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ECHAHID CHEIKH LARBI TEBESSI UNIVERSITY

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Course

Physiqucs1 (Mechanics of material point)

Matter : Physics1 (Mechanics)
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Preface

The concepts offered in this course is an introductory in point mechanics and material point systems. Specifically addressed to first-year science and technology students. Our concern while writing this book has been to refer to the knowledge that students have acquired in high school classes in order to ensure the most continuous transition as possible.

Certainly the main difficulty we encountered during this course was mathematical. Mechanics is a science that requires precision and concepts gained during high school learning are dealt with here in a more formal and rigorous manner. Therefore, we focused a lot on the mathematical tools that we believe are necessary for a good understanding of the physics course.

Chapter I: Mathematical notions

1. Dimensional analysis

Or dimensional equations. We call the equation with dimensions for the amount G the equation that takes the form

$$[G] = M^{\alpha} L^{\beta} T^{\gamma}$$

Where M, L, T stand for mass, length and temps. The aim of this equation is to reach the phrase unite in the international sentence of units (SI).

Three basic elements enter the description of any physical quantity: its spatial properties, how much matter is involved, and its temporal elements. All descriptions of matter, relationships, and events involve combinations of these three basic characteristics, and all measurements can be reduced ultimately to the measurement of three fundamental physical quantities: length, time, and mass. Any physical quantity. no matter how complex, can be expressed as an algebraic combination of these three basic quantities. velocity, for example, is a length per time.

Length, time, and mass therefore have significance far beyond that of providing the basis of a system of units. They specify the three primary dimensions. We use the abbreviations $[L]$, $[T]$, and $[M]$ for the se primary dimensions (the square brackets here and below indicate that we are dealing with dimensions). The dimension of a physical quantity is the algebraic combination of $[L]$, $[T]$, and $[M]$ from which the quantity is formed. The speed v provides an example. The dimension of v is

$$[v] = [L/T]$$

and

$$[v] = [LT^{-1}]$$

Study of the dimensions of an equation-dimensional analysis-is an important exercise with several different roles in science and engineering. Any equation that relates physical quantities must have consistent dimensions; that is, the dimensions on one side of an equation must be the same as those on the other side. This provides a valuable check for any calculation.

Example 01

we can derive a the this relation of the acceleration. The relation reads

$$a = \frac{v}{t}$$

So,

$$[a] = \frac{[v]}{[t]}$$

$$[a] = \frac{LT^{-1}}{T} = LT^{-2}$$

Example 02

The volume of cylinder

$$V = \pi\, r^2 h$$

We have

$$[V] = [\pi\,][\,r^2][h]$$

Where $[\pi\,] = 1$ we have

$$[V] = [\,r^2][h] = [\,L]^2[L] = [\,L]^3$$

Example 03

The student uses the following relationship to determine the distance of a car from him as a function of time.

$$s = \frac{1}{2}\, v_0 t^2 + vt$$

s is the distance of the car and v is its speed.

$$[s] = [\,v_0][\,t]^2 + [\,v][\,t]$$

$$[L] = \frac{[L]}{[T]}[\,T]^2 + \frac{[L]}{[T]}[\,T] = [L][\,T] + [L]$$

We note that the dimensions of the right side differ from the dimensions of the left side, so the equation is wrong.

The following table provides examples of other non-core units with a dimensional analysis of them.

Quantity	symbol	Unit	Dimension
Velocity	v	ms^{-1}	LT^{-1}
Acceleration	a	ms^{-2}	LT^{-2}
Volume	V	m^3	L^3
Frequency	f	Hz	T^{-1}
Force	F	N	MLT^{-2}
Volumetric mass	r	kgm^{-3}	ML^{-3}
Work	W	J	ML^2T^{-2}
Power	P	w	ML^2T^{-3}
angular momentum	L	Nm	ML^2T^{-2}
Pressure	p	Pa	$ML^{-1}T^{-2}$
Surface tension	A	kgs^{-2}	MT^{-2}
Heat capacity	C	JK^{-1}	$ML^2T^{-2}Q^{-2}$

Chapter I: Mathematical notions

2. Vector Analysis

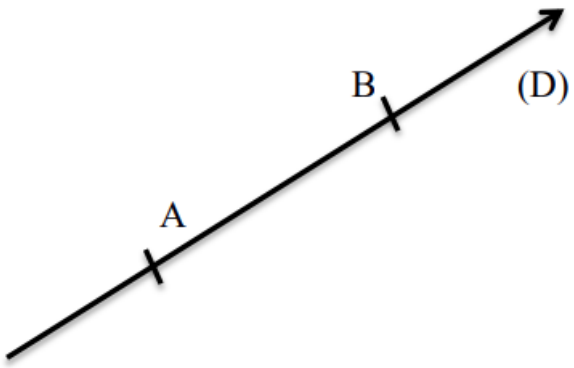
Physics is the science that is concerned with the study of natural phenomena and tries to find a scientific explanation for them by deducing basic laws and physical quantities that govern these phenomena. Physical quantities are classified into scalar and vector quantities.

The scalar quantity is an ordinary algebraic quantity. A scalar quantity has no direction associated with it. Other examples of scalar quantities include temperature. the energy of a moving body, and electric charge.

Some scalar quantities. such as mass, are always positive, whereas others, such as electric charge, can be positive or negative.

There are physical quantities, however, that cannot be described by scalars; a direction is needed for a complete description of these quantities. Specifying the velocity of a ball requires specifying not only the speed (how fast it is going) but also the direction in which it is traveling. Quantities that must be described by both a magnitude (always positive) and a direction are called vectors. Vectors describe displacement, velocity $\vec{v}$, acceleration $\vec{a}$, force $\vec{F}$, electric fields $\vec{E}$, and numerous other quantities. They play an important role in physics.

A vector may be represented geometrically by an oriented distance, i.e., by a distance associated with a direction, such that holds; for example: Let A be the initial point and B the endpoint of the vector $\vec{V}$ (compare figure).



Chapter I: Mathematical notions

Vector $\vec{V}$ pointing from A to B. where

$$\vec{V} = \overrightarrow{AB} = -\overrightarrow{BA}$$

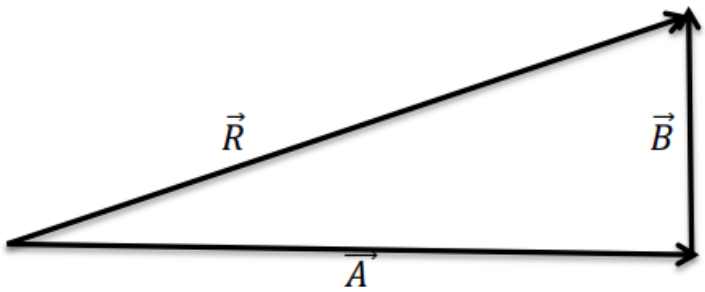
The magnitude of the vector $\vec{V}$ is then represented by the length of the distance AB. The magnitude of a vector $\vec{V}$ is written as

$$\|\vec{V}\| = \|\overrightarrow{AB}\|$$

2.1 Addition of Vectors

If two vectors $\vec{A}$ and $\vec{B}$ are added, the initial point of the one vector is brought by a parallel shift to coincide with the endpoint of the other one. The sum $\vec{A} + \vec{B}$, also called the resultant $\vec{R}$, then corresponds to the distance from the initial point of the first vector to the endpoint of the second one. This sum may also be found as the diagonal of the parallelogram formed by $\vec{A}$ and $\vec{B}$.

$$\vec{R} = \vec{A} + \vec{B}$$



with the two properties of commutation law and association law

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}$$

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$$

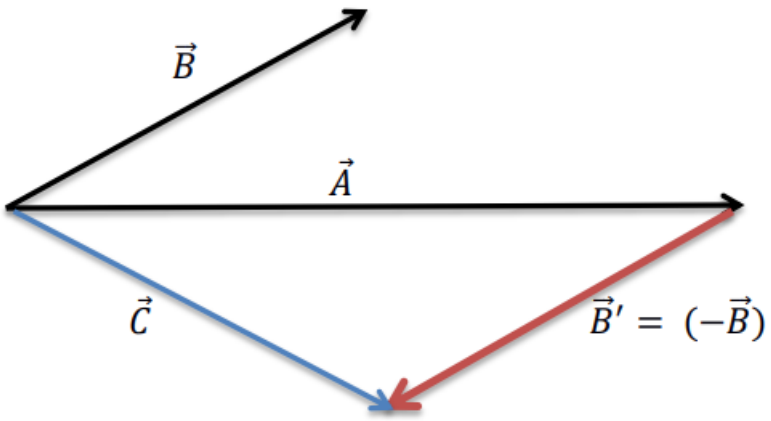
2.2 Subtraction of Vectors

The difference of two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\vec{C} = \vec{A} - \vec{B}$$

Subtracting the vector $\vec{B}$ from the vector $\vec{A}$ is the same as adding the vectors $\vec{A}$ and $(-\vec{B})$.

$$\vec{C} = \vec{A} - \vec{B}$$
$$= \vec{A} + (-\vec{B})$$
$$= \vec{A} + \vec{B}'$$



We draw a vector $\vec{B}'$ that has the same intensity as vector $\vec{B}$ and opposite in direction.

2.3 Multiplication of a vector by a scalar

The product $\alpha\vec{V}$ of a vector $\vec{V}$ by a scalar α, where α is a real number, is understood as the vector having the same orientation as $\vec{V}$ and the magnitude

$$\|\alpha\vec{V}\| = |\alpha|\|\vec{V}\|$$

Properties

$\vec{V}$ and $\vec{C}$ are vectors, α and β are real numbers. We have the following properties

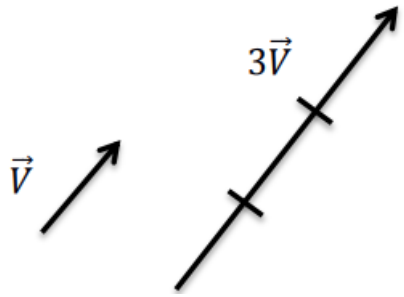
$$\alpha(\beta\vec{V}) = \beta(\alpha\vec{V}) = \alpha\beta\vec{V}$$

$$(\alpha + \beta)\vec{V} = \alpha\vec{V} + \beta\vec{V}$$

$$\alpha(\vec{V} + \vec{C}) = \alpha\vec{V} + \alpha\vec{C}$$

Example

The multiplication of a vector $\vec{V}$ by the number 3



2.4 The analytical expression for a vector

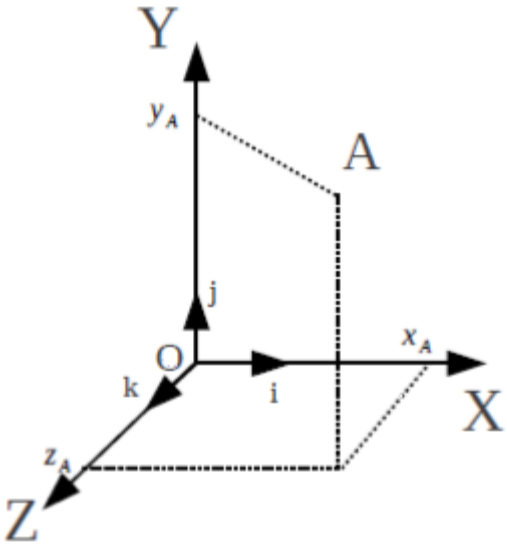
The vector $\vec{A}$ can be represented in three-dimensional space by three components A_x, A_y, A_z on the perpendicular axes (OX), (OY), (OZ) with unit $\vec{\imath}, \vec{\jmath}, \vec{k}$.

We know the vector $\vec{A}$ as follows

$$\vec{A} = A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$

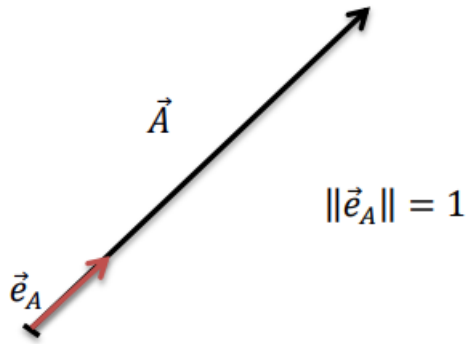
The magnitude of the vector $\vec{A}$ at these coordinates

$$\|\vec{A}\| = \sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}$$



2.5 Unit vector

Unit vector $\vec{e}_A$ of the vector $\vec{A}$ is understood as vector of magnitude 1. If $\|\vec{A}\| \neq 0$, then



where

$$\vec{e}_A = \frac{\vec{A}}{\|\vec{A}\|}$$

2.6 Addition and subtraction using analytical expression

Let the vectors $\vec{A}$ and $\vec{B}$ be

$$\vec{A} = A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$

$$\vec{B} = B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k}$$

We have

$$\vec{A} + \vec{B} = (A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}) + (B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k})$$

$$\vec{A} + \vec{B} = (A_x + B_x)\,\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k}$$

And

$$\vec{A} - \vec{B} = (A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}) - (B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k})$$

$$\vec{A} - \vec{B} = (A_x - B_x)\,\vec{\imath} + (A_y - B_y)\vec{\jmath} + (A_z - B_z)\vec{k}$$

2.7 The Scalar Product

The scalar product $\vec{A}.\vec{B}$ of two vectors is understood as

$$\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\|cos(\theta)$$

where θ is the angle enclosed by $\vec{A}$ and $\vec{B}$. The scalar product $\vec{A}.\vec{B}$ is a real number.

Properties

The scalar product is commutative $\vec{A}.\vec{B} = \vec{B}.\vec{A}$

The scalar product is distributive $\vec{A}.(\vec{B} + \vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$

If $\vec{A}$ and $\vec{B}$ are perpendicular $\vec{A}.\vec{B} = 0$

If $\vec{A}$ and $\vec{B}$ are parallel $\vec{A}.\vec{B} = \|\vec{A}\|.\|\vec{B}\|$

$\vec{A}.\vec{A} = \|\vec{A}\|^2$

$\vec{e}_A.\vec{e}_B = cos(\theta)$

The scalar product using the analytical expression

we have two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A}.\vec{B} = (A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}).(B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k})$$

$$= A_x\,B_x\vec{\imath}.\vec{\imath} + A_xB_y\vec{\imath}.\vec{\jmath} + A_xB_z\,\vec{\imath}.\vec{k} + A_y\,B_x\vec{\jmath}.\vec{\imath} + A_yB_y\vec{\jmath}.\vec{\jmath} + A_yB_z\,\vec{\jmath}.\vec{k}$$

$$+A_z\,B_x\vec{k}.\vec{\imath} + A_zB_y\vec{k}.\vec{\jmath} + A_zB_z\vec{k}.\vec{k}$$

To simplify the calculation, we use the properties of the scalar product with respect to unit vectors

$$\vec{\imath}.\vec{\imath} = \|\vec{\imath}\|.\|\vec{\imath}\|\,cos(\theta) = cos(\theta)$$

Where

$$\theta = 0$$

We have

$$\begin{cases} \vec{\imath}.\vec{\imath} = 1, & \vec{\jmath}.\vec{\jmath} = 1, & \vec{k}.\vec{k} = 1 \\ \vec{\imath}.\vec{\jmath} = 0, & \vec{\imath}.\vec{k} = 0, & \vec{\jmath}.\vec{k} = 0 \end{cases}$$

Using the commutative property of the scalar product, we get

$$\vec{A}.\vec{B} = A_x\,B_x + A_yB_y + A_zB_z$$

On the other hand we find

$$cos(\theta) = \frac{A_x\,B_x + A_yB_y + A_zB_z}{\sqrt{{A_x}^2 + {A_y}^2 + {A_z}^2}\sqrt{{B_x}^2 + {B_y}^2 + {B_z}^2}}$$

We use this relationship to calculate the angle between the two vectors $\vec{A}$ and $\vec{B}$.

Example

We define the vectors

$$\begin{cases} \vec{V}_1 = 2\vec{\imath} + 3\vec{\jmath} + 5\vec{k} \\ \vec{V}_2 = -\vec{\imath} + 2\vec{\jmath} - \vec{k} \\ \vec{V}_3 = 3\vec{\imath} - 2\vec{\jmath} + \vec{k} \end{cases}$$

1. Calculate $\vec{D} = \vec{V}_1 + \vec{V}_2 + \vec{V}_3$, $\vec{V}_1.\vec{V}_2$ and $\vec{V}_3.\vec{V}_2$

2. Calculate the angle between the two vectors $\vec{V}_1$ and $\vec{V}_2$.

3. Find the unit vector of the vector $\vec{V}_3$.

Solution

1.
$$\vec{D} = 4\vec{\imath} + 3\vec{\jmath} + 5\vec{k}$$
$$\vec{V}_1.\vec{V}_2 = 2(-1) + 3(2) + 5(-1) = -2 + 6 - 5 = -1$$
$$\vec{V}_3.\vec{V}_2 = 3(-1) - 2(2) + (-1) = -3 - 4 - 1 = -8$$

2.
$$\vec{V}_1.\vec{V}_2 = \|\vec{V}_1\|.\|\vec{V}_2\|\cos(\theta) = -1 \qquad \Rightarrow \cos(\theta) = -\frac{1}{\|\vec{V}_1\|.\|\vec{V}_2\|}$$

So,

$$\cos(\theta) = -\frac{1}{\sqrt{2^2 + 3^2 + 5^2}\sqrt{(-1)^2 + 2^2 + (-1)^2}} = -\frac{1}{\sqrt{38}\sqrt{6}}$$

We have

$$\theta = 93.79^0$$

3.
$$\vec{e}_{\vec{V}_3} = \frac{\vec{V}_3}{\|\vec{V}_3\|} = \frac{3\vec{\imath} - 2\vec{\jmath} + \vec{k}}{\sqrt{(3)^2 + (-2)^2 + 1^2}}$$

we have

$$\vec{e}_{\vec{V}_3} = \frac{3}{\sqrt{14}}\vec{\imath} - \frac{2}{\sqrt{14}}\vec{\jmath} + \frac{1}{\sqrt{14}}\vec{k}$$

2.8 The Vector Product

One may define a further product between vectors. Here a new vector arises that is defined as follows. The vector product of two vectors $\vec{A}$ and $\vec{B}$ is the vector

$$\vec{A} \times \vec{B} = \|\vec{A}\|.\|\vec{B}\| sin(\theta)\vec{e}_n$$

Or

$$\vec{A} \wedge \vec{B} = \|\vec{A}\|.\|\vec{B}\| sin(\theta)\vec{e}_n$$

where $\vec{e}_n$ is the unit vector being perpendicular to the plane fixed by $\vec{A}$ and $\vec{B}$, and pointing out of the plane as a right-handed helix when rotating the first vector of the product into the second vector.

Properties

The vector product has the following properties

No commutativity $\vec{A} \times \vec{B} = -\vec{B}.\vec{A}$

If $\vec{A}$ and $\vec{B}$ are parallel $\vec{A} \times \vec{B} = \vec{0}$

If $\vec{A}$ and $\vec{B}$ are perpendicular $\vec{A} \times \vec{B} = \|\vec{A}\|.\|\vec{B}\|\vec{e}_n$

Distributivity $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Note

$$\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$$

The vector product using the analytical expression

we have two vectors $\vec{A}$ and $\vec{B}$

$$\vec{A} \times \vec{B} = (A_x\,\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}) \times (B_x\,\vec{\imath} + B_y\vec{\jmath} + B_z\vec{k})$$

$$= A_x\,B_x\vec{\imath} \times \vec{\imath} + A_xB_y\vec{\imath} \times \vec{\jmath} + A_xB_z\,\vec{\imath} \times \vec{k} + A_y\,B_x\vec{\jmath} \times \vec{\imath} + A_yB_y\vec{\jmath} \times \vec{\jmath}$$

$$+A_yB_z\,\vec{\jmath} \times \vec{k} + A_z\,B_x\vec{k} \times \vec{\imath} + A_zB_y\vec{k} \times \vec{\jmath} + A_zB_z\vec{k} \times \vec{k}$$

To simplify the calculation, we use the properties of the vector product with respect to unit vectors.

Where $\theta = 0$. We have

$$\vec{\imath} \times \vec{\imath} = \vec{0}, \quad \vec{\jmath} \times \vec{\jmath} = \vec{0}, \quad \vec{k} \times \vec{k} = \vec{0}$$

using our right-hand rule

$$\vec{\imath} \times \vec{\jmath} = \vec{k}, \quad \vec{\jmath} \times \vec{k} = \vec{\imath}, \quad \vec{k} \times \vec{\imath} = \vec{\jmath}$$

With the use of the non-commutative property of the vector product, we find

$$\vec{A} \times \vec{B} = (A_y\,B_z - A_zB_y)\vec{\imath} + (A_z\,B_x - A_xB_z)\vec{\jmath} + (A_x\,B_y - A_yB_x)\vec{k}$$

Note

We call the determinant $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}$ the following formula

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{12}A_{21}$$

We calculate the determinant of 3×3 as follows

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} = \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} A_{11} - \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} A_{12} + \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} A_{13}$$

From which the expression the vector product of two vectors $\vec{A}$ and $\vec{B}$ becomes

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

2.9 Mixed product

It is a scalar value written from the form

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{C}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{C} \times \vec{A})$$

Example

We define the vectors $\vec{V}_1$, $\vec{V}_2$ and $\vec{V}_3$

$$\begin{cases} \vec{V}_1 = 3\vec{\imath} + 3\vec{\jmath} \\ \vec{V}_2 = \vec{\imath} + 3\vec{\jmath} + \vec{k} \\ \vec{V}_3 = \vec{\imath} - \vec{\jmath} + 2\vec{k} \end{cases}$$

Prove that

$$\vec{V}_1.(\vec{V}_2 \times \vec{V}_3) = \vec{V}_2.(\vec{V}_3 \times \vec{V}_1) = \vec{V}_3.(\vec{V}_1 \times \vec{V}_2) = 18$$

1. We have

$$\vec{V}_1.(\vec{V}_2 \times \vec{V}_3) = \vec{V}_1.\left(\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & 3 & 1 \\ 1 & -1 & 2 \end{vmatrix}\right)$$

$$\vec{V}_1.(\vec{V}_2 \times \vec{V}_3) = \vec{V}_1.\left(\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix}\vec{k}\right)$$

$$\vec{V}_1.(\vec{V}_2 \times \vec{V}_3) = (3\vec{\imath} + 3\vec{\jmath}).(7\vec{\imath} - \vec{\jmath} - 4\vec{k}) = 18$$

2. $$\vec{V}_2.(\vec{V}_3 \times \vec{V}_1) = \vec{V}_2.\left(\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 2 \\ 3 & 3 & 0 \end{vmatrix}\right)$$

$$\vec{V}_2.(\vec{V}_3 \times \vec{V}_1) = \vec{V}_2.\left(\begin{vmatrix} -1 & 2 \\ 3 & 0 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix}\vec{k}\right)$$

$$\vec{V}_2.(\vec{V}_3 \times \vec{V}_1) = (\vec{\imath} + 3\vec{\jmath} + \vec{k}).(-6\vec{\imath} + 6\vec{\jmath} + 6\vec{k}) = 18$$

3. $$\vec{V}_3.(\vec{V}_1 \times \vec{V}_2) = \vec{V}_3.\left(\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 3 & 3 & 0 \\ 1 & 3 & 1 \end{vmatrix}\right)$$

$$\vec{V}_3.(\vec{V}_1 \times \vec{V}_2) = \vec{V}_3.\left(\begin{vmatrix} 3 & 0 \\ 3 & 1 \end{vmatrix}\vec{\imath} - \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}\vec{\jmath} + \begin{vmatrix} 3 & 3 \\ 1 & 3 \end{vmatrix}\vec{k}\right)$$

$$\vec{V}_3.(\vec{V}_1 \times \vec{V}_2) = (\vec{\imath} - \vec{\jmath} + 2\vec{k}).(3\vec{\imath} - 3\vec{\jmath} + 6\vec{k}) = 18$$

2.10 The double vector product

is defined as follows

$$\vec{D} = \vec{A} \times (\vec{B} \times \vec{C})$$

2.11 Derivative of the vector

Let us have the vector

$$\vec{V}(t) = x(t)\,\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

The derivative of the vector $\vec{V}(t)$ with respect to time is given by the following expression

$$\frac{d\vec{V}(t)}{dt} = \frac{dx(t)}{dt}\vec{\imath} + \frac{dy(t)}{dt}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k}$$

Where

$$\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = 0$$

Properties

$\vec{V}$ and $\vec{C}$ are vectors, α and β are real numbers. We have the following properties

$$\frac{d}{dt}\left(\alpha\vec{V}+\beta\vec{C}\right)=\alpha\frac{d\vec{V}}{dt}+\beta\frac{d\vec{C}}{dt}$$

$$\frac{d\left(\vec{V}.\vec{C}\right)}{dt}=\frac{d\vec{V}}{dt}.\vec{C}+\vec{V}.\frac{d\vec{C}}{dt}$$

$$\frac{d\left(\vec{V}\times\vec{C}\right)}{dt}=\frac{d\vec{V}}{dt}\times\vec{C}+\vec{V}\times\frac{d\vec{C}}{dt}$$

Example

We define the vectors $\vec{V}_1$ and $\vec{V}_2$

$$\begin{cases}\vec{V}_1=\sin(t)\,\vec{\imath}-\cos(t)\,\vec{\jmath}+3t\vec{k}\\ \vec{V}_2=5t^3\vec{\imath}+3t\vec{\jmath}-2t^4\vec{k}\end{cases}$$

. Calculate $\frac{d}{dt}(\vec{V}_1.\vec{V}_2)$ and $\frac{d}{dt}(\vec{V}_1\times\vec{V}_2)$

1. We have

$$\frac{d}{dt}\left(\vec{V}_1.\vec{V}_2\right)=\frac{d}{dt}(5t^3sin(t)-3tcos(t)-6t^5)$$

$$\frac{d}{dt}\left(\vec{V}_1.\vec{V}_2\right)=15t^2sin(t)+5t^3cos(t)-3cos(t)+3tsin(t)-30t^5$$

2. $$\left(\vec{V}_1\times\vec{V}_2\right)=\begin{vmatrix}\vec{\imath}&\vec{\jmath}&\vec{k}\\ sin(t)&-cos(t)&3t\\ 5t^3&3t&-2t^4\end{vmatrix}$$

$$=\begin{vmatrix}-cos(t)&3t\\ 3t&-2t^4\end{vmatrix}\vec{\imath}-\begin{vmatrix}sin(t)&3t\\ 5t^3&-2t^4\end{vmatrix}\vec{\jmath}+\begin{vmatrix}sin(t)&-cos(t)\\ 5t^3&3t\end{vmatrix}\vec{k}$$

$$=(2t^4\,cos(t)-9t^2)\vec{\imath}-(-2t^4\,sin(t)-15t^4)\vec{\jmath}+(3t\,sin(t)+5t^3\,cos(t))\vec{k}$$

So,

$$\frac{d}{dt}\left(\vec{V}_1 \times \vec{V}_2\right) = (8t^3\, cos(t) - 2t^4 sin(t) - 18t)\vec{\imath}$$
$$-(-8t^3\, sin(t) - 2t^4\, cos(t) - 60t^3)\vec{\jmath}$$
$$+(3\, sin(t) + 3t\, cos(t) + 15t^2\, cos(t) - 5t^3\, sin(t))\vec{k}$$

3. Calculation of uncertainties

The objective of this notion is to know how to determine the value of the error of experimental measurement carried out in practical work in order to better compare the experiment with the theoretical model.

The origins of error in most cases is the measuring device and the experimenter, the error calculation, or uncertainty calculation has been very important to become aware of the degree of approximation with which one is working, the errors which may have been committed in the various measurements are estimated and their consequences in the results obtained are calculated.

Consider a physical quantity x to be determined experimentally. To do this, it is necessary to carry out one or more measurements (n times) of this quantity by an experimental protocol that is as suitable as possible. We then obtain at the end of the experiment an average result

$$\bar{x} = \frac{\sum_i^n x_i}{n}$$

(the average value of the measured values) to be matched with an absolute uncertainty Δx, the results of the measurements carried out of the quantity x must be presented in the form

$$x = \bar{x} \pm \Delta x$$

Chapter I: Mathematical notions

3.1. Absolute uncertainty

we define the absolute uncertainty Δx as the absolute value of the error $x_m - \bar{x}$ and we write

$$\Delta x = max|x_m - \bar{x}|$$

x_m is the measured value. If the quantity to be measured depends on several variables $x(a, b, c)$, the absolute uncertainty will be given by the following expression

$$\Delta x = \left|\frac{\partial x}{\partial a}\right| \Delta a + \left|\frac{\partial x}{\partial b}\right| \Delta b + \left|\frac{\partial x}{\partial c}\right| \Delta c$$

3.2. Relative uncertainty

It gives us the precision of the measurement and is expressed by the ratio

$$\varepsilon(\%) = \frac{\Delta x}{x}.100$$

3.3. Calculation of uncertainty

We use the method of the differential of the logarithmic function as follows:

1- We insert the logarithm function on the physical quantity function.

2- We derive the function.

3- Convert the differential symbol dx to the uncertainty symbol Δx.

4- We convert the negative signs into positive and we enter the absolute value on the values.

The following relationships can be found

1) $x = a + b$ the absolute uncertainty

$$\Delta x = \Delta a + \Delta b$$

2) $x = a - b$ the absolute uncertainty

$$\Delta x = \Delta a + \Delta b$$

3) $x = a.b$

We have

$$lnx = ln\,(a.b) = ln\,(a) + ln\,(b)$$

and

$$\frac{dx}{x} = \frac{da}{a} + \frac{db}{b}$$

so the relative uncertainty

$$\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

4) $x = \frac{a}{b}$ the relative uncertainty

$$\frac{\Delta x}{x} = \frac{\Delta a}{a} + \frac{\Delta b}{b}$$

5) $x = a^n.b^m$ the relative uncertainty

$$\frac{\Delta x}{x} = n\frac{\Delta a}{a} + m\frac{\Delta b}{b}$$

Example 01

For the experimental relation

$$x = (2u+1)\sqrt[3]{\frac{y}{z^2}}$$

the calculation by the logarithms gives

$$lnx = \ln\left[(2u+1)\sqrt[3]{\frac{y}{z^2}}\right]$$

$$= \ln[(2u+1)] + \ln\left[\sqrt[3]{\frac{y}{z^2}}\right]$$

$$= ln\,[(2u+1)] + \frac{1}{3}ln\,\left[\frac{y}{z^2}\right]$$

We have

$$lnx = ln\,[(2u+1)] + \frac{1}{3}ln\,[y] - \frac{2}{3}ln\,[z]$$

and

$$\frac{dx}{x} = \frac{2du}{2u+1} + \frac{1}{3}\frac{dy}{y} - \frac{2}{3}\frac{dz}{z}$$

the relative uncertainty is

$$\frac{\Delta x}{x} = \frac{2\Delta u}{2u+1} + \frac{1}{3}\left(\frac{\Delta y}{y}\right) + \frac{2}{3}\left(\frac{\Delta z}{z}\right)$$

Example 02

Identification of the experimental law: an electrical wire of diameter d, length l and electrical resistance R is made of copper whose electrical conductivity is given by

$$\rho = \frac{\pi}{4} R \frac{d^2}{l}$$

With the measurement results

$$d = (0.30 \pm 0.01)mm$$

$$l = (2 \pm 0.001)m$$

and

$$R = (0.4562 \pm 0.0002)\Omega.$$

Calculation by logarithms gives

$$\frac{\Delta\rho}{\rho} = \left(\frac{\Delta R}{R}\right) + 2\left(\frac{\Delta d}{d}\right) + \left(\frac{\Delta l}{l}\right)$$

So

$$\frac{\Delta\rho}{\rho} = \left(\frac{0.0002}{0.4562}\right) + 2\left(\frac{0.01}{0.30}\right) + \left(\frac{0.001}{2}\right) = 0.071$$

and

$$\rho = \frac{3.14}{4}(0.4562)\frac{(0.30)^2}{2} = 1.6 \times 10^{-8}$$

we have

$$\rho = (1.6 \pm 0.1) \times 10^{-8}\Omega$$

Example 03

We measure the diameter and the mass of a ball.

$$r = (5 \pm 0.01)mm$$

and

$$m = (9.9 \pm 0.1)g$$

1) Volume $V = \frac{4}{3}\pi r^3$ the relative uncertainty

$$\frac{\Delta V}{V} = 3\left(\frac{\Delta r}{r}\right)$$

$$\frac{\Delta V}{V} = 3\left(\frac{0.01}{10}\right) = 0.003$$

With

$$V = \frac{4}{3}3.14(5)^3 = 523.33$$

We write

$$V = (523.33 \pm 1.57)mm^3$$

2) The density

$$\rho = \frac{m}{V} = \frac{9.9g}{0.52333cm^3} = 18{,}92\frac{g}{cm^3}$$

the relative uncertainty

$$\frac{\Delta\rho}{\rho} = \left(\frac{\Delta m}{m}\right) + \left(\frac{\Delta V}{V}\right) = \left(\frac{0.1}{9.9}\right) + 0.003 = 0.013$$

We write

$$\rho = (18.92 \pm 0.25)\frac{g}{cm^3}$$

Chapter II: Kinematics

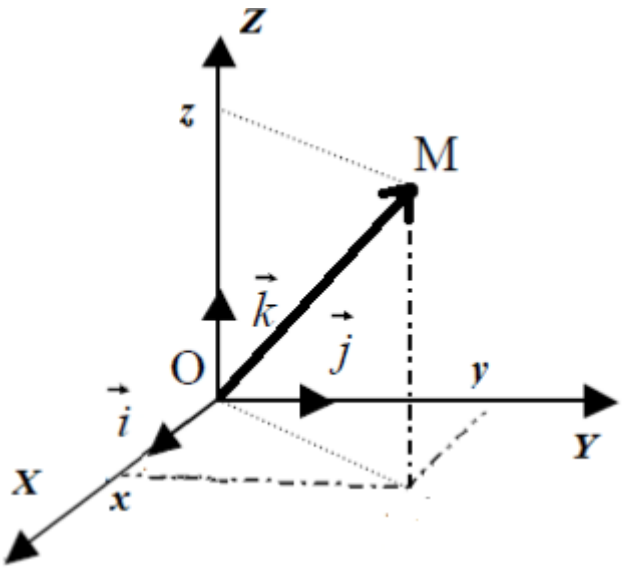
Kinematics is a descriptive engineering study of motion. It describes motion without linking it to its causes, and depends on the concept of space and time. In this chapter, we discuss how to write each of the position vector and the velocity vector and acceleration in various coordinates. At the end of the chapter we study the relativity of motion.

1. Cartesian coordinates

1.1 Position vector

We define the position vector $\overrightarrow{OM}$ in Cartesian coordinates as follows

$$\overrightarrow{OM} = \vec{r} = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

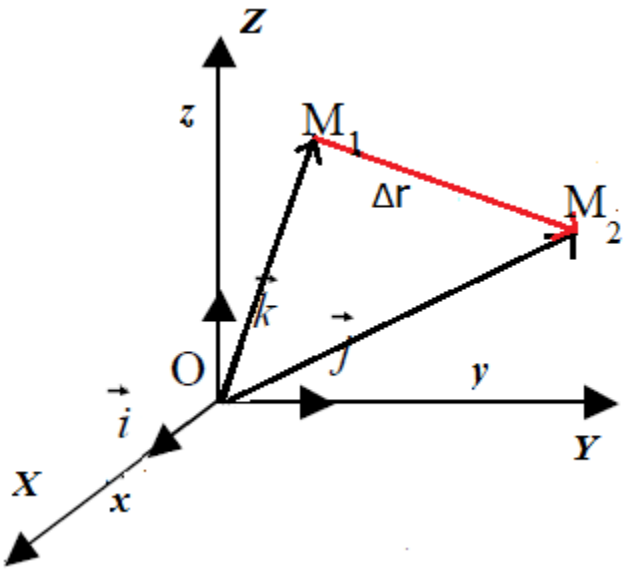


The displacement vector

$\overrightarrow{\Delta r}$ is also known as the transition vector from point M_1 at moment t_1 to point M_2 at moment t_2.

From the figure below, and according to the Chasles relationship

$$\overrightarrow{OM_1} + \overrightarrow{\Delta r} = \overrightarrow{OM_2}$$



So,

$$\overrightarrow{\Delta r} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$$

In the Cartesian coordinates, the displacement vector is

$$\overrightarrow{\Delta r} = (x_2 - x_1)\vec{\imath} + (y_2 - y_1)\vec{\jmath} + (z_2 - z_1)\vec{k}$$

Trajectory

It is the sum of the consecutive points occupied by the mobile. To find the path equation, we look for a mathematical relationship between the representations $x(t), y(t), z(t)$ independent of time.

Example 01

Coordinates are given for the movement of a material point as

$$\begin{cases} x = V_0 t \\ y = -\frac{1}{2} g t^2 \end{cases}$$

To find the equation of the path (**Trajectory**) we get rid of the time as follows

$$t = \frac{x}{V_0}$$

We have the equation

$$y = -\frac{1}{2}\frac{g}{{V_0}^2}x^2$$

Example 02

If the coordinates

$$\begin{cases} x(t) = cos(t^2) \\ y(t) = sin(t^2) \end{cases}$$

To find the path equation, we square $x(t)$ and $y(t)$ and then add.

$$x^2 + y^2 = cos(t^2) + sin(t^2)$$

We have the equation

$$x^2 + y^2 = 1$$

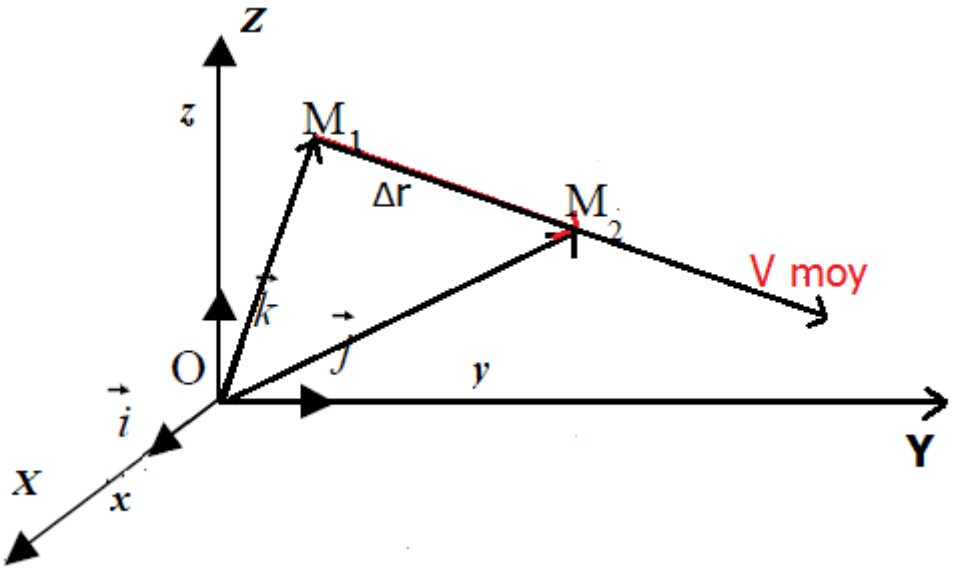
The path in this case is a circle with radius 1 and center (0,0).

1.2 The velocity vector

The average velocity

The average velocity $\vec{v}_{moy}$ of a moving M between the moments t_1 and t_2 is defined by the following relation

$$\vec{v}_{moy} = \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{\overrightarrow{OM_2}(t_2) - \overrightarrow{OM_1}(t_1)}{t_2 - t_1}$$



For the Cartesian coordinates, the analytical expression is given for the average velocity

$$\vec{v}_{moy} = \frac{(x_2 - x_1)}{\Delta t}\vec{\imath} + \frac{(y_2 - y_1)}{\Delta t}\vec{\jmath} + \frac{(z_2 - z_1)}{\Delta t}\vec{k}$$

Instantaneous velocity vector

Instantaneous velocity vector $\vec{v}(t)$ of a mover M

$$\vec{v}(t) = \lim_{t_2 \to t_1} \vec{v}_{moy}$$

and

$$\vec{v}(t) = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

For the Cartesian coordinates,

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x(t)\vec{\imath} + \ y(t)\vec{\jmath} + z(t)\vec{k})$$

$$\vec{v}(t) = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

We write

$$\vec{v}(t) = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

And we calculate The magnitude of the instantaneous velocity vector using the relation

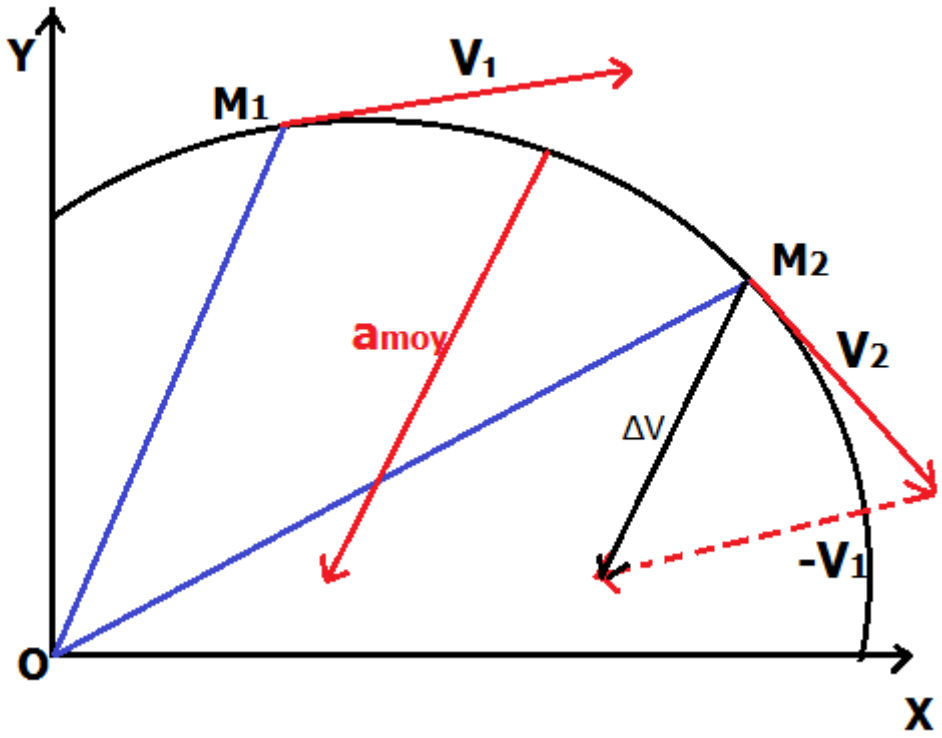
$$|\vec{v}(t)| = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$$

1.3 The acceleration vector

The average acceleration

The average acceleration $\vec{a}_{moy}$ of a moving M between the moments t_1 and t_2 is defined by the following relation

$$\vec{a}_{moy} = \frac{\vec{V}_2(t_2) - \vec{V}_1(t_1)}{t_2 - t_1} = \frac{\overrightarrow{\Delta V}}{\Delta t}$$



Instantaneous acceleration vector

Instantaneous acceleration vector $\vec{a}(t)$ of a mover M

$$\vec{a}(t) = \lim_{t_2 \to t_1} \frac{\overrightarrow{\Delta V}}{\Delta t} = \frac{d\vec{v}}{dt}$$

and

$$\vec{a}(t) = \frac{d}{dt}(V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k})$$

For the Cartesian coordinates,

$$\vec{a}(t) = \frac{dV_x}{dt}\vec{\imath} + \frac{dV_y}{dt}\vec{\jmath} + \frac{dV_z}{dt}\vec{k}$$

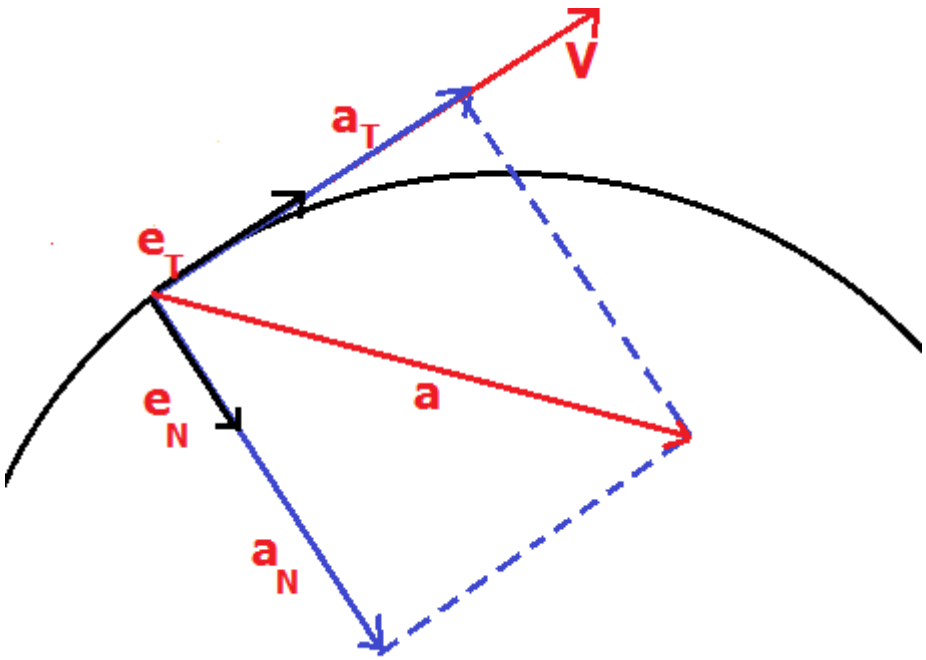
$$\vec{a}(t) = a_x\vec{\imath} + a_y\vec{\jmath} + a_z\vec{k}$$

The magnitude of the instantaneous acceleration vector

$$|\vec{a}(t)| = \sqrt{{a_x}^2 + {a_y}^2 + {a_z}^2}$$

2. Circular Motion

In physics, we say that a body has acceleration when there is a change in the velocity vector, either in magnitude or direction.



$$\vec{a} = \vec{a}_T + \vec{a}_N$$

Where,

$\vec{a}_T$ is the **Tangential Acceleration**. Associated with the changes of the magnitude of velocity.

$\vec{a}_N$ is the **Normal Acceleration.** Related to the change in direction of velocity.

To find the two components $\vec{a}_T$ and $\vec{a}_N$ we do the following. From the figure above

$$\vec{e}_T = \frac{\vec{V}(t)}{|\vec{V}(t)|} \quad \Rightarrow \vec{V}(t) = |\vec{V}(t)|\, \vec{e}_T$$

We have

$$\vec{a}(t) = \frac{d\vec{V}(t)}{dt} = \frac{d|\vec{V}(t)|}{dt}\, \vec{e}_T + |\vec{V}(t)| \frac{d\vec{e}_T}{dt}$$

Where

$$\begin{cases} \vec{e}_T = cos(\varphi)\vec{\imath} + sin(\varphi)\vec{\jmath} \\ \vec{e}_N = -sin(\varphi)\vec{\imath} + cos(\varphi)\vec{\jmath} \end{cases}$$

So,

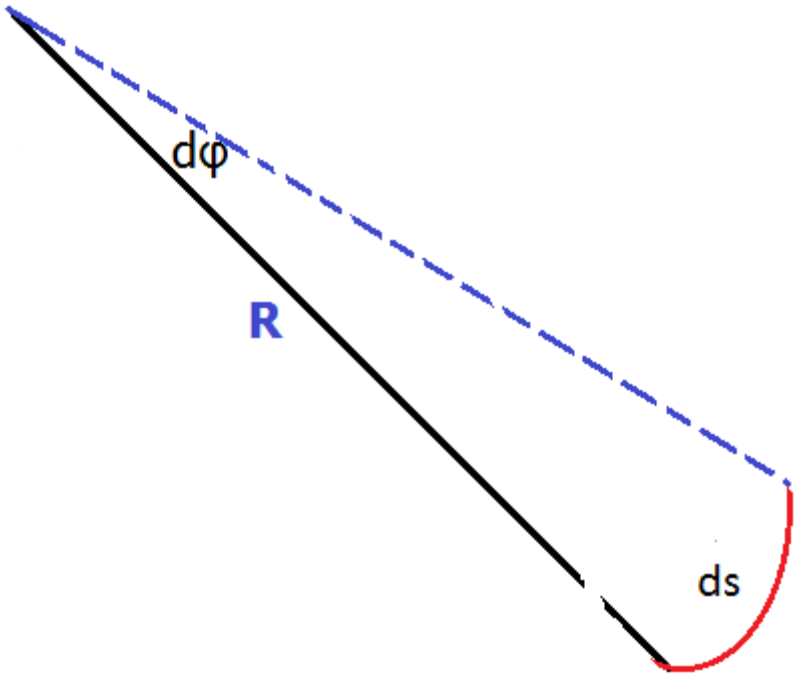
$$\frac{d\vec{e}_T}{dt} = \frac{d}{dt}\{cos(\varphi)\vec{\imath} + sin(\varphi)\vec{\jmath}\} = \frac{d\varphi}{dt}\vec{e}_T$$

The acceleration relation becomes

$$\vec{a}(t) = \frac{d|\vec{V}|}{dt}\,\vec{e}_T + |\vec{V}|\frac{d\varphi}{dt}\vec{e}_T$$

We calculate $\frac{d\varphi}{dt}$ as follows

$$\frac{d\varphi}{dt} = \frac{ds}{dt}\frac{d\varphi}{ds} = |\vec{V}|\,\frac{d\varphi}{ds}$$



R is the radius of curvature, and ds is the radius of curvature.

Where,

$$ds = Rd\varphi \Rightarrow \frac{d\varphi}{ds} = \frac{1}{R}$$

Substituting in the expression $\frac{d\varphi}{dt}$ we find

$$\frac{d\varphi}{dt} = \frac{|\vec{V}|}{R}$$

The acceleration relation becomes

$$\vec{a}(t) = \frac{d|\vec{V}|}{dt}\vec{e}_T + \frac{|\vec{V}|^2}{R}\vec{e}_T$$

Finally, the two components a_T and a_N are

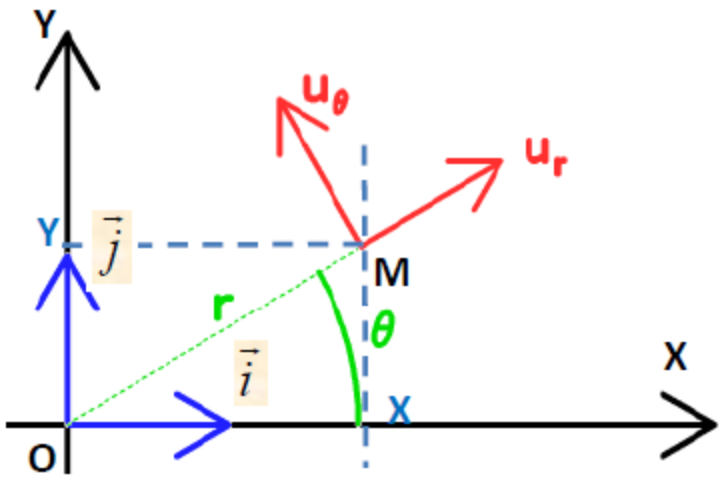
$$\begin{cases} a_T = \dfrac{d|\vec{V}|}{dt} \\ a_N = \dfrac{|\vec{V}|^2}{R} \end{cases}$$

3. Polar coordinates

3.1 Position vector

We define the position vector $\overrightarrow{OM}$ in Polar coordinates as follows

$$\overrightarrow{OM} = \vec{r} = r\vec{u}_r$$



Where

$$\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \\ r = \sqrt{x^2 + y^2} \end{cases}$$

we can write the unit vectors in the polar coordinates $\vec{u}_r$ and $\vec{u}_\theta$ in terms of the unit vectors in the Cartesian coordinates $\vec{\imath}$ and $\vec{\jmath}$.

$$\begin{cases} \vec{u}_r = cos(\theta)\vec{\imath} + sin(\theta)\vec{\jmath} \\ \vec{u}_\theta = -sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath} \end{cases}$$

3.2 The velocity vector

To find the expression for the velocity vector $\vec{v}(t)$, we differentiate the expression for the position vector $\overrightarrow{OM}$ with respect to time.

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$

We have

$$\vec{v}(t) = \frac{d}{dt}(r\vec{u}_r) = \frac{dr}{dt}\vec{u}_r + r\frac{d\vec{u}_r}{dt}$$

Now, we calculate the expression $\frac{d\vec{u}_r}{dt}$

$$\frac{d\vec{u}_r}{dt} = \frac{d\vec{u}_r}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}(cos(\theta)\vec{\imath} + sin(\theta)\vec{\jmath})$$

And

$$\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt}(-sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath}) = \frac{d\theta}{dt}\vec{u}_\theta$$

So we write the expression for the velocity vector as

$$\vec{v}(t) = \frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta$$

3.3 The acceleration vector

To find the expression for the acceleration vector $\vec{a}(t)$, we differentiate the expression for the velocity vector $\vec{v}(t)$ with respect to time.

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

We have

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{dr}{dt}\vec{u}_r + r\frac{d\theta}{dt}\vec{u}_\theta\right)$$

and

$$\vec{a}(t) = \frac{d^2r}{dt^2}\vec{u}_r + \frac{dr}{dt}\frac{d\vec{u}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{u}_\theta + r\frac{d^2\theta}{dt^2}\vec{u}_\theta + r\frac{d\theta}{dt}\frac{d\vec{u}_\theta}{dt}$$

Now, we calculate the expression $\frac{d\vec{u}_\theta}{dt}$

$$\frac{d\vec{u}_\theta}{dt} = \frac{d\vec{u}_\theta}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\frac{d}{d\theta}(-sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath})$$

We have

$$\frac{d\vec{u}_r}{dt} = \frac{d\theta}{dt}(-cos(\theta)\vec{\imath} - sin(\theta)\vec{\jmath}) = -\frac{d\theta}{dt}\vec{u}_r$$

The acceleration vector expression becomes

$$\vec{a}(t) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{u}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\vec{u}_\theta$$

4. Cylindrical coordinates

4.1 Position vector

We define the position vector $\overrightarrow{OM}$ in Cylindrical coordinates as follows

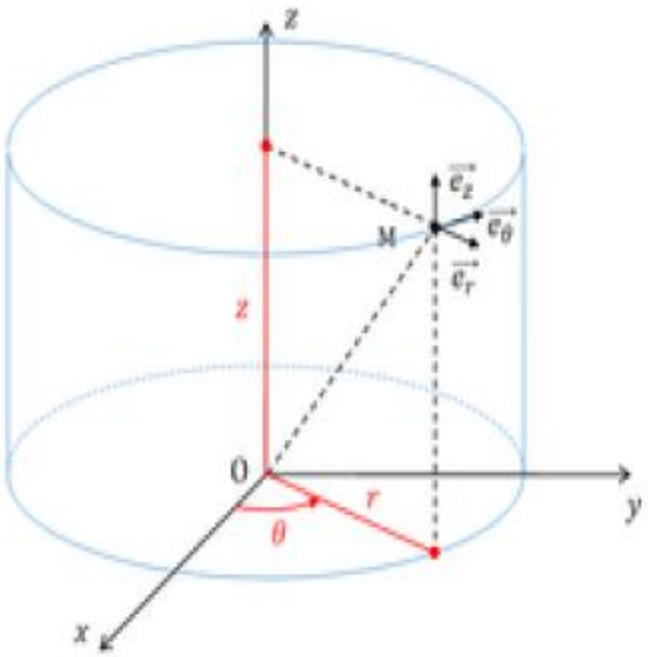
$$\overrightarrow{OM} = \vec{r} = r\vec{e}_r + z\vec{k}$$

Where

$$\begin{cases} x = rcos(\theta) \\ y = rsin(\theta) \\ z = z \end{cases}$$

And

$$r = \sqrt{x^2 + y^2}$$



we can write the unit vectors in the Cylindrical coordinates $\vec{e}_r$ and $\vec{e}_\theta$ in terms of the unit vectors in the Cartesian coordinates $\vec{\imath}$ and $\vec{\jmath}$.

$$\begin{cases} \vec{e}_r = cos(\theta)\vec{\imath} + sin(\theta)\vec{\jmath} \\ \vec{e}_\theta = -sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath} \end{cases}$$

4.2 The velocity vector

To find the expression for the velocity vector $\vec{v}(t)$, we differentiate the expression for the position vector $\overrightarrow{OM}$ with respect to time.

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$

We have

$$\vec{v}(t) = \frac{d}{dt}\left(r\vec{e}_r + z\vec{k}\right) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} + \frac{dz}{dt}\vec{k}$$

Now, we calculate the expression $\frac{d\vec{e}_r}{dt}$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \frac{d\theta}{dt}\vec{e}_\theta$$

The velocity vector expression becomes

$$\vec{v}(t) = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \frac{dz}{dt}\vec{k}$$

4.3 The acceleration vector

To find the expression for the acceleration vector $\vec{a}(t)$, we differentiate the expression for the velocity vector $\vec{v}(t)$ with respect to time.

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

We have

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + \frac{dz}{dt}\vec{k}\right)$$

And

$$\vec{a}(t) = \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$

Now, we calculate the expression $\frac{d\vec{e}_\theta}{dt}$

$$\frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\frac{d\theta}{dt}\vec{e}_r$$

The acceleration vector expression becomes

$$\vec{a}(t) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\vec{e}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\vec{e}_\theta + \frac{d^2z}{dt^2}\vec{k}$$

5. Spherical coordinates

5.1 Position vector

We define the position vector $\overrightarrow{OM}$ in Spherical coordinates as follows

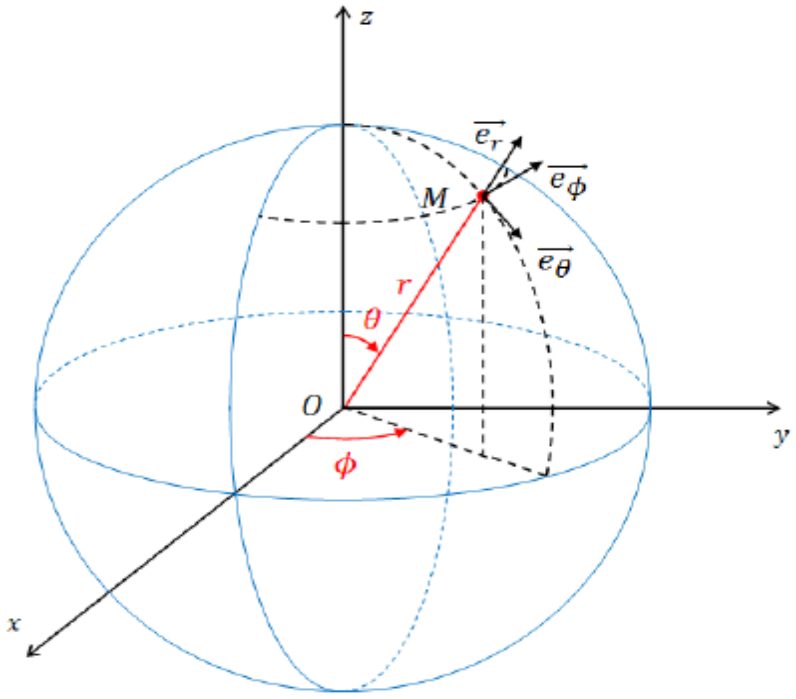
$$\overrightarrow{OM} = \vec{r} = r\vec{e}_r$$

Where

$$\begin{cases} x = rsin(\theta)cos(\phi) \\ y = rsin(\theta)sin(\phi) \\ z = rcos(\theta) \end{cases}$$

and

$$r = \sqrt{x^2 + y^2 + z^2}$$



we can write the unit vectors in the Cylindrical coordinates $\vec{e}_r$ and $\vec{e}_\theta$ in terms of the unit vectors in the Cartesian coordinates $\vec{\imath}$ and $\vec{\jmath}$.

$$\begin{cases} \vec{e}_r = sin(\theta)cos(\phi)\vec{\imath} + sin(\theta)sin(\phi)\vec{\jmath} + cos(\theta)\vec{k} \\ \vec{e}_\theta = cos(\theta)cos(\phi)\vec{\imath} + cos(\theta)sin(\phi)\vec{\jmath} - sin(\theta)\vec{k} \\ \vec{e}_\phi = -sin(\phi)\vec{\imath} + cos(\phi)\vec{k} \end{cases}$$

5.2 The velocity vector

To find the expression for the velocity vector $\vec{v}(t)$, we differentiate the expression for the position vector $\overrightarrow{OM}$ with respect to time.

$$\vec{v}(t) = \frac{d\overrightarrow{OM}}{dt}$$

We have

$$\vec{v}(t) = \frac{d}{dt}(r\vec{e}_r) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt}$$

Now, we calculate the expression $\frac{d\vec{e}_r}{dt}$

$$\frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} + \frac{d\vec{e}_r}{d\phi}\frac{d\phi}{dt} = \frac{d\theta}{dt}\vec{e}_\theta + sin(\theta)\frac{d\phi}{dt}\vec{e}_\phi$$

The velocity vector expression becomes

$$\vec{v}(t) = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + rsin(\theta)\frac{d\phi}{dt}\vec{e}_\phi$$

5.3 The acceleration vector

To find the expression for the acceleration vector $\vec{a}(t)$, we differentiate the expression for the velocity vector $\vec{v}(t)$ with respect to time.

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

We have

$$\vec{a}(t) = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta + rsin(\theta)\frac{d\phi}{dt}\vec{e}_\phi\right)$$

The acceleration vector expression becomes

$$\vec{a}(t) = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - rsin(\theta)\left(\frac{d\phi}{dt}\right)^2\right]\vec{e}_r$$

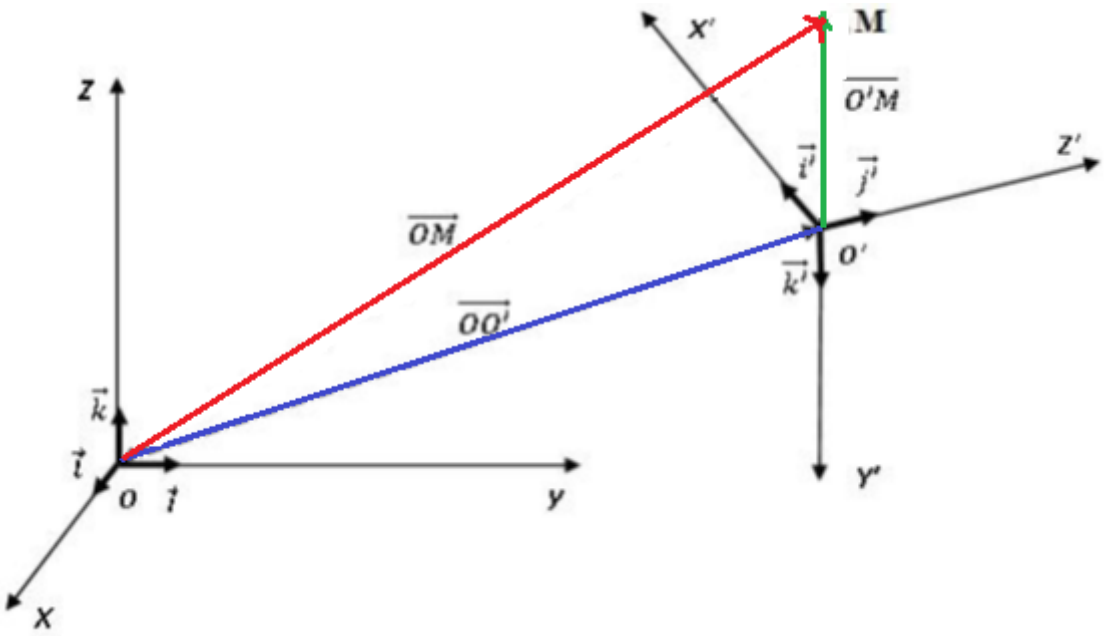
$$+\left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} - rsin(\theta)cos(\theta)\left(\frac{d\phi}{dt}\right)^2\right]\vec{e}_\theta$$

$$+\left[2rcos(\theta)\frac{d\theta}{dt}\frac{d\phi}{dt} + 2\frac{dr}{dt}\frac{d\theta}{dt}sin(\theta) + rsin(\theta)\frac{d^2\phi}{dt^2}\right]\vec{e}_\phi$$

6. Relative motion

We now learn how to describe motion when we move from one reference to another and how to deduce the laws of the composition of velocities and the composition of accelerations.

Let us have two references, a fixed reference $R(x, y, z)$ (absolute reference) and a moving reference $R'(x', y', z')$ (relative reference) as in the figure.



We know the position vectors as follows

$$\begin{cases} \overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \\ \overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M} \end{cases}$$

where

$$\begin{cases} \overrightarrow{O'M} = x'\overrightarrow{\imath'} + y'\overrightarrow{\jmath'} + z'\overrightarrow{k'} \\ \overrightarrow{OO'} = x_0\vec{\imath} + y_0\vec{\jmath} + z_0\vec{k} \end{cases}$$

We define velocity and acceleration in absolute frames $R(x, y, z)$and relative frames $R'(x', y', z')$ as follows.

Absolute velocity $\vec{v}_a$: which is the velocity with respect to the reference R.

$$\vec{v}_a = \left(\frac{d\overrightarrow{OM}}{dt}\right)_R$$

Relative velocity $\vec{v}_r$: which is the velocity with respect to the reference R'.

$$\vec{v}_r = \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{R'}$$

Absolute acceleration $\vec{\gamma}_a$: which is the acceleration with respect to the reference R.

$$\vec{\gamma}_a = \left(\frac{d\vec{v}_a}{dt}\right)_R$$

Relative acceleration $\vec{\gamma}_r$: which is the acceleration with respect to the reference R'.

$$\vec{\gamma}_a = \left(\frac{d\vec{v}_r}{dt}\right)_{R'}$$

6.1 Law of composition of velocities

We now calculate the absolute velocity $\vec{v}_a$ as we know it.

So,

$$\vec{v}_a = \left(\frac{d\overrightarrow{OM}}{dt}\right)_R = \left(\frac{d(\overrightarrow{OO'} + \overrightarrow{O'M})}{dt}\right)_R = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \left(\frac{d\overrightarrow{O'M}}{dt}\right)_R$$

Where

$$\left(\frac{d\overrightarrow{O'M}}{dt}\right)_R = \left(\frac{d(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'})}{dt}\right)_R$$

We have

$$\left(\frac{d\overrightarrow{O'M}}{dt}\right)_R = \left(\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right) + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}$$

And

$$\left(\frac{d\overrightarrow{O'M}}{dt}\right)_R = \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{R'} + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}$$

$$= \vec{v}_r + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}$$

We find

$$\vec{v}_a = \left(\frac{d\overrightarrow{OM}}{dt}\right)_R = \left(\frac{d(\overrightarrow{OO'} + \overrightarrow{O'M})}{dt}\right)_R$$

$$= \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{v}_r + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt}$$

If the movement in the case of retracting motion (translation)

$$\frac{d\vec{\imath'}}{dt} = \frac{d\vec{\jmath'}}{dt} = \frac{d\vec{k'}}{dt} = 0$$

We have

$$\vec{v}_a = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{v}_r$$

If the movement is rotational

Where

$$\theta = \omega t$$

And

$$\begin{cases} \overrightarrow{\imath'} = cos(\theta)\vec{\imath} + sin(\theta)\vec{\jmath} \\ \overrightarrow{\jmath'} = -sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath} \end{cases}$$

We have

$$\begin{cases} \dfrac{d\overrightarrow{\imath'}}{dt} = \dfrac{d\overrightarrow{\imath'}}{d\theta}\dfrac{d\theta}{dt} = \omega\dfrac{d\overrightarrow{\imath'}}{d\theta} \\ \dfrac{d\overrightarrow{\jmath'}}{dt} = \dfrac{d\overrightarrow{\jmath'}}{d\theta}\dfrac{d\theta}{dt} = -\omega\dfrac{d\overrightarrow{\jmath'}}{d\theta} \end{cases}$$

Note

$$\frac{d\overrightarrow{k'}}{dt} = 0$$

Because the reference rotates on the oz axis. Here $\vec{k}$ and $\overrightarrow{k'}$ are parallel.

Substituting into the expression for absolute velocity $\vec{v}_a$, where $\vec{\omega} = \omega\overrightarrow{k'}$ we find

$$\vec{v}_a = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{v}_r + x'\omega\overrightarrow{\jmath'} - y'\omega\overrightarrow{\imath'} = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{v}_r + \vec{\omega} \times \overrightarrow{O'M}$$

Because

$$\vec{\omega} \times \overrightarrow{O'M} = \begin{vmatrix} \overrightarrow{\imath'} & \overrightarrow{\jmath'} & \overrightarrow{k'} \\ 0 & 0 & \omega \\ x' & y' & z' \end{vmatrix} = x'\omega\overrightarrow{\jmath'} - y'\omega\overrightarrow{\imath'}$$

Finally, we write the formula for composition of velocities as follows

$$\vec{v}_a = \vec{v}_r + \vec{v}_e$$

Where $\vec{v}_r$ is the relative velocity. And $\vec{v}_e$ is the **traction velocity**.

$$\vec{v}_e = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{\omega} \times \overrightarrow{O'M}$$

6.2 Law of composition of accelerations

If we derive the law of composition of speeds with respect to time in the fixed frame of reference R, we obtain the law of composition of accelerations,

As we defined at the beginning we have absolute acceleration $\vec{\gamma}_a$ calculated by the relation

$$\vec{\gamma}_a = \left(\frac{d\vec{v}_a}{dt}\right)_R = \left(\frac{d(\vec{v}_r + \vec{v}_e)}{dt}\right)_R = \left(\frac{d\vec{v}_r}{dt}\right)_R + \left(\frac{d\vec{v}_e}{dt}\right)_R$$

Where

$$\vec{v}_r = v_x\vec{\imath'} + v_y\vec{\jmath'} + v_z\vec{k'}$$

which is written

$$\left(\frac{d\vec{v}_r}{dt}\right)_R = \left(\frac{dv_x}{dt}\vec{\imath'} + \frac{dv_y}{dt}\vec{\jmath'} + \frac{dv_z}{dt}\vec{k'}\right) + v_x\frac{d\vec{\imath'}}{dt} + v_y\frac{d\vec{\jmath'}}{dt} + v_z\frac{d\vec{k'}}{dt}$$

We have

$$\left(\frac{d\vec{v}_r}{dt}\right)_R = \left(\frac{d\vec{v}_r}{dt}\right)_{R'} + v_x\omega\vec{\jmath'} - v_y\omega\vec{\imath'} = \left(\frac{d\vec{v}_r}{dt}\right)_{R'} + \vec{\omega}\times\vec{v}_r$$

and

$$\left(\frac{d\vec{v}_r}{dt}\right)_R = \vec{\gamma}_r + \vec{\omega}\times\vec{v}_r$$

On the other hand we have

$$\left(\frac{d\vec{v}_e}{dt}\right)_R = \frac{d}{dt}\left(\left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{\omega}\times\overrightarrow{O'M}\right)_R$$

and

$$\left(\frac{d\vec{v}_e}{dt}\right)_R = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \frac{d}{dt}\left(\vec{\omega}\times\overrightarrow{O'M}\right)_R$$

$$\left(\frac{d\vec{v}_e}{dt}\right)_R = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt}\times\overrightarrow{O'M}\right)_R + \left(\vec{\omega}\times\frac{d\overrightarrow{O'M}}{dt}\right)_R$$

$$\left(\frac{d\vec{v}_e}{dt}\right)_R = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)_R + \vec{\omega} \times \left(\vec{v}_r + \vec{\omega} \times \overrightarrow{O'M}\right)$$

We have

$$\left(\frac{d\vec{v}_e}{dt}\right)_R = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)_R + \vec{\omega} \times \vec{v}_r + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{O'M}\right)$$

Substituting in the absolute acceleration expression $\vec{\gamma}_a$ we find

$$\vec{\gamma}_a = \vec{\gamma}_r + 2\vec{\omega} \times \vec{v}_r + \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)_R + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{O'M}\right)$$

Finally, we write the law of composition of accelerations as follows

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_c + \vec{\gamma}_e$$

Where $\vec{\gamma}_r$ is the relative acceleration. And $\vec{v}_e$ is the **Coriolis acceleration**.

$$\vec{\gamma}_c = 2\vec{\omega} \times \vec{v}_r$$

$\vec{\gamma}_e$ is the traction acceleration

$$\vec{\gamma}_e = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)_R + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{O'M}\right)$$

Example

In reference R' we know the coordinates of point M as follows

$$\begin{cases} x' = a\cos(2\omega t) \\ y' = -a\sin(2\omega t) \end{cases}$$

The reference R' rotates around the point O' with an angular speed $\vec{\omega} = \omega\overrightarrow{k'}$ according to the following data.

$$\begin{cases} \overrightarrow{OO'} = a\theta\vec{j} \\ \theta = \omega t \end{cases}$$

Calculate the absolute velocity $\vec{v}_a$ and absolute acceleration $\vec{\gamma}_a$.

Answer

1. Calculate the absolute velocity $\vec{v}_a$.

According to the law of combination of velocities.

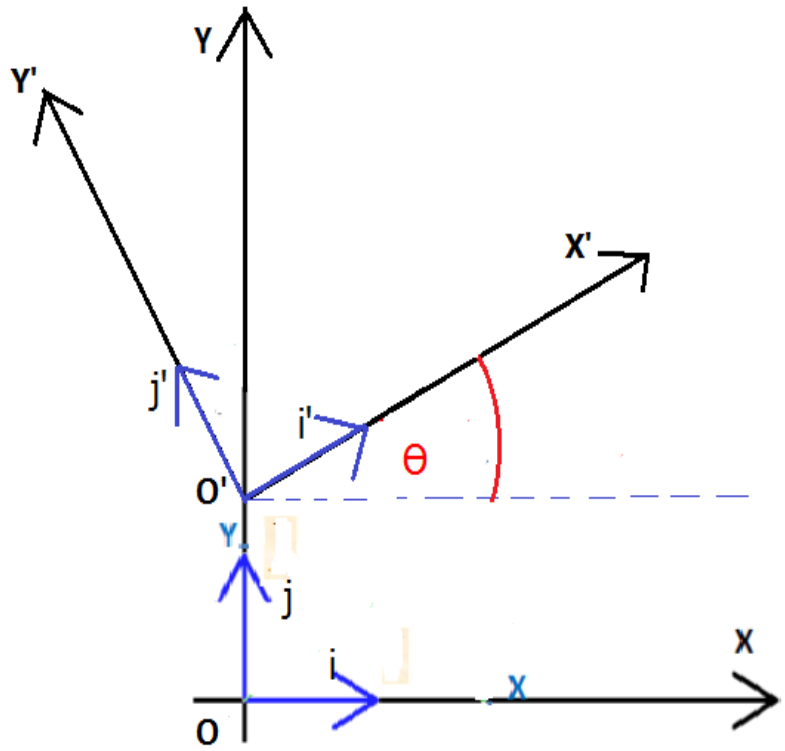
$$\vec{v}_a = \vec{v}_r + \vec{v}_e$$

Where

$$\vec{v}_r = \left(\frac{d\overrightarrow{O'M}}{dt}\right)_{R'}$$

And

$$\overrightarrow{O'M} = acos(2\omega t)\vec{\imath'} - asin(2\omega t)\vec{\jmath'}$$



So,

$$\vec{v}_r = -2a\omega sin(2\omega t)\vec{\imath'} - 2a\omega cos(2\omega t)\vec{\jmath'}$$

Where

$$\begin{cases} \overrightarrow{\imath'} = cos(\theta)\vec{\imath} + sin(\theta)\vec{\jmath} \\ \overrightarrow{\jmath'} = -sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath} \end{cases}$$

We have

$$\vec{v}_r = -2a\omega(sin(2\theta)cos(\theta) - cos(2\theta)sin(\theta))\vec{\imath} - 2a\omega(sin(2\theta)sin(\theta) + cos(2\theta)cos(\theta))\vec{\jmath}$$

Using conversion constitutions

$$\begin{cases} sin(2\theta)cos(\theta) - cos(2\theta)sin(\theta) = sin(\theta) \\ sin(2\theta)sin(\theta) + cos(2\theta)cos(\theta) = cos(\theta) \end{cases}$$

We write the expression for relative velocity $\vec{v}_r$

$$\vec{v}_r = -2a\omega(sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath})$$

Traction velocity $\vec{v}_e$.

$$\vec{v}_e = \left(\frac{d\overrightarrow{OO'}}{dt}\right)_R + \vec{\omega} \times \overrightarrow{O'M}$$

We have

$$\vec{v}_e = \frac{d}{dt}(a\theta\vec{\jmath}) + \begin{vmatrix} \overrightarrow{\imath'} & \overrightarrow{\jmath'} & \overrightarrow{k'} \\ 0 & 0 & \omega \\ acos(\theta) & -asin(\theta) & 0 \end{vmatrix}$$

And

$$\vec{v}_e = a\omega\vec{\jmath} + a\omega(sin(2\theta)\overrightarrow{\imath'} + cos(2\theta)\overrightarrow{\jmath'})$$

We use the same method as before, we have

$$\vec{v}_e = a\omega\vec{\jmath} + a\omega(sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath})$$

Finally, the term absolute velocity $\vec{v}_a$.

$$\vec{v}_a = a\omega\vec{\jmath} - a\omega(sin(\theta)\vec{\imath} + cos(\theta)\vec{\jmath})$$

2. Calculate the absolute acceleration $\vec{\gamma}_a$.

The first method: Here we use the direct calculation.

$$\vec{\gamma}_a = \left(\frac{d\vec{v}_a}{dt}\right)_R$$

So,

$$\vec{\gamma}_a = -a\omega^2(cos(\theta)\vec{\imath} - sin(\theta)\vec{\jmath})$$

The second method: Here we use the law of composition of accelerations.

$$\vec{\gamma}_a = \vec{\gamma}_r + \vec{\gamma}_c + \vec{\gamma}_e$$

Relative acceleration $\vec{\gamma}_r$

$$\vec{\gamma}_r = \frac{d}{dt}(\vec{v}_r) = -4a\omega^2 cos(\theta)\overrightarrow{\imath'} + 4a\omega^2 sin(\theta)\overrightarrow{\jmath'}$$

Coriolis acceleration $\vec{\gamma}_c$

$$\vec{\gamma}_c = 2\vec{\omega} \times \vec{v}_r = 2\begin{vmatrix} \overrightarrow{\imath'} & \overrightarrow{\jmath'} & \overrightarrow{k'} \\ 0 & 0 & \omega \\ -2a\omega sin(2\omega t) & -2a\omega cos(2\omega t) & 0 \end{vmatrix}$$

We have

$$\vec{\gamma}_c = 4a\omega^2 cos(2\theta)\overrightarrow{\imath'} - 4a\omega^2 sin(2\theta)\overrightarrow{\jmath'}$$

Traction acceleration $\vec{\gamma}_e$

$$\vec{\gamma}_e = \left(\frac{d^2\overrightarrow{OO'}}{dt^2}\right)_R + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)_R + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{O'M}\right)$$

We have

$$\vec{\gamma}_e = 0 + 0 + \vec{\omega} \times \left(\vec{\omega} \times \overrightarrow{O'M}\right)$$

and

$$\vec{\gamma}_e = \vec{\omega} \times (a\omega sin(2\theta)\overrightarrow{\imath'} + a\omega cos(2\theta)\overrightarrow{\jmath'})$$

$$\vec{\gamma}_e = \begin{vmatrix} \vec{i'} & \vec{j'} & \vec{k'} \\ 0 & 0 & \omega \\ a\omega sin(2\theta) & a\omega cos(2\theta) & 0 \end{vmatrix}$$

and

$$\vec{\gamma}_e = -a\omega^2 cos(2\theta)\vec{i'} + a\omega^2 sin(2\theta)\vec{j'}$$

So absolute acceleration $\vec{\gamma}_a$

$$\vec{\gamma}_a = -a\omega^2 cos(2\theta)\vec{i'} + a\omega^2 sin(2\theta)\vec{j'}$$

Using conversion constitutions, we find

$$\vec{\gamma}_a = -a\omega^2(cos(\theta)\vec{i} - sin(\theta)\vec{j})$$

Chapter III : Dynamics

Dynamics is the study of why objects move. In this chapter we try to answer the questions: Why do objects move? What determines the shape of the path? How do we deduce the laws that describe movement? To answer here, we use what are called Newton's laws. Before that, we give a summary of the most important works that led to the formulation of these laws:

Galilei (1564-1642): He proved experimentally that all bodies have a constant acceleration as they fall, unlike Aristotle, who believed that the speed of a falling body is proportional to its mass, as a heavier body falls faster than a lighter body. Galileo also formulated the concept of the principle of inertia in a primitive way.

Kepler (1571-1630): He found the explanation that links the movement of the planets in the solar system, as he concluded that each planet travels in a specific path around the sun in the form of an ellipse, and also gave the beginnings of the principle of mutual attraction between the different masses.

Isaac Newton (1643-1727): Newton generalized Galilei's principle of inertia (first law), in his book "Mathematical Principles of Natural Philosophy" in 1687, which contains three topics: the principle of inertia, the basic law, and the law of action and reaction. Newton generalized the concept of force, defined for the first time the concept of mass, and formulated the law of universal gravitation.

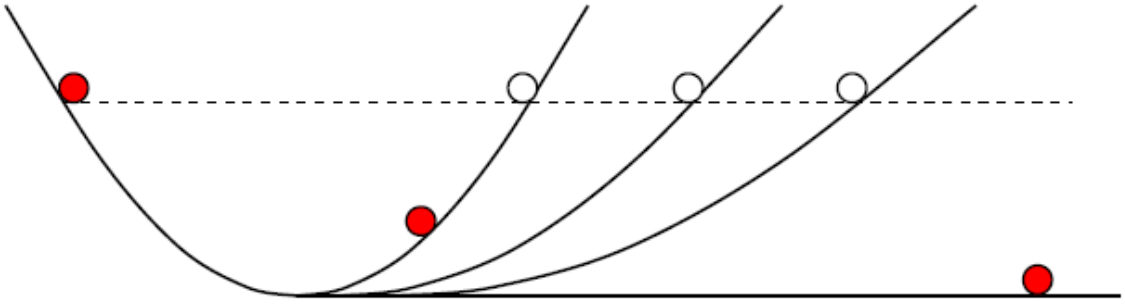
1. Physical body

We consider the body (or **object**) a mass of infinitesimal dimensions, for example we consider the movement of a planet around the sun or the movement of a proton as material bodies.

2. The principle of inertia (first law)

Aristotle: We have a static body on a flat surface, when we pull or push it, the body moves, when the effect of force on the body is removed, it becomes static as it was. From this observation, Aristotle believed that uniform rectilinear motion requires a constant force acting on this body. I also believe that a heavy body falls faster than a lighter body.

Galilei: The idea began when Galilei conducted an experiment that led him to the isolated body model, which is a model that is difficult to achieve in practice, and which led him to discover the principle of inertia.



Galileo concluded that the ball will always reach the same height regardless of the inclination of the plane, and to achieve this we must imagine that the body is isolated. An isolated body is a body that does not interact with other bodies. This happens when the rest of the bodies are very far from it, or all the results of these interactions are non-existent.

Newton: Newton generalized Galilei's principle and declared the first law (the principle of inertia), which states: "Every body remains in its state of rest, or uniform, straight motion, unless acted upon by a force that changes its state." In other words, we write

$$\sum \vec{F} = \vec{0}$$

$$\vec{F} = \vec{0} \quad \Longrightarrow \quad \vec{v} = \vec{0} \quad \Longrightarrow \quad \vec{a} = \vec{0}$$

From the principle of inertia, it becomes clear to us that if a body has non-zero acceleration (accelerating body), then this body is not isolated and is subject to a force.

3. Référentiel galiléen (Inertial frame of reference)

is the frame in which the principle of inertia is realized (the body is isolated).

1- Every frame that moves at a constant velocity relative to a Galilean frame is also a Galilean frame.

2- The velocity and acceleration measured by a Galilean parameter are called absolute velocity and absolute acceleration, and those measured by a non-Galilean parameter are called relative velocity and relative acceleration.

We now mention the well-known Galilean frames:

1- **The Copernic frame**: It is a frame in which the principle is the center of mass of the sun and its three axes are fixed stars very far from the sun. This frame is used to study the movement of planets and spacecraft.

2- **The geocentric frame**: The principle in this frame is the center of mass of the Earth and its axes are the same as the axes of the Copernican frame. This frame is used to study the movement of the moon and satellites surrounding the planet Earth.

3- **The ground frame**: The frame attached to the ground can be considered an inertial frame if it comes to experiments of a short period of time at heights that do not exceed tens of meters.

Chapter III : Dynamics

Mass

Mass is defined as "the amount of matter contained in a body," but this definition does not give an explanation of mass because it defines mass by mass.

- When a person confirms the mass of an object, he weighs it. In fact, this person is measuring the force of the weight or heaviness, and this force would not appear if it were not for the presence of the Earth (another mass) pulling it downward. Therefore, we say that the object has mass if it is able to influence it. on other masses by the force of attraction.

- If we have a car and a truck traveling at the same speed, it would be easier to stop the car than to stop the truck, because the mass of the truck is greater than the mass of the car. The mass of the truck prevents stopping more than the mass of the car prevents it. From here we can define mass, as it is "the physical quantity that expresses Bodies refrain from changing their motion. Mass is a constant, positive quantity that is not related to the parameter.

Force

It is an effect that changes the state of motion of the body. If the body is at rest, it becomes moving, and if the body is moving, it changes its velocity. There are two types of force, contact and dimensional.

Force is symbolized by F and its unit of measurement is the newton N, which is derived from international units.

$$1N = 1Kg\frac{m}{s^2}$$

1- We say that the body is statically balanced

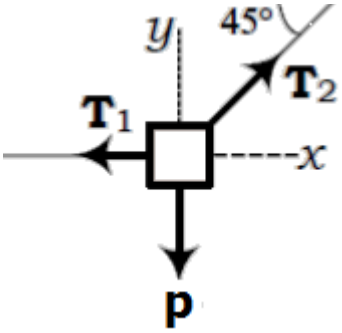
$$\vec{F} = \vec{0} \quad , \quad \vec{v} = \vec{0} \quad \Longrightarrow \quad \vec{a} = \vec{0}$$

2- We say that the body is kinematically balanced

$$\vec{F} = \vec{0} \quad , \quad \vec{v} = \text{const} \quad \Longrightarrow \quad \vec{a} = \vec{0}$$

Example

Calculate the values of T_1 and T_2 if the body is balanced. If $p = 50N$.



Our body is balanced meaning:

$$\sum \vec{F} = \vec{0}$$

$$\vec{p} + \vec{T}_1 + \vec{T}_2 = \vec{0}$$

By projecting onto the (ox) and (oy) axes, we find

$$\begin{cases} T_2 cos(45^0) = T_1 \\ T_2 sin(45^0) = p \end{cases}$$

After the calculation

$$\begin{cases} T_1 = 50N \\ T_2 = 70N \end{cases}$$

Basic forces (basic effects)

All forces existing in nature are derived from the following four basic forces:

1- The force of attraction: This force is described by Newton's law of universal gravitation.

2- Electromagnetic force: This force results from the interaction between electrical charges. The electromagnetic force exists in all phenomena that contain electric or magnetic force.

3- The strong nuclear force: It is the interaction between the nucleons (protons and neutrons) that make up the nucleus of the atom. This force works to prevent the nucleus from collapsing. This force works over a very short distance and is responsible for the basic cohesion of the nucleus. The strong nuclear force is the strongest of the four fundamental forces found in nature.

4- The weak nuclear force: It applies to all particles of matter, especially neutrons. The weak interaction manifests itself in certain types of nuclear reactions, such as radioactivity.

4. Momentum

Motion is not only related to velocity, but also to mass. For example, we have two bodies with different masses that move at the same velocity. How can we differentiate between them?

For this reason, we know here the mathematical expression that expresses momentum.

$$\vec{P} = m\vec{v}$$

If the moving body is free, its momentum is constant.

The change in momentum is due to the presence of an external force.

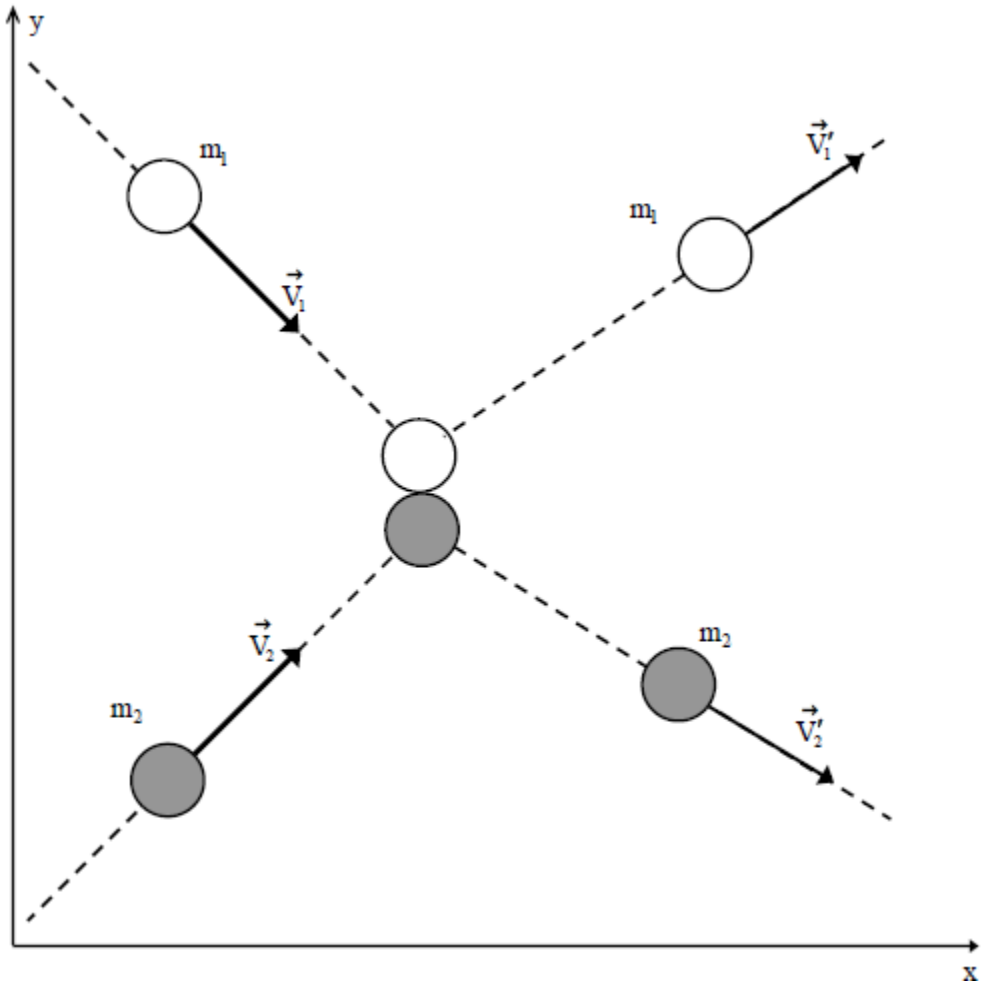
The principle of conservation of momentum

If we have a system consisting of n particles, then the total momentum of this system is the sum of the momentum of the particles as follows:

$$\vec{P} = \vec{P}_1 + \vec{P}_2 + \vec{P}_3 + \cdots = \sum_{i=1}^{n} \vec{P}_n$$

1- The case of a single body: The principle of conservation of momentum states that if a body is free, it moves with a constant momentum.

2- The case of two bodies: Two bodies with masses m_1 and m_2 and velocities $\vec{V}_1$ and $\vec{V}_2$ collide (here friction is non-existent) as in the figure:



After the collision, we notice that the velocities of the two bodies became $\overrightarrow{V'}_1$ and $\overrightarrow{V'}_2$.

The principle of conservation of momentum states that the momentum before the collision is the same as the momentum after the collision. So,

$$\vec{P} = \vec{P}'$$

This means that

$$m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V'}_1 + m_2\vec{V'}_2$$

By projecting onto the (ox) and (oy) axes, we find

$$\begin{cases} m_1V_{1x} + m_2V_{2x} = m_1V'_{1x} + m_2V'_{2x} \\ m_1V_{1y} + m_2V_{2y} = m_1V'_{1y} + m_2V'_{2y} \end{cases}$$

On the other hand, returning to the first statement:

$$\vec{P} = \vec{P}'$$

We have

$$\vec{P}_1 + \vec{P}_2 = \vec{P}'_1 + \vec{P}'_2$$

And

$$\vec{P}'_1 - \vec{P}_1 = -(\vec{P}'_2 - \vec{P}_2) \quad \Longrightarrow \quad \Delta\vec{P}_1 = -\Delta\vec{P}_2$$

5. Newton's laws

5.1 The first law

This is the law that states the principle of inertia that we mentioned previously, which we can know as follows, and this is after learning about the concept of momentum (a body is free if its momentum is constant).

5.2 The second law

It is the basic law of motion, which states (in an inertial parameter, the change in the momentum of the body is proportional to the resultant of the external forces to which it is subjected).

$$\sum \vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v})$$

And

$$\sum \vec{F} = \frac{dm}{dt}\vec{v} + m\frac{d\vec{v}}{dt}$$

If the mass of the body is constant, we write the second law in the expression

$$\sum \vec{F} = m\vec{a}$$

5.3 The third law

which is the law of action and reaction, which states (If an object A exerts a force on object B with a force $\vec{F}_{AB}$, then object B must exert a force of equal magnitude and opposite direction back on object A with a force $\vec{F}_{BA}$).

$$\vec{F}_{AB} = -\vec{F}_{BA}$$

This law represents a certain symmetry in nature: forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as action-reaction, where the force exerted is the action and the force experienced as a consequence is the reaction.

Example 1

A body with a mass of $2Kg$ moves from rest in a straight line with a constant acceleration. It travels a distance of $8m$ in $2s$, and then this body moves at a constant velocity. Calculate the force intensity at each stage.

Solution

From the data we notice that we have two phases for the moving body.

The first stage: We have

$$a = \frac{dv}{dt} \quad \Longrightarrow \quad a\,dt = dv$$

And

$$\int_{v_0}^{v} dv = \int_{0}^{t} a\ dt \quad \Longrightarrow \quad v - v_0 = at \quad \Longrightarrow \quad v = at + v_0$$

On the other hand we have

$$v = \frac{dx}{dt} \quad \Longrightarrow \quad v\,dt = dx$$

and

$$\int_{x_0}^{x} dx = \int_{0}^{t} (at + v_0)\,dt \quad \Longrightarrow \quad x - x_0 = a\int_{0}^{t} t dt + \int_{0}^{t} v_0\,dt$$

$$\Longrightarrow \quad x = \frac{1}{2}\,a\,t^2 + v_0 t + x_0$$

By substituting, we find

$$8 = \frac{1}{2}\,a\,2^2 \quad \Longrightarrow \quad a = 4\frac{m}{s^2}$$

Applying Newton's second law

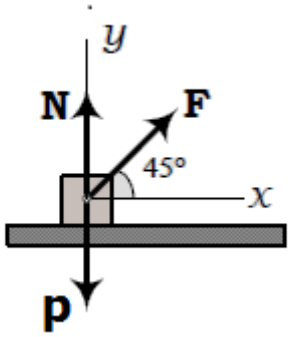
$$F = ma = 8N$$

The second stage: we have the velocity constant

$$F = 0$$

Example 2

A body with a mass of $10Kg$ moves as shown in the figure.



Calculate the value of the force F and the force of reaction N. If $a = 2\frac{m}{s^2}$.

Solution

Apply Newton's second law

$$\sum \vec{F} = m\vec{a}$$

And

$$\vec{N} + \vec{P} + \vec{F} = m\vec{a}$$

By projecting onto the (ox) and (oy) axes, we find

$$\begin{cases} Fcos(45^0) = ma \\ P - N - Fsin(45^0) = 0 \end{cases}$$

After the calculation, we have

$$\begin{cases} F = 28.3N \\ N = 78N \end{cases}$$

Example 3

A mass m moves in space, its position vector is given by the following expression

$$\overrightarrow{OM} = (4t^2 - t^3)\vec{\imath} - 5t\vec{\jmath} + (t^4 - 2)\vec{k}$$

Calculate at the moment $t = 1s$ the velocity, acceleration, momentum, and force applied to the mass.

Solution

1- Calculating velocity

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt}$$

$$\vec{v} = (8t - 3t^2)\vec{\imath} - 5\vec{\jmath} + 4t^3\vec{k}$$

2- Calculating acceleration

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$\vec{a} = (8 - 6t)\vec{\imath} + 12t^2\vec{k}$$

3- Calculating the momentum

$$\vec{p} = m\vec{v}$$

$$\vec{p} = m(8t - 3t^2)\vec{\imath} - 5m\vec{\jmath} + 4mt^3\vec{k}$$

4- Calculating the force

$$\vec{F} = m\vec{a}$$

$$\vec{F} = m(8 - 6t)\vec{\imath} + 12mt^2\vec{k}$$

at the moment $t = 1s$ we have

$$\vec{v} = 5\vec{\imath} - 5\vec{\jmath} + 4\vec{k}$$

$$\vec{a} = 2\vec{\imath} + 12\vec{k}$$

$$\vec{p} = 5m\vec{\imath} - 5m\vec{\jmath} + 4m\vec{k}$$

$$\vec{F} = 2m\vec{\imath} + 12m\vec{k}$$

Example 4

The momentum of a mass m is given by the following expression

$$\vec{p} = 3e^{-t}\vec{\imath} - 2\,cos(t)\,\vec{\jmath} - 3sin(t)\vec{k}$$

Calculate the force applied to the mass. Then determine the velocity and acceleration.

Solution

1- Calculate the force applied to the mass

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\vec{F} = -3e^{-t}\vec{\imath} + 2sin(t)\vec{\jmath} - 3\,cos(t)\,\vec{k}$$

2- Calculating acceleration

$$\vec{a} = \frac{\vec{F}}{m}$$

$$\vec{a} = -\frac{3}{m}e^{-t}\vec{\imath} + \frac{2}{m}sin(t)\vec{\jmath} - \frac{3}{m}cos(t)\,\vec{k}$$

3- Calculating velocity

$$\vec{v} = \frac{\vec{p}}{m}$$

$$\vec{v} = \frac{3}{m}e^{-t}\vec{\imath} - \frac{2}{m}cos(t)\vec{\jmath} - \frac{3}{m}sin(t)\,\vec{k}$$

6. Examples of forces and their types

Now we will learn about the most important forces that we encounter in the study of dynamics.

6.1 The force of gravity p

We know the force of gravity p as follows

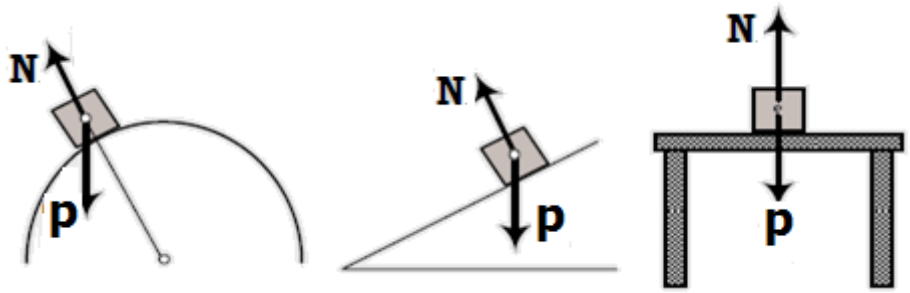
$$p = mg$$

Where g is the Earth's gravity.

$\vec{p}$ Always pointing downwards.

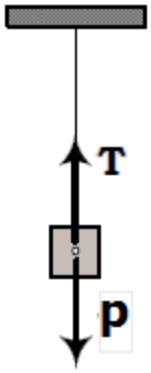
6.2 Reaction force $\vec{N}$

The reaction force $\vec{N}$ is always perpendicular to the surface.



6.3 Tension force $\vec{T}$

If we suspend a body by two ends of a rope, it produces a force equal to the force of gravity and opposite to it in the direction $\vec{T}$ as in the figure.



6.4 Frictional force

The frictional force results from the movement of a solid body on a solid surface (solid friction) or from the movement of a solid body on a liquid surface (viscous friction), and the frictional force is parallel to the surface and opposite the direction of movement.

Static friction f_s

is the force that keeps a body at rest, even in the presence of an external force. Where we have

$$f_s \leq u_s N$$

u_s is the coefficient of static friction. And N is the reaction force.

Example

We place a body with a weight of $80N$ on a horizontal rough surface. We apply a force of $20N$ to it that creates an angle of 30^0 with the horizon. The coefficient of static friction is 0.30.

1- Calculate the intensity of the static friction force.

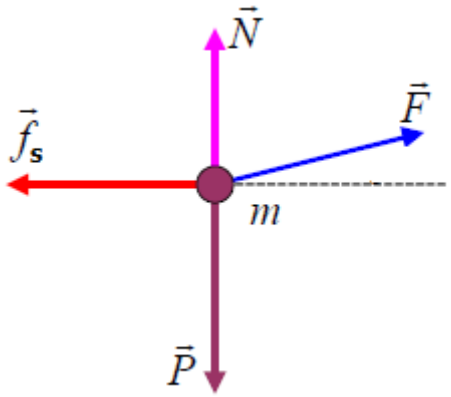
2- Calculate the intensity of the reaction force.

3- Calculate the maximum force of static friction.

4- How strong must the force be for the body to move?

Solution

The body is at rest, applying Newton's second law



$$\sum \vec{F} = \vec{0}$$

$$\vec{N} + \vec{F} + \vec{p} + \vec{f_s} = \vec{0}$$

By projecting onto the (ox) and (oy) axes, we find

$$\begin{cases} Fcos(30^0) - f_s = 0 \\ P - N - Fsin(30^0) = 0 \end{cases}$$

1- Calculate the intensity of the static friction force

$$f_s = Fcos(30^0)$$

We have

$$f_s = 20 \times 0.86 = 17.2N$$

2- Calculate the intensity of the reaction force

$$N = P - Fsin(30^0)$$

$$N = 80 - 20 \times 0.5 = 70N$$

3- Calculating the maximum force of static friction

$$f_s = u_s N = 0.30 \times 70 = 21N$$

4- Calculating the intensity of the force until the body moves

$$f_s = F cos(30^0)$$

$$F = \frac{f_s}{cos(30^0)} = \frac{21}{0.86} = 24.24N$$

Kinetic friction f_c

is the force that resists movement when an object moves over a rough surface. Where we have

$$f_c = u_c N$$

u_c is the coefficient of static friction. And N is the reaction force.

Example

A body with a mass of $10.2Kg$ moves on a horizontal rough surface under the influence of a force of $20N$, making an angle of 45^0 with the horizon. The coefficient of kinetic friction is 0.15.

Here we take $g = 9.81$.

1- Calculate the intensity of the reaction force.

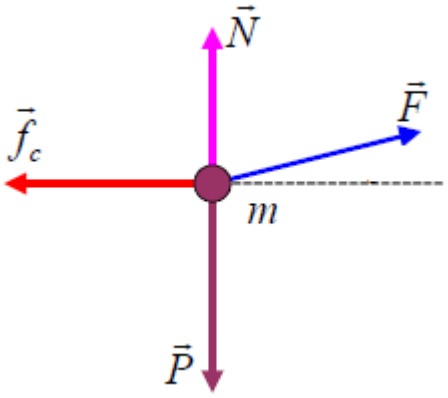
2- Calculate the intensity of the kinetic friction force.

3- What should the value of the acquired acceleration be?

Solution

The body is at rest, applying Newton's second law

$$\sum \vec{F} = m\vec{a}$$



$$\vec{N} + \vec{F} + \vec{p} + \vec{f_c} = m\vec{a}$$

By projecting onto the (ox) and (oy) axes, we find

$$\begin{cases} Fcos(30^0) - f_c = ma \\ P - N - Fsin(30^0) = 0 \end{cases}$$

1- Calculating the intensity of the reaction force: From the second equation we have

$$N = P - Fsin(30^0)$$

$$N = 10.2 \times 9.81 - 20 \times \frac{\sqrt{2}}{2} = 85.86N$$

2- Calculating the intensity of the kinetic friction force: We have

$$f_c = u_c N = 0.15 \times 85.86 = 12.9N$$

3- Calculate the value of the acquired acceleration: from the first equation

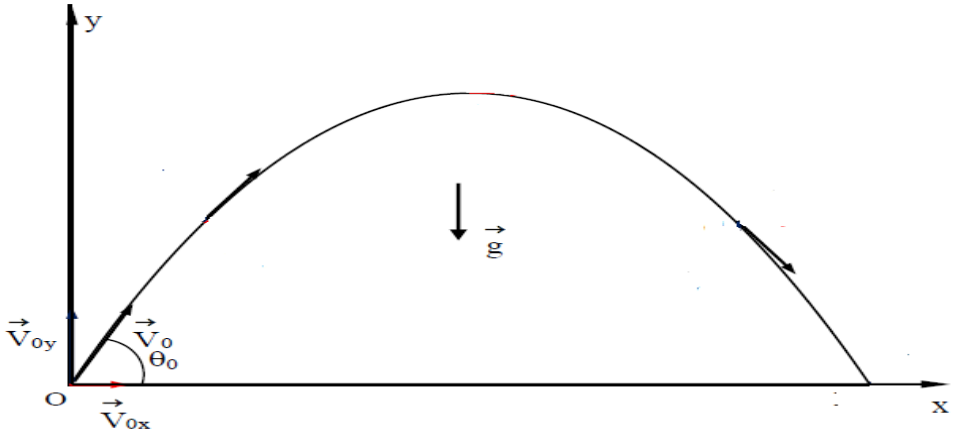
$$a = \frac{Fcos(30^0) - f_c}{m}$$

$$a = \frac{20 \times \frac{\sqrt{2}}{2} - 12.9}{10.2} = 0.12 \frac{m}{s^2}$$

7. Movement applications

7.1 Projectile movement

We throw a projectile with an initial speed of $\vec{V}_0$, making an angle θ_0 with the horizon, as in the figure



The components of acceleration are written as follows

$$\begin{cases} a_x = 0 \\ a_y = -g \end{cases}$$

To find velocity components, we follow the following steps

$$a_x = \frac{dV_x}{dt} \quad \Longrightarrow \quad a_x\, dt = dV_x$$

$$\int_{V_{0x}}^{V_x} dV_x = \int_0^t a_x\, dt \quad \Longrightarrow \quad V_x - V_{0x} = \int_0^t a_x\, dt$$

We have

$$V_x = \int_0^t a_x\, dt + V_{0x}$$

In the same way we find

$$V_y = \int_0^t a_y\, dt + V_{0y}$$

We have

$$\begin{cases} V_{0x} = V_0 \cos(\theta_0) \\ V_{0y} = V_0 \sin(\theta_0) \end{cases}$$

The components of velocity

$$\begin{cases} V_x = \int_0^t 0 \ dt + V_0 \cos(\theta_0) = V_0 \cos(\theta_0) \\ V_y = \int_0^t -g \ dt + V_0 \sin(\theta_0) = -gt + V_0 \sin(\theta_0) \end{cases}$$

On the other hand we have

$$V_X = \frac{dX}{dt}$$

$$\Longrightarrow \quad V_X \, dt = dX$$

and

$$\int_{X_0}^{X} dX = \int_0^t V_X \, dt \quad \Longrightarrow \quad X - X_0 = \int_0^t V_X \, dt$$

we have

$$X = \int_0^t V_X \, dt + X_0$$

In the same way we find

$$Y = \int_0^t V_Y \, dt + Y_0$$

where

$$\begin{cases} X_0 = 0 \\ Y_0 = 0 \end{cases}$$

The position compounds

$$\begin{cases} X = \int_0^t V_0 \cos(\theta_0) \ dt + X_0 \\ Y = \int_0^t (-gt + V_0 \sin(\theta_0)) \, dt + Y_0 \end{cases}$$

So,

$$\begin{cases} X = (V_0 \cos(\theta_0))t \\ Y = -\frac{1}{2} g t^2 + (V_0 \sin(\theta_0))t \end{cases}$$

We have

$$t = \frac{X}{V_0 \cos(\theta_0)}$$

By substituting into the second equation, we find the trajectory equation for the projectile's motion as follows

$$Y = -\frac{g}{2(V_0 \cos(\theta_0))^2} X^2 + tan(\theta_0)X$$

1- Calculating the maximum height: It is the greatest height the projectile reaches, and this is when it is

$$V_Y = 0$$

We have

$$-gt + V_0 \sin(\theta_0) = 0$$

And

$$t = \frac{V_0 \sin(\theta_0)}{g}$$

By substituting into the Y equation, we find the greatest height the projectile reaches

$$Y_{Max} = \frac{(V_0 \sin(\theta_0))^2}{2g}$$

2- Calculating the maximum distance: It is the greatest distance the projectile travels, and this is when it is $Y = 0$. By substituting into the path equation, we find the greatest distance traveled by the projectile

$$X_{Max} = tan(\theta_0) \frac{2(V_0 \cos(\theta_0))^2}{2g}$$

$$X_{Max} = \frac{2{V_0}^2 \cos(\theta_0) \sin(\theta_0)}{g} = \frac{2{V_0}^2 \sin(2\theta_0)}{g}$$

Example

We launch a projectile from ground level vertically upward at a speed of $10\,\frac{m}{s}$.

1- How high does the projectile reach?

2- What is the velocity of the projectile after 1.5 seconds?

3- What is the time interval between the moment of ejection and the moment the projectile reaches the ground?

Solution

1- The highest account that the projectile reaches.

$$Y_{Max} = \frac{(V_0 \sin(\theta_0))^2}{2g} = \frac{(10 \sin\left(\frac{\pi}{2}\right))^2}{2 \times 9.8} = 5.1m$$

2- Calculate the speed of the projectile after 1.5 seconds.

$$\begin{cases} V_x = V_0 \cos(\theta_0) = V_0 \cos\left(\frac{\pi}{2}\right) = 0 \\ V_y = -gt + V_0 \sin(\theta_0) = -9.8 \times 1.5 + 10 \sin\left(\frac{\pi}{2}\right) = -4.7 \end{cases}$$

So,

$$\mathrm{V} = \sqrt{(V_x)^2 + (V_y)^2} = 4.7\,\frac{m}{s}$$

3- Calculating the time interval between the moment of ejection and the moment the projectile reaches the ground. Its meaning

$$-gt + V_0 \sin(\theta_0) = 0$$

We have

$$t = \frac{V_0 \sin(\theta_0)}{g}$$

And

$$t = \frac{10 \sin\left(\frac{\pi}{2}\right)}{9.8} = 2.04s$$

7.2 Newton's Law of Universal Gravitation

In some references, it is mentioned as Newton's fourth law

$$F = \frac{Gm_1m_2}{r^2}$$

Where G is the general gravitational constant.

$$G = 6.67 \times 10^{-11} \frac{Nm^2}{Kg^2}$$

And r is the distance between the two objects m_1 and m_2.

Example

If we have a body m on the surface of the Earth.

$$F = p = mg = \frac{GmM_T}{{R_T}^2}$$

M_T is the mass of the Earth, and R_T is the radius of the Earth. So we have it

$$g = \frac{GM_T}{{R_T}^2}$$

where

$$M_T = 5.98 \times 10^{24} Kg$$

$$R_T = 6.37 \times 10^{6} m$$

We have

$$g = 9.8 NKg^{-1}$$

7.3 Kepler's third law

It states that (the square of the period of motion of a planet or moon is proportional to the cube of the large radius of its path). If T is the period of motion and a is the large radius, then

$$T^2 \propto a^3$$

How do we calculate the constant of proportionality? applying Newton's second law

$$\sum \vec{F} = m\vec{a}$$

Where $\vec{a}$ is the centripetal acceleration.

$$F = \mathrm{G}\frac{mM}{r^2} = m\frac{v^2}{a}$$

We have circular motion

$$v = \omega a = \frac{2\pi}{T}a$$

Substituting, we find the mathematical expression for Kepler's third law

$$T^2 = \frac{4\pi^2}{GM}a^3$$

7.4 Satellite movement

We have a satellite of mass m orbiting on Earth. How do we find the distance of the moon on the surface of the earth? Applying Newton's second law

$$F = \frac{GmM}{r^2} = m\frac{v^2}{r}$$

The motion is circular $v = \omega r$, so

$$F = \frac{GmM}{r^2} = m\omega^2 r$$

Finally, we find the distance of the satellite on the surface of the Earth

$$r = (G\frac{M}{\omega^2})^{\frac{1}{3}}$$

Example

What is the velocity of a satellite rotating at an altitude of $1000km$ above the Earth's surface?

Solution

Apply Newton's second law

$$F = \frac{GmM}{r^2} = m\frac{v^2}{r}$$

where

$$v^2 = \frac{GM}{r}$$

Where R is the radius of the Earth, h is the distance of the Moon on the Earth's surface

$$r = R + h$$

We have

$$r = 6370 + 1000 = 7370 \times 10^3 m$$

and

$$v^2 = \frac{6.67 \times 10^{-11} \frac{Nm^2}{Kg^2} 5.98 \times 10^{24} Kg}{7370 \times 10^3 m}$$

Finally, we find the velocity of the satellite

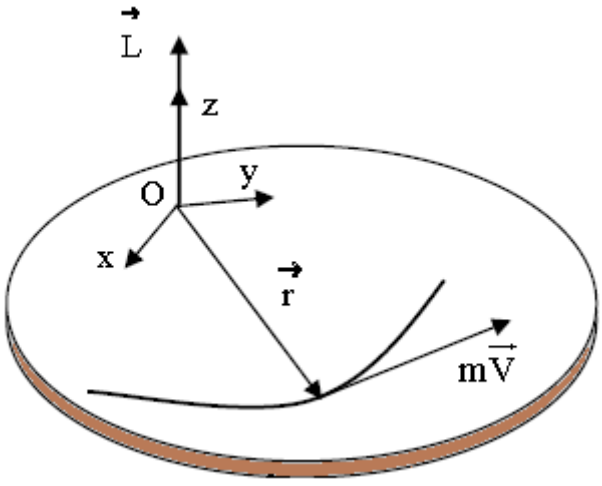
$$v = 7.35 \times 10^3 \frac{m}{s}$$

8. Angular momentum and Torque

We know it by the following expression

$$\vec{L} = \vec{r} \times \vec{p}$$

Where $\vec{r}$ is the position vector and $\vec{p}$ is the momentum vector.



Through our definition of the vector product, we can write the expression for the kinetic moment as follows

$$\vec{L} = |\vec{r}||\vec{p}|\sin(\vartheta)$$

ϑ is the angle between the two vectors $\vec{r}$ and $\vec{p}$.

On the other hand we have

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d}{dt}(\vec{r} \times m\vec{V})$$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times m\vec{V} + \vec{r} \times m\frac{d\vec{V}}{dt} = \vec{V} \times m\vec{V} + \vec{r} \times m\frac{d\vec{V}}{dt}$$

Where

$$\vec{V} \times m\vec{V} = \vec{0}$$

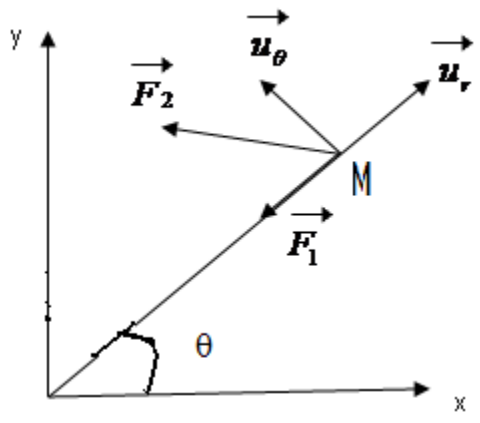
$$\vec{r} \times m\frac{d\vec{V}}{dt} = \vec{r} \times m\vec{a} = \vec{r} \times \vec{F}$$

We now know the physical concept $\vec{\tau}$, which is called the moment of force

$$\vec{\tau} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$$

Example

The mass m moves on the plane (oxy) under the influence of the two forces $\vec{F}_1 = -\overrightarrow{OM}$ and $\vec{F}_2 = -2m\vec{v}$ as in the figure.



1 - We use polar coordinates and the basic principle of dynamic. Prove that the time equation of motion is written by the relation $r = ae^{-t}$.

Given: $\theta = \omega t$, a and ω are constant. And at $t = 0$ we have $r = a$.

2 - Use the angular moment theory to find the relationship $r = ae^{-t}$.

Solution

1 - Applying the basic principle of dynamics.

$$\sum \vec{F} = m\vec{a}$$

$$\vec{F}_1 + \vec{F}_2 = m\vec{a}$$

We use polar coordinates

$$\begin{cases} \vec{F}_1 = -r\vec{u}_r \\ \vec{F}_2 = -2m\vec{v} = -2m(r\vec{u}_r + r\dfrac{d\theta}{dt}\vec{u}_\theta) \end{cases}$$

On the other hand

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right)\vec{u}_r + (r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt})\vec{u}_\theta$$

Where

$$\begin{cases} \dfrac{d\theta}{dt} = \omega \\ \dfrac{d^2\theta}{dt^2} = 0 \end{cases}$$

We have

$$\begin{cases} \vec{F}_1 = -r\vec{u}_r \\ \vec{F}_2 = -2m(r\vec{u}_r + r\omega\vec{u}_\theta) \end{cases}$$

And

$$\vec{a} = \left(\frac{d^2r}{dt^2} - r\omega^2\right)\vec{u}_r + (2\omega\frac{dr}{dt})\vec{u}_\theta$$

We find

$$-r\vec{u}_r - 2m(r\vec{u}_r + r\omega\vec{u}_\theta) = m\left(\frac{d^2r}{dt^2} - r\omega^2\right)\vec{u}_r + m(2\omega\frac{dr}{dt})\vec{u}_\theta$$

And

$$-(r + 2mr)\vec{u}_r - 2mr\omega\vec{u}_\theta = m\left(\frac{d^2r}{dt^2} - r\omega^2\right)\vec{u}_r + m(2\omega\frac{dr}{dt})\vec{u}_\theta$$

Comparison between the two parties

$$-2mr\omega = m(2\omega\frac{dr}{dt})$$

We have

$$-r = \frac{dr}{dt}$$

$$-dt = \frac{dr}{r}$$

We integrate both sides

$$-t = ln(r)$$

And

$$r = cont.e^{-t}$$

From the data at $t = 0$ we have $r = a$. We find

$$r = ae^{-t}$$

2 - Use the angular moment theory to find the relationship $r = ae^{-t}$.

We have

$$\frac{d\vec{L}}{dt} = \sum \overrightarrow{OM} \times \vec{F} = \overrightarrow{OM} \times \vec{F}_1 + \overrightarrow{OM} \times \vec{F}_2$$

Where

$$\overrightarrow{OM} \times \vec{F}_1 = -r\vec{u}_r \times r\vec{u}_r = 0$$

$$\overrightarrow{OM} \times \vec{F}_2 = -r\vec{u}_r \times 2m(r\vec{u}_r + r\omega\vec{u}_\theta) = -2mr^2\omega\vec{k}$$

On the other hand

$$\vec{L} = \overrightarrow{OM} \times m\vec{v} = r\vec{u}_r \times m(r\vec{u}_r + r\omega\vec{u}_\theta) = mr^2\omega\vec{k}$$

So,

$$\frac{d\vec{L}}{dt} = 2mr\omega\frac{dr}{dt}\vec{k}$$

In the end we find

$$-r = \frac{dr}{dt}$$

And

$$r = ae^{-t}$$

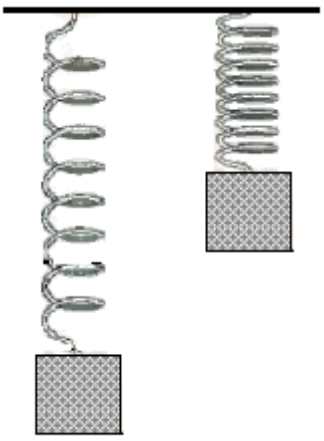
9. Equation of motion

Example 1

We have an elastic spring of length l_0 at rest. If its length changes due to the weight of the mass, its length becomes l. It gains the force of return (Hooke's law)

$$\vec{F} = -k(l_e - l_0)$$

Where k is the spring tension constant and l_e is the length of the spring in equilibrium.



Applying the basic principle of motion in equilibrium.

$$\sum \vec{F} = \vec{0}$$

So,

$$\vec{P} + \vec{F} = \vec{0}$$

$$mg - k(l_e - l_0) = 0$$

In the case of motion, the length of the spring becomes $l_e + x$, applying the basic principle of motion

$$\sum \vec{F} = m\vec{a}$$

$$\vec{P} + \vec{F} = m\vec{a}$$

We have

$$mg - k(l_e + x - l_0) = ma$$

$$mg - k(l_e - l_0) - kx = ma$$

and

$$-kx = ma$$

We find

$$m\frac{d^2x}{dt^2} + kx = 0$$

Or we write

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

The equation of motion is a second-order differential equation

$$\ddot{x} + \omega^2 x = 0$$

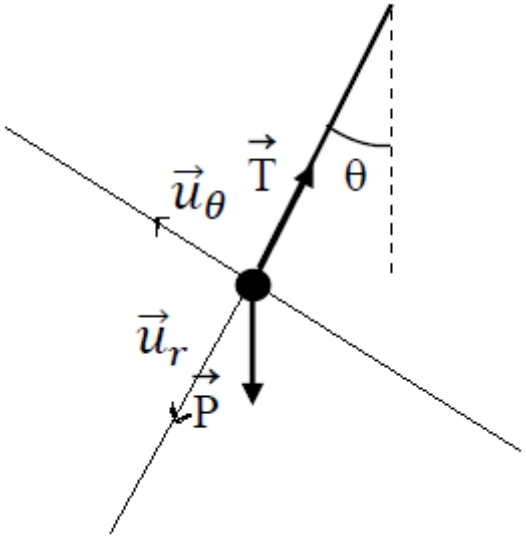
Where $\omega = \sqrt{\frac{k}{m}}$ is the pulse of motion, the solution to the equation of motion is of the form

$$x = Acos(\omega t + \varphi)$$

A is the amplitude of the motion, A and φ can be determined from the initial conditions.

Example 2

We have a simple pendulum as shown in the figure.



applying the basic principle of motion

$$\vec{P} + \vec{T} = m\vec{a}$$

Using polar coordinates we find

$$mgcos(\theta)\vec{u}_r - mgsin(\theta)\vec{u}_\theta - T\vec{u}_r = m\vec{a}$$

We write the acceleration in polar coordinates

$$\vec{a} = l\ddot{\theta}\vec{u}_\theta - l\dot{\theta}^2\vec{u}_r$$

We have

$$mgcos(\theta)\vec{u}_r - mgsin(\theta)\vec{u}_\theta - T\vec{u}_r = ml\ddot{\theta}\vec{u}_\theta - ml\dot{\theta}^2\vec{u}_r$$

Projecting onto the $\vec{u}_\theta$ axis, we find

$$-mgsin(\theta) = ml\ddot{\theta}$$

and

$$l\ddot{\theta} + gsin(\theta) = 0$$

The equation of motion is a second-order differential equation. If θ is small, then

$$sin(\theta) = \theta$$

finally

$$\ddot{\theta} + \omega^2\theta = 0$$

Where $\omega = \sqrt{\frac{k}{m}}$ is the pulse of motion, the solution to the equation of motion is of the form

$$\theta = Acos(\omega t)$$

Chapter IV: Work and energy

We previously used the principles of motion (Newton's laws) to deduce the equation of motion from our knowledge of the expression of forces affecting the system. In this chapter, we study motion without the need to use these forces and without knowing the details of motion such as path, relying on the concept of work and energy, where The most important characteristics of movement can be deduced.

1. Work of a constant force

The work of a constant force $\vec{F}$ when it moves from point A to point B is given by the scalar product.

$$W_{AB} = \vec{F}.\overrightarrow{AB}$$

Where W_{AB} is work, its unit of measurement is the joule J.

$\overrightarrow{AB}$ is the displacement vector from point A to point B.

Using the definition of a scalar product we can write

$$W_{AB} = \|\vec{F}\|.\|\overrightarrow{AB}\|\cos(\alpha)$$

α is the angle between the force $\vec{F}$ and the displacement vector $\overrightarrow{AB}$.

Note

1- The work of the force is not related to the path followed, that is, the work W_{AB} of the force $\vec{F}$ has the same value regardless of the path.

2- If $\overrightarrow{AB} \perp \vec{F}$ then $W_{AB}(\vec{F}) = 0$.

3- If $W_{AB} > 0$, we say that work is a motive, and this is when $0 < \alpha < 90^0$.

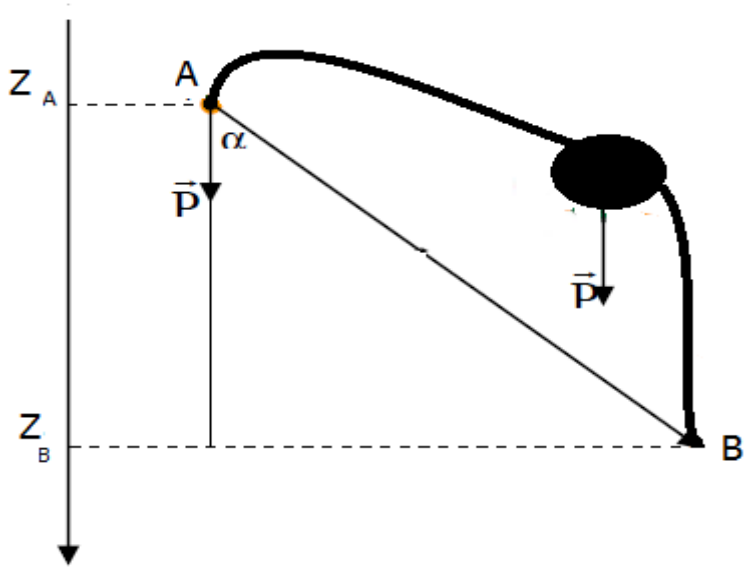
4- If $W_{AB} < 0$, we say that the work is resistant, and this is when $90^0 < \alpha < 180^0$.

Example

The work of the force of gravity.

We now calculate the work of the force of gravity $\vec{P} = m\vec{g}$ when the center of mass is from point A to point B.

$$W_{AB}(\vec{P}) = \|\vec{P}\|.\|\overrightarrow{AB}\|\cos(\alpha)$$



From our figure, we have

$$\cos(\alpha) = \frac{Z_B - Z_A}{AB}$$

So the work of the force of gravity becomes as follows

$$W_{AB}(\vec{P}) = P.AB\frac{Z_B - Z_A}{AB}$$

We have

$$W_{AB}(\vec{P}) = mg(Z_B - Z_A)$$

2. The work of an unstable force

If the force $\vec{F}$ is variable in intensity, then its work is not equal to $\vec{F}.\overrightarrow{AB}$, here we divide the path into elemental transitions $d\vec{r}$ and we call the work of the force during the elemental transition dW, which is calculated as follows

$$dW = \vec{F}.d\vec{r}$$

The total work during the elemental transition from point A to point B is the sum of the elemental works dW. And we write

$$W_{AB} = \int_A^B dW = \int_{r_A}^{r_B} \vec{F}.d\vec{r}$$

We use this relationship if the force and elemental displacement are defined as follows

$$\begin{cases} \vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \\ d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \end{cases}$$

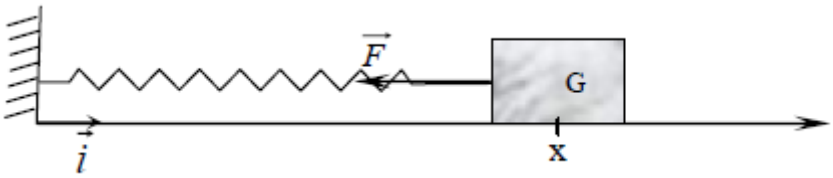
So,

$$dW = \vec{F}.d\vec{r} = F_x dx + F_y dy + F_z y$$

Example

The operation of the return force

Calculate the work required for a spring stretched by x as shown in the figure



1- Calculate the work required W_1, to move the mass from x to 0.

We have

$$\begin{cases} \vec{F} = -kx\vec{\imath} \\ d\vec{r} = dx\vec{\imath} \end{cases}$$

so

$$dW_1 = \vec{F}.d\vec{r} = -kxdx$$

We find

$$W_1 = \int_x^0 dW_1 = \int_x^0 -kxdx = \frac{1}{2}kx^2$$

2- Calculate the work required W_2 to move the mass from 0 to x.

$$W_2 = \int_0^x dW_2 = \int_0^x -kxdx = -\frac{1}{2}kx^2$$

Example

In this example, we calculate the work of the force of gravity using the expression dW.

We have

$$W_{AB}(\vec{P}) = \int -mg\vec{k}(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

and

$$W_{AB}(\vec{P}) = \int_{Z_B}^{Z_A} -mgdz = mg(Z_B - Z_A)$$

3. Power

We know the instantaneous power by the following relationship

$$P = \frac{dW}{dt}$$

We know the average power by the following relationship

$$P = \frac{W_{AB}}{\Delta t}$$

Where W is work in joules, t is time in seconds. The unit of measurement for power P is watts.

5. Kinetic energy E_c

We apply the basic principle of motion with the expression elemental work.

We have

$$dW = F.dr = m\frac{dv}{dt}dr$$

and

$$dW = m\frac{dr}{dt}dv = mvdv$$

$$W = m\int_A^B vdv$$

In the end we find

$$W_{AB}(\vec{F}) = \frac{1}{2}m{v_B}^2 - \frac{1}{2}m{v_A}^2$$

The term $\frac{1}{2}mv^2$ is called kinetic energy and is written

$$E_c = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

The theory of kinetic energy states that the change in the kinetic energy of a body between positions A and B is equal to the work of the net force applied to the body. And we write

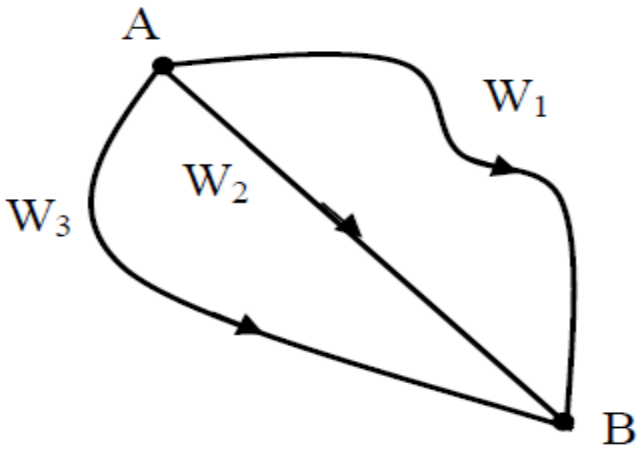
$$W = \Delta E_c$$

6. Conservative force

We say that force $\vec{F}$ is conservative if it fulfills the following two conditions.

The first condition: The work of the force $\vec{F}$ between points A and B is independent of the path.

Example



We have

$$W_1(\vec{F}) = W_2(\vec{F}) = W_3(\vec{F}) = W_{AB}(\vec{F})$$

and

$$\int_A^B \vec{F}.d\vec{r}_1 = \int_A^B \vec{F}.d\vec{r}_2 = \int_A^B \vec{F}.d\vec{r}_3$$

The second condition

The work of the force $\vec{F}$ on a closed path is zero.

We start with the first condition, where the work is independent of the path.

$$\int_A^B \vec{F}.d\vec{r}_1 = \int_A^B \vec{F}.d\vec{r}_2$$

and

$$\int_A^B \vec{F}.d\vec{r}_1 - \int_A^B \vec{F}.d\vec{r}_2 = 0$$

$$\int_A^B \vec{F}.d\vec{r}_1 + \int_B^A \vec{F}.d\vec{r}_2 = 0$$

We find

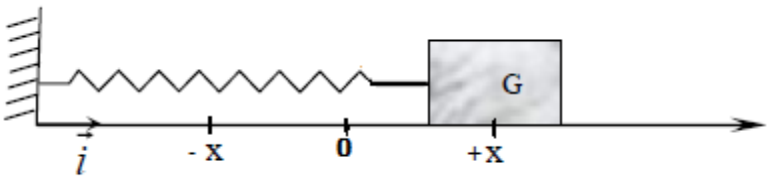
$$W_1(\vec{F})\big|_A^B + W_1(\vec{F})\big|_B^A = 0$$

7. Potential energy E_p

Example 1

Elastic potential energy

We attract a mass m connected to a spring to the $+x$ position and then let it go. The center of mass moves from the $+x$ position to the $-x$ position, then it searches for equilibrium at position 0, as in the figure



When the center of mass is at the $+x$ position, it has no kinetic energy $E_c = 0$ because its velocity is zero through this position. We say here that the mass has potential energy E_p and the same applies at the $-x$ position. In the end, the mass stops at position 0 (spring balance). Here the mass has neither kinetic energy nor potential energy. The following table shows how energy is transformed from kinetic energy to potential energy and vice versa.

Mass in $-x$	Mass between 0 and $-x$	Mass in 0	Mass between $+x$ and 0	Mass in $+x$	position / energy
0	E_c	Max E_c	E_c	0	E_c
Max E_p	E_c	0	E_c	Max E_p	E_c

Energy is transformed from kinetic energy to potential energy and vice versa, but their sum remains constant. If kinetic energy increases, potential energy decreases and vice versa until $E_c + E_p = const$

We call the expression $E_c + E_p$ mechanical energy or total energy.

$$E_T = E_c + E_p$$

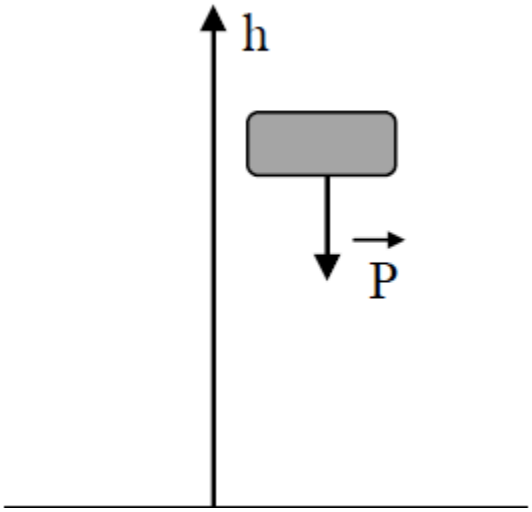
where

$$\Delta E_c + \Delta E_p = 0$$

Example 2

Gravitational potential energy

1- We study the free fall of a body at a height h from the surface of the Earth, as in the figure.



At the beginning of the fall, there is no kinetic energy for the body because its velocity is zero. The sentence (earth + mass) has gravitational potential energy, which after falling is transformed into kinetic energy.

2- We now throw this body upward with an initial velocity v_0, thus $E_c = \frac{1}{2}mv_0{}^2$ When the body reaches the highest height, this energy is transformed into gravitational potential energy and then transformed again into kinetic energy.

Calculating potential energy E_p

As we mentioned previously, the change in total energy between points A and B is zero.

We write

$$\Delta E_c|_A^B + \Delta E_p\big|_A^B = 0$$

and

$$\Delta E_p\big|_A^B = -\Delta E_c|_A^B = -W_{AB}(\vec{F}) = -\int_A^B \vec{F}.d\vec{r}$$

We have

$$E_p(B) - E_p(A) = -\int_A^B \vec{F}.d\vec{r}$$

7.1 Calculating the elastic potential energy

From the previous example when we calculated the work of the spring's return force. we've got

$$\begin{cases}\vec{F} = -kx\vec{\imath} \\ d\vec{r} = dx\vec{\imath}\end{cases}$$

When the spring is at $x = 0$, it has no potential energy

$$E_p(x) - E_p(0) = -\int_0^x \vec{F}.d\vec{r}$$

Where $E_p(0) = 0$, we have

$$E_p(x) = -\int_0^x -kxdx = \frac{1}{2}kx^2$$

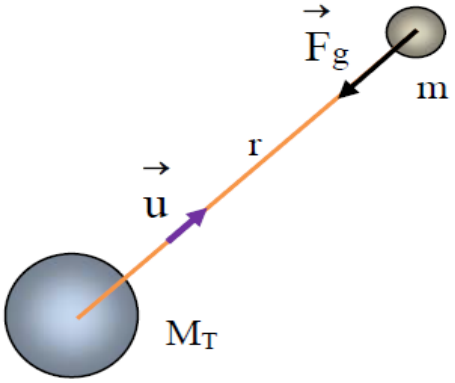
In the general case we write the expression for the elastic potential energy of the spring

$$E_p(\Delta x) = \frac{1}{2}k\Delta x^2$$

7.2 Calculating gravitational potential energy

From our figure.

$$\vec{F}_g = -G\frac{M_T m}{r^2}\vec{u}$$



From the force relationship, it becomes clear to us that the force goes to zero $\vec{F}_g \to 0$ when $r \to \infty$.

So,

$$E_p(r) - E_p(\infty) = -\int_{\infty}^{r} \vec{F}_g . d\vec{r}$$

$E_p(\infty) = 0$, we have

$$E_p(r) = -\int_{\infty}^{r} -G\frac{M_T m}{r^2}\vec{u}.d\vec{r}$$

Where

$$\vec{u}.d\vec{r} = \|\vec{u}\|.\|d\vec{r}\|\cos(\emptyset) = dr$$

We have

$$E_p(r) = GM_T m \int_{\infty}^{r} \frac{dr}{r^2} = -G\frac{M_T m}{r}$$

7.3 Calculating the potential energy of a body under the influence of the force of gravity

Returning to the free fall example.

Since the force of gravity on the ox axis is non-existent.

$$\Delta E_p\big|_A^B = -\Delta E_c\big|_A^B = -W_{AB}(\vec{P})$$

And because

$$P_y = -mg$$

We have

$$\Delta E_p\big|_A^B = -W_{AB}(\vec{P}) = mg(y_{B-}y_A) = mg\Delta y$$

And

$$E_p = mgy$$

h is the height. Finally, we write

$$E_p(h) = mgh$$

Note

We say that the force $\vec{F}(x,y,z)$ is derived from a potential if

$$\vec{F}(x,y,z) = -\overrightarrow{grad}E_p(x,y,z) = \vec{\nabla}E_p(x,y,z) = -\frac{\partial E_p}{\partial x}\vec{\imath} - \frac{\partial E_p}{\partial y}\vec{\jmath} - \frac{\partial E_p}{\partial z}\vec{k}$$

And

$$\overrightarrow{rot}\vec{\mathrm{F}}(\mathrm{x},\mathrm{y},\mathrm{z}) = \vec{\nabla} \times E_p(\mathrm{x},\mathrm{y},\mathrm{z}) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

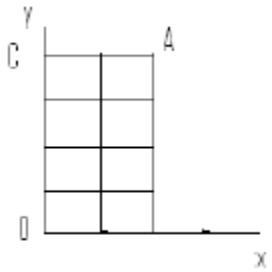
Example 1

We have a material point under the influence of a force $\vec{\mathrm{F}}$ where

$$\vec{\mathrm{F}} = (x - ay)\vec{\imath} + (3y - 2x)\vec{\jmath}$$

1- Calculate the work necessary for the force $\vec{\mathrm{F}}$ to move the point from the principle point $A(2,4)$ to through point $C(0,4)$.

2- Calculate the value of a so that the force $\vec{\mathrm{F}}$ is conservative, then extract the potential energy resulting from the force $\vec{\mathrm{F}}$.



Solution

1- Calculate the work of the force $\vec{\mathrm{F}}$. we've got

$$dW = \vec{F}.d\vec{r} = F_x dx + F_y dy$$

$$dW = (x - ay)dx + (3y - 2x)dy = W_{OC} + W_{CA}$$

For the path from O to C we have

$$x = 0 \rightarrow dx = 0$$

We have

$$W_{OC} = \int_0^4 3ydy = 24$$

For the path from C to A we have

$$y = 4 \rightarrow dy = 0$$

We have

$$W_{CA} = \int_0^2 (x - 4a)dx = 2 - 8a$$

The work of the force $\vec{F}$

$$W = (26 - 2a)\ joule$$

2- Calculate the value of a so that the force $\vec{F}$ is conservative.

$$\nabla \times \vec{F}(x, y, z) = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x - ay) & (3y - 2x) & 0 \end{vmatrix} = (-2 + a)\vec{k} = 0$$

So,

$$a = 2$$

We now calculate the potential energy resulting from the force $\vec{F}$.

$$\vec{F}(x,y,z) = -\nabla E_p(x,y,z) = -\frac{\partial E_p}{\partial x}\vec{\imath} - \frac{\partial E_p}{\partial y}\vec{\jmath} - \frac{\partial E_p}{\partial z}\vec{k}$$

We have

$$\begin{cases} \dfrac{\partial E_p}{\partial x} = -(x-2y) \\ \dfrac{\partial E_p}{\partial y} = -(3y-2x) \end{cases}$$

and

$$E_p = -\int (x-2y)dx = -\frac{x^2}{2} + 2xy + C(y)$$

$$\frac{\partial E_p}{\partial y} = -(3y-2x) = 2x + \frac{\partial C(y)}{\partial y}$$

We have

$$C(y) = -3\,\frac{y^2}{2} + canst$$

In the end we find

$$E_p = -\int (x-2y)dx$$

$$E_p = -\frac{x^2}{2} + 2xy - 3\,\frac{y^2}{2} + canst$$

Differential Calculus

1. Cartesian coordinates

The unit vectors of the Cartesian basis are ($\vec{\imath}, \vec{\jmath}, \vec{k}$) . we define the scalar function $f(x, y, z)$ and the vector $\vec{A}$

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}$$

The expression for the differential element is

$$dr = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

and the volume element is

$$dV = dxdydz$$

1.1 Gradient

from which we deduce the expression of the gradient of the function $f(x, y, z)$

$$\nabla f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$

and therefore of the operator ∇ (called **del**)

$$\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

We also deduce the **Laplacian** ($\Delta = \nabla^2$) form of the function $f(x, y, z)$

$$\Delta = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

1.2 The Divergence

The divergence of the vector $\vec{A}$ is a scalar that can be represented as

$$\nabla . \vec{A} = \left(\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right).(A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k})$$

$$\nabla . \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

1.3 The Curl

The Curl of the vector $\vec{A}$ is a vector that can be represented as

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}$$

Example

We have the position vector $\vec{r}$

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

and the magnitude of the position vector $\vec{r}$

$$r = \sqrt{x^2 + y^2 + z^2}$$

1- Find the gradient of $\vec{r}$.

2- Calculate the divergence of $\vec{r}$.

3- Calculate the divergence of $\vec{u}_r$.

4- Calculate the curl of $\vec{r}$.

5- Calculate the curl of $\vec{u}_r$.

Solution

1- The gradient

$$\nabla r = \frac{\partial r}{\partial x}\vec{\imath} + \frac{\partial r}{\partial y}\vec{\jmath} + \frac{\partial r}{\partial z}\vec{k}$$

Where

$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2x}{2\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2y}{2\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial r}{\partial z} = \frac{\partial}{\partial z}\left(\sqrt{x^2 + y^2 + z^2}\right) = \frac{2z}{2\sqrt{x^2 + y^2 + z^2}}$$

So,

$$\nabla r = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\vec{\imath} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\vec{\jmath} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\vec{k}$$

$$\nabla r = \frac{x\vec{\imath} + y\vec{\jmath} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

And

$$\nabla r = \frac{\vec{r}}{r} = \vec{u}_r$$

2- The divergence

$$\frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = \frac{\partial z}{\partial z} = 1$$

We have

$$\nabla . \vec{r} = 1 + 1 + 1 = 3$$

3- The divergence of $\vec{u}_r$

We have

$$\vec{u}_r = \frac{\vec{r}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{\sqrt{x^2 + y^2 + z^2}}$$

And

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{\sqrt{x^2 + y^2 + z^2} - \frac{2x^2}{2\sqrt{x^2+y^2+z^2}}}{x^2 + y^2 + z^2}$$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{y^2 + z^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}$$

And

$$\frac{\partial}{\partial y}\left(\frac{y}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{x^2 + z^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) = \frac{x^2 + y^2}{(x^2 + y^2 + z^2)\sqrt{x^2 + y^2 + z^2}}$$

So,

$$\nabla.\vec{u}_r = \frac{y^2+z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$+\frac{x^2+z^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}+\frac{x^2+y^2}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

And

$$\nabla.\vec{u}_r = \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)\sqrt{x^2+y^2+z^2}}$$

$$\nabla.\vec{u}_r = \frac{2}{\sqrt{x^2+y^2+z^2}} = \frac{2}{r}$$

4- The curl of $\vec{r}$.

$$\nabla \times \vec{r} = \left(\frac{\partial z}{\partial y}-\frac{\partial y}{\partial z}\right)\vec{\imath}+\left(\frac{\partial x}{\partial z}-\frac{\partial z}{\partial x}\right)\vec{\jmath}+\left(\frac{\partial y}{\partial x}-\frac{\partial x}{\partial y}\right)\vec{k}$$

We have

$$\nabla \times \vec{r} = (0)\vec{\imath}+(0)\vec{\jmath}+(0)\vec{k}$$

And

$$\nabla \times \vec{r} = \vec{0}$$

5- The curl of $\vec{u}_r$.

$$\nabla \times \vec{u}_r = \left(\frac{\partial}{\partial y}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial z}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)\right)\vec{\imath}$$

$$+\left(\frac{\partial}{\partial z}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial x}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)\right)\vec{\jmath}$$

$$+\left(\frac{\partial}{\partial x}\left(\frac{y}{\sqrt{x^2+y^2+z^2}}\right)-\frac{\partial}{\partial y}\left(\frac{x}{\sqrt{x^2+y^2+z^2}}\right)\right)\vec{k}$$

And

$$\nabla\times\vec{u}_r=\left(-\frac{\frac{yz}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}+\frac{\frac{yz}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right)\vec{\imath}$$

$$+\left(-\frac{\frac{xz}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}+\frac{\frac{xz}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right)\vec{\jmath}$$

$$+\left(-\frac{\frac{xy}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}+\frac{\frac{xy}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}\right)\vec{k}$$

We have

$$\nabla\times\vec{u}_r=\vec{0}$$

2. Cylindrical coordinates

The unit vectors of the Cylindrical basis are $(\vec{e}_\rho,\vec{e}_\theta,\vec{k})$. we define the scalar function f and the vector $\vec{A}$

$$\vec{A}=A_\rho\vec{e}_\rho+A_\theta\vec{e}_\theta+A_z\vec{k}$$

The expression for the differential element is

$$dr=d\rho\vec{e}_\rho+\rho d\theta\vec{e}_\theta+dz\vec{k}$$

and the volume element is

$$dV=\rho d\rho d\theta dz$$

2.1 Gradient

from which we deduce the expression of the gradient of the function f is

$$\nabla f = \frac{\partial f}{\partial \rho}\vec{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{\partial f}{\partial z}\vec{k}$$

We also deduce the **Laplacian** ($\Delta = \nabla^2$) form of the function f is

$$\Delta = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

2.2 The Divergence

The divergence of the vector $\vec{A}$ is a scalar that can be represented as

$$\nabla.\vec{A} = \frac{1}{\rho}\frac{\partial\left(\rho A_\rho\right)}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

2.3 The Curl

The Curl of the vector $\vec{A}$ is a vector that can be represented as

$$\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\vec{e}_\rho + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{e}_\theta + \frac{1}{\rho}\left(\frac{\partial(\rho A_\theta)}{\partial \rho} - \frac{\partial A_\rho}{\partial \theta}\right)\vec{k}$$

3. Spherical coordinates

The unit vectors of the Spherical basis are $(\vec{e}_r, \vec{e}_\theta, \vec{e}_\varphi)$. we define the scalar function f and the vector $\vec{A}$

$$\vec{A} = A_r\vec{e}_r + A_\theta\vec{e}_\theta + A_\varphi\vec{e}_\varphi$$

The expression for the differential element is

$$d\boldsymbol{r} = dr\vec{e}_r + rd\theta\vec{e}_\theta + r\sin\theta d\varphi\,\vec{e}_\varphi$$

and the volume element is

$$dV = r^2 \sin\theta \, dr d\theta d\varphi$$

2.1 Gradient

from which we deduce the expression of the gradient of the function f is

$$\nabla f = \frac{\partial f}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\vec{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\vec{e}_\varphi$$

We also deduce the **Laplacian** ($\Delta = \nabla^2$) form of the function f is

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin\theta^2}\frac{\partial^2 f}{\partial \varphi^2}$$

2.2 The Divergence

The divergence of the vector $\vec{A}$ is a scalar that can be represented as

$$\nabla.\vec{A} = \frac{1}{r^2}\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

2.3 The Curl

The Curl of the vector $\vec{A}$ is a vector that can be represented as

$$\nabla \times \vec{A} = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(A_\varphi \sin\theta) - \frac{\partial A_\theta}{\partial \varphi}\right)\vec{e}_r$$

$$+\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}(rA_\varphi)\right)\vec{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right)\vec{e}_\varphi$$

Terms

Vector	شعاع	energy	طاقة
Dimension	بعد	Temperature	درجة حرارة
Equation	معادلة	Unit vector	شعاع وحدة
Mass	كتلة	Scalar Product	جداء سلمي
Length	طول	Vector Product	جداء شعاعي
Temps	زمن	Mixed product	جداء مختلط
Quantity	كمية	Derivative	اشتقاق
Element	عنصر	Commutative	تبديلي
Relationship	علاقة	Distributive	توزيعي
Fundamental	اساسي	Perpendicular	عمودي
Position	موضع	Parallel	موازي
Velocity- speed	سرعة	Determinant	محدد
Acceleration	تسارع	Uncertaintie	ارتياب
Characteristic	خصائص	Experiment	تجربة
Measurement	قياس	Absolute	مطلق
Cartesian	ديكارتي- كارتيزي	Relative	نسبي
Polar	قطبي	Kinematics	حركيات
Cylinder	اسطوانة	Motion	حركة
Sphere	كرة	displacement vector	شعاع الانتقال
Distance	مسافة	Moment	لحظة
Function	دالة	Trajectory- path	مسار
Magnitude	شدة- طويلة	Radius	نصف قطر
Properties	خصائص	Average	متوسط
Analysis	تحليل	Instantaneous	لحظي
Calculation	عملية حسابية	Circular Motion	حركة دائرية
Power	استطاعة	Tangential Acceleration	تسارع مماسي
Volume	حجم	Normal Acceleration	تسارع ناظمي
Force	قوة	Relative motion	حركة نسبية
Work	عمل	fixed reference	معلم ثابت
Phenomena	ظاهرة	moving	معلم متحرك

Traction	جر	reference	معلم
Dynamics	تحريك	infinitesimal	متناهي الصغر
object- body	جسم	planet	كوكب
principle of inertia	مبدا العطالة	sun	شمس
heavy body	جسم اثقل	moon	قمر
lighter body	جسم اخف	falls	سقوط
Isolated	معزول	Momentum	كمية الحركة
Friction	احتكاك	Static	ساكن
Collision	تصادم	Projectile movement	حركة قذيفة
Gravity	جاذبية	surface	سطح
Reaction	رد فعل	Angular momentum	عزم الحركة
Tension	توتر	equation of motion	معادلة الحركة
Unstable	غير مستقر	Potential energy	طاقة كامنة
Conservative	محفوظ	kinetic energy	طاقة حركية
Elastic	مرن	increases	يتزايد
Spring	نابض	decreases	يتناقص

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