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شُكر وعِرفان

الحمد لله الذي ما انتهى درب وختم جهد ولا تم سعي الا بفضله, والصلاة والسلام على اشرف المرسلين سيدنا وحبيبنا محمد صلى الله عليه وسلم.

قال رسول الله صلى الله عليه وسلم "من لم يشكر الناس لم يشكر الله ومن اهدى اليكم معروفا فكافئوه فان لم تستطيعوا فادعوا له ".

وعملا بهذا الحديث, احمد الله عزوجل واشكره على ان وفقني لاتمام هذا العمل واتقدم بجزيل الشكر الى الأستاذة المشرفة "زديري صونية" التي رافقتني طيلة هذا العمل وامدتني بالمعلومات والنصائح القيمة وارجوا من الله عزوجل ان يسدد خطاها ويحقق مُناها فجزاها الله عني كل خير.

كما اتوجّه بالشكر الى أعضاء لجنة المناقشة الافاضل لقبولهم مناقشة مذكرتي, كل من الأستاذ "عكروت كمال" رئيسًا والاستاذة "دقايشية حكيمة" ممتحنًا اللذين لا شك انهما سيفيضون عليا بتوجيهاتهم القيمة وملاحظاتهم السديدة.

ولا يفوتني ان أتقدّم بجزيل الشكر و الامتنان الى كل الأساتذة اللذين تعلمت منهم خلال سنوات دراستي, وكل من كان له فضل ولو صغير, في مسيرتي العلمية.

اخيرًا, امدّ شكري وامتناني لكل من كان له اثر وكل من ساهم بكلمة, فكرة, دعم, او حتى دعاء, فلهم مني جزيل الشكر والعرفان.

لكم مني فائق التقدير والاحترام, وجزاكم الله عني كل خير.



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مهما كتبت من عبارات لن اجد اصدق من قوله تعالى "يرفع الله اللذين امنوا منكم واللذين اوتوا العلم درجات".

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اليكم جميعًا, اهدي ثمرة جهدي.

Abstract

In this work, we discuss the existence of positive solutions of a class of differential equations called double phase with Dirichlet condition, that involves both a singular and a superlinear parametric term, posed by the discontinuous weights and the the structure of the differential operator.

Using Nehari's method and critical point theory, it is shown that for small values of the argument of the parameter λ, the equation admits at least two positive solutions.

Keywords: Double phase problem, singular term, Nehari manifold, positive solutions, discontinuous weight

Résumé

Dans ce travail, nous discutons de l'existence de solutions positives d'une classe d'équations différentielles dite de double phase avec condition de Dirichlet, qui combine un terme singulier et un terme paramétrique superlinéaire, face aux défis posés par la présence de poids discontinus et à la structure de l'opérateur différentiel.

À l'aide de la méthode de Nehari et la théorie des points critiques, on démontre que pour des petites valeurs de l'argument du paramètre λ, l'équation admet au moins deux solutions positives.

Mot clé: Problème de double phase, terme singulier, variétés de Nehari, solutions positives, poids discontinu

الملخص

في هذا العمل نقوم بدراسة وجود حلول وموجبة لفئة من المعادلات تفاضلية الجزئية تدعى ذات المدخلين، التي تجمع بين وجود حدود بقيم حرجة وحدود فوق خطية، تحت تحديات يطرحها عدم استمرار دوال الثقل وطبيعة المؤثر التفاضلي المزدوج.

باستخدام طريقة منوعات نهاري ونظرية النقاط الحرجة نثبت انه من اجل قيم صغيرة للوسيط λ المعادلة تقبل على الاقل حلين موجبين.

كلمات مفتاحية: إشكالية بمدخلين، حدود بقيم حرجة، منوعات نهاري، حلول موجبة، دوال ثقل غير مستمرة.

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Introduction

Elliptic partial differential equations (EPDEs) are important mathematical modeling tools in describing steady-state physical phenomena, such as the Laplace equation $\Delta u = 0$ and the Poisson equation $\Delta u = f(x)$, these equations are used in areas such as heat transfer, stress analysis, gravity, and incompressible fluid transport, they are also used in solid mechanics to analyze stress and strain, and in wave physics to describe wave propagation using the Helmholtz equation. These equations are often solved using Dirichlet or Neumann-type boundary conditions.

One of the elliptic partial differential equations that authors have been interested in studying in recent years is double phase problems, the idea of tackling these problems originally came from Zhikov (see [24], [26], [25]), who through his research developed binary methods for analyzing averaging problems in nonlinear variational problems, focusing on applications of stochastic models and their challenges in mechanics, such as plasticity and random impurities. It also introduces new techniques for guaranteeing the existence of solutions even in cases where the uniqueness and regularity of the solutions are lost. We also refer to Zhikov-Kozlov-Oleinik's study (see [23]). Double phase equations have been studied by Cencelj-Rădulescu-Repovš (see [9]), they dealt with problems of variable growth, Colasuonno-Squassina (see [3]) treated an eigenvalue problem in the framework of biphasic variational integration and provide a series of nonlinear eigenvalues through a minimization procedure, Colombo-Mingione (see [11], [10]) introduced regularity theorems for minimizers of a class of variational integrals in which the (integrand) switches between two different types of degenerate elliptic phases, according to the zero set of the modulation coefficient $a(.)$. The typical case is given by the function $\omega \to \int \left(|D\omega|^p + |D\omega|^q\right) dx$ where $q > p$ and $a(.) \geqq 0$. When $a(.) \in C^{0,\alpha}$, $0 \leq a(.)$, and $1 < p < q$, the bounded minimisers of this functional are $C^{1,\beta}$ regular, provided the sharp bound $q \leq p + \alpha$ holds.

And there are other works, Baroni-Colombo-Mingione (see [20]), and Liu-Dai (see [28]) (problems with a differential operator which exhibits unbalanced growth). A nice survey of the recent works on such equations can be found in Rădulescu (see [27]). We also mention the recent works on (p, q)-equations (equations driven by the sum of a p-Laplacian and of a q-Laplacian) with singular terms of Papageorgiou-Rădulescu-Repovš (see [17]) and Papageorgiou-Vetro-Vetro (see [14]). For such differential operators, the integrand of the energy functional is $k(t) = \frac{1}{p}t^p + \frac{1}{q}t^q$ for all $t > 0$ (that is, $\xi(x) = 1$) and so the use of the global regularity theory of Lieberman (see [4]) and the nonlinear maximum principle of Pucci-Serrin (see [21]) is possible. This fact in turn, permits the use of truncation and comparison techniques, which make it possible to bypass the singularity in the reaction.

In this work, we study the nonlinear singular double phase problem, and we depend in our work Papageorgiou- Repovš-Vetro (see [13]).

In this memoir, we study the existence of positive solutions for a class of double phase Dirichlet equations which have the combined effects of a singular term. The differential operator of the equation is the sum of a p-Laplacian and of a weighted q-Laplacian $(q < p)$ with discontinuous weight. This study takes place in Sobolev space $W_0^{1,p}(\Omega)$.

Given that the presence of the weight $\xi(.)$, which is discontinuous and not bounded away from zero, does not permit the use of the global regularity theory of Lieberman (see [4]) and of the nonlinear strong maximum principle of Pucci-Serrin (see [21]) (p. 111, 120). The absence of these basic tools leads to a different approach based on the Nehari method. In the reaction (the right hand side), we have the combined effects of a singular term and of a parametric $(p-1)$-superlinear perturbation. We are looking for positive solutions and by using the Nehari method and the critical point theory, we show that for all small values $\lambda > 0$ of the parameter, problem has at least two positive solutions.

We have divided our work into three chapters:

Chapter one: We introduce the fundamental concepts of Lebesgue spaces, the Sobolev spaces, convergence criteria and we introduce also some inequalities in functional spaces.

Chapter two: In this chapter, we recall the concept of critical points, the Nehari manifold theorem, the concept of Bounded from below functionnelle, that help us study and analyze a singular double phase problem.

Chapter three: We study the existence of two positive solutions for the following singular double phase problem

$$\begin{cases} -\Delta_p u - \operatorname{div}\left(\xi(x)\left|\nabla u\right|^{q-2}\nabla u\right) = a(x)\,u^{-\gamma} + \lambda u^{r-1}, \text{ in } \Omega \\ u\big|_{\partial\Omega} = 0,\ 1 < q < p < r < p^*,\ 0 < \gamma < 1,\ u \geq 0,\ \lambda > 0. \end{cases} \tag{P_λ}$$

such that $\Omega \subseteq \mathbb{R}^N$ is a bounded domain with Lipschitz boundary $\partial\Omega$, and Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p = \operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right), \text{ for all } u \in W_0^{1,p}(\Omega).$$

and $\xi : \Omega \to \mathbb{R}_+$ is essentially bounded weight.

Chapter 1

Preliminaires

In this chapter we introduce the fundamental concepts, notations, and preliminary results that will be used later in the memoir.

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1.1 *[6] Let $p \in \mathbb{R}$ with $1 < p < \infty$; we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; \text{ measurable and } \mid f \mid^p \in L^1(\Omega) \right\},$$

equipped with norm

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, d\mu \right]^{\frac{1}{p}}.$$

Definition 1.2 *[6] We set*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R}; \text{ measurable and } \exists C > 0 \text{ such that } |f(x)| \leq C \text{ a.e. on } \Omega \}.$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C; \mid f(x)| \leq C \text{ a.e.on } \Omega \}.$$

Definition 1.3 *[6] We define the Lebesgue space $L^p(\Omega; \mathbb{R}^N)$ by*

$$L^p(\Omega; \mathbb{R}^N) = \left\{ u : \Omega \to \mathbb{R}^N; \text{ measurable and } \left(\int_\Omega |\nabla u|^p \, dx \right)^{\frac{1}{p}} < +\infty \right\}.$$

Proposition 1.1 *[6] Assume that Ω is a measurable subset of $\mathbb{R}^N$. Then*

- The space $L^p(\Omega)$ is reflexive and separable,for any p, $1 < p < \infty$, and its dual is $L^{p'}(\Omega)$.
- If $p = 1$, the space $L^1(\Omega)$ is non-reflexive and separable, and its dual is $L^\infty(\Omega)$.
- If $p = \infty$, the space $L^\infty(\Omega)$ is non-reflexive and non-separable, and its dual is its dual strictly bigger than $L^1(\Omega)$.

1.1.2 Sobolev spaces

Let $\Omega \subset \mathbb{R}$ be an open set and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$

Definition 1.4 *[6] The sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) \text{ such that } \nabla u \in L^p(\Omega)\}.$$

For $u \in W^{1,p}(\Omega)$ we write

$$\nabla u = \text{grad } u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right).$$

The space $W^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p.$$

or sometimes with the equivalent norm

$$\left(\|u\|_p^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}, \text{ if } 1 \leq p < \infty.$$

For $p = 2$, we set

$$W^{1,2}(\Omega) = H^1(\Omega).$$

The space $H^1(\Omega)$ is equipped with the scalar product

$$(u,v)_{H^1(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2(\Omega)} = \int_\Omega uv + \sum_{i=1}^{N} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i}.$$

The associated norm is

$$\|u\|_{H^1(\Omega)} = \left(\|u\|_2^2 + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_2^2 \right)^{\frac{1}{2}}.$$

is equivalent to the $W^{1,2}(\Omega)$ norm.

Proposition 1.2 *The space $W^{1,p}(\Omega)$ is*

i) Banach space for every $1 \leq p \leq \infty$.

ii) Reflexive for $1 < p < \infty$.

iii) Separable for $1 \leq p < \infty$.

$H^1(\Omega)$ is a separable Hilbert space.

Definition 1.5 ***(The space*** $W_0^{1,p}(\Omega)$***)*** *[6] Let* $1 \leq p < \infty$*;* $W_0^{1,p}(\Omega)$ *denotes the closure of* $C_c^1(\Omega)$ *in* $W^{1,p}(\Omega)$.

Set

$$H_0^1(\Omega) = W_0^{1,2}(\Omega).$$

The space $W_0^{1,p}(\Omega)$ equipped with the $W^{1,p}(\Omega)$ norm, is a separable Banach space; it is reflexive if $1 < p < \infty$.
$H_0^1(\Omega)$, equipped with the $H^1(\Omega)$ scalar product, is a Hilbert space.

Let $\Omega \subset \mathbb{R}^N$ and we suppose that Ω is an open set of class C^1 with bounded or else that $\Omega = \mathbb{R}_+^N$, we introduce the following definitions.

Definition 1.6 ***(Continuous and compact embedding)*** *[6] Let* $1 \leq p \leq \infty$*, We have the following injections:*

• *For* $p < N$*, we have*

$$W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega), \text{ where } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N}.$$

and there exists a constant C *such that*

$$\|u\|_{L^{p^*}(\Omega)} \leq C \|\nabla u\|_{W^{1,p}(\Omega)}, \forall u \in W^{1,p}(\Omega).$$

• *For* $p = N$*, we have*

$$W^{1,p}(\Omega) \hookrightarrow L^q(\Omega), \ \forall q \in [p, +\infty).$$

such that

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{W^{1,p}(\Omega)}, \ \forall u \in W^{1,p}(\Omega).$$

• *For* $p > N$*, we have*

$$W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega).$$

for all $u \in W^{1,p}(\Omega)$*, we have*

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,p}(\Omega)} |x - y|^\alpha, \text{ a.e. } x, y \in \Omega.$$

with $\alpha = 1 - \left(\frac{N}{p}\right)$ *and* C *depends only on* Ω, p, *and* N. *Inparticular,* $W^{1,p}(\Omega) \hookrightarrow C\left(\bar{\Omega}\right)$.
And all these injections are continuous.

Definition 1.7 ***(Rellich–Kondrachov)*** *[6] Suppose that* Ω *is bounded and of class* C^1. *Then we have the following compact injections:*

$$\begin{aligned} &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [1, p^*) \quad \text{where } \tfrac{1}{p^*} = \tfrac{1}{p} - \tfrac{1}{N}, \text{ if } p < N, \\ &W^{1,p}(\Omega) \hookrightarrow L^q(\Omega) \quad \forall q \in [p, +\infty), \text{ if } p = N, \\ &W^{1,p}(\Omega) \hookrightarrow C\left(\bar{\Omega}\right), \quad \text{if } p > N. \end{aligned}$$

In particular, $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ *with compact injection for all* p *(and all* N*).*

1.1.3 Convergence criteria

Lemma 1.1 ***(Weak convergence)*** *[19] Let* $1 < p < \infty$ *and* $(f_n)_n$ *be a bounded sequence of* $L^p(\Omega)$ *converging a.e. to* f. *Then* $f_n \rightharpoonup f$ *in* $L^p(\Omega)$.

Theorem 1.1 *[6] Let* $(x_n)_{n\in\mathbb{N}}$ *a sequence of* E. *We have the following properties*

- $x_n \rightharpoonup x$ iff $T(x_n) \to T(x)$ for all $T \in E'$.

- If $x_n \to x$ then $x_n \rightharpoonup x$.

- If $x_n \rightharpoonup x$ then $(x_n)_{n\in\mathbb{N}}$ is bounded in E and

$$\|x\|_E \leq \liminf \|x_n\|_E .$$

- $T_n \to T$ in E' and $x_n \rightharpoonup x$ then $T(x_n) \to T(x)$. We can also note $\langle T_n, x_n\rangle \to \langle T, x\rangle$.

Theorem 1.2 ***(dominated convergence theorem, Lebesgue).*** *[6] Let* (f_n) *be a sequence of functions in* $L^1(\Omega)$ *that satisfy*

a) $f_n(x) \to f(x)$ a.e. on Ω,

b) there is a function $g \in L^1(\Omega)$ such that for all n, $|f_n(x)| \leq g(x)$ a.e. on Ω.

Then $f \in L^1(\Omega)$ and $\|f_n - f\|_1 \to 0$.

Definition 1.8 *[22]* ***(Carathéodory function)***
Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, *a non-empty measurable set, and let* $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$ *and* $u : \Omega \to \mathbb{R}^m$ *function. The function* f *is called a Carathéodory function if the following two conditions are satisfied:*

i) $x \to f(x,s)$ is measurable for any $s \in \mathbb{R}^m$.

ii) $s \to f(x,s)$ is continuous on $\mathbb{R}^m$, a.e. $x \in \Omega$.

Lemma 1.2 *[22] Let $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$, $m \geq 1$; a Carathéodory function that satisfies a growth condition of the form :*

$$|f(x,s)| \leq k(x) + c\sum_{i=1}^{m} |s_i|^{\frac{p_i}{q}}, \ (s_{1z}, ..., s_m), \ \textit{a.e.} \ x \in \Omega; \ 1 \leq i \leq m.$$

for a positive constant c and a certain $k \in L^q(\Omega)$, and $1 \leq q, p_i < \infty, \forall i = 1, ..., m$.
Then the operator F defined by :

$$Fu(x) = f(x, u_1(x), ..., u_m(x)),$$

is continuous and bounded by $L^{p_1}(\Omega) \times ... \times L^{p_m}(\Omega)$ to $L^q(\Omega)$. Here, u denotes the vector function $u = (u_1, ..., u_m)$. What's more,

$$\|Fu\|_{L^q(\Omega)} \leq c\left(\|k\|_{L^q(\Omega)} + \sum_{i=1}^{m} \|u_i\|_{L^{p_i}(\Omega)}^{\frac{p_i}{q}}\right).$$

1.1.4 Some inequalities

1. **Hölder's inequality:** [6] Let $1 \leq p \leq \infty$; we denote by p' the conjugate exponent of p, and $\frac{1}{p} + \frac{1}{p'} = 1$. Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$. then $fg \in L^1(\Omega)$ and

$$\int |fg| \leq \|f\|_p \|g\|_{p'}.$$

2. **Poincaré's inequality:** [6] Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then there exists a constant C (depending on Ω and p) such that

$$\|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}, \forall u \in W_0^{1,p}(\Omega).$$

In particular, the expression $\|\nabla u\|_{L^p(\Omega)}$ is a norm on $W_0^{1,p}(\Omega)$, and it is equivalent to the norm $\|u\|_{W^{1,p}(\Omega)}$; on $H_0^1(\Omega)$ the expression $\sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}$ is a scalar product that induces the norm $\|\nabla u\|_{L^p(\Omega)}$ and it is equivalent to the norm $\|u\|_{H^1(\Omega)}$.

Chapter 2

Critical point theory and Nehari manifold

In this chapter, we introduce the concept of critical points and Nehari manifold theory in the study of solutions to some elliptic problems. We also review some of the basic properties that are used to apply these concepts effectively.

2.1 Critical point theory

2.1.1 Lower semi-continuous function

If K is a topological space, recall that a function J from K to $\mathbb{R}$ is said to be lower semi-continuous (abbreviated to l.s.c.) if for all $\lambda \in \mathbb{R}$ the set $[J \leq \lambda] := \{x \in K; J(x) \leq \lambda\}$ is closed. J is said to be upper semi-continuous (abbreviated u.s.c.) if $-J$ is l.s.c.
Let's introduce the following definition:

Definition 2.1 *[19] Let X be a Banach space and ω a part of X. A function $J : \omega \to \mathbb{R}$ is said to be weakly sequentially l.s.c. if for any sequence $(x_n)_n$ of ω converging weakly to $x \in \omega$ we have*

$$J(x) \leq \liminf_{n\to\infty} J(x_n).$$

Definition 2.2 *[18]* ***(Essentially Bounded Function)*** *Let (X, Σ, μ) be a measure space. And let $f : X \to \mathbb{R}$ be a Σ-measurable function.*
We say that f is essentially bounded if and only if there exists a real number c such that:

$$\mu(\{x \in X : |f(x)| > c\}) = 0.$$

2.1.2 Gâteaux derivative

Definition 2.3 *[19] Let ω be a part of a Banach space X and $J : \omega \to \mathbb{R}$.*
If $u \in \omega$, J is said to be derivable in the sense of Gâteaux (or G-derivable or G-differentiable) in u, if there exists $l \in X'$ such that in each direction $z \in X$ where $J(u+tz)$ exists for $t > 0$ small enough, the directional derivative $J'_z(u)$ exists and we have:

$$\lim_{t \to 0^+} \frac{J(u+tz) - J(u)}{t} = \langle l, z \rangle .$$

Let $J'(u) := l$.

2.1.3 Notion of critical point

Definition 2.4 ***(Critical point and regular point)*** *[19] Let X be a Banach space, $\omega \subset X$ an open and $J \in C^1(\omega, \mathbb{R})$. We say that $u \in \omega$ is a critical point of J, if $J'(u) = 0$. If u is not a critical point, we say that u is a regular point of J.*
If $c \in \mathbb{R}$, we say that c is a critical value of J, if there exists u such that $J(u) = c$ and $J'(u) = 0$. If c is not a critical value, we say that c is a regular value.

Often, when X is a functional space and the equation $J'(u) = 0$ corresponds to a partial differential equation, we say that $J'(u) = 0$ is the Euler equation satisfied by the critical point u.

Definition 2.5 *[12] Many boundary value problems are equivalent to*

$$Au = 0. \tag{2.1}$$

where $A : X \to Y$ is a mapping between two Banach spaces. When the problem is variational, there exists a differentiable functional $J : X \to \mathbb{R}$ such that $A = J'$, i.e.

$$\langle Au, v \rangle = \lim_{t \to 0} \frac{J(u+tv) - J(u)}{t}.$$

The space Y corresponds then to the topological dual X' of X and equation (2.1) is equivalent to $J'(u) = 0$, i.e.

$$\langle J'(u), v \rangle = 0, \quad \forall v \in X. \tag{2.2}$$

A critical point of J is a solution u of (2.2) and the value of J at u is a critical value of J. When J is bounded from below, the infimum

$$c := \inf_X J.$$

Definition 2.6 ***(coercivity)*** *[19] If H is a Hilbert space, let $\|\ .\ \|$ denote its norm and $(.\mid .)$ its scalar product. Let $a(.,.)$ be a bilinear form defined on $H\times H$ with real values.*
We say that $a(.,.)$ is coercive or uniformly elliptic on H if there exists $\alpha>0$ such that:

$$\forall u\in H,\ a(u,u)\geq\alpha\|u\|^2.$$

Theorem 2.1 ***(Implicit Function Theorem)*** *[7] If $W\subseteq\mathbb{R}^m$ is open, $F:W\to\mathbb{R}$ is continuously differentiable, and $p=(p',q)\in\mathbb{R}^{m-1}\times\mathbb{R}$, where $p\in W$ is a point for which*

$$F(p)=0,\text{ and }\frac{\partial F}{\partial x_m}(p)\neq 0.$$

then there exists an open set $W'\subset\mathbb{R}^{m-1}$ with $p'\in W'$ and a unique continuously differentiable function $\psi:W'\to\mathbb{R}$ such that $q=\psi(p')$ and

$$F[x',\psi(x')]=0.$$

holds for $x'\in W'$.

Proof. Without loss of generality, we may assume that

$$\frac{\partial F}{\partial x_m}(p)>0.$$

By the continuity of $\frac{\partial F}{\partial x_m}$, and by passing to a smaller neighborhood W of p if necessary, but without changing notation, we may assume that

$$\frac{\partial F}{\partial x_m}(x)>0. \tag{2.3}$$

holds for all $x\in W$.
Since (by (2.3)) $F(p',.)$ is an increasing function in an interval about q, we can find $q_1<q<q_2$ so that

$$F(p',q_1)<0<F(p',q_2).$$

holds, Using the continuity of F, we can find a neighborhood W' of p' so that $W'\times[q_1,q_2]\subseteq W$ and

$$F(x',q_1)<0<F(x',q_2).$$

holds for $x'\in W'$ It follows from the intermediate value theorem that for each $x'\in W'$ there is a number y with $q_1<y<q_2$ so that $F(x',y)=0$ and by (2.3) that number y is unique. We let that y be $\psi(x')$ and note that the uniqueness of this value y also implies that ψ is continuous.
To complete the proof, we need to show that

$$\frac{\partial\psi}{\partial x_j}=\frac{-\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_m}}. \tag{2.4}$$

hols, for $j = 1, 2, ..., m-1$. We do this with a very direct argument Specifically, fixing the point $x' \in W'$ and setting $y = \psi(x')$, we have

$$F(x' + se_j, y + t) - F(x', y) = s\frac{\partial F}{\partial x_j}(x', y) + t\frac{\partial F}{\partial x_m}(x', y) + \varepsilon\sqrt{s^2 + t^2}. \tag{2.5}$$

where $\varepsilon = \varepsilon(s, t)$ approaches 0 as $\sqrt{s^2 + t^2}$ approaches 0. Now, taking $t = \psi(x' + se_j) - \psi(x')$ in (2.5), we find that

$$t\frac{\partial F}{\partial x_m}(x', y) = -s\frac{\partial F}{\partial x_j}(x', y) - \varepsilon\sqrt{s^2 + t^2}.$$

So we have

$$|t|\left|\frac{\partial F}{\partial x_m}(x', y)\right| \leq |s|\left|\frac{\partial F}{\partial x_j}(x', y)\right| + |\varepsilon||s| + |\varepsilon||t|.$$

For small enough choice of $|s|$ we have $|\varepsilon| \leq \frac{1}{2}\left|\frac{\partial F}{\partial x_m}(x', y)\right|$ and $|\varepsilon| \leq 2\left|\frac{\partial F}{\partial x_j}(x', y)\right|$, so that

$$|t| \leq \frac{6|s|\left|\frac{\partial F}{\partial x_j}(x', y)\right|}{\left|\frac{\partial F}{\partial x_m}(x', y)\right|}.$$

holds. Thus we see that

$$\begin{aligned} &\left|\frac{\psi(x' + se_j) - \psi(x')}{s} + \left[\frac{\partial F}{\partial x_j}(x', y)\right]\left[\frac{\partial F}{\partial x_m}(x', y)\right]^{-1}\right| \\ \leq\ & |\varepsilon|\left(1 + 6\left|\left[\frac{\partial F}{\partial x_j}(x', y)\right]\right|\left|\frac{\partial F}{\partial x_m}(x', y)\right|^{-1}\right). \end{aligned} \tag{2.6}$$

Taking the limit as $s \to 0$ in (2.6), we see that $\frac{\partial \psi}{\partial x_j}$ exists at x' and is given by the formulain (2.4). The continuous differentiability of ψ follows. ■

2.2 Bounded from below functionnelle

Definition 2.7 ***(Convex functions)*** *[19] Let X be a vector space. A part K of X is said to be convex if :*

$$\forall x, y \in K, \quad \forall \theta \in [0, 1], \quad \theta x + (1 - \theta) y \in K.$$

When K is convex and $J : K \to \mathbb{R}$ is a function, J is said to be convex if :

$$\forall x, y \in K, \quad \forall \theta \in [0, 1], \quad J(\theta x + (1 - \theta) y) \leq \theta J(x) + (1 - \theta) J(y).$$

We say that J is strictly convex if for all $x, y \in K$ with $x \neq y$ and $\theta \in]0, 1[$ we have :

$$J(\theta x + (1 - \theta) y) < \theta J(x) + (1 - \theta) J(y).$$

We can show that, in reflexive Banach spaces, weakly sequentially l.s.c. functions reach their minimum, provided they tend to $+\infty$ at infinity.

Proposition 2.1 *[19] Let X be a reflexive Banach space, $K \subset X$ a closed convex and $J : K \to \mathbb{R}$ a weakly sequentially l.s.c. function. Moreover, if K is unbounded, we assume that for any sequence $(x_n)_n$ of K such that $\|x_n\| \to \infty$, we have $J(x_n) \to +\infty$. Then J is inferiorly bounded and reaches its minimum i.e.*

$$\exists u \in K,\ J(u) = \inf_{v \in K} J(v) = \min_{v \in K} J(v).$$

Proof. Indeed, if $\alpha := \inf_{v \in K} J(v)$ and $(u_n)_n$ is a minimizing sequence, as J tends to infinity at infinity, $(u_n)_n$ is bounded. Since X is reflexive, there exists a sub-suite $(u_{n_i})_i$ and $u \in X$ such that $u_{n_i} \rightharpoonup u$ in weak X; Since K is a closed convex, it is weakly closed and $u \in K$. Finally:

$$\alpha \leq J(u) \leq \liminf_{i \to \infty} J(u_{n_i}) = \alpha,$$

because J is weakly sequentially l.s.c. So $\alpha = J(u) \in \mathbb{R}$ and J reaches its minimum in $u \in K$. ■

Lemma 2.1 *[19] Let X be a Banach space, K a closed convex of X and J a convex l.s.c. function from K to $\mathbb{R}$. Then J is weakly l.s.c.*

Proof. Since J is convex l.s.c., for $\lambda \in \mathbb{R}$ the set $[J \leq \lambda]$ convex and closed: proposition (2.1) then shows that $[J \leq \lambda]$ is weakly closed and therefore J is weakly l.s.c. ■

Corollary 2.1 *[19] Let X be a reflexive Banach space, K a closed convex of X and $J : K \to \mathbb{R}$ a convex l.s.c. function. If K is unbounded, suppose that for any sequence $(u_n)_n$ of K such that $\|x_n\| \to \infty$, when $n \to \infty$, we have $J(x_n) \to +\infty$. Then J is inferiorly bounded and reaches its minimum on K :*

$$\exists u \in K, \quad J(u) = \inf_{v \in K} J(v) = \min_{v \in K} J(v).$$

Moreover, if J is strictly convex, u is unique.

Proof. Using Lemma (2.1) and Proposition (2.1), we conclude that J reaches its minimum at a point $u \in K$.
If J is strictly convex and $u_0 \in K$ convex and $u_0 \in K$ verifies $J(u_0) = \alpha := J(u)$, we note that if $u \neq u_0$ we would have

$$\alpha \leq J\left(\frac{u + u_0}{2}\right) < \frac{1}{2}\left(J(u) + J(u_0)\right) = \alpha,$$

which establishes the uniqueness of the minimum point. ■

Remark 2.1 *[19] When the aim is to minimize a non-convex functional J defined on a closed convex K of a Banach space, we try to decompose it in the form $J = J_1 + J_2$ where J_1 is convex l.s.c. and J_2 is in some sense controlled or dominated by J_1, and see if J_2 is weakly sequentially l.s.c.*

A useful characterization of the point $u \in K$, where a convex function J reaches its minimum, is contained in the following proposition. is contained in the following proposition.

Proposition 2.2 *[19] Let $K \subset X$ be a convex and $J : K \to \mathbb{R}$ a convex function, Gâteaux-differentiable at any point in K. Then the following two properties are equivalent:*

(i) $u \in K$ and for all $v \in K$ we have $J(u) \leq J(v)$;

(ii) $u \in K$ and for all $v \in K$ we have $\langle J'(u), v - u \rangle \geq 0$.

If for all $v, w \in K, \tau \to J'(v + \tau(w - v))$ is continuous on $[0, 1]$, then properties (i) and (ii) are equivalent to:

(iii) $u \in K$ and for all $v \in K$ we have $\langle J'(v), v - u \rangle \geq 0$.

Proof. (i) $\Longrightarrow$ (ii): As u is the minimum of J on K, for $0 < \tau < 1$ we have $0 \leq J(u + \tau(v - u)) - J(u)$, which, dividing by $\tau > 0$ and making τ tend towards zero, gives (ii).
: From Lemma (2.1), we have

$$J(v) - J(u) \geq \langle J'(u), v - u \rangle \geq 0.$$

(ii) $\Longrightarrow$ (iii): From Lemma (2.1),

$$\langle J'(v), v - u \rangle \geq \langle J'(u), v - u \rangle \geq 0.$$

(iii) $\Longrightarrow$ (ii): For $0 < \tau < 1$ we have

$$\langle J'(u + \tau(v - u)), (u + \tau(v - u)) \rangle \geq 0.$$

which, by dividing by τ, making it tend towards zero and using the continuity of J' on the half-lines (the so-called hemicontinuity of J') gives (ii). ■

Remark 2.2 *[19] Under the above assumptions, when K is a subspace of X, by choosing $v := u + tw$ with $t \in \mathbb{R}$ and $w \in K$ in characterization (ii) we see that $\langle J'(u), w \rangle = 0$ for all $w \in K$, i.e. that $J'(u) \in K^{\perp} \subset X'$. In particular, if $K = X$, we have $J'(u) = 0$; moreover, if J is strictly convex, we know that u is the only solution to the equation $J'(v) = 0$.*

Remark 2.3 *[19] When dealing with variational problems on $W^{1,p}(\Omega)$ spaces, there's a strong link between weakly l.s.c. functionals and convex functionals.*
Let $N \geq 1$ an integer and Ω a bounded open of class C^1 in $\mathbb{R}^N$. Consider a function $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ and a closed convex K of $W^{1,p}(\Omega)$, for example for $\varphi \in W^{1,p}(\Omega)$ given:

$$K := \left\{ v; v \in W^{1,p}(\Omega), \quad v = \varphi \text{ on } \partial\Omega \right\}.$$

So if $J(v) := \int_\Omega F(x, \nabla v(x))\, dx$, and if F satisfies a growth condition of the type:

$$\forall \xi \in \mathbb{R}^N, \text{ a.e on } \Omega, \quad -C_1 + C_2 |\xi|^p \leq F(x, \xi) \leq C_3 (1 + |\xi|^p),$$

we can easily see that J is minimized on $W^{1,p}(\Omega)$ and that any minimizing sequence $(u_n)_n$ of K such that $J(u_n) \to \inf_{v \in K} J(v)$, is bounded in $W^{1,p}(\Omega)$.
What's remarkable is that J is weakly sequentially l.s.c. if and only if, for almost any $x \in \Omega$, the function $\xi \to F(x, \xi)$ is convex on $\mathbb{R}^N$.

2.3 Nehari manifold

Definition 2.8 *[12] Assume that $J \in C^1(X, \mathbb{R})$ is such that $J'(0) = 0$. A necessary condition for $u \in X$ to be a critical point of J is that $\langle J'(u), u \rangle = 0$. This condition defines the Nehari manifold*

$$N := \{ u \in X : \langle J'(u), u \rangle = 0, u \neq 0 \}.$$

A critical point $u \neq 0$ of J is a ground state or a least energy critical point if

$$J(u) = \inf_N J.$$

we give a minimax characterization of $\inf_n J$ It is simpler to prove that the minimax value is a critical value. On the other hand, it is easier to obtain the qualitative properties of $\inf_N J$. Finally, we use the Nehari manifold to prove the existence of solutions.

To make this definition clearer, we provide the following example.

Example 2.1 *[1] we consider in the general case the problem:*

$$\begin{cases} -\Delta u(x) = f(x, u(x)) & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

Whose equation turns out to be the Euler equation of the functional :

$$J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega F(x, u(x)) \text{ where } F(x, u(x)) = \int_0^u f(x, s)\, ds.$$

that we consider, taking into account the edge condition, on the functional space $H = W_0^{1,2}(\Omega)$. *Obviously the functional* $J(u)$ *may not be bounded on the whole space but may be bounded on certain subsets of* H. *A good candidate for such a subset is the so-called Nehari variety :*

$$N := \{u \in H : \langle J'(u), u\rangle = 0\}.$$

The following theorem introduces a class of applications that serve nehari manifold.

Theorem 2.2 *[1] Let* $u \in H_{\setminus\{0\}}$ *and* $t > 0$. *Then* $tu \in N$ *iff* $\omega_u'(t) = 0$ *where* $\omega_u(t) = J(tu)$.

Proof. By definition

$$\omega_u(t) = J(tu).$$

and therefore:

$$\omega_u'(t) = \langle J'(u), u\rangle = \frac{1}{t}\langle J'(tu), tu\rangle.$$

If $\omega_u'(t) = 0$, then $\langle J'(tu), tu\rangle = 0$, i.e. $tu \in N$.

In other words, the points of the variety N correspond to the stationary points of the ω_u applications.

we divide N into three subsets N^+, N^- and N^0 which correspond to the local minima, local maxima and inflection points of the ω_u applications.

To do this, we look at the second derivative :

$$\begin{aligned}\omega_u'(t) &= \langle J'(tu), u\rangle \\ &= \int_\Omega |\nabla(tu)|\,|\nabla u|\,dx - \lambda\int_\Omega f(x,tu)\,u dx \\ &= t\int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f(x,tu)\,u dx,\end{aligned}$$

hence

$$\begin{aligned}\omega_u''(t) &= \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'((x,tu)\,u)\,u dx \\ &= \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'((x,tu))\,u^2 dx.\end{aligned}$$

And we can define the subsets:

$$\begin{aligned}N^+ &= \left\{u \in N : \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'((x,tu))\,u^2 dx > 0\right\}, \\ N^0 &= \left\{u \in N : \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'((x,tu))\,u^2 dx < 0\right\}, \\ N^- &= \left\{u \in N : \int_\Omega |\nabla u|^2\,dx - \lambda\int_\Omega f_u'((x,tu))\,u^2 dx = 0\right\}.\end{aligned}$$

and it is $\omega_u''(1)$ that is used for these definitions, since it is clear that if u is a local minimum for J, then $\omega_u(t)$ has a local minimum in $t = 1$. ■

Chapter 3

Study of singular double phase problem

Introduction

Why is it called double phase problem?

The double phase problems are drived from some studies on (p, q)-equations, for example Zhikov found that its hardening

properties drastically change with at certain points which is called the Lavrentiev's phenomenon, (see [24], [25]) In order to describe this phenomenon, he introduced the functional

$$I(u) = \int \left(|\nabla u(x)|^p + \xi(x) |\nabla u(x)^q|\right) dx, \ p < q$$

whose integrands change their ellipticity rate according to the point. The function $\xi(\cdot)$ was used as an aid to regulating the mixture between two different materials, with power hardening of rates p and q, respectively, ξ can be continuous, bounded or not. The functional I belongs to the class of the integral functionals with nonstandard growth condition.

The main characteristic of the functional I is the change of ellipticity on the set where the weight function is zero, that is, on the set $\{x \in \Omega, \xi(x) = 0\}$, indeed, the energy density of I exhibits ellipticity in the gradient of order q in the set $\{x \in \Omega, \xi(x) > \varepsilon\}$ for any fixed $\varepsilon > 0$, and of order p on the points x where $\xi(x)$ vanishes. So the integrand I switches between two different phases of elliptic behaviours. This is the reason why it is called double phase.

In this chapter, we study the existence of positive solutions for a class of double phase Dirichlet equations which have the combined effects of a singular term $a(x) u^{-\gamma}$. The differential operator of the equation is the sum of a p-Laplacian and of a weighted q-Laplacian $(q < p)$ with discontinuous weight $\xi(x)$. Using the Nehari method intervolution, we show that for all small values of the parameter $\lambda > 0$, the equation has at least two positive solutions.

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. In this chapter we study the following singular double phase problem

$$\begin{cases} -\Delta_p u - \operatorname{div}\left(\xi(x)\left|\nabla u\right|^{q-2}\nabla u\right) = a(x)\,u^{-\gamma} + \lambda u^{r-1}, \text{ in } \Omega \\ u\big|_{\partial\Omega} = 0,\ 1 < q < p < r < p^*,\ 0 < \gamma < 1,\ u \geq 0,\ \lambda > 0. \end{cases} \tag{P_λ}$$

Here, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p = \operatorname{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right), \text{ for all } u \in W_0^{1,p}(\Omega).$$

The weight $\xi : \Omega \to \mathbb{R}_+$ is essentially bounded. Thus the differential operator in (P_λ) is the sum of a p-Laplacian and of a weighted q-Laplacian $(q < p)$.
The integrand in the energy functional of this operator is

$$k(x,t) = \frac{1}{p}t^p + \frac{1}{q}\xi(x)\,t^q, \text{ for all } t > 0.$$

This is a Carathéodory function (that is, for all $t > 0$, $x \to k(x,t)$ is measurable, and for a.e. $x \in \Omega$, $t \to k(x,t)$ is continuous) which exhibits balanced growth in the $t > 0$ variable, such that

$$\frac{1}{p}t^p \leq k(x,t) \leq c_0\left[1 + t^p\right], \text{ for a.e } x \in \Omega, \text{ for all } t > 0.$$

3.1 Preliminaries

we denote by $W_0^{1,p}(\Omega)$ the usual Dirichlet Sobolev space, it is equipped with the norm

$$\|u\| = \|\nabla u\|_p, \text{ for all } u \in W_0^{1,p}(\Omega).$$

Let $1 \leq s \leq +\infty$, by $\|\,.\,\|_s$ we denote the norm of $L^s(\Omega, \mathbb{R}^m)$, $m \in \mathbb{N}$.
Also, by $|\,.\,|$ we denote the norm of $\mathbb{R}^N$ and by p^* the critical Sobolev exponent corresponding to p, that is

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

Hypotheses:
We assume the following hypotheses on the data of (P_λ)
• $H_1 : \xi \in L^\infty(\Omega)$ and $\xi(x) > 0$ for a.e. $x \in \Omega$.
• $H_2 : a \in L^\infty(\Omega)$ and $a(x) \geq 0$ for a.e. $a \in \Omega$, a a non zero function.
The energy functional for this problem $\varphi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ is given by:

$$\varphi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \xi(x)\left|\nabla u\right|^q dx - \frac{1}{1-\gamma}\int_\Omega a(x)\left|u\right|^{1-\gamma} dx - \frac{\lambda}{r}\|u\|_r^r, \forall u \in W_0^{1,p}(\Omega).$$

On account of the singular term $a(x)\,u^{-\gamma}$, this functional is not C^1. So, the use of variational methods based on the critical point theory presents difficulties which are compounded by the fact that the weight $\xi(\,.\,)$ is discontinuous and not bounded away from zero. For this reason our approach is based on the Nehari method.

Definition 3.1 ***(Weak solution)*** *[13]*

We call $u \in W_0^{1,p}(\Omega)$ *a weak solution of* (P_λ)*, if* $u(x) \geq 0$ *for a.e.* $x \in \Omega$, u *a non zero function, and*

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2} \nabla u \nabla h dx + \int_\Omega \xi(x) \left(|\nabla u|^{q-2} \nabla u \nabla h \right) dx \\ = &\int_\Omega a(x) u^{-\gamma} h dx + \lambda \int_\Omega u^{r-1} h dx, \forall h \in W_0^{1,p}(\Omega). \end{aligned}$$

Nehari manifold approach

For every $\lambda > 0$, we introduce the Nehari manifold for problem (P_λ) defined by

$$N_\lambda = \left\{ u \in W_0^{1,p}(\Omega) : \|\nabla u\|_p^p + \int_\Omega \xi(x) |\nabla u|^q dx = \int_\Omega a(x) |u|^{1-\gamma} dx - \lambda \|u\|_r^r \ , \ u \neq 0 \right\}.$$

Evidently, the Nehari manifold contains the weak solutions of (P_λ) and as we will see in the sequel, for small $\lambda > 0$ one has $N_\lambda \neq \emptyset$. The Nehari manifold is much smaller than $W_0^{1,p}(\Omega)$ and so $\varphi_\lambda|_{N_\lambda}$ can have nice properties which fail to be true globally.

It will be helpful to decompose N_λ into three disjoint parts:

$$\begin{aligned} N_\lambda^+ &= \left\{ u \in N_\lambda : (p+\gamma-1)\|\nabla u\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q dx - \lambda(r+\gamma-1)\|u\|_r^r > 0 \right\}, \\ N_\lambda^0 &= \left\{ u \in N_\lambda : (p+\gamma-1)\|\nabla u\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q dx = \lambda(r+\gamma-1)\|u\|_r^r \right\}, \\ N_\lambda^- &= \left\{ u \in N_\lambda : (p+\gamma-1)\|\nabla u\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q dx - \lambda(r+\gamma-1)\|u\|_r^r < 0 \right\}. \end{aligned}$$

3.2 Main result

The main result of our work is the following multiplicity theorem for problem (P_λ), (see [13])

Theorem 3.1 *[13] If hypotheses* H_1, H_2 *hold, then there exists* $\hat{\lambda}_0^* > 0$ *such that for all* $\lambda \in (0, \hat{\lambda}_0^*]$ *problem* (P_λ) *has at least two positive solutions* $u^*, v^* \in W_0^{1,p}(\Omega)$ *such that*
$\varphi_\lambda(u^*) < 0 \leq \varphi_\lambda(v^*)$.

In the following, we need to prove the next results, under the pretext of proving the previous theorem (3.1).

The proof of Theorem (3.1) (see [13])

In this section, using the Nehari method, we shall prove our result, Theorem (3.1), which asserts that for all small $\lambda > 0$, problem (P_λ) has at least two positive solutions. Our proof will be broken down in a sequence of propositions.

Proposition 3.1 *[13] If hypotheses H_1, H_2 hold and $\lambda > 0$, then $\varphi_\lambda|_{N_\lambda}$ is coercive.*

Proof. Let $u \in N_\lambda$. From the definition of N_λ, we have

$$-\frac{1}{r}\|\nabla u\|_p^p - \frac{1}{r}\int_\Omega \xi(x)|\nabla u|^q\,dx + \frac{1}{r}\int_\Omega a(x)|u|^{1-\gamma}\,dx + \frac{\lambda}{r}\|u\|_r^r = 0. \tag{3.1}$$

Using (3.1), we have

$$\varphi_\lambda(u) = \left[\frac{1}{p}-\frac{1}{r}\right]\|\nabla u\|_p^p + \left[\frac{1}{q}-\frac{1}{r}\right]\int_\Omega \xi(x)|\nabla u|^q\,dx + \left[\frac{1}{r}-\frac{1}{1-\gamma}\right]\int_\Omega a(x)|u|^{1-\gamma}\,dx.$$

Then

$$\varphi_\lambda(u) \geq c_1\|u\|^p - c_2\|u\|^{1-\gamma} \text{ for some } c_1, c_2 > 0 \text{ (since } q < p < r).$$

Here we have used the Poincaré's inequality, Theorem 13.17 on p.196 of Hewitt-Stromberg [2] and the Sobolev embedding theorem. From the last inequality and since $p > 1 > 1-\gamma$, we can conclude that $\varphi_\lambda|_{N_\lambda}$ is coercive. ■

Let $m_\lambda^+ = \inf_{N_\lambda^+}\varphi_\lambda$.

Proposition 3.2 *[13] If hypotheses H_1, H_2 hold and $N_\lambda^+ \neq \emptyset$, then $m_\lambda^+ < 0$.*

Proof. By the definition of N_λ^+, we have

$$\lambda\|u\|_r^r < \frac{p+\gamma-1}{r+\gamma-1}\|\nabla u\|_p^p + \frac{q+\gamma-1}{r+\gamma-1}\int_\Omega \xi(x)|\nabla u|^q\,dx \quad \text{for all } u \in N_\lambda^+. \tag{3.2}$$

We know that $N_\lambda^+ \subseteq N_\lambda$. So,we have

$$-\frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}\,dx = -\frac{1}{1-\gamma}\|\nabla u\|_p^p - \frac{1}{1-\gamma}\int_\Omega \xi(x)|\nabla u|^q\,dx + \frac{\lambda}{1-\gamma}\|u\|_r^r \tag{3.3}$$

for all $u \in N_\lambda^+$. Now , for all $u \in N_\lambda^+$, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \xi(x)|\nabla u|^q\,dx - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}\,dx - \frac{\lambda}{r}\|u\|_r^r \\
&= \left[\frac{1}{p}-\frac{1}{1-\gamma}\right]\|\nabla u\|_p^p + \left[\frac{1}{q}-\frac{1}{1-\gamma}\right]\int_\Omega \xi(x)|\nabla u|^q\,dx + \lambda\left[\frac{1}{1-\gamma}-\frac{1}{r}\right]\|u\|_r^r \quad \text{(see (3.3))} \\
&\leq \left[\frac{-(p+\gamma-1)}{p(1-\gamma)} + \frac{p+\gamma-1}{r+\gamma-1}\frac{r+\gamma-1}{r(1-\gamma)}\right]\|\nabla u\|_p^p \\
&\quad + \left[\frac{-(q+\gamma-1)}{q(1-\gamma)} + \frac{q+\gamma-1}{r+\gamma-1}\frac{r+\gamma-1}{r(1-\gamma)}\right]\int_\Omega \xi(x)|\nabla u|^q\,dx \qquad \text{(see (3.2))} \\
&= \frac{p+\gamma-1}{1-\gamma}\left[\frac{1}{r}-\frac{1}{p}\right]\|\nabla u\|_p^p + \frac{q+\gamma-1}{1-\gamma}\left[\frac{1}{r}-\frac{1}{q}\right]\int_\Omega \xi(x)|\nabla u|^q\,dx < 0
\end{aligned}$$

(see hypothesis H_1 and recall that $q < p < r$),
and

$$\varphi_\lambda|_{N_\lambda^+} < 0,$$

then

$$m_\lambda^+ < 0.$$

■

Proposition 3.3 *[13] If hypotheses H_1, H_2 hold, then there exists $\lambda^* > 0$ such that $N_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$.*

Proof. *Arguing by contradiction, suppose that $N_\lambda^0 \neq \emptyset$ for all $\lambda > 0$, we can find $u \in N_\lambda$ such that*

$$(p+\gamma-1)\|\nabla u\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q\,dx = \lambda(r+\gamma-1)\|u\|_r^r. \tag{3.4}$$

Since $u \in N_\lambda$, we also have

$$(r+\gamma-1)\|\nabla u\|_p^p + (r+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q\,dx - (r+\gamma-1)\int_\Omega a(x)|u|^{1-\gamma}\,dx = \lambda(r+\gamma-1)\|u\|_r^r. \tag{3.5}$$

We subtract (3.4) from (3.5) and obtain

$$(r-p)\|\nabla u\|_p^p + (r-q)\int_\Omega \xi(x)|\nabla u|^q\,dx = (r+\gamma-1)\int_\Omega a(x)|u|^{1-\gamma}\,dx,$$

then we have

$$\|u\|^p \leq c_3\|u\|, \text{ for some } c_3 > 0$$

then

$$\|u\|^{p-1} \leq c_3. \tag{3.6}$$

From (3.4) and the Sobolev embedding theorem, we have

$$\|u\|^p \leq \lambda c_4\|u\|^r, \text{ for some } c_4 > 0,$$

and finally

$$\left[\frac{1}{\lambda c_4}\right]^{\frac{1}{r-p}} \leq \|u\|.$$

If $\lambda \to 0^+$, then $\|u\| \to +\infty$ and this contradicts (3.6). This shows that there exists $\lambda^* > 0$ such that $N_\lambda^0 = \emptyset$ for all $\lambda \in (0, \lambda^*)$. ■

Now let $u \in W_0^{1,p}(\Omega)$ and consider the function $\hat{\omega}_u : (0,+\infty) \to \mathbb{R}$ defined by

$$\hat{\omega}_u(t) = t^{p-r} \|\nabla u\|_p^p - t^{-r-\gamma+1} \int_\Omega a(x) |u|^{1-\gamma} dx, \text{ for all } t > 0.$$

Since $r - p < r + \gamma - 1$, we see that there exists $\hat{t}_0 > 0$ such that

$$\hat{\omega}_u(\hat{t}_0) = \max_{t>0} \hat{\omega}_u.$$

Then we have

$$\hat{\omega}_u'(\hat{t}_0) = 0$$

then

$$(p-r)\, \hat{t}_0^{p-r-1} \|\nabla u\|_p^p + (r+\gamma-1)\, \hat{t}_0^{-r-\gamma} \int_\Omega a(x) |u|^{1-\gamma} dx = 0,$$

such that

$$\hat{t}_0 = \left[\frac{(r+\gamma-1) \int_\Omega a(x) |u|^{1-\gamma} dx}{(r-p) \|\nabla u\|_p^p} \right]^{\frac{1}{p+\gamma-1}}.$$

Therefore we have

$$\begin{aligned}
\hat{\omega}_u(\hat{t}_0) &= \frac{\left[(r-p)\|\nabla u\|_p^p\right]^{\frac{r-p}{p+\gamma-1}}}{\left[(r+\gamma-1)\int_\Omega a(x)|u|^{1-\gamma}dx\right]^{\frac{r-p}{p+\gamma-1}}} \|\nabla u\|_p^p \\
&\quad - \frac{\left[(r-p)\|\nabla u\|_p^p\right]^{\frac{r+\gamma-1}{p+\gamma-1}}}{\left[(r+\gamma-1)\int_\Omega a(x)|u|^{1-\gamma}dx\right]^{\frac{r+\gamma-1}{p+\gamma-1}}} \int_\Omega a(x)|u|^{1-\gamma}dx \\
&= \frac{(r-p)^{\frac{r-p}{p+\gamma-1}} \|\nabla u\|_p^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{\left[(r+\gamma-1)\int_\Omega a(x)|u|^{1-\gamma}dx\right]^{\frac{r-p}{p+\gamma-1}}} \\
&\quad - \frac{(r-p)^{\frac{r+\gamma-1}{p+\gamma-1}} \|\nabla u\|_p^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{(r+\gamma-1)^{\frac{r+\gamma-1}{p+\gamma-1}} \left[\int_\Omega a(x)|u|^{1-\gamma}dx\right]^{\frac{r-p}{p+\gamma-1}}} \\
&= \frac{p+\gamma-1}{r-p} \left[\frac{r-p}{r+\gamma-1}\right]^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{\|\nabla u\|_p^{\frac{p(r+\gamma-1)}{p+\gamma-1}}}{\left[\int_\Omega a(x)|u|^{1-\gamma}dx\right]^{\frac{r-p}{p+\gamma-1}}}. \qquad (3.7)
\end{aligned}$$

If S denotes the best Sobolev constant,we have

$$S \|u\|_{p^*}^p \leq \|\nabla u\|_p^p. \qquad (3.8)$$

Also,we have

$$\int_{\Omega} a(x)|u|^{1-\gamma}dx \leq c_5 \|u\|_{p^*}^{1-\gamma}, \text{ for some } c_5 > 0. \tag{3.9}$$

Then we have

$$\begin{aligned} & \hat{\omega}_u(\hat{t}_0) - \lambda\|u\|_r^r \\ \geq & \frac{p+\gamma-1}{r-p}\left[\frac{r-p}{r+\gamma-1}\right]^{\frac{r+\gamma-1}{p+\gamma-1}} \frac{S^{\frac{p(r+\gamma-1)}{p+\gamma-1}}\left(\|u\|_{p^*}^p\right)^{\frac{r+\gamma-1}{p+\gamma-1}}}{\left(c_5\|u\|_{p^*}^{1-\gamma}\right)^{\frac{r-p}{p+\gamma-1}}} - \lambda c_6\|u\|_{p^*}^r, \end{aligned}$$

for some $c_6 > 0$ (see (3.7), (3.8), (3.9) and recall $r < p^*$), we get

$$\hat{\omega}_u(\hat{t}_0) - \lambda\|u\|_r^r \geq [c_7 - \lambda c_6]\|u\|_{p^*}^r, \text{ for some } c_6 > 0.$$

So, there exists $\hat{\lambda}^* \in (0, \lambda^*]$ independent of u, such that

$$\hat{\omega}_u(\hat{t}_0) - \lambda\|u\|_r^r > 0, \text{ for all } \lambda \in \left(0, \hat{\lambda}^*\right). \tag{3.10}$$

Proposition 3.4 *[13] If hypotheses H_1, H_2 hold , then there exists $\hat{\lambda}^* \in (0, \lambda^*]$ such that for every $\lambda \in (0, \lambda^*)$ we can find $u^* \in N_\lambda^+$ such that $\varphi_\lambda(u^*) = m_\lambda^+ < 0$ and $u^*(x) \geq 0$ for a.e. $x \in \Omega$.*

Proof. For $u \in W_0^{1,p}(\Omega)$ we consider the function $\omega_u : (0, +\infty) \to \mathbb{R}$ defined by

$$\omega_u(t) = t^{p-r}\|\nabla u\|_p^p + t^{q-r}\int_{\Omega}\xi(x)|\nabla u|^q dx - t^{-r-\gamma+1}\int_{\Omega} a(x)|u|^{1-\gamma}dx, \text{ for all } t > 0.$$

Since $r - p < r - q < r + \gamma - 1$, we can find $t_0 > 0$ such that

$$\omega_u(t_0) = \max_{t>0}\omega_u.$$

Evidently, we have $\omega_u \geq \hat{\omega}_u$ and so from (3.10) we see that we can find $\hat{\lambda}^* \in (0, \lambda^*]$ such that

$$\omega_u(t_0) - \lambda\|u\|_r^r > 0 \text{ for all } \lambda \in \left(0, \hat{\lambda}^*\right).$$

Consequently, we can find $t_1 < t_0 < t_2$ such that

$$\omega_u(t_1) = \lambda\|u\|_r^r = \omega_u(t_2) \quad \text{and} \quad \omega_u'(t_2) < 0 < \omega_u'(t_1). \tag{3.11}$$

Now we see that

$$t_1 u \in N_\lambda^+ \text{ and } t_2 u \in N_\lambda^-.$$

Therfore for all $\lambda \in \left(0, \hat{\lambda}^*\right)$, we have $N_\lambda^\pm \neq \emptyset$ while $N_\lambda^0 = \emptyset$ (see proposition 3.3).

Now consider a minimizing sequence $\{u_n\}_{n\geq 1} \subseteq N_\lambda^+$, that is,

$$\varphi_\lambda(u_n) \downarrow m_\lambda^+ \text{ as } n \to +\infty.$$

On account of Proposition (3.1), we have that

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (recall that } N_\lambda^+ \subseteq N_\lambda).$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$u_n \rightharpoonup u^* \text{ in } W_0^{1,p}(\Omega) \text{ and } u_n \to u^* \text{ in } L^r(\Omega).$$

We consider the function $\omega_{u^*}(.)$ and let $t_1 < t_0$ be as in (3.11) (with $u = u^*$).
From the first part of the proof we know that $t_1 u^* \in N_\lambda^+$.
We claim that $u_n \to u^*$ in $W_0^{1,p}(\Omega)$ as $n \to +\infty$. Arguing by contradiction, suppose that $u_n \not\to u^*$ in $W_0^{1,p}(\Omega)$. Then we will have

$$\liminf_{n\to+\infty} \|\nabla u\|_p^p > \|\nabla u\|_p^p. \tag{3.12}$$

For $u \in W_0^{1,p}(\Omega)$ we consider the fibering function $\mu_u(t) : (0,+\infty) \to \mathbb{R}$ defined by

$$\mu_u(t) = \varphi_\lambda(tu) \quad \text{for all } t > 0.$$

Using (3.12) (see also [5], [15]), we have

$$\begin{aligned}
& \liminf_{n\to+\infty} \mu'_{u_n}(t_1) \\
= & \liminf_{n\to+\infty} \left[t_1^{p-1} \|\nabla u_n\|_p^p + t_1^{q-1} \int_\Omega \xi(x) |\nabla u_n|^q \, dx - t_1^{-\gamma} \int_\Omega a(x) |u_n|^{1-\gamma} \, dx - \lambda t_1^{r-1} \|u_n\|_r^r \right] \\
> & \; t_1^{p-1} \|\nabla u^*\|_p^p + t_1^{q-1} \int_\Omega \xi(x) |\nabla u^*|^q \, dx - t_1^{-\gamma} \int_\Omega a(x) |u^*|^{1-\gamma} \, dx - \lambda t_1^{r-1} \|u^*\|_r^r \qquad \text{(see(3.12))} \\
= & \; \mu'_{u^*}(t_1) = 0 \quad \text{(see(3.11))}.
\end{aligned} \tag{3.13}$$

Then it follows from (3.13) that we can find $n_0 \in \mathbb{N}$ such that

$$\mu'_{u^*}(t_1) > 0 \text{ for all } n \geq n_0. \tag{3.14}$$

Since $u_n \in N_\lambda^+ \subseteq N_\lambda$ and

$$\mu'_{u_n}(t) = t^{r-1} \left[\omega_{u_n}(t) - \lambda \|u_n\|_r^r \right],$$

we have

$$\mu'_{u_n}(t) < 0 \text{ for all } t \in (0,1) \text{ and } \mu'_{u_n}(1) = 0,$$

then

$$t_1 > 0 \text{ (see (3.14))}.$$

The function $\mu_{u^*}(.)$ is decreasing on $(0,t_1)$. Hence we have

$$\varphi_\lambda(t_1u^*) \leq \varphi_\lambda(u^*) < m_\lambda^+ \text{ (see (3.12)).} \tag{3.15}$$

However, $t_1u^* \in N_\lambda^+$. Hence

$$m_\lambda^+ \leq \varphi_\lambda(tu^*) < m_\lambda^+ \quad \text{see(3.15)),}$$

a contradiction. This proves that our initial claim holds and we have

$$u_n \to u^* \text{ in } W_0^{1,p}(\Omega), \tag{3.16}$$

then

$$\varphi_\lambda(u_n) \to \varphi_\lambda(u^*),$$

and finally

$$\varphi_\lambda(u^*) = m_\lambda^+.$$

Since $u_n \in N_\lambda^+$ for all $n \in \mathbb{N}$, we have

$$(p+\gamma-1)\|\nabla u_n\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u_n|^q dx > \lambda(r+\gamma-1)\|u_n\|_r^r,$$

then

$$(p+\gamma-1)\|\nabla u^*\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u^*|^q dx \quad > \quad \lambda(r+\gamma-1)\|u^*\|_r^r \tag{3.17}$$

(see (3.16))

However, $\lambda \in \left(0,\hat{\lambda}^*\right)$ and $\hat{\lambda}^* \leq \lambda^*$. So, from Proposition (3.3), we know that $N_\lambda^0 = \emptyset$. Therefore in (3.17) equality cannot hold and so we can conclude that $u^* \in N_\lambda^+$.
Clearly, we can replace u^* by $|u^*|$ and so we can say that $u^*(x) \geq 0$ for a.e. $x \in \Omega$. ■
We will need the following lemma which was inspired by Lemma 3 of SunWu-Long [29].
In what follows. Let

$$B_\varepsilon(0) = \left\{\omega \in W_0^{1,p}(\Omega) : \|\omega\| < \varepsilon\right\}, \quad \varepsilon > 0.$$

Lemma 3.1 *[13] If hypotheses H_1, H_2 hold and $u \in N_\lambda^\pm$, then there exist $\varepsilon > 0$ and a continuous function $\beta : B_\varepsilon(0) \to \mathbb{R}_+$ such that $\beta(0) = 1$, $\beta(\omega)(u+\omega) \in N_\lambda^+$ for all $\omega \in B_\varepsilon(0)$.*

Proof. We shall only give the proof for $u \in N_\lambda^+$, the proof for $u \in N_\lambda^-$ is similar.
Consider the function $E : W_0^{1,p}(\Omega) \times \mathbb{R} \to \mathbb{R}$ defined by

$$E(\omega,t) = t^{p+\lambda-1}\|\nabla(u+\omega)\|_p^p + t^{q+\lambda-1}\int_\Omega \xi(x)|\nabla(u+\omega)|^q dx$$
$$-\int_\Omega a(x)|u+\omega|^{1-\gamma} dx - \lambda t^{r+\lambda-1}\|u+\omega\|_r^r, \text{ for all } \omega \in W_0^{1,p}(\Omega).$$

We have

$$\begin{aligned}
&E(0,1) = 0 \quad (\text{since } u \in N_\lambda^+ \subseteq N_\lambda),\\
&E_t'(0,1) = (p+\lambda-1)\|\nabla u\|_p^p + (q+\lambda-1)\int_\Omega \xi(x)|\nabla u|^q\,dx\\
&-\lambda(r+\lambda-1)\|u\|_r^r \ > 0 \ \ (\text{since } u \in N_\lambda^+).
\end{aligned}$$

Invoking the implicit function theorem (see Gasiński-Papageorgiou [8], p. 481), we can find $\varepsilon > 0$ and continuous

$$\beta : B_\varepsilon(0) \to \mathbb{R}_+ = (0,+\infty).$$

such that

$$\beta(0) = 1,\ \beta(\omega)(u+\omega) \in N_\lambda \text{ for all } \omega \in B_\varepsilon(0).$$

Taking $\varepsilon > 0$ even smaller if necessary, we can also have

$$\beta(\omega)(u+\omega) \in N_\lambda^+ \text{ for all } \omega \in B_\varepsilon(0).$$

■

Proposition 3.5 *[13] If hypotheses H_1, H_2 hold, $\lambda \in \left(0, \hat{\lambda}^*\right]$, and $h \in W_0^{1,p}(\Omega)$, then we can find $b > 0$ such that $\varphi_\lambda(u^*) \le \varphi_\lambda(u^*+th)$ for all $t \in [0,b]$.*

Proof. We introduce the function $\eta_h : [0,+\infty) \to \mathbb{R}$ defined by

$$\begin{aligned}
\eta_h(t) \ &= \ (p-1)\|\nabla u^* + t\nabla h\|_p^p + (q-1)\int_\Omega \xi(x)|\nabla u^* + t\nabla h|^q\,dx\\
&\quad +\gamma\int_\Omega a(x)|u^*+th|^{1-\gamma}\,dx - \lambda(r-1)\|u^*+th\|_r^r. \qquad (3.18)
\end{aligned}$$

Since $u^* \in N_\lambda^+ \subseteq N_\lambda$ (see proposition (3.4)), we have

$$\gamma\int_\Omega a(x)|u^*|^{1-\gamma}\,dx = \gamma\|\nabla u^*\|_p^p + \gamma\int_\Omega \xi(x)|u^*|^q\,dx - \lambda\gamma\|u^*\|_r^r, \qquad (3.19)$$

$$(p+\gamma-1)\|\nabla u^*\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u^*|^q\,dx - \lambda(r+\gamma-1)\|u^*\|_r^r \ > 0. \qquad (3.20)$$

It follows from (3.18), (3.19), (3.20) that $\eta_h(0) > 0$.
The continuity of $\eta_h(.)$ implies that we can find $b_0 > 0$ such that

$$\eta_h(t) > 0 \text{ for all } t \in [0,b_0].$$

On account of Lemma (3.1), we can find $\vartheta(t) > 0$, $t \in [0,b_0]$ such that

$$\vartheta(t)(u^*+th) \in N_\lambda^+,\ \vartheta(t) \to 1 \text{ as } t \to 0^+. \qquad (3.21)$$

Therefore we have

$$m_\lambda^+ = \varphi_\lambda(u^*) \leq \varphi_\lambda(\vartheta(t)(u^* + th)) \text{ for all } t \in [0, b_0].$$

then

$$m_\lambda^+ \leq \varphi_\lambda(u^*) \leq \varphi_\lambda(u^* + th) \text{ for all } t \in [0, b] \text{ with } 0 < b < b_0 \text{ (see (3.21))}.$$

■

The next proposition shows that N_λ^+ is in fact, a natural constraint for the energy functional φ_λ (see Papageorgiou-Rădulescu-Repovš [16], p. 425).

Proposition 3.6 *[13] If hypotheses H_1, H_2 hold, and $\lambda \in \left(0, \hat{\lambda}^*\right)$, then u^* is a weak solution of (P_λ).*

Proof. Let $h \in W_0^{1,p}(\Omega)$ and let $b > 0$ as postulated by Proposition (3.5). For $0 \leq t \leq b$ we have

$$0 \leq \varphi_\lambda(u^* + th) - \varphi_\lambda(u^*) \quad \text{(see Proposition (3.5))},$$

then

$$\begin{aligned} & \frac{1}{1-\gamma}\int_\Omega a(x)\left[|u^* + th|^{1-\gamma} - |u^*|^{1-\gamma}\right]dx \\ \leq\ & \frac{1}{p}\left[\|\nabla u^* + t\nabla h\|_p^p - \|\nabla u^*\|_p^p\right] + \frac{1}{q}\int_\Omega \xi(x)\left[|\nabla u^* + t\nabla h|^q - |\nabla u^*|^q\right]dx \\ & -\frac{\lambda}{r}\left[\|u^* + th\|_r^r - \|u^*\|_r^r\right]. \end{aligned}$$

We divide by $t > 0$ and then let $t \to 0^+$. We obtain

$$\begin{aligned} & \int_\Omega a(x)(u^*)^{-\gamma} h dx \\ \leq\ & \int_\Omega |\nabla u^*|^{p-2}(\nabla u^*, \nabla h)_{\mathbb{R}^N} dx + \int_\Omega \xi(x)|\nabla u^*|^{q-2}(\nabla u^*, \nabla h)_{\mathbb{R}^N} dx \\ & -\lambda\int_\Omega (u^*)^{r-1} h dx. \end{aligned}$$

Since $h \in W_0^{1,p}(\Omega)$ is arbitrary, equality must hold and so u^* is a weak solution of (P_λ), $\lambda \in \left(0, \hat{\lambda}^*\right)$. ■

This proposition leads to the first positive solution of (P_λ), $\lambda \in \left(0, \hat{\lambda}^*\right)$.

Proposition 3.7 *[13] If hypotheses H_1, H_2 hold and $\lambda \in \left(0, \hat{\lambda}^*\right)$, then problem (P_λ) admits a positive solution $u^* \in W_0^{1,p}(\Omega)$ such that $\varphi_\lambda(u^*) < 0$ and $u^*(x) \geq 0$ for a.e. $x \in \Omega$, u^* a non zero function .*

Next, using the set N_λ^- , we will generate a second positive solution for problem (P_λ) when $\lambda > 0$ is small.

Proposition 3.8 *[13] If hypotheses H_1, H_2 hold, then we can find $\hat{\lambda}_0^* \in \left(0, \hat{\lambda}^*\right]$ such that $\varphi_\lambda|_{N_\lambda^-} \geq 0$ for all $\lambda \in \left(0, \hat{\lambda}^*\right]$.*

Proof. Let $u \in N_\lambda^-$. We have

$$(p+\gamma-1)\|\nabla u\|_p^p + (q+\gamma-1)\int_\Omega \xi(x)|\nabla u|^q\,dx < \lambda(r+\gamma-1)\|u\|_r^r,$$

and we have

$$(p+\gamma-1)\,c_8\|u\|_r^p < \lambda(r+\gamma-1)\|u\|_r^r, \text{ for some } c_8 > 0$$

(here we have used the fact that $W_0^{1,p}(\Omega) \hookrightarrow L^r(\Omega)$),
then

$$\left[\frac{(p+\gamma-1)\,c_8}{\lambda(r+\gamma-1)}\right]^{\frac{1}{r-p}} \leq \|u\|_r. \tag{3.22}$$

Arguing by contradiction, suppose that the proposition is not true. Then we can find $u \in N_\lambda^-$ such that

$$\varphi_\lambda(u) < 0,$$

then

$$\frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\int_\Omega \xi(x)|\nabla u|^q\,dx - \frac{1}{1-\gamma}\int_\Omega a(x)|u|^{1-\gamma}\,dx - \frac{\lambda}{r}\|u\|_r^r < 0. \tag{3.23}$$

We know that $u \in N_\lambda^-$. Therefore

$$\|\nabla u\|_p^p = \int_\Omega a(x)|u|^{1-\gamma}\,dx + \lambda\|u\|_r^r - \int_\Omega \xi(x)|\nabla u|^q\,dx. \tag{3.24}$$

We use (3.24) in (3.23) and obtain

$$\begin{aligned}&\left[\tfrac{1}{p}-\tfrac{1}{1-\gamma}\right]\int_\Omega a(x)|u|^{1-\gamma}\,dx + \left[\tfrac{1}{q}-\tfrac{1}{p}\right]\int_\Omega \xi(x)|\nabla u|^q\,dx\\ &+\lambda\left[\tfrac{1}{q}-\tfrac{1}{r}\right]\|u\|_r^r < 0,\end{aligned}$$

then

$$\lambda\left[\frac{1}{q}-\frac{1}{r}\right]\|u\|_r^r < \frac{(p+\gamma-1)}{p(1-\gamma)}c_9\|u\|_r^{1-\gamma} \text{ for some } c_9 > 0 \text{ (recall that } q < p < r),$$

then

$$\|u\|_r^{r+\gamma-1} \leq \frac{(p+\gamma-1)\,r c_9}{\lambda(r-p)(1-\gamma)},$$

then we have

$$\|u\|_r \leq c_{10}\left(\frac{1}{\lambda}\right)^{\frac{1}{r+\gamma-1}} \quad \text{for some } c_{10}>0. \tag{3.25}$$

We use (3.25) in (3.22) and obtain

$$c_{11}\left(\frac{1}{\lambda}\right)^{\frac{1}{r-p}} \leq c_{10}\left(\frac{1}{\lambda}\right)^{\frac{1}{r+\gamma-1}} \quad \text{with } c_{11}=\left[\frac{(p+\gamma-1)\,c_8}{r+\gamma-1}\right]^{\frac{1}{r-p}}>0,$$

and, therfore

$$c_{12} \leq \lambda^{\frac{1}{r-p}-\frac{1}{r+\gamma-1}} \quad \text{with } c_{12}=\frac{c_{11}}{c_{10}}>0,$$

then

$$c_{12} \leq \lambda^{\frac{p+\gamma-1}{(r-p)(r+\gamma-1)}} \to 0 \ \text{ as } \lambda \to 0^+ \ \ (\text{since } 1<p<r,\ \gamma \in (0,1)),$$

a contradiction. Therefore we conclude that we can find $\hat{\lambda}_0^* \in \left(0, \hat{\lambda}^*\right]$ such that $\varphi_\lambda|_{N_\lambda^-} \geq 0$ for all $\lambda \in \left(0, \hat{\lambda}^*\right]$. ■

Proposition 3.9 *[13] If hypotheses H_1, H_2 hold and $\lambda \in \left(0, \hat{\lambda}^*\right]$, then there exists $v^* \in N_\lambda^-$, $v^* \geq 0$ such that*

$$m_\lambda^- = \inf_{N_\lambda^-} \varphi_\lambda = \varphi_\lambda\left(v^*\right).$$

Proof. The reasoning is similar to that in the proof of Proposition (3.4). If $\{v_n\}_{n\geq 1} \subseteq N_\lambda^-$ is a minimizing sequence, then on account of Proposition (3.1), we have that $\{v_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$v_n \rightharpoonup v^* \text{ in } W_0^{1,p}(\Omega) \text{ and } v_n \to v^* \text{ in } L^r(\Omega) \text{ as } n \to +\infty.$$

From the proof of Proposition (3.4) we can find $t_0 < t_2$ such that

$$\omega'_{v^*}(t_2) < 0 \text{ and } \omega_{v^*}(t_2) = \lambda\|v^*\|_r^r \ \text{(see (3.11))} \tag{3.26}$$

($t_0 > 0$ being the maximizer of ω_{v^*}). We argue as in the proof of Proposition (3.4) and using (3.26), we obtain that $v^* \in N_\lambda^-$, $v^* \geq 0$, $m_\lambda^- = \varphi_\lambda(v^*)$. ■

Using Lemma (3.1) and reasoning as in the proofs of Propositions (3.5) and (3.6), we can also prove the following proposition.

Proposition 3.10 *[13] If hypotheses H_1, H_2 hold and $\lambda \in \left(0, \hat{\lambda}^*\right)$, then v^* is a weak solution of (P_λ).*

Finally, from propositions (3.1) to (3.10) and lemma (3.1) we complete the proof of our main result (theorem (3.1)), and we conclude that there exists $\hat{\lambda}_0^* > 0$ such that for all $\lambda \in (0, \hat{\lambda}_0^*]$ problem (P_λ) has at least two positive solutions $u^*, v^* \in W_0^{1,p}(\Omega)$ such that $u^* \in N_\lambda^+$ and $v^* \in N_\lambda^-$ with $\varphi_\lambda(u^*) < 0 \leq \varphi_\lambda(v^*)$.

Conclusion

In this memoire, we studied the existence of positive solutions for singular double phase problem involving a singular term and a superlinear term, using Nehari manifold and critical point theory. The study can be generalized to double phase equations with unbalanced growth, which require advanced analytical tools such as Musielak-Orlicz-Sobolev spaces. The research can be extended by studying the behavior of solutions at large values of the coefficient, or analyzing their regularity and multiplicity. Other possible directions include: Numerical generalization, physical applications, and sensitivity analysis.

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