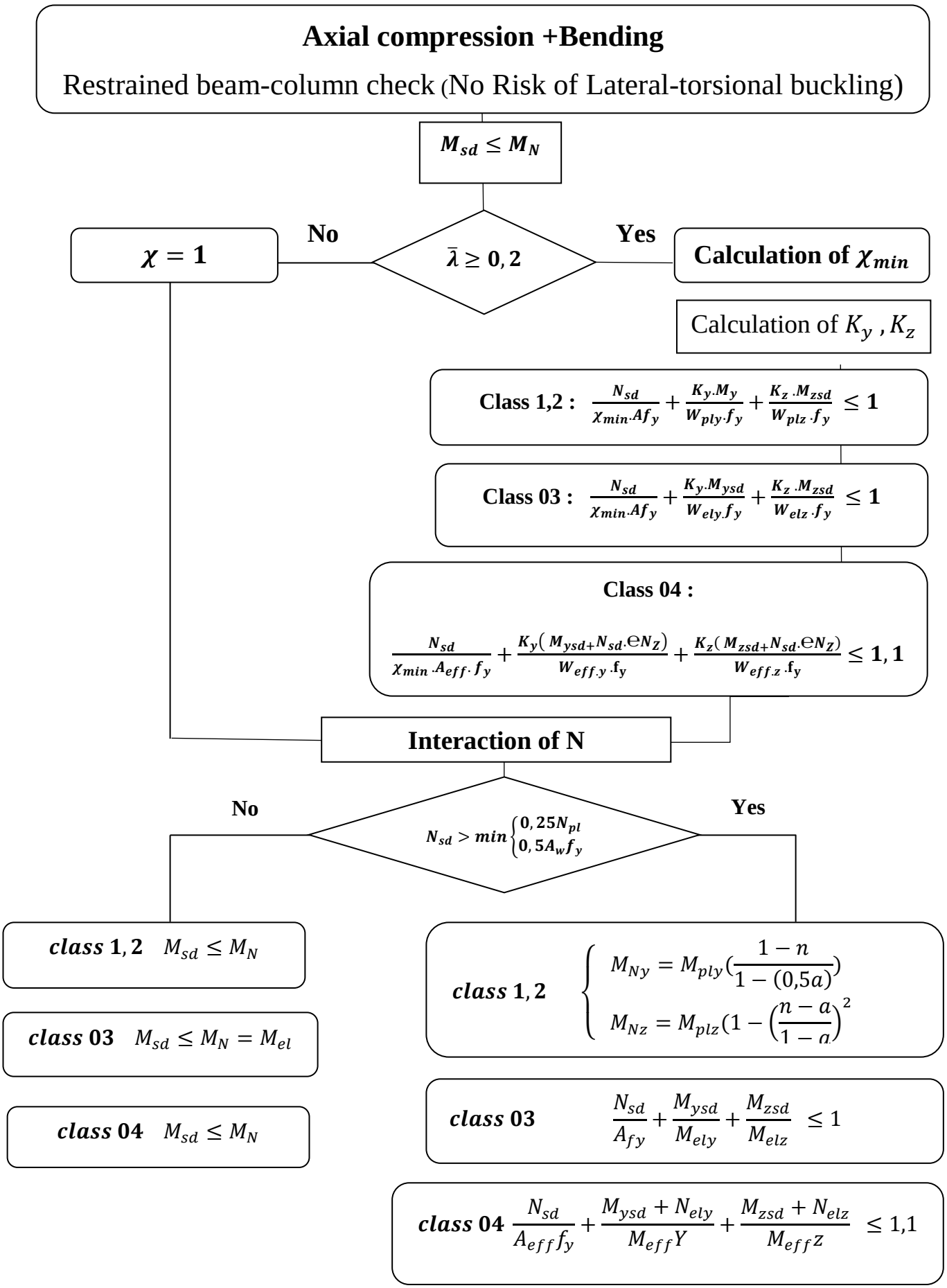
- W_{el} is the elastic modulus

- It, A, Ap, and p are as defined before.

Appendix B: Calculation Flowcharts




Bending
Lateral-torsional buckling check
$M_{sd} \leq M_{b.Rd} = \chi_{LT} \frac{W_y f_y}{\gamma_M}$
$\bar{\lambda}_{LT} \geq 0,4$
No
Yes
No Risk of Lateral-torsional buckling
Risk of Lateral-torsional buckling
$\chi_{LT} = 1$
Calculation of $\chi_{I.T}$
class 1, 2
$M_{b.Rd} = \chi_{LT} \frac{W_{pl.y} f_y}{\gamma_{M1}}$
class 1, 2 $M_{sd} \leq M_{pl}$
class 03
$M_{b.Rd} = \chi_{LT} \frac{W_{el.y} f_y}{\gamma_{M1}}$
class 03 $M_{sd} \leq M_{el}$
class 04
$M_{b.Rd} = \chi_{LT} \frac{W_{eff.y} f_y}{\gamma_{M1}}$
class 04 $M_{sd} \leq M_{eff}$


Axial compression +Bending
Restrained beam-column check (No Risk of Lateral-torsional buckling)
$M_{sd} \leq M_N$
$\bar{\lambda} \geq 0,2$
No
Yes
$\chi = 1$
Calculation of χ_{min}
Calculation of K_y , K_z
Class 1,2 : $\frac{N_{sd}}{\chi_{min}.Af_y} + \frac{K_y.M_y}{W_{ply}.f_y} + \frac{K_z.M_{zsd}}{W_{plz}.f_y} \leq 1$
Class 03 : $\frac{N_{sd}}{\chi_{min}.Af_y} + \frac{K_y.M_{ysd}}{W_{ely}.f_y} + \frac{K_z.M_{zsd}}{W_{elz}.f_y} \leq 1$
Class 04 :
$\frac{N_{sd}}{\chi_{min}.A_{eff}.f_y} + \frac{K_y(M_{ysd}+N_{sd}.eN_Z)}{W_{eff.y}.f_y} + \frac{K_z(M_{zsd}+N_{sd}.eN_Z)}{W_{eff.z}.f_y} \leq 1,1$
Interaction of N
No
Yes
$N_{sd} > min\begin{cases}0,25N_{pl}\\0,5A_wf_y\end{cases}$
class 1, 2 $M_{sd} \leq M_N$
class 03 $M_{sd} \leq M_N = M_{el}$
class 04 $M_{sd} \leq M_N$
class 1, 2 $\begin{cases} M_{Ny} = M_{ply}(\frac{1-n}{1-(0,5a)}) \\ M_{Nz} = M_{plz}(1-(\frac{n-a}{1-a})^2 \end{cases}$
class 03 $\frac{N_{sd}}{A_{fy}} + \frac{M_{ysd}}{M_{ely}} + \frac{M_{zsd}}{M_{elz}} \leq 1$
class 04 $\frac{N_{sd}}{A_{eff}f_y} + \frac{M_{ysd}+N_{ely}}{M_{eff}Y} + \frac{M_{zsd}+N_{elz}}{M_{eff}z} \leq 1,1$

Axial compression +Bending

Lateral-torsional buckling check

$\bar{\lambda}_{LT} \geq 0,4$

No

No Risk of Lateral-torsional buckling

Go to Calculation Flowchart

Axial compression +Bending

Restrained beam-column check

Yes

Risk of Lateral-torsional buckling

Calculation of χ_{LT}, χ_z

Calculation of K_{LT} , K_z

Class 1,2 : $\frac{N_{sd}}{\chi_z . A f_y} + \frac{K_{LT} . M_y}{\chi_{LT} . W_{ply} . f_y} + \frac{K_z . M_{zsd}}{W_{plz} . f_y} \leq 1,1$

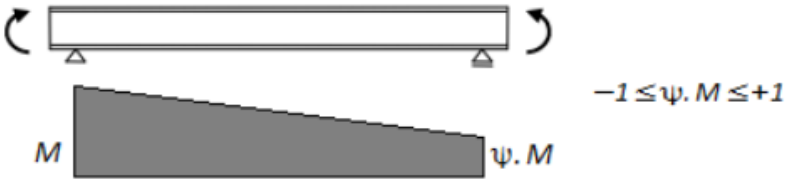
Class 03 : $\frac{N_{sd}}{\chi_z . A f_y} + \frac{K_{LT} . M_{ysd}}{\chi_{LT} . W_{ely} f_y} + \frac{K_z . M_{zsd}}{W_{elz} . f_y} \leq 1,1$

Class 04: $\frac{N_{sd}}{\chi_z . A_{eff} . f_y} + \frac{K_{LT}(M_{ysd} + N_{sd} . e N_Z)}{\chi_{LT} . W_{eff.y} . \mathrm{f_y}} + \frac{K_z(M_{zsd} + N_{sd} . e N_Z)}{W_{eff.z} . \mathrm{f_y}} \leq 1,1$

Appendix C: Elastic critical moment factors for LTB checks for Constant and double symmetrical section beams

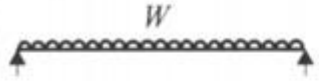
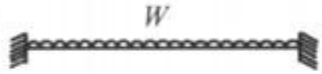
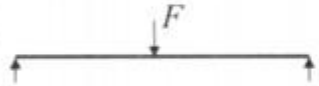
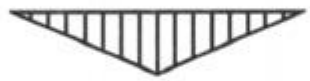
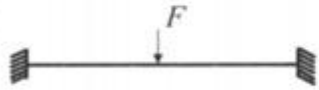
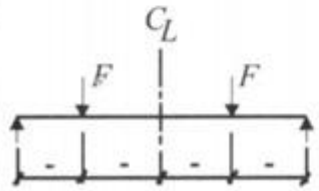
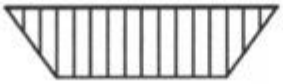
C1 and C2 are factors that depend on the end conditions and the loading conditions of the member. The NCCI provides values for C1 and C2 for the different cases as given in the tables below:

C.1 Beams subjected to extreme moments:



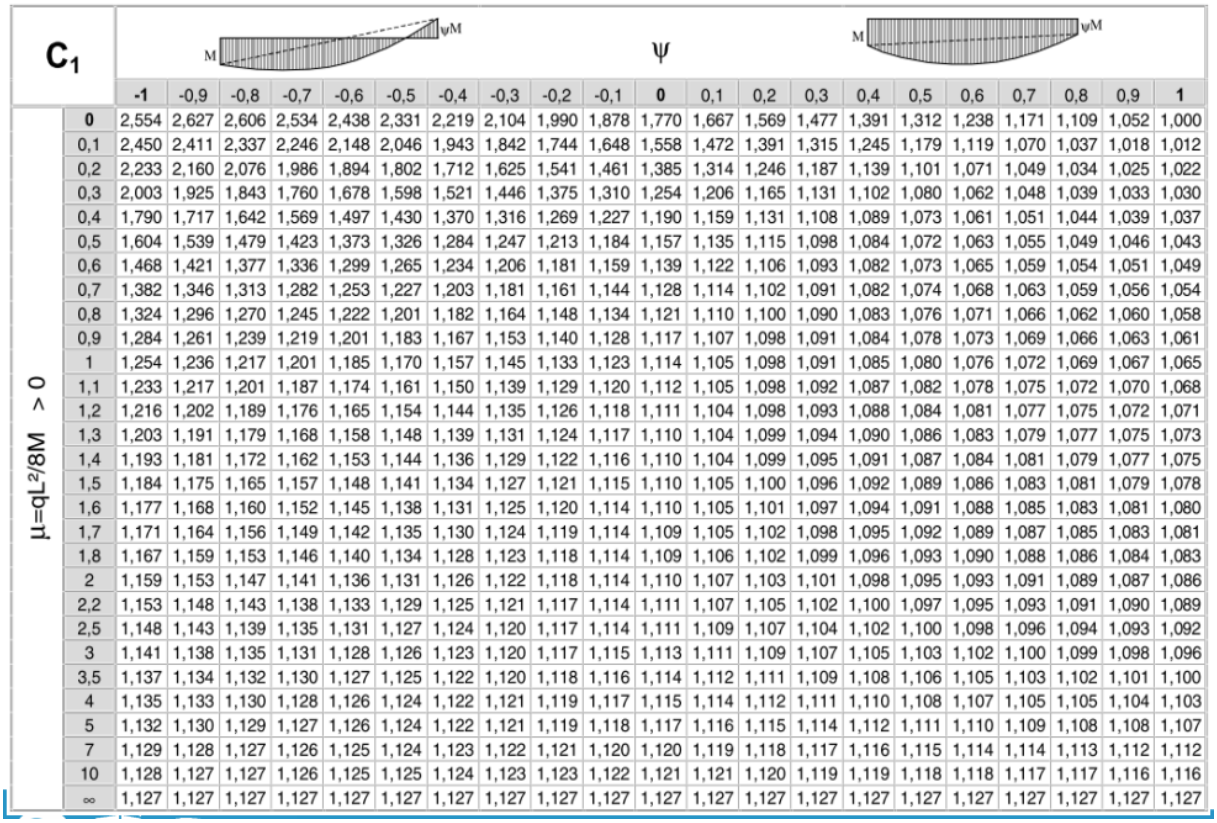
ψ	C1
+1,00	1,00
+0,75	1,14
+0,50	1,31
+0,25	1,52
0,00	1,77
-0,25	2,05
-0,50	2,33
-0,75	2,57
-1,00	2,55

C.2 Beams subjected to a transverse load:

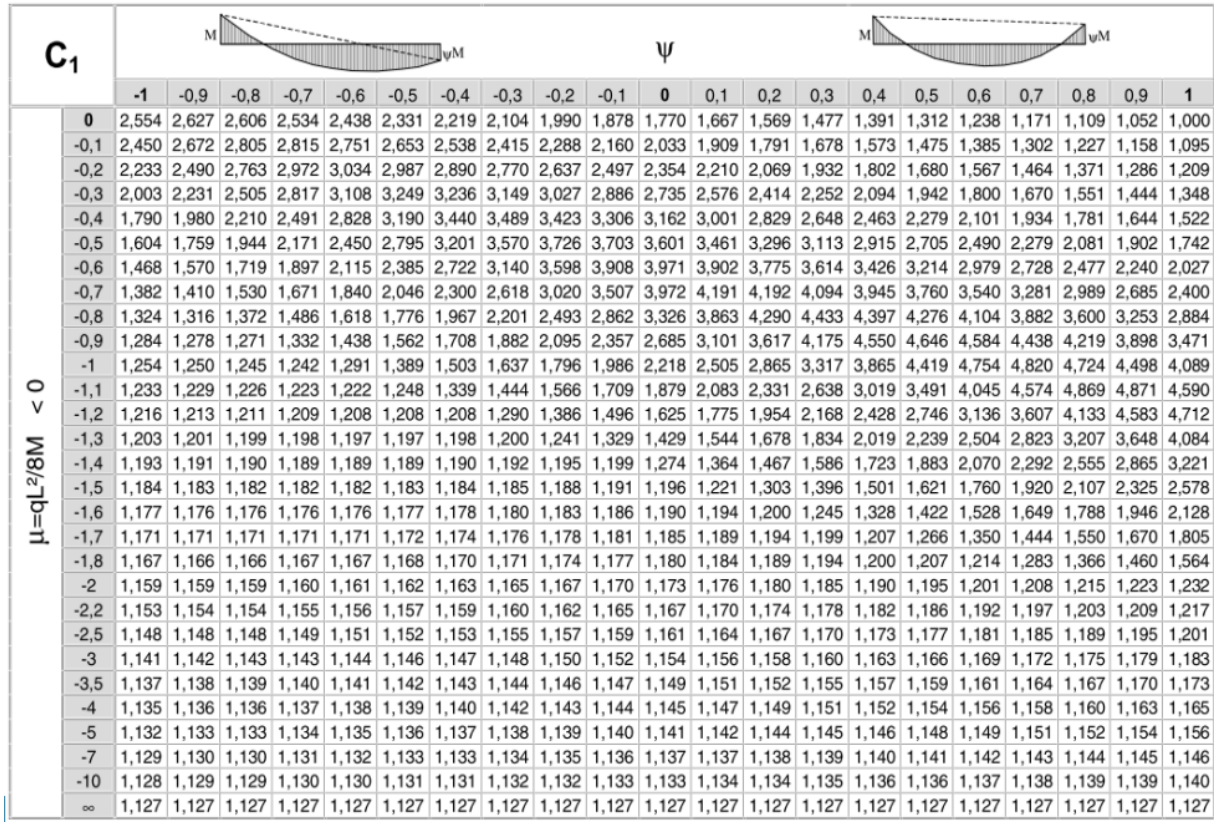
Loading and Boundry Conditions	Bending Moment Diagram	k	Coefficients	
			C_1	C_2
		1,0 0,5	1,132 0,972	0,459 0,304
		1,0 0,5	2,57 0,712	1,562 0,652
		1,0 0,5	1,365 1,070	0,553 0,432
		1,0 0,5	1,565 0,938	1,267 0,715
		1,0 0,5	1,046 1,010	0,430 0,410

C.3 Beams subjected to transverse load and end moments:

C.3.1 C1 values:



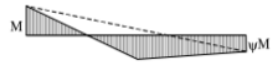
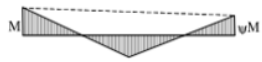
C_1	ψ	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\mu=qL^2/8M > 0$	0	2,554	2,627	2,606	2,534	2,438	2,331	2,219	2,104	1,990	1,878	1,770	1,667	1,569	1,477	1,391	1,312	1,238	1,171	1,109	1,052	1,000
	0,1	2,450	2,411	2,337	2,246	2,148	2,046	1,943	1,842	1,744	1,648	1,558	1,472	1,391	1,315	1,245	1,179	1,119	1,070	1,037	1,018	1,012
	0,2	2,233	2,160	2,076	1,986	1,894	1,802	1,712	1,625	1,541	1,461	1,385	1,314	1,246	1,187	1,139	1,101	1,071	1,049	1,034	1,025	1,022
	0,3	2,003	1,925	1,843	1,760	1,678	1,598	1,521	1,446	1,375	1,310	1,254	1,206	1,165	1,131	1,102	1,080	1,062	1,048	1,039	1,033	1,030
	0,4	1,790	1,717	1,642	1,569	1,497	1,430	1,370	1,316	1,269	1,227	1,190	1,159	1,131	1,108	1,089	1,073	1,061	1,051	1,044	1,039	1,037
	0,5	1,604	1,539	1,479	1,423	1,373	1,326	1,284	1,247	1,213	1,184	1,157	1,135	1,115	1,098	1,084	1,072	1,063	1,055	1,049	1,046	1,043
	0,6	1,468	1,421	1,377	1,336	1,299	1,265	1,234	1,206	1,181	1,159	1,139	1,122	1,106	1,093	1,082	1,073	1,065	1,059	1,054	1,051	1,049
	0,7	1,382	1,346	1,313	1,282	1,253	1,227	1,203	1,181	1,161	1,144	1,128	1,114	1,102	1,091	1,082	1,074	1,068	1,063	1,059	1,056	1,054
	0,8	1,324	1,296	1,270	1,245	1,222	1,201	1,182	1,164	1,148	1,134	1,121	1,110	1,100	1,090	1,083	1,076	1,071	1,066	1,062	1,060	1,058
	0,9	1,284	1,261	1,239	1,219	1,201	1,183	1,167	1,153	1,140	1,128	1,117	1,107	1,098	1,091	1,084	1,078	1,073	1,069	1,066	1,063	1,061
	1	1,254	1,236	1,217	1,201	1,185	1,170	1,157	1,145	1,133	1,123	1,114	1,105	1,098	1,091	1,085	1,080	1,076	1,072	1,069	1,067	1,065
	1,1	1,233	1,217	1,201	1,187	1,174	1,161	1,150	1,139	1,129	1,120	1,112	1,105	1,098	1,092	1,087	1,082	1,078	1,075	1,072	1,070	1,068
	1,2	1,216	1,202	1,189	1,176	1,165	1,154	1,144	1,135	1,126	1,118	1,111	1,104	1,098	1,093	1,088	1,084	1,081	1,077	1,075	1,072	1,071
	1,3	1,203	1,191	1,179	1,168	1,158	1,148	1,139	1,131	1,124	1,117	1,110	1,104	1,099	1,094	1,090	1,086	1,083	1,079	1,077	1,075	1,073
	1,4	1,193	1,181	1,172	1,162	1,153	1,144	1,136	1,129	1,122	1,116	1,110	1,104	1,099	1,095	1,091	1,087	1,084	1,081	1,079	1,077	1,075
	1,5	1,184	1,175	1,165	1,157	1,148	1,141	1,134	1,127	1,121	1,115	1,110	1,105	1,100	1,096	1,092	1,089	1,086	1,083	1,081	1,079	1,078
	1,6	1,177	1,168	1,160	1,152	1,145	1,138	1,131	1,125	1,120	1,114	1,110	1,105	1,101	1,097	1,094	1,091	1,088	1,085	1,083	1,081	1,080
	1,7	1,171	1,164	1,156	1,149	1,142	1,135	1,130	1,124	1,119	1,114	1,109	1,105	1,102	1,098	1,095	1,092	1,089	1,087	1,085	1,083	1,081
	1,8	1,167	1,159	1,153	1,146	1,140	1,134	1,128	1,123	1,118	1,114	1,109	1,106	1,102	1,099	1,096	1,093	1,090	1,088	1,086	1,084	1,083
	2	1,159	1,153	1,147	1,141	1,136	1,131	1,126	1,122	1,118	1,114	1,110	1,107	1,103	1,101	1,098	1,095	1,093	1,091	1,089	1,087	1,086
	2,2	1,153	1,148	1,143	1,138	1,133	1,129	1,125	1,121	1,117	1,114	1,111	1,107	1,105	1,102	1,100	1,097	1,095	1,093	1,091	1,090	1,089
	2,5	1,148	1,143	1,139	1,135	1,131	1,127	1,124	1,120	1,117	1,114	1,111	1,109	1,107	1,104	1,102	1,100	1,098	1,096	1,094	1,093	1,092
	3	1,141	1,138	1,135	1,131	1,128	1,126	1,123	1,120	1,117	1,115	1,113	1,111	1,109	1,107	1,105	1,103	1,102	1,100	1,099	1,098	1,096
	3,5	1,137	1,134	1,132	1,130	1,127	1,125	1,122	1,120	1,118	1,116	1,114	1,112	1,111	1,109	1,108	1,106	1,105	1,103	1,102	1,101	1,100
	4	1,135	1,133	1,130	1,128	1,126	1,124	1,122	1,121	1,119	1,117	1,115	1,114	1,112	1,111	1,110	1,108	1,107	1,105	1,105	1,104	1,103
	5	1,132	1,130	1,129	1,127	1,126	1,124	1,122	1,121	1,119	1,118	1,117	1,116	1,115	1,114	1,112	1,111	1,110	1,109	1,108	1,108	1,107
	7	1,129	1,128	1,127	1,126	1,125	1,124	1,123	1,122	1,121	1,120	1,120	1,119	1,118	1,117	1,116	1,115	1,114	1,114	1,113	1,112	1,112
	10	1,128	1,127	1,127	1,126	1,125	1,125	1,124	1,123	1,123	1,122	1,121	1,121	1,120	1,119	1,119	1,118	1,118	1,117	1,117	1,116	1,116
	∞	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127



C_1	ψ	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\mu=qL^2/8M < 0$	0	2,554	2,627	2,606	2,534	2,438	2,331	2,219	2,104	1,990	1,878	1,770	1,667	1,569	1,477	1,391	1,312	1,238	1,171	1,109	1,052	1,000
	-0,1	2,450	2,672	2,805	2,815	2,751	2,653	2,538	2,415	2,288	2,160	2,033	1,909	1,791	1,678	1,573	1,475	1,385	1,302	1,227	1,158	1,095
	-0,2	2,233	2,490	2,763	2,972	3,034	2,987	2,890	2,770	2,637	2,497	2,354	2,210	2,069	1,932	1,802	1,680	1,567	1,464	1,371	1,286	1,209
	-0,3	2,003	2,231	2,505	2,817	3,108	3,249	3,236	3,149	3,027	2,886	2,735	2,576	2,414	2,252	2,094	1,942	1,800	1,670	1,551	1,444	1,348
	-0,4	1,790	1,980	2,210	2,491	2,828	3,190	3,440	3,489	3,423	3,306	3,162	3,001	2,829	2,648	2,463	2,279	2,101	1,934	1,781	1,644	1,522
	-0,5	1,604	1,759	1,944	2,171	2,450	2,795	3,201	3,570	3,726	3,703	3,601	3,461	3,296	3,113	2,915	2,705	2,490	2,279	2,081	1,902	1,742
	-0,6	1,468	1,570	1,719	1,897	2,115	2,385	2,722	3,140	3,598	3,908	3,971	3,902	3,775	3,614	3,426	3,214	2,979	2,728	2,477	2,240	2,027
	-0,7	1,382	1,410	1,530	1,671	1,840	2,046	2,300	2,618	3,020	3,507	3,972	4,191	4,192	4,094	3,945	3,760	3,540	3,281	2,989	2,685	2,400
	-0,8	1,324	1,316	1,372	1,486	1,618	1,776	1,967	2,201	2,493	2,862	3,326	3,863	4,290	4,433	4,397	4,276	4,104	3,882	3,600	3,253	2,884
	-0,9	1,284	1,278	1,271	1,332	1,438	1,562	1,708	1,882	2,095	2,357	2,685	3,101	3,617	4,175	4,550	4,646	4,584	4,438	4,219	3,898	3,471
	-1	1,254	1,250	1,245	1,242	1,291	1,389	1,503	1,637	1,796	1,986	2,218	2,505	2,865	3,317	3,865	4,419	4,754	4,820	4,724	4,498	4,089
	-1,1	1,233	1,229	1,226	1,223	1,222	1,248	1,339	1,444	1,566	1,709	1,879	2,083	2,331	2,638	3,019	3,491	4,045	4,574	4,869	4,871	4,590
	-1,2	1,216	1,213	1,211	1,209	1,208	1,208	1,208	1,290	1,386	1,496	1,625	1,775	1,954	2,168	2,428	2,746	3,136	3,607	4,133	4,583	4,712
	-1,3	1,203	1,201	1,199	1,198	1,197	1,197	1,198	1,200	1,241	1,329	1,429	1,544	1,678	1,834	2,019	2,239	2,504	2,823	3,207	3,648	4,084
	-1,4	1,193	1,191	1,190	1,189	1,189	1,189	1,190	1,192	1,195	1,199	1,274	1,364	1,467	1,586	1,723	1,883	2,070	2,292	2,555	2,865	3,221
	-1,5	1,184	1,183	1,182	1,182	1,182	1,183	1,184	1,185	1,188	1,191	1,196	1,221	1,303	1,396	1,501	1,621	1,760	1,920	2,107	2,325	2,578
	-1,6	1,177	1,176	1,176	1,176	1,176	1,177	1,178	1,180	1,183	1,186	1,190	1,194	1,200	1,245	1,328	1,422	1,528	1,649	1,788	1,946	2,128
	-1,7	1,171	1,171	1,171	1,171	1,171	1,172	1,174	1,176	1,178	1,181	1,185	1,189	1,194	1,199	1,207	1,266	1,350	1,444	1,550	1,670	1,805
	-1,8	1,167	1,166	1,166	1,167	1,167	1,168	1,170	1,171	1,174	1,177	1,180	1,184	1,189	1,194	1,200	1,207	1,214	1,283	1,366	1,460	1,564
	-2	1,159	1,159	1,159	1,160	1,161	1,162	1,163	1,165	1,167	1,170	1,173	1,176	1,180	1,185	1,190	1,195	1,201	1,208	1,215	1,223	1,232
	-2,2	1,153	1,154	1,154	1,155	1,156	1,157	1,159	1,160	1,162	1,165	1,167	1,170	1,174	1,178	1,182	1,186	1,192	1,197	1,203	1,209	1,217
	-2,5	1,148	1,148	1,148	1,149	1,151	1,152	1,153	1,155	1,157	1,159	1,161	1,164	1,167	1,170	1,173	1,177	1,181	1,185	1,189	1,195	1,201
	-3	1,141	1,142	1,143	1,143	1,144	1,146	1,147	1,148	1,150	1,152	1,154	1,156	1,158	1,160	1,163	1,166	1,169	1,172	1,175	1,179	1,183
	-3,5	1,137	1,138	1,139	1,140	1,141	1,142	1,143	1,144	1,146	1,147	1,149	1,151	1,152	1,155	1,157	1,159	1,161	1,164	1,167	1,170	1,173
	-4	1,135	1,136	1,136	1,137	1,138	1,139	1,140	1,142	1,143	1,144	1,145	1,147	1,149	1,151	1,152	1,154	1,156	1,158	1,160	1,163	1,165
	-5	1,132	1,133	1,133	1,134	1,135	1,136	1,137	1,138	1,139	1,140	1,141	1,142	1,144	1,145	1,146	1,148	1,149	1,151	1,152	1,154	1,156
	-7	1,129	1,130	1,130	1,131	1,132	1,133	1,133	1,134	1,135	1,136	1,137	1,137	1,138	1,139	1,140	1,141	1,142	1,143	1,144	1,145	1,146
	-10	1,128	1,129	1,129	1,130	1,130	1,131	1,131	1,132	1,132	1,133	1,133	1,134	1,134	1,135	1,136	1,136	1,137	1,138	1,139	1,139	1,140
	∞	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127	1,127

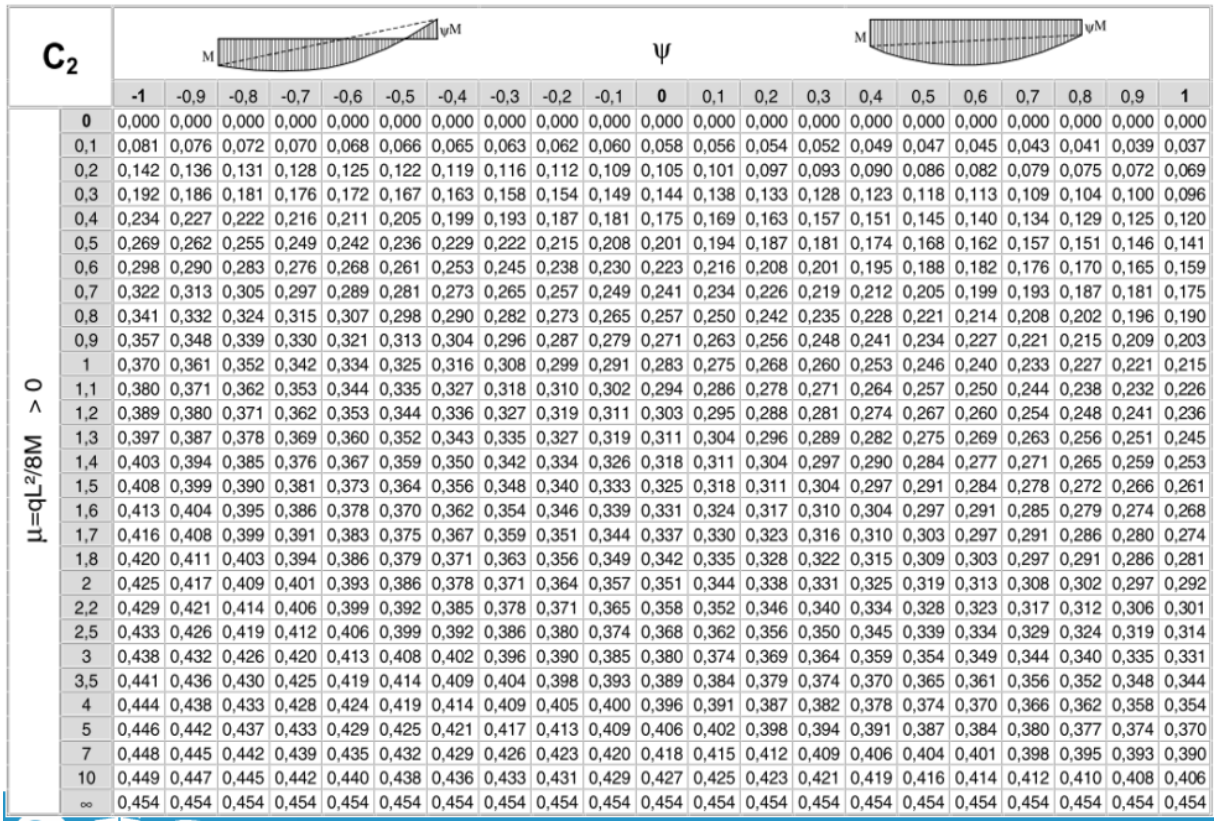
C_1	 ψ 																					
	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	**0**	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	**1**
0	2,554	2,627	2,606	2,534	2,438	2,331	2,219	2,104	1,990	1,878	1,770	1,667	1,569	1,477	1,391	1,312	1,238	1,171	1,109	1,052	1,000
0,1	2,494	2,475	2,408	2,317	2,216	2,109	2,002	1,896	1,792	1,693	1,597	1,507	1,423	1,344	1,271	1,203	1,140	1,082	1,029	1,028	1,027
0,2	2,348	2,285	2,200	2,105	2,006	1,906	1,807	1,711	1,619	1,532	1,449	1,371	1,298	1,231	1,168	1,109	1,055	1,055	1,055	1,053	1,051
0,3	2,168	2,089	2,000	1,908	1,815	1,724	1,636	1,551	1,470	1,394	1,322	1,255	1,192	1,134	1,079	1,080	1,080	1,078	1,077	1,074	1,072
0,4	1,983	1,901	1,816	1,730	1,646	1,564	1,486	1,412	1,342	1,276	1,213	1,155	1,101	1,102	1,103	1,102	1,101	1,098	1,096	1,093	1,090
0,5	1,809	1,730	1,651	1,573	1,498	1,426	1,358	1,293	1,231	1,174	1,120	1,122	1,124	1,124	1,123	1,121	1,118	1,116	1,112	1,109	1,105
0,6	1,650	1,577	1,505	1,436	1,370	1,306	1,246	1,189	1,136	1,140	1,142	1,143	1,143	1,141	1,140	1,137	1,134	1,131	1,127	1,123	1,119
0,7	1,508	1,442	1,378	1,317	1,258	1,203	1,150	1,155	1,158	1,160	1,161	1,160	1,159	1,157	1,154	1,151	1,148	1,144	1,140	1,136	1,132
0,8	1,383	1,324	1,267	1,213	1,161	1,168	1,173	1,175	1,177	1,177	1,177	1,175	1,173	1,170	1,167	1,163	1,160	1,156	1,152	1,147	1,143
0,9	1,273	1,221	1,170	1,179	1,185	1,189	1,191	1,192	1,192	1,192	1,190	1,188	1,185	1,182	1,178	1,174	1,170	1,166	1,162	1,158	1,153
1	1,177	1,187	1,194	1,200	1,203	1,206	1,207	1,206	1,205	1,204	1,201	1,199	1,195	1,192	1,188	1,184	1,180	1,176	1,171	1,167	1,163
1,1	1,202	1,209	1,214	1,217	1,219	1,220	1,219	1,218	1,217	1,214	1,211	1,208	1,205	1,201	1,197	1,193	1,188	1,184	1,180	1,175	1,171
1,2	1,222	1,227	1,230	1,231	1,232	1,231	1,230	1,229	1,226	1,223	1,220	1,217	1,213	1,209	1,205	1,201	1,196	1,192	1,187	1,183	1,179
1,3	1,238	1,241	1,243	1,243	1,243	1,241	1,240	1,237	1,235	1,231	1,228	1,224	1,220	1,216	1,212	1,208	1,203	1,199	1,194	1,190	1,186
1,4	1,252	1,253	1,254	1,253	1,252	1,250	1,248	1,245	1,242	1,238	1,235	1,231	1,227	1,223	1,218	1,214	1,210	1,205	1,201	1,196	1,192
1,5	1,263	1,263	1,263	1,262	1,260	1,258	1,255	1,252	1,248	1,245	1,241	1,237	1,233	1,229	1,224	1,220	1,216	1,211	1,207	1,202	1,198
1,6	1,273	1,272	1,271	1,269	1,267	1,264	1,261	1,258	1,254	1,250	1,246	1,242	1,238	1,234	1,230	1,225	1,221	1,217	1,212	1,208	1,204
1,7	1,280	1,279	1,277	1,275	1,273	1,270	1,266	1,263	1,259	1,255	1,251	1,247	1,243	1,239	1,235	1,230	1,226	1,222	1,217	1,213	1,209
1,8	1,287	1,286	1,283	1,281	1,278	1,275	1,271	1,268	1,264	1,260	1,256	1,252	1,248	1,243	1,239	1,235	1,231	1,226	1,222	1,218	1,214
2	1,298	1,296	1,293	1,290	1,287	1,283	1,279	1,276	1,272	1,268	1,264	1,260	1,256	1,251	1,247	1,243	1,239	1,235	1,231	1,226	1,222
2,2	1,306	1,303	1,300	1,297	1,294	1,290	1,286	1,282	1,278	1,274	1,270	1,266	1,262	1,258	1,254	1,250	1,246	1,242	1,238	1,234	1,230
2,5	1,315	1,312	1,309	1,305	1,302	1,298	1,294	1,290	1,287	1,283	1,279	1,275	1,271	1,267	1,263	1,259	1,255	1,251	1,247	1,244	1,240
3	1,325	1,322	1,318	1,315	1,311	1,307	1,304	1,300	1,296	1,293	1,289	1,285	1,282	1,278	1,274	1,271	1,267	1,264	1,260	1,257	1,253
3,5	1,331	1,328	1,324	1,321	1,317	1,314	1,311	1,307	1,304	1,300	1,297	1,293	1,290	1,287	1,283	1,280	1,277	1,273	1,270	1,267	1,263
4	1,335	1,332	1,328	1,325	1,322	1,319	1,316	1,312	1,309	1,306	1,303	1,300	1,296	1,293	1,290	1,287	1,284	1,281	1,278	1,275	1,272
5	1,339	1,337	1,334	1,331	1,328	1,325	1,322	1,320	1,317	1,314	1,311	1,308	1,306	1,303	1,300	1,298	1,295	1,292	1,290	1,287	1,284
7	1,343	1,341	1,339	1,337	1,334	1,332	1,330	1,328	1,326	1,323	1,321	1,319	1,317	1,315	1,313	1,311	1,308	1,306	1,304	1,302	1,300
10	1,346	1,344	1,342	1,341	1,339	1,337	1,336	1,334	1,332	1,331	1,329	1,327	1,326	1,324	1,322	1,321	1,319	1,318	1,316	1,315	1,313
∞	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348

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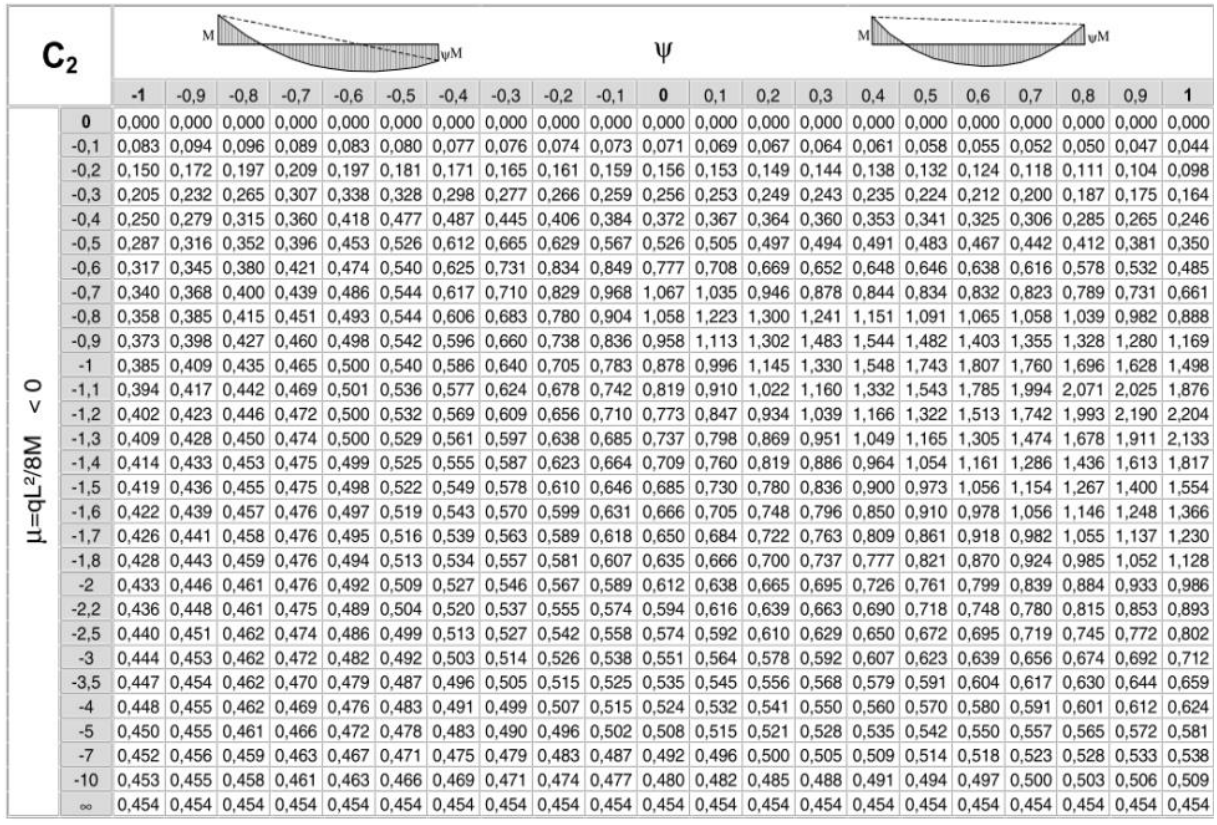
C_1	 ψ 																					
	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	**0**	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	**1**
0	2,554	2,627	2,606	2,534	2,438	2,331	2,219	2,104	1,990	1,878	1,770	1,667	1,569	1,477	1,391	1,312	1,238	1,171	1,109	1,052	1,000
-0,1	2,494	2,682	2,760	2,737	2,663	2,563	2,451	2,333	2,212	2,090	1,970	1,852	1,740	1,633	1,533	1,439	1,353	1,274	1,202	1,136	1,075
-0,2	2,348	2,597	2,804	2,889	2,868	2,792	2,690	2,575	2,451	2,324	2,195	2,066	1,939	1,817	1,700	1,590	1,489	1,395	1,310	1,232	1,161
-0,3	2,168	2,417	2,689	2,916	3,012	2,994	2,919	2,817	2,700	2,573	2,440	2,304	2,166	2,029	1,896	1,769	1,649	1,538	1,437	1,344	1,261
-0,4	1,983	2,207	2,472	2,763	3,012	3,124	3,114	3,044	2,943	2,826	2,697	2,560	2,417	2,270	2,123	1,978	1,838	1,707	1,586	1,476	1,377
-0,5	1,809	2,000	2,231	2,505	2,811	3,083	3,219	3,225	3,163	3,067	2,950	2,821	2,681	2,532	2,377	2,218	2,059	1,906	1,762	1,631	1,512
-0,6	1,650	1,811	2,004	2,236	2,514	2,828	3,122	3,291	3,322	3,274	3,185	3,072	2,943	2,801	2,647	2,483	2,311	2,138	1,970	1,813	1,671
-0,7	1,508	1,643	1,802	1,992	2,222	2,496	2,811	3,122	3,332	3,399	3,372	3,295	3,189	3,062	2,919	2,760	2,586	2,400	2,210	2,025	1,855
-0,8	1,383	1,496	1,627	1,783	1,968	2,189	2,453	2,760	3,080	3,332	3,449	3,453	3,394	3,297	3,175	3,032	2,868	2,681	2,477	2,267	2,066
-0,9	1,273	1,368	1,478	1,605	1,754	1,931	2,141	2,390	2,681	2,998	3,283	3,458	3,508	3,475	3,393	3,278	3,135	2,963	2,760	2,533	2,302
-1	1,177	1,258	1,349	1,454	1,576	1,718	1,884	2,080	2,312	2,583	2,885	3,185	3,414	3,523	3,529	3,469	3,363	3,220	3,035	2,807	2,554
-1,1	1,202	1,220	1,239	1,327	1,427	1,542	1,675	1,831	2,012	2,225	2,472	2,753	3,048	3,313	3,485	3,544	3,515	3,422	3,273	3,066	2,806
-1,2	1,222	1,239	1,258	1,279	1,301	1,396	1,505	1,629	1,773	1,940	2,133	2,357	2,612	2,889	3,163	3,382	3,500	3,512	3,434	3,273	3,031
-1,3	1,238	1,255	1,273	1,293	1,314	1,338	1,363	1,465	1,581	1,713	1,865	2,040	2,242	2,470	2,722	2,984	3,222	3,386	3,439	3,374	3,193
-1,4	1,252	1,268	1,285	1,304	1,324	1,346	1,369	1,395	1,424	1,531	1,652	1,791	1,949	2,129	2,333	2,559	2,798	3,029	3,210	3,291	3,233
-1,5	1,263	1,278	1,295	1,312	1,331	1,351	1,374	1,397	1,423	1,451	1,481	1,592	1,718	1,861	2,022	2,203	2,403	2,616	2,827	3,003	3,090
-1,6	1,273	1,287	1,302	1,319	1,337	1,356	1,376	1,398	1,422	1,447	1,474	1,503	1,534	1,648	1,777	1,921	2,082	2,258	2,446	2,633	2,792
-1,7	1,280	1,294	1,309	1,325	1,341	1,359	1,378	1,398	1,420	1,443	1,468	1,494	1,522	1,551	1,581	1,698	1,827	1,969	2,125	2,289	2,453
-1,8	1,287	1,300	1,314	1,329	1,345	1,361	1,379	1,398	1,418	1,439	1,461	1,485	1,511	1,537	1,565	1,594	1,623	1,739	1,866	2,003	2,147
-2	1,298	1,310	1,323	1,336	1,350	1,364	1,380	1,396	1,413	1,431	1,451	1,471	1,492	1,514	1,537	1,561	1,585	1,611	1,636	1,660	1,683
-2,2	1,306	1,317	1,329	1,341	1,353	1,366	1,380	1,394	1,409	1,425	1,441	1,458	1,476	1,495	1,515	1,535	1,556	1,577	1,599	1,621	1,642
-2,5	1,315	1,325	1,335	1,345	1,356	1,367	1,379	1,391	1,403	1,416	1,430	1,444	1,458	1,473	1,489	1,505	1,522	1,539	1,556	1,574	1,592
-3	1,325	1,333	1,341	1,349	1,358	1,367	1,376	1,386	1,395	1,405	1,416	1,426	1,437	1,448	1,460	1,472	1,484	1,497	1,509	1,522	1,535
-3,5	1,331	1,338	1,344	1,351	1,359	1,366	1,373	1,381	1,389	1,397	1,406	1,414	1,423	1,432	1,441	1,450	1,460	1,469	1,479	1,489	1,499
-4	1,335	1,341	1,346	1,352	1,359	1,365	1,371	1,378	1,384	1,391	1,398	1,405	1,412	1,420	1,427	1,435	1,442	1,450	1,458	1,466	1,475
-5	1,339	1,344	1,348	1,353	1,358	1,363	1,367	1,372	1,377	1,382	1,388	1,393	1,398	1,404	1,409	1,414	1,420	1,426	1,432	1,437	1,443
-7	1,343	1,347	1,350	1,353	1,356	1,359	1,363	1,366	1,369	1,372	1,376	1,379	1,383	1,386	1,390	1,393	1,397	1,401	1,404	1,408	1,411
-10	1,346	1,348	1,350	1,352	1,354	1,356	1,358	1,361	1,363	1,365	1,367	1,369	1,372	1,374	1,376	1,379	1,381	1,383	1,386	1,388	1,390
∞	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348	1,348

μ=FL/4M < 0

C.3.2 C2 values:



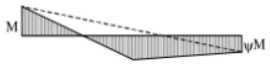
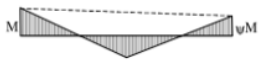
C_2	ψ	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\mu=qL^2/8M > 0$	0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	0,1	0,081	0,076	0,072	0,070	0,068	0,066	0,065	0,063	0,062	0,060	0,058	0,056	0,054	0,052	0,049	0,047	0,045	0,043	0,041	0,039	0,037
	0,2	0,142	0,136	0,131	0,128	0,125	0,122	0,119	0,116	0,112	0,109	0,105	0,101	0,097	0,093	0,090	0,086	0,082	0,079	0,075	0,072	0,069
	0,3	0,192	0,186	0,181	0,176	0,172	0,167	0,163	0,158	0,154	0,149	0,144	0,138	0,133	0,128	0,123	0,118	0,113	0,109	0,104	0,100	0,096
	0,4	0,234	0,227	0,222	0,216	0,211	0,205	0,199	0,193	0,187	0,181	0,175	0,169	0,163	0,157	0,151	0,145	0,140	0,134	0,129	0,125	0,120
	0,5	0,269	0,262	0,255	0,249	0,242	0,236	0,229	0,222	0,215	0,208	0,201	0,194	0,187	0,181	0,174	0,168	0,162	0,157	0,151	0,146	0,141
	0,6	0,298	0,290	0,283	0,276	0,268	0,261	0,253	0,245	0,238	0,230	0,223	0,216	0,208	0,201	0,195	0,188	0,182	0,176	0,170	0,165	0,159
	0,7	0,322	0,313	0,305	0,297	0,289	0,281	0,273	0,265	0,257	0,249	0,241	0,234	0,226	0,219	0,212	0,205	0,199	0,193	0,187	0,181	0,175
	0,8	0,341	0,332	0,324	0,315	0,307	0,298	0,290	0,282	0,273	0,265	0,257	0,250	0,242	0,235	0,228	0,221	0,214	0,208	0,202	0,196	0,190
	0,9	0,357	0,348	0,339	0,330	0,321	0,313	0,304	0,296	0,287	0,279	0,271	0,263	0,256	0,248	0,241	0,234	0,227	0,221	0,215	0,209	0,203
	1	0,370	0,361	0,352	0,342	0,334	0,325	0,316	0,308	0,299	0,291	0,283	0,275	0,268	0,260	0,253	0,246	0,240	0,233	0,227	0,221	0,215
	1,1	0,380	0,371	0,362	0,353	0,344	0,335	0,327	0,318	0,310	0,302	0,294	0,286	0,278	0,271	0,264	0,257	0,250	0,244	0,238	0,232	0,226
	1,2	0,389	0,380	0,371	0,362	0,353	0,344	0,336	0,327	0,319	0,311	0,303	0,295	0,288	0,281	0,274	0,267	0,260	0,254	0,248	0,241	0,236
	1,3	0,397	0,387	0,378	0,369	0,360	0,352	0,343	0,335	0,327	0,319	0,311	0,304	0,296	0,289	0,282	0,275	0,269	0,263	0,256	0,251	0,245
	1,4	0,403	0,394	0,385	0,376	0,367	0,359	0,350	0,342	0,334	0,326	0,318	0,311	0,304	0,297	0,290	0,284	0,277	0,271	0,265	0,259	0,253
	1,5	0,408	0,399	0,390	0,381	0,373	0,364	0,356	0,348	0,340	0,333	0,325	0,318	0,311	0,304	0,297	0,291	0,284	0,278	0,272	0,266	0,261
	1,6	0,413	0,404	0,395	0,386	0,378	0,370	0,362	0,354	0,346	0,339	0,331	0,324	0,317	0,310	0,304	0,297	0,291	0,285	0,279	0,274	0,268
	1,7	0,416	0,408	0,399	0,391	0,383	0,375	0,367	0,359	0,351	0,344	0,337	0,330	0,323	0,316	0,310	0,303	0,297	0,291	0,286	0,280	0,274
	1,8	0,420	0,411	0,403	0,394	0,386	0,379	0,371	0,363	0,356	0,349	0,342	0,335	0,328	0,322	0,315	0,309	0,303	0,297	0,291	0,286	0,281
	2	0,425	0,417	0,409	0,401	0,393	0,386	0,378	0,371	0,364	0,357	0,351	0,344	0,338	0,331	0,325	0,319	0,313	0,308	0,302	0,297	0,292
	2,2	0,429	0,421	0,414	0,406	0,399	0,392	0,385	0,378	0,371	0,365	0,358	0,352	0,346	0,340	0,334	0,328	0,323	0,317	0,312	0,306	0,301
	2,5	0,433	0,426	0,419	0,412	0,406	0,399	0,392	0,386	0,380	0,374	0,368	0,362	0,356	0,350	0,345	0,339	0,334	0,329	0,324	0,319	0,314
	3	0,438	0,432	0,426	0,420	0,413	0,408	0,402	0,396	0,390	0,385	0,380	0,374	0,369	0,364	0,359	0,354	0,349	0,344	0,340	0,335	0,331
	3,5	0,441	0,436	0,430	0,425	0,419	0,414	0,409	0,404	0,398	0,393	0,389	0,384	0,379	0,374	0,370	0,365	0,361	0,356	0,352	0,348	0,344
	4	0,444	0,438	0,433	0,428	0,424	0,419	0,414	0,409	0,405	0,400	0,396	0,391	0,387	0,382	0,378	0,374	0,370	0,366	0,362	0,358	0,354
	5	0,446	0,442	0,437	0,433	0,429	0,425	0,421	0,417	0,413	0,409	0,406	0,402	0,398	0,394	0,391	0,387	0,384	0,380	0,377	0,374	0,370
	7	0,448	0,445	0,442	0,439	0,435	0,432	0,429	0,426	0,423	0,420	0,418	0,415	0,412	0,409	0,406	0,404	0,401	0,398	0,395	0,393	0,390
	10	0,449	0,447	0,445	0,442	0,440	0,438	0,436	0,433	0,431	0,429	0,427	0,425	0,423	0,421	0,419	0,416	0,414	0,412	0,410	0,408	0,406
	∞	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454



C_2	ψ	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	0	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	1
$\mu=qL^2/8M < 0$	0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
	-0,1	0,083	0,094	0,096	0,089	0,083	0,080	0,077	0,076	0,074	0,073	0,071	0,069	0,067	0,064	0,061	0,058	0,055	0,052	0,050	0,047	0,044
	-0,2	0,150	0,172	0,197	0,209	0,197	0,181	0,171	0,165	0,161	0,159	0,156	0,153	0,149	0,144	0,138	0,132	0,124	0,118	0,111	0,104	0,098
	-0,3	0,205	0,232	0,265	0,307	0,338	0,328	0,298	0,277	0,266	0,259	0,256	0,253	0,249	0,243	0,235	0,224	0,212	0,200	0,187	0,175	0,164
	-0,4	0,250	0,279	0,315	0,360	0,418	0,477	0,487	0,445	0,406	0,384	0,372	0,367	0,364	0,360	0,353	0,341	0,325	0,306	0,285	0,265	0,246
	-0,5	0,287	0,316	0,352	0,396	0,453	0,526	0,612	0,665	0,629	0,567	0,526	0,505	0,497	0,494	0,491	0,483	0,467	0,442	0,412	0,381	0,350
	-0,6	0,317	0,345	0,380	0,421	0,474	0,540	0,625	0,731	0,834	0,849	0,777	0,708	0,669	0,652	0,648	0,646	0,638	0,616	0,578	0,532	0,485
	-0,7	0,340	0,368	0,400	0,439	0,486	0,544	0,617	0,710	0,829	0,968	1,067	1,035	0,946	0,878	0,844	0,834	0,832	0,823	0,789	0,731	0,661
	-0,8	0,358	0,385	0,415	0,451	0,493	0,544	0,606	0,683	0,780	0,904	1,058	1,223	1,300	1,241	1,151	1,091	1,065	1,058	1,039	0,982	0,888
	-0,9	0,373	0,398	0,427	0,460	0,498	0,542	0,596	0,660	0,738	0,836	0,958	1,113	1,302	1,483	1,544	1,482	1,403	1,355	1,328	1,280	1,169
	-1	0,385	0,409	0,435	0,465	0,500	0,540	0,586	0,640	0,705	0,783	0,878	0,996	1,145	1,330	1,548	1,743	1,807	1,760	1,696	1,628	1,498
	-1,1	0,394	0,417	0,442	0,469	0,501	0,536	0,577	0,624	0,678	0,742	0,819	0,910	1,022	1,160	1,332	1,543	1,785	1,994	2,071	2,025	1,876
	-1,2	0,402	0,423	0,446	0,472	0,500	0,532	0,569	0,609	0,656	0,710	0,773	0,847	0,934	1,039	1,166	1,322	1,513	1,742	1,993	2,190	2,204
	-1,3	0,409	0,428	0,450	0,474	0,500	0,529	0,561	0,597	0,638	0,685	0,737	0,798	0,869	0,951	1,049	1,165	1,305	1,474	1,678	1,911	2,133
	-1,4	0,414	0,433	0,453	0,475	0,499	0,525	0,555	0,587	0,623	0,664	0,709	0,760	0,819	0,886	0,964	1,054	1,161	1,286	1,436	1,613	1,817
	-1,5	0,419	0,436	0,455	0,475	0,498	0,522	0,549	0,578	0,610	0,646	0,685	0,730	0,780	0,836	0,900	0,973	1,056	1,154	1,267	1,400	1,554
	-1,6	0,422	0,439	0,457	0,476	0,497	0,519	0,543	0,570	0,599	0,631	0,666	0,705	0,748	0,796	0,850	0,910	0,978	1,056	1,146	1,248	1,366
	-1,7	0,426	0,441	0,458	0,476	0,495	0,516	0,539	0,563	0,589	0,618	0,650	0,684	0,722	0,763	0,809	0,861	0,918	0,982	1,055	1,137	1,230
	-1,8	0,428	0,443	0,459	0,476	0,494	0,513	0,534	0,557	0,581	0,607	0,635	0,666	0,700	0,737	0,777	0,821	0,870	0,924	0,985	1,052	1,128
	-2	0,433	0,446	0,461	0,476	0,492	0,509	0,527	0,546	0,567	0,589	0,612	0,638	0,665	0,695	0,726	0,761	0,799	0,839	0,884	0,933	0,986
	-2,2	0,436	0,448	0,461	0,475	0,489	0,504	0,520	0,537	0,555	0,574	0,594	0,616	0,639	0,663	0,690	0,718	0,748	0,780	0,815	0,853	0,893
	-2,5	0,440	0,451	0,462	0,474	0,486	0,499	0,513	0,527	0,542	0,558	0,574	0,592	0,610	0,629	0,650	0,672	0,695	0,719	0,745	0,772	0,802
	-3	0,444	0,453	0,462	0,472	0,482	0,492	0,503	0,514	0,526	0,538	0,551	0,564	0,578	0,592	0,607	0,623	0,639	0,656	0,674	0,692	0,712
	-3,5	0,447	0,454	0,462	0,470	0,479	0,487	0,496	0,505	0,515	0,525	0,535	0,545	0,556	0,568	0,579	0,591	0,604	0,617	0,630	0,644	0,659
	-4	0,448	0,455	0,462	0,469	0,476	0,483	0,491	0,499	0,507	0,515	0,524	0,532	0,541	0,550	0,560	0,570	0,580	0,591	0,601	0,612	0,624
	-5	0,450	0,455	0,461	0,466	0,472	0,478	0,483	0,490	0,496	0,502	0,508	0,515	0,521	0,528	0,535	0,542	0,550	0,557	0,565	0,572	0,581
	-7	0,452	0,456	0,459	0,463	0,467	0,471	0,475	0,479	0,483	0,487	0,492	0,496	0,500	0,505	0,509	0,514	0,518	0,523	0,528	0,533	0,538
	-10	0,453	0,455	0,458	0,461	0,463	0,466	0,469	0,471	0,474	0,477	0,480	0,482	0,485	0,488	0,491	0,494	0,497	0,500	0,503	0,506	0,509
	∞	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454	0,454

C_2	 ψ 																					
	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	**0**	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	**1**
0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
0,1	0,060	0,057	0,054	0,053	0,053	0,054	0,055	0,055	0,056	0,056	0,055	0,054	0,053	0,051	0,050	0,048	0,046	0,044	0,042	0,040	0,038
0,2	0,109	0,105	0,103	0,103	0,104	0,105	0,106	0,106	0,106	0,105	0,103	0,101	0,099	0,095	0,092	0,089	0,085	0,081	0,078	0,075	0,071
0,3	0,153	0,151	0,150	0,151	0,152	0,153	0,153	0,152	0,151	0,149	0,145	0,142	0,138	0,133	0,129	0,124	0,119	0,114	0,110	0,105	0,101
0,4	0,196	0,195	0,195	0,196	0,196	0,196	0,195	0,193	0,190	0,186	0,182	0,177	0,172	0,166	0,160	0,154	0,149	0,143	0,138	0,132	0,127
0,5	0,237	0,236	0,236	0,236	0,235	0,234	0,231	0,228	0,223	0,219	0,213	0,207	0,201	0,194	0,188	0,181	0,175	0,168	0,162	0,156	0,151
0,6	0,275	0,274	0,273	0,272	0,270	0,267	0,263	0,258	0,253	0,247	0,240	0,233	0,226	0,219	0,212	0,205	0,198	0,191	0,184	0,178	0,172
0,7	0,310	0,308	0,306	0,303	0,300	0,295	0,290	0,284	0,278	0,271	0,264	0,256	0,249	0,241	0,233	0,226	0,218	0,211	0,204	0,198	0,191
0,8	0,340	0,337	0,334	0,330	0,325	0,320	0,314	0,307	0,300	0,292	0,284	0,276	0,268	0,260	0,252	0,245	0,237	0,229	0,222	0,215	0,208
0,9	0,367	0,363	0,358	0,353	0,347	0,341	0,334	0,326	0,319	0,311	0,303	0,294	0,286	0,277	0,269	0,261	0,253	0,246	0,238	0,231	0,224
1	0,390	0,385	0,379	0,373	0,366	0,359	0,351	0,343	0,335	0,327	0,318	0,310	0,301	0,293	0,285	0,276	0,268	0,261	0,253	0,246	0,239
1,1	0,410	0,404	0,397	0,390	0,383	0,375	0,367	0,359	0,350	0,341	0,333	0,324	0,315	0,307	0,298	0,290	0,282	0,274	0,267	0,259	0,252
1,2	0,426	0,420	0,413	0,405	0,397	0,389	0,380	0,372	0,363	0,354	0,345	0,337	0,328	0,319	0,311	0,302	0,294	0,287	0,279	0,271	0,264
1,3	0,441	0,434	0,426	0,418	0,410	0,401	0,392	0,383	0,375	0,366	0,357	0,348	0,339	0,331	0,322	0,314	0,306	0,298	0,290	0,283	0,276
1,4	0,453	0,446	0,437	0,429	0,420	0,412	0,403	0,394	0,385	0,376	0,367	0,358	0,349	0,341	0,333	0,324	0,316	0,308	0,301	0,293	0,286
1,5	0,464	0,456	0,448	0,439	0,430	0,421	0,412	0,403	0,394	0,385	0,376	0,368	0,359	0,350	0,342	0,334	0,326	0,318	0,310	0,303	0,296
1,6	0,473	0,465	0,456	0,448	0,439	0,430	0,421	0,412	0,403	0,394	0,385	0,376	0,368	0,359	0,351	0,343	0,335	0,327	0,320	0,312	0,305
1,7	0,482	0,473	0,464	0,455	0,446	0,437	0,428	0,419	0,410	0,401	0,393	0,384	0,376	0,367	0,359	0,351	0,343	0,335	0,328	0,321	0,313
1,8	0,489	0,480	0,471	0,462	0,453	0,444	0,435	0,426	0,417	0,408	0,400	0,391	0,383	0,375	0,366	0,358	0,351	0,343	0,336	0,328	0,321
2	0,500	0,491	0,482	0,473	0,465	0,456	0,447	0,438	0,429	0,421	0,412	0,404	0,396	0,388	0,380	0,372	0,365	0,357	0,350	0,343	0,336
2,2	0,509	0,500	0,491	0,483	0,474	0,465	0,457	0,448	0,440	0,431	0,423	0,415	0,407	0,399	0,392	0,384	0,377	0,369	0,362	0,355	0,349
2,5	0,519	0,511	0,502	0,494	0,485	0,477	0,469	0,460	0,452	0,444	0,436	0,429	0,421	0,414	0,407	0,399	0,392	0,385	0,379	0,372	0,365
3	0,530	0,523	0,514	0,507	0,499	0,491	0,484	0,476	0,469	0,461	0,454	0,447	0,440	0,433	0,426	0,420	0,413	0,407	0,400	0,394	0,388
3,5	0,538	0,530	0,523	0,515	0,508	0,501	0,494	0,487	0,480	0,474	0,467	0,460	0,454	0,448	0,441	0,435	0,429	0,423	0,417	0,412	0,406
4	0,542	0,535	0,529	0,522	0,515	0,509	0,502	0,496	0,490	0,483	0,477	0,471	0,465	0,459	0,454	0,448	0,442	0,437	0,431	0,426	0,420
5	0,548	0,542	0,536	0,531	0,525	0,519	0,514	0,508	0,503	0,497	0,492	0,487	0,482	0,477	0,472	0,467	0,462	0,457	0,452	0,447	0,443
7	0,553	0,549	0,544	0,540	0,535	0,531	0,527	0,522	0,518	0,514	0,510	0,506	0,502	0,498	0,494	0,490	0,486	0,482	0,478	0,475	0,471
10	0,556	0,552	0,549	0,546	0,543	0,539	0,536	0,533	0,530	0,527	0,524	0,521	0,518	0,515	0,512	0,509	0,506	0,503	0,500	0,497	0,494
∞	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630

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C_2	 ψ 																					
	-1	-0,9	-0,8	-0,7	-0,6	-0,5	-0,4	-0,3	-0,2	-0,1	**0**	0,1	0,2	0,3	0,4	0,5	0,6	0,7	0,8	0,9	**1**
0	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000	0,000
-0,1	0,067	0,073	0,069	0,063	0,059	0,057	0,057	0,058	0,059	0,061	0,061	0,061	0,061	0,060	0,058	0,056	0,053	0,051	0,048	0,046	0,044
-0,2	0,134	0,154	0,163	0,151	0,135	0,125	0,121	0,120	0,122	0,125	0,128	0,129	0,130	0,128	0,126	0,121	0,116	0,111	0,105	0,099	0,094
-0,3	0,202	0,231	0,263	0,277	0,253	0,222	0,202	0,193	0,191	0,194	0,198	0,203	0,206	0,206	0,204	0,198	0,191	0,181	0,172	0,161	0,152
-0,4	0,267	0,304	0,349	0,396	0,416	0,382	0,332	0,298	0,280	0,275	0,277	0,282	0,289	0,293	0,293	0,288	0,278	0,265	0,250	0,235	0,220
-0,5	0,326	0,368	0,419	0,482	0,548	0,581	0,543	0,475	0,421	0,391	0,378	0,376	0,381	0,388	0,392	0,390	0,381	0,365	0,345	0,322	0,300
-0,6	0,377	0,420	0,474	0,541	0,621	0,707	0,761	0,736	0,658	0,583	0,533	0,508	0,499	0,501	0,506	0,507	0,500	0,483	0,457	0,427	0,396
-0,7	0,419	0,463	0,516	0,582	0,662	0,757	0,862	0,943	0,947	0,875	0,787	0,717	0,674	0,652	0,646	0,645	0,639	0,623	0,592	0,552	0,509
-0,8	0,452	0,496	0,548	0,610	0,684	0,775	0,882	1,002	1,110	1,156	1,113	1,026	0,942	0,880	0,842	0,822	0,808	0,789	0,754	0,703	0,645
-0,9	0,480	0,522	0,571	0,629	0,697	0,779	0,876	0,991	1,122	1,252	1,341	1,346	1,282	1,197	1,121	1,065	1,026	0,993	0,949	0,886	0,808
-1	0,502	0,542	0,588	0,642	0,704	0,777	0,863	0,965	1,084	1,220	1,364	1,490	1,552	1,532	1,463	1,383	1,313	1,252	1,189	1,108	1,005
-1,1	0,520	0,558	0,601	0,651	0,707	0,773	0,849	0,938	1,041	1,161	1,299	1,449	1,598	1,710	1,748	1,716	1,647	1,567	1,481	1,376	1,243
-1,2	0,535	0,571	0,611	0,657	0,709	0,767	0,835	0,913	1,002	1,105	1,224	1,360	1,512	1,670	1,815	1,910	1,930	1,887	1,803	1,683	1,523
-1,3	0,547	0,581	0,619	0,661	0,708	0,762	0,822	0,890	0,968	1,057	1,159	1,276	1,408	1,556	1,716	1,875	2,008	2,079	2,070	1,983	1,824
-1,4	0,557	0,589	0,624	0,664	0,707	0,756	0,810	0,871	0,940	1,017	1,105	1,205	1,318	1,446	1,588	1,744	1,905	2,053	2,155	2,174	2,084
-1,5	0,565	0,596	0,629	0,665	0,706	0,750	0,799	0,854	0,915	0,984	1,060	1,146	1,243	1,352	1,474	1,610	1,759	1,915	2,064	2,177	2,209
-1,6	0,572	0,601	0,632	0,666	0,704	0,745	0,790	0,839	0,894	0,955	1,022	1,098	1,182	1,275	1,380	1,496	1,625	1,765	1,913	2,055	2,167
-1,7	0,578	0,606	0,635	0,667	0,702	0,740	0,781	0,826	0,876	0,930	0,991	1,057	1,131	1,212	1,302	1,402	1,513	1,635	1,766	1,904	2,036
-1,8	0,584	0,609	0,637	0,667	0,700	0,735	0,773	0,815	0,860	0,909	0,963	1,023	1,088	1,159	1,238	1,325	1,421	1,526	1,641	1,764	1,891
-2	0,592	0,615	0,640	0,667	0,696	0,726	0,760	0,795	0,833	0,875	0,920	0,968	1,021	1,078	1,140	1,207	1,281	1,361	1,449	1,543	1,645
-2,2	0,598	0,620	0,642	0,666	0,692	0,719	0,748	0,779	0,812	0,848	0,886	0,926	0,970	1,017	1,068	1,123	1,181	1,245	1,314	1,388	1,467
-2,5	0,605	0,624	0,644	0,665	0,686	0,710	0,734	0,760	0,788	0,817	0,847	0,880	0,915	0,952	0,991	1,033	1,077	1,125	1,176	1,230	1,287
-3	0,612	0,628	0,644	0,661	0,679	0,698	0,717	0,737	0,759	0,781	0,804	0,828	0,854	0,881	0,909	0,938	0,969	1,002	1,036	1,073	1,111
-3,5	0,617	0,630	0,644	0,658	0,673	0,689	0,705	0,721	0,739	0,756	0,775	0,794	0,814	0,835	0,857	0,880	0,903	0,928	0,953	0,980	1,008
-4	0,620	0,632	0,644	0,656	0,669	0,682	0,695	0,709	0,724	0,739	0,754	0,770	0,787	0,804	0,821	0,839	0,858	0,878	0,898	0,919	0,941
-5	0,624	0,633	0,642	0,652	0,662	0,672	0,682	0,693	0,704	0,715	0,727	0,738	0,750	0,763	0,775	0,788	0,802	0,815	0,830	0,844	0,859
-7	0,627	0,633	0,640	0,647	0,653	0,660	0,667	0,675	0,682	0,689	0,697	0,705	0,712	0,720	0,728	0,737	0,745	0,754	0,762	0,771	0,780
-10	0,628	0,633	0,637	0,642	0,647	0,651	0,656	0,661	0,666	0,671	0,676	0,681	0,686	0,691	0,696	0,701	0,707	0,712	0,718	0,723	0,729
∞	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630	0,630

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