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بسم الله الرحمن الرحيم

{يرفع الله الذين آمنوا منكم والذين أوتوا العلم درجات}

شكر:

قال الله تعالى -{فاذكروني اذكركم واشكروا لي ولا تكفرون}-
الحمد والشكر لله تعالى، له الفضل والمنة على توفيقه لإتمام هذا العمل
فإن أصبنا فمن الله ، وإن أخطأنا فمن أنفسنا.
في حضرة العلم، تتضاءل الكلمات، وتتسامى المعاني...
وفي حضرة العرفان، يصبح الصمت خجولًا أمام مقام من كانوا نورًا لطريقي، وركنًا ثابتًا في مسيرتي.
إلى أولئك الذين لم يكونوا مجرّد أسماء على ورق، بل كانوا أثرًا خالدًا في مساري الأكاديمي، أتوجه بجزيل الشكر والتقدير.
أولهم أستاذي الذي لا يضاهى، البروفيسور نوري بومعزة اسم لا يختزل في منصب أو صفة، بل هو ذاكرة حية في مسيرتي الجامعية. رافقني منذ أول يوم بخطوات واثقة، بتشجيع صامت، وبدعم دون حدود، دمت ذخرًا للعلم، رفعك الله درجات وجعل ما تُقدمه نورًا لا ينطفئ.
كما أرفع جزيل الشكر والامتنان إلى مشرفتي القديرة الدكتورة كماش حورية ، التي واكبت هذا العمل بكل مهنية واهتمام، وخصّتني بتوجيهاتها وملاحظاتها التي ساهمت في بناء هذا الجهد المتواضع.
إلى عمي العزيز، البروفيسور مسلوب السعيد الذي لم يبخل علي يومًا بعلمه ولا بتوجيهه، كنت سندي الصادق وملجأي في كل عثرة، أعود إليك كلما استعصى علي أمر، فأجد لديك دومًا الجواب والدعم. شكرًا بحجم السماء لوقوفك الدائم إلى جانبي، ووفقك الله وأدامك نورًا يضيء دربي
كما أخص بالشكر والتقدير البروفيسور زارعي عبد الرحمن،
الذي أفادني بعلمه وخبرته، ولم يبخل علي بالمعلومة ولا بالإشارة، فكان أستاذًا قلّ نظيره، له مني كل الاحترام والتقدير والدعاء بالتوفيق والبركة.
وختاما لا يسعني إلا أن أعبّر عن خالص امتناني وخضوعي بالعرفان لخالتي،
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التي لم تكن فقط سندًا عائليًا، بل كانت ركيزة علمية حقيقية، مدّتني بما يفوق التوجيه، فكانت حريصة على كل تفصيلة، داعمة دون كلل. فشكرًا من القلب يا من كان لعطائك طعم الأمان والمعرفة.
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Abstract

The main goal of this thesis is studing the Asymptotic behavior of solutions to a viscoelastic Kirchhoff-type plate equation with memory and delay terms, under classical initial and boundary conditions.first we present necessary mathematical preliminaries. Then,by using energy estimates and suitable assumptions on the damping and relaxation functions, we prove the global existence. Finally, we establish a general decay result for the total energy by constructing an appropriate Lyapunov functional. The result shows that the energy decays over time, illustrating the combined stabilizing effect of damping and memory, even in the presence of delay.

Résumé

Le principal objectif de ce mémoire est d'étudier le comportement asymptotique des solutions d'une équation de plaque viscoélastique de type Kirchhoff avec des termes de mémoire et de retard, sous des conditions initiales et aux limites classiques. Nous présentons d'abord les notions mathématiques préliminaires nécessaires. Ensuite, en utilisant des estimations d'énergie et des hypothèses appropriées sur les fonctions d'amortissement et de relaxation, nous prouvons l'existence globale. Enfin, nous établissons un résultat général de décroissance de l'énergie totale en construisant la foncrionnelle de Lyapunov appropriée. Le résultat montre que l'énergie décroît au cours du temps, illustrant l'effet stabilisateur combiné de la mémoire et du retard.

الملخص

تهدف هذه المذكرة إلى دراسة السلوك التقاربي لحلول معادلة لوحة لزجة من نوع كيرشوف تحتوي على حدود ذاكرة وأعضاء تأخير، وذلك تحت شروط ابتدائية وحدية كلاسيكية. في البداية، نعرض التمهيدات الرياضية الضرورية. ثم، وباستخدام تقديرات الطاقة وتحت فرضيات مناسبة على دوال التخميد والاسترخاء، نثبت الوجود العام للحلول. بعد ذلك، نقوم بإنشاء دالة ليابونوف ملائمة ونبرهن من خلالها على نتيجة للاضمحلال العام للطاقة الكلية. وتبين النتيجة أن الطاقة تتناقص مع الزمن، مما يدل على التأثير التثبيتي المشترك للتخميد والذاكرة، رغم وجود التأخير الزمني.

Contents

Introduction 8

1 Preliminary **14**

1.1 Normed Space 14

1.2 Banach space 15

1.2.1 Definitions and Properties 15

1.2.2 The Weak and Weak-Star Topologies 17

1.3 Espace de Hilbert 18

1.4 Function Spaces 20

1.4.1 The Space $L^p(\Omega)$ 20

1.5 Sobolev Spaces 24

1.5.1 Weak Derivatives 24

1.5.2 The Space $W^{1,p}(\Omega)$ 25

1.5.3 Integration by Parts and Some Inequalities 25

1.6 Some Useful Inequalities 26

1.7 Existence and asymptotic behavior of solution 28

2 Global existence of solution **29**

2.1 Problem Setting 29

2.2 Global Existence 33

3 General decay of solution **36**

3.1 Assumptions and Preliminary Lemmas 37

3.2 General decay 39

Bibliography **48**

Introduction

Wave equations are fundamental tools in modeling various physical phenomena such as acoustics, elasticity, and electromagnetism. The classical linear wave equation provides a simple yet effective description of wave propagation in homogeneous media. However, real-world materials often exhibit more complex behaviors such as memory effects and nonlinear responses that go beyond the scope of the classical model. These complexities necessitate the development of more advanced mathematical models.

One such refinement involves viscoelastic materials, which exhibit both elastic and viscous properties. These materials do not respond instantaneously to external forces; instead, they have a delayed response, often referred to as "memory." To capture this effect, mathematical models incorporate memory terms typically expressed as convolution integrals involving a relaxation function. These terms model how the past states of the system influence its present behavior, adding a layer of realism to wave propagation in such materials.

Another extension is provided by Kirchhoff-type wave equations, which introduce a nonlinearity depending on the gradient of the displacement. Unlike the classical wave equation, where the wave speed is constant, the Kirchhoff-type model reflects situations in which the tension or stiffness of the medium varies with deformation. This is relevant for systems such as vibrating strings and membranes where the internal force depends on the total strain energy.

Additionally, delay effects play a significant role in many physical and engineering systems. Delays may arise due to signal transmission time, processing lag, or feedback mechanisms. Even small delays can significantly affect system dynamics and stability. Including delay terms in mathematical models allows for a more accurate and comprehensive understanding of such systems. In the context of wave equations, delays can alter the decay rate of the energy and even destabilize the system under certain conditions.

This thesis focuses on the general decay behavior of solutions to a viscoelastic Kirchhoff-type wave equation with a delay term. Our aim is to study how the total energy of the system evolves over time and under what conditions it decays. The concept of general decay includes both exponential and polynomial decay as special cases, depending on the properties of the relaxation function and the delay weight. Understanding the decay rate is essential for predicting the long-term behavior of solutions and ensuring the stability of the system.

Some mathematical problems have been previously studied by several authors [16, 23]. many authors studied the following equation with appropriate boundary and initial conditions:

$$u_{tt} + \alpha\Delta^2 u - M\left(\int_\Omega |\nabla u|^2 dx\right)\Delta u - \sigma(t)\int_0^t g(t-s)\Delta^2 u(s)\,ds = 0, \tag{0.1}$$

where M and g are functions satisfying certain conditions (see, e.g., [10,28]). Menezes et al. [7] proved existence and uniform decay results for the nonlinear degenerate equation (0.1) with $g = 0$. Park et al. [24] established decay rate estimates for the problem (0.1) with $\alpha = 1$, $g = 0$, and boundary damping. These works focus on exponential decay rates.

The purpose of the present work is to establish a general decay result for equation (0.1) with time-varying delay. Due to the presence of the delay term $a_1 u_t(x, t-\tau(t))$, the problem under consideration differs from those studied in the existing literature. Time delays naturally appear in many practical applications, where the current state depends not only on the present but also on past events. The presence of such delays can potentially cause instability (see, e.g., [6, 19]). Consequently, the control of partial differential equations with time-delay effects has recently attracted significant research interest (see [11,14,19,20,27] and references therein). Kafini et al. [14] and Nicaise and Pignotti [19] studied, respectively, a Timoshenko-type thermoelasticity system of type III and a wave equation with constant delay. Mustafa [11] analyzed a thermoelastic system of second sound involving an internal time-varying delay. Nicaise and Pignotti [19] also derived decay rates for a wave equation with time-varying delay. These contributions are primarily concerned with exponential or polynomial decay behavior.
Delays are commonly encountered in numerous real-world phenomena and can have a significant impact on the behavior of dynamical systems. In fact, the presence of delay may destabilize an otherwise stable system. This highlights the importance of studying the stability of systems that include delay terms, from both theoretical and practical perspectives.

By taking into account the effects of damping and delay, the equation (0.1) is modified to the following form:

$$u_{tt}(x,t) + \Delta^2 u(x,t) - \sigma(t)\int_0^t g(t-s)\Delta^2 u(x,s)\,ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \tag{0.2}$$

where μ_1 and μ_2 are real constants, and $\tau > 0$ denotes the time delay.

In this memory We consider a linear viscoelastic wave equation that incorporates both a linear damping effect and a delay term:

$$\begin{cases} v_{tt}(x,t) + \Delta^2 v(x,t) - M(\|\nabla v\|^2)\Delta v(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta^2 v(x,s)\,ds \\ \quad - \mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau) = \mu|v|^{p-2}v, & x \in \Omega,\ t > 0, \\ v(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ v_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{0.3}$$

Here, $v = v(x,t)$ with $t \geq 0$ and $x \in \Omega$; Δ and Δ^2 denote the Laplace and biharmonic operators with respect to the spatial variable x, respectively. The domain $\Omega \subset \mathbb{R}^n$ is bounded with a sufficiently smooth boundary $\partial\Omega$, and $n \geq 1$. The constants μ,μ_1 and μ_2 are positive, and $\tau > 0$ represents the delay time. The functions v_0, v_1, and f_0 are given initial and history data belonging to suitable functional spaces. The term $\mu_2 v_t(x,t-\tau)$ models the delayed damping effect. The objective of this dissertation is to investigate the existence and asymptotic stability of problem (0.3), which includes a delay term appearing in the control component of the main equation.

Our primary motivation is to investigate the energy decay properties of solutions for the model (0.1). We aim to establish sufficient conditions under which the energy of this system decays over time. This topic has recently attracted considerable attention. In the case of Kirchhoff-type equations, numerous results concerning energy decay have been obtained (see, for instance, [1,2] and references therein). However, studies concerning viscoelastic plate models remain limited, particularly those involving delay effects.

In this direction, we refer to the work of Lagnese [9], who investigated a viscoelastic plate equation under nonlinear feedback conditions and proved that the solution decays to zero as time tends to infinity. Furthermore, exponential (or polynomial) decay of the energy was established for linear viscoelastic Kirchhoff plates, under the assumption that the relaxation function g decays exponentially (or polynomially). Subsequent works addressed viscoelastic plate equations with boundary conditions, obtaining existence and uniform decay rates of the energy by considering nonlinear and nonlocal boundary feedbacks.

In [18], Cavalcanti et al. considered the following model:

$$u_{tt} + \Delta^2 u - \int_0^t g(t-s)\Delta^2 u(s)\, ds + a(t)u_t = 0, \tag{0.4}$$

and established energy decay results under appropriate conditions on the damping function $a(t)$.

In $\Omega \times (0, \infty)$, where $a(t)$ is a nonlocal nonlinear function, the authors proved the exponential decay of the energy in general domains. In the context of localized damping, Park et al. [25,26] investigated the following model:

$$u_{tt} + \Delta^2 u - M(\|\nabla u\|^2)\Delta u - \int_0^t g(t-s)\Delta u(s)\, ds + \rho(x, u_t) = 0, \tag{0.5}$$

where $M \in C^1(\mathbb{R}^+)$. Energy decay results were obtained by employing certain difference inequalities and analytical techniques developed in [17, 22].

A distinguishing aspect of the present work lies in its focus on a viscoelastic equation incorporating both internal damping and a delay term. In particular, this study emphasizes the effect of the delay term $\mu_2 u_t(x, t-\tau)$. Delay phenomena have been studied in the context of various partial differential equations. For instance, Nicaise and Pignotti [19] examined the wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0, \tag{0.6}$$

and established stability results under the condition $0 < \mu_2 < \mu_1$.

Furthermore, Kirane and Said-Houari [12] considered the viscoelastic wave equation:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)\, ds + \mu_1 u_t(x,t) + \mu_2 u_t(x, t-\tau) = 0, \tag{0.7}$$

with appropriate initial and boundary conditions, and proved well-posedness and energy decay of solutions under the condition $0 < \mu_2 \leq \mu_1$. These results were later improved by Dai and the present author [5]improved the results of [12] under weaker conditions assumptions.

In this sence, we provide a deeper and more comprehensive understanding of the intricate interaction between viscoelastic damping and time delay, which represents a significant and novel aspect of our contribution. This interplay plays a crucial role in influencing the long-term behavior of the solutions and highlights the importance of considering both effects simultaneously in the analysis.

Our thesis consists of three chapters.

Chapter 1:

In this chapter, we recall the main concepts that we will need. We begin with normed spaces, Banach spaces, and Hilbert spaces. Then, we introduce L^p spaces and Sobolev spaces. Finally, we present the necessary inequalities that are used in this thesis, and we also mention some useful theorems.

Chapter 2:

In this chapter, we study the global existence using energy estimates. We consider the following linear problem for $v \in C([0,T]; H_0^1(\Omega))$:

$$\begin{aligned} v_{tt}(x,t) + \Delta^2 v(x,t) - M(\|\nabla v\|^2)\Delta v(x,t) - \sigma(t)\int_0^t g(t-s)\Delta^2 v(x,s)\,ds \\ - \mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau) = \mu|v|^{p-2}v, \end{aligned} \qquad x \in \Omega,\ t > 0,$$

with the initial conditions: $v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in \Omega,$
and the boundary condition: $v(x,t) = \frac{\partial v}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega,\ t > 0.$
Using appropriate energy functionals and under suitable assumptions on the nonlinear and memory terms, we derive a priori estimates that ensure the solution is globally defined in time. A contradiction argument confirms that the energy remains positive, and thus the solution exists for all $t > 0$ and stays bounded.

Chapter 3:

This chapter investigates the general decay of energy for a viscoelastic Kirchhoff-type plate equation with delay. The approach relies on constructing Lyapunov functionals and auxiliary tools to handle the nonlinear damping and delay terms. By proving the equivalence between the Lyapunov functional and the system's energy, and deriving suitable differential inequalities, we show that the energy decreases over time. The result covers various decay types, including exponential decay, depending on the relaxation function. These findings highlight the importance of damping, memory, and delay in the system's term stability.

Notations

- Ω : Bounded open subset of $\mathbb{R}^n$
- $\partial\Omega$: The smooth boundary of Ω
- $x = (x_1, x_2, \ldots, x_n)$: The generic point of an open set Ω in $\mathbb{R}^n$
- ∇v : Gradient of v
- Δv : Laplacian of v
- v, $v(t)$: $v(x,t)$
- $v_t = v'$; $v_{tt} = v''$
- w_j : $w_j(x)$
- a.e. : Almost everywhere convergence
- $\overset{!}{\to}$: Strong convergence
- $\overset{*}{\to}$: Weak convergence
- $\overset{*}{\to}$ (with a star) : Weak-* convergence
- $\langle \cdot, \cdot \rangle$: Inner product
- $\frac{\partial v}{\partial t}$: Partial derivative of v with respect to t
- $C^k(\Omega)$: Space of k-times continuously differentiable functions in Ω
- $C^0(\Omega)$: Space of continuous functions vanishing on the boundary of Ω
- H : Hilbert space
- $V_n'' = V_{tt}^n$
- $V_n' = V_t^n$
- $H_0^1 = W_0^{1,2}$

Chapter 1

Preliminary

1.1 Normed Space

Definition 1.1 A linear vector space E is called a *normed space* if for each element $v \in E$, there exists a real number denoted by $\|v\|$ satisfying the following axioms:

1. $\|v\| = 0 \iff v = 0$;
2. $\|v + w\| \leq \|v\| + \|w\|$ for all $v, w \in E$ (triangle inequality);
3. $\|\lambda v\| = |\lambda| \cdot \|v\|$ for all $v \in E$ and all $\lambda \in \mathbb{R}$ or $\mathbb{C}$.

Remark 1.1 $(E, \|\cdot\|_E)$ is a normed space.

Definition 1.2(Equivalence of Norms)
Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. These two norms are said to be *equivalent* if there exist two strictly positive constants $c_1, c_2 > 0$ such that

$$\forall v \in V, \quad c_1\|v\|_1 \leq \|v\|_2 \leq c_2\|v\|_1.$$

Proposition 1.1 Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two equivalent norms on V. Then, we have the equivalence:

$$v_n \to v \text{ with respect to } \|\cdot\|_1 \iff v_n \to v \text{ with respect to } \|\cdot\|_2.$$

Definition 1.3 (Cauchy Sequence)

Let $(E, \|\cdot\|)$ be a normed space and $(v_n)_{n\in\mathbb{N}}$ a sequence of elements in E. The sequence (v_n) is called a *Cauchy sequence* if:

$$\forall \varepsilon > 0, \exists n_0 = n_0(\varepsilon) \in \mathbb{N}, \forall n, m \geq n_0 \Rightarrow \|v_n - v_m\| < \varepsilon.$$

Definition 1.4 (Complete Space)

Let E be a vector space. We say that E is a *complete space* if every Cauchy sequence $(v_n)_{n\in\mathbb{N}}$ in E converges to an element $v \in E$.

1.2 Banach space

1.2.1 Definitions and Properties

Definition 1.5(Banach Space)

Let $(E, \|\cdot\|)$ be a normed space. We say that E is a *Banach space* if E is complete.

Definition 1.6

Let E' be the linear space of all continuous linear functionals $f : E \to \mathbb{R}$, called the *dual space* of E.

Proposition 1.2 [8]

The space E', equipped with the norm

$$\|f\|_{E'} = \sup\{|f(v)| : \|v\| \leq 1\},$$

is also a Banach space.

We denote the value of $f \in E'$ at $v \in E$ by $f(v)$ or $\langle f, v\rangle_{E,E'}$.

Definition 1.7

Let $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ be two Banach spaces such that there exists an injective linear map from E to F. This map allows us to consider E as a vector subspace of F, denoted by

$E \hookrightarrow F$ or $E \subseteq F$.

We say that this inclusion is:

1. **continous:** If there exists a constant $C > 0$ such that

$$\|v\|_F \leq C\|v\|_E,$$

we write

$$E \hookrightarrow F.$$

2. **Compact:** If for every bounded sequence in E (with respect to the norm of E), one can extract a subsequence that converges in F (with respect to the norm of F). We denote this by:

$$E \overset{\text{compact}}{\hookrightarrow} F.$$

3. **Dense:** If for every $v \in E$, there exists a sequence $(v_n)_{n\in\mathbb{N}} \subset E$ such that

$$\lim_{n\to\infty} v_n = v,$$

where the convergence is with respect to the norm of F.

Theorem 1.1 [4]

Let E be a Banach space. Then E is reflexive if and only if

$$B_E = \{v \in E : \|v\| \leq 1\}$$

is compact with respect to the weak topology $\sigma(E, E')$.

Definition 1.8

Let E be a Banach space and $(v_n)_{n\in\mathbb{N}}$ a sequence in E. We say that v_n converges strongly to v in E if and only if

$$\lim_{n\to\infty} \|v_n - v\| = 0,$$

which we denote by $v_n \to v$ or $\lim_{n\to\infty} v_n = v$.

1.2.2 The Weak and Weak-Star Topologies

Let E be a Banach space and $f \in E'$. We define the map:

$$\varphi_f : E \longrightarrow \mathbb{R}, \quad v \mapsto \varphi_f(v),$$

thus, for $f \in E'$, we obtain the family of maps $\{\varphi_f\}_{f \in E'}$ from E to $\mathbb{R}$.

Definition 1.9

The weak topology on E is the coarsest topology on E, denoted $\sigma(E, E')$, such that every map φ_f with $f \in E'$ is continuous.

Now, we define the weak-star topology on E', denoted $\sigma(E', E)$. For each $v \in E$, we define:

$$\varphi_v : E' \to \mathbb{R}, \quad f \mapsto \varphi_v(f) = \langle f, v \rangle_{E',E}.$$

Thus, for each $v \in E$, we obtain the family of maps $\{\varphi_v\}_{v \in E}$ from E' to $\mathbb{R}$.

Definition 1.10

The weak-star topology on E' is the coarsest topology on E' such that each map φ_v is continuous for every $v \in E$.

Remark 1.2 [4]

Since $E \subset E''$, it is clear that the weak-star topology $\sigma(E', E)$ is weaker than the topology $\sigma(E', E'')$.

Theorem 1.2

Let E be a reflexive Banach space. Then any bounded sequence in E has at least one weakly convergent subsequence.

Definition 1.11

A sequence $(v_n)_{n \in \mathbb{N}}$ in E weakly converges to v if and only if

$$\lim_{n \to \infty} f(v_n) = f(v), \quad \text{for all } f \in E',$$

which is denoted by

$$v_n \to v.$$

Remark 1.3 [4]

1) If the weak limit exists, it is unique.

2) If $v_n \to v$ (strongly), then $v_n \to v$ (weakly).

3) If $\dim(E) < +\infty$, then weak convergence implies strong convergence.

Proposition 1.3 [8]

On compactness in the three topologies in the Banach space E:

1) The unit ball

$$B = \{v \in E : \|v\| \leq 1\}$$

in E is compact if and only if $\dim(E) < \infty$.

2) The unit ball B_0 in E' is weakly compact in E' if and only if E is reflexive.

3) B_0 is always weak-star compact in the weak-star topology of E'.

Proposition 1.4 [4]

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in E'. We have:

1) $f_n \overset{*}{\to} f$ in $\sigma(E', E)$ if and only if $f_n(v) \to f(v)$ for all $v \in E$.

2) If $f_n \to f$ (strongly), then $f_n \to f$ in $\sigma(E', E'')$.

3) If $f_n \to f$ in $\sigma(E', E'')$, then $f_n \overset{*}{\to} f$ in $\sigma(E', E)$.

4) If $f_n \overset{*}{\to} f$ in $\sigma(E', E)$, then $\|f_n\|$ is bounded and $\|f\| \leq \liminf \|f_n\|$.

5) If $f_n \overset{*}{\to} f$ in $\sigma(E', E)$ and $v_n \to v$ (strongly) in E, then $f_n(v_n) \to f(v)$.

1.3 Espace de Hilbert

Definition 1.12

Let E be a vector space. A map $h : E \times E \to K = \mathbb{C}$ defined by $\langle \cdot, \cdot \rangle$ is called an inner product if:

1. $\langle v, w \rangle = \langle w, v \rangle$, for all $v, w \in E$;
2. $\langle \alpha v_1 + v_2, w \rangle = \alpha \langle v_1, w \rangle + \langle v_2, w \rangle$, for all $v, w \in E$ and $\alpha \in \mathbb{C}$;
3. $\langle v, \alpha w \rangle = \alpha \langle v, w \rangle$, for all $\alpha \in \mathbb{C}$;
4. $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \iff v = 0$.

Definition 1.13

A Hilbert space is a Banach space $(E, \|\cdot\|_E)$ (a complete normed vector space) equipped with

an inner product for the associated norm:

$$\|v\|_E = \sqrt{\langle v, v\rangle}$$

Definition 1.14 (Orthonormal System)

Let E be a Hilbert space. The sequence $(e_n)_{n\in\mathbb{N}}$ in E is called an orthonormal system if

$$\langle e_n, e_m\rangle = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m, \\ 0, & \text{if } n \neq m. \end{cases}$$

1. If $\{e_n\} \perp \{e_m\}$, we say that the system $\{e_n\}$ is orthogonal.
2. If the system $\{e_n\}$ is orthogonal, then the system $\left\{\frac{e_n}{\|e_n\|}\right\}_{n\in\mathbb{N}}$ is orthonormal.

Definition 1.15 (Hilbert Basis)

Let E be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. A countable set $\{e_n\}_{n\in\mathbb{N}}$ is called a Hilbert basis for E if it is orthonormal for the inner product, and the vector space spanned by this family is dense in E.

Definition 1.16 (Separable Space)

A normed vector space that contains a countable dense subset is called a separable space.

Example 1.1 The spaces $\mathbb{R}$ and $\mathbb{C}$ are separable (for example, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$).

Theorem 1.3

Every separable Hilbert space has a Hilbert basis.

Proposition 1.5

Let E be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let $(e_n)_{n\geq 1}$ be a Hilbert basis of E. There exists a unique sequence $(v_n)_{n\geq 1}$ defined by

$$v_n = \langle v, e_n\rangle,$$

such that the partial sums

$$\sum_{n=1}^{p} v_n e_n$$

converge to v as $p \to \infty$.

Moreover, we have:

$$\|v\|^2 = \langle v, v\rangle = \sum_{n=1}^{\infty} |\langle v, e_n\rangle|^2,$$

thus we write:

$$v = \sum_{n=1}^{\infty} \langle v, e_n\rangle e_n.$$

Theorem 1.4 [21]

Let $(v_n)_{n\in\mathbb{N}}$ be a bounded sequence in the Hilbert space E. Then, one can extract a subsequence that converges in the weak topology.

Theorem 1.5 [21]

In a Hilbert space, every sequence that converges in the weak topology is bounded.

Theorem 1.6 [21]

Let $(v_n)_{n\in\mathbb{N}}$ be a sequence that converges weakly to v in E, and let $(w_n)_{n\in\mathbb{N}}$ be another sequence that converges weakly to w. Then,

$$\lim_{n\to\infty} \langle w_n, v_n\rangle = \langle w, v\rangle.$$

Theorem 1.7 [21]

Let E be a normed space. Then the unit ball

$$B_0 = \{v \in E' : \|v\| \leq 1\}$$

of E' is compact in the weak topology $\sigma(E', E)$.

Proposition 1.6 [21]

Let E and F be two Hilbert spaces, and let $(v_n)_{n\in\mathbb{N}} \subset E$ be a sequence that converges weakly to $v \in E$. Let $A \in \mathcal{L}(E, F)$ (a bounded linear operator from E to F). Then the sequence $(A(v_n))_{n\in\mathbb{N}}$ converges weakly to $A(v)$ in the weak topology of F.

1.4 Function Spaces

1.4.1 The Space $L^p(\Omega)$

Definition 1.17

Let $1 \leq p \leq \infty$ and let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \in \mathbb{N}$. The space of equivalence

classes of functions $L^p(\Omega)$ is defined by:

$$L^p(\Omega) = \left\{ v : \Omega \to \mathbb{R} \;\middle|\; v \text{ is measurable and } \int_\Omega |v(x)|^p \, dx < +\infty \right\}.$$

For $1 \leq p < \infty$, the norm is defined by:

$$\|v\|_p = \left(\int_\Omega |v(x)|^p \, dx \right)^{1/p}.$$

For $p = \infty$, we define:

$$L^\infty(\Omega) = \{ v : \Omega \to \mathbb{R} \,|\, v \text{ measurable}, \exists\, C > 0 \text{ s.t. } |v(x)| < C \text{ a.e. in } \Omega \}$$

and it is equipped with the essential supremum norm:

$$\|v\|_\infty = \text{ess sup}_{x \in \Omega} |v(x)| = \inf \{ C > 0 : |v(x)| < C \text{ a.e. in } \Omega \}.$$

Proposition 1.7

The space $L^p(\Omega)$, equipped with its norm $\| \cdot \|_{L^p}$, is a Banach space for all $1 \leq p \leq \infty$.

Theorem 1.8

The space $L^p(\Omega)$ is separable for $1 < p < \infty$.

Proposition 1.8

The space $L^\infty(\Omega)$ is a non-separable Banach space.

Theorem 1.9 [8]

The space $L^p(\Omega)$ is reflexive for $1 < p < \infty$.

Lemma 1.1

Let Ω be a bounded open subset of $\mathbb{R}^n$ and $1 \leq p \leq \infty$. Let $v_n, v \in L^p(\Omega)$ such that $v_n \to v$ strongly in $L^p(\Omega)$, then:

$$v_n \to v \text{ a.e. in } \Omega.$$

Lemma 1.2

Let Ω be a bounded open subset of $\mathbb{R}^n$, and $1 \leq p \leq \infty$. Suppose that (v_n) is a bounded sequence in $L^p(\Omega)$ and that $v_n \to v$ almost everywhere in Ω, then

1) $v \in L^p(\Omega)$

2) $v_n \rightharpoonup v$ weakly in $L^p(\Omega)$

Remark 1.4

Suppose $1 < p < \infty$ and E is a reflexive Banach space, then the weak* convergence is equivalent to weak convergence.

Theorem 1.10

For every sequence $(v_n)_{n\in\mathbb{N}}$ that converges in $L^p(\Omega)$, one can extract a subsequence that converges almost everywhere in Ω.

If $v \in L^1(\Omega)$, then:

$$|v(x)| \leq \|v\|_{L^1} \quad \text{a.e. in } \Omega.$$

Remark 1.5

1. The strong or weak limit of a sequence of functions is always unique.
2. In the case $p = 1$, the symbol * is used to emphasize that the definition of weak convergence in L^1 is not exactly the same as in the spaces $L^p(\Omega)$ for $1 < p < \infty$. Indeed, the dual of $L^1(\Omega)$ is strictly larger than $L^1(\Omega)$.
3. Strong convergence in $L^p(\Omega)$ implies weak convergence in $L^p(\Omega)$, for all $1 \leq p \leq \infty$.

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

1) If $v_n \rightharpoonup^* v$ in $L^1(\Omega)$, then $v_n \rightharpoonup v$ in $L^p(\Omega)$, for all $p \geq 1$.

2) If $v_n \rightharpoonup v$ in $L^p(\Omega)$, then $\|v_n\|_{L^p(\Omega)} \rightharpoonup \|v\|_{L^p(\Omega)}$ in $\mathbb{R}$, for all $1 \leq p \leq \infty$.

3) If $1 \leq p < \infty$ and $v_n \rightharpoonup v$ in $L^p(\Omega)$, then there exists $K > 0$ such that $\|v_n\|_{L^p(\Omega)} \leq K$, and:

$$\|v\|_{L^p(\Omega)} \leq \liminf_{n\to\infty} \|v_n\|_{L^p(\Omega)}.$$

This result is also valid for $p = 1$ when $v_n \rightharpoonup^* v$ in $L^1(\Omega)$.

4) If $1 < p < \infty$ and there exists $K > 0$ such that $\|v_n\|_{L^p} \leq K$, then there exists a subsequence v_{n_i} and $v \in L^p(\Omega)$ such that $v_{n_i} \to v$ in $L^p(\Omega)$.
The result also holds for $p = 1$ with $v_n \rightharpoonup^* v$ in $L^1(\Omega)$.

5) If $1 \leq p \leq \infty$ and $v_n \rightharpoonup v$ in $L^p(\Omega)$, then there exists a subsequence v_{n_i} such that v

Proposition 1.9

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, v_m and v functions in $L^p(\Omega)$, with $1 < p < \infty$, such that:

$$\|v_m\|_{L^p(\Omega)} \leq C \quad \text{and} \quad v_m \to v \text{ a.e. in } \Omega.$$

Then $v_m \rightharpoonup v$ weakly in $L^p(\Omega)$.

Lemma 1.3

The space $L^2(\Omega)$ is a Hilbert space with the inner product:

$$\langle v, w \rangle_{L^2(\Omega)} = \int_\Omega v(x)w(x)\,dx, \quad \forall v, w \in L^2(\Omega).$$

The Space $L^p((0,T);E)$

Definition 1.18

Let E be a Banach space, $1 \leq p \leq \infty$, and $[0,T] \subset \mathbb{R}$. The Lebesgue space of E-valued functions, denoted $L^p((0,T);E)$, consists of measurable functions $v : (0,T) \to E$ such that:

i) If $1 \leq p < \infty$, then

$$\|v\|_{L^p((0,T);E)} = \left(\int_0^T \|v(t)\|_E^p \, dt \right)^{1/p} < \infty;$$

ii) If $p = \infty$, then

$$\|v\|_{L^\infty((0,T);E)} = \text{ess sup}_{t\in(0,T)} \|v(t)\|_E < \infty.$$

Theorem 1.11 [21]

The space $L^p((0,T);E)$ is complete.

Lemma 1.4 [15]

Let $v \in L^p((0,T);E)$ and $\frac{\partial v}{\partial t} \in L^p((0,T);E)$, with $1 \leq p \leq \infty$. Then $v \in C([0,T];E)$; that is, v is continuous from $[0,T]$ into E.

Properties:

- For $1 \leq p \leq \infty$, $L^p((0,T);E)$ is a Banach space; in particular, $L^2((0,T);E)$ is a Hilbert space if E is a Hilbert space.
- For $1 \leq p < \infty$, if E is reflexive, then $L^p((0,T);E)$ is reflexive.

- For $1 \leq p < \infty$, if E is separable, then $L^p((0,T);E)$ is separable.

Lemma 1.5

Let $\varphi = (0,T) \times \Omega$ be an open subset of $\mathbb{R} \times \mathbb{R}^n$, and let $g_w, g \in L^q([0,T];L^q(\Omega))$, with $1 < q < \infty$, such that:

$$\|g_w\|_{L^q([0,T];L^q(\Omega))} \leq C, \quad \forall w \in \mathbb{N}.$$

and

$$g_w \to g \quad \text{a.e. in } \varphi,$$

then

$$g_w \rightharpoonup g \quad \text{weakly in } L^q(\varphi).$$

Proposition 1.10 [18]

The space $L^p((0,T);E)$, endowed with the norm $\|\cdot\|_{L^p((0,T);E)}$, is a Banach space.

Proposition 1.11 [7]

Let E be a reflexive Banach space with dual E', and let $1 < p < \infty$, $1 < q < \infty$ such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

then the dual of $L^p((0,T);E)$ is algebraically and topologically isomorphic to $L^q((0,T);E')$.

Proposition 1.12 [18]

Let E and F be Banach spaces such that $E \hookrightarrow F$ with continuous embedding, then

$$L^p((0,T);E) \hookrightarrow L^p((0,T);F)$$

also holds with continuous embedding.

1.5 Sobolev Spaces

1.5.1 Weak Derivatives

Definition 1.19

Let Ω be an open subset of $\mathbb{R}^n$, $1 \leq i \leq n$, and let $v \in L^1_{\text{loc}}(\Omega)$. We say that v has a weak i-th partial derivative in $L^1_{\text{loc}}(\Omega)$ if there exists a function $f_i \in L^1_{\text{loc}}(\Omega)$ such that for all $\varphi \in C_c^\infty(\Omega)$,

the following holds:

$$\int_{\Omega} v(x)\, \partial_i \varphi(x)\, dx = -\int_{\Omega} f_i(x)\, \varphi(x)\, dx.$$

In this case, f_i is called the i-th weak derivative of v, and we write:

$$\partial_i v = \frac{\partial v}{\partial x_i} = f_i.$$

1.5.2 The Space $W^{1,p}(\Omega)$

Definition 1.20

Let Ω be an arbitrary open subset of $\mathbb{R}^n$, and let $p \in \mathbb{R}$, with $1 \leq p \leq \infty$. The Sobolev space $W^{1,p}(\Omega)$ is defined as:

$$W^{1,p}(\Omega) = \{v \in L^p(\Omega) \mid \partial_i v \in L^p(\Omega) \text{ for } i = 1, \ldots, n\},$$

where $\partial_i v$ denotes the weak partial derivative of v.

1.5.3 Integration by Parts and Some Inequalities

Definition 1.23 (Integration by Parts)

Let $(v, w) \in H^1(\Omega)$, and for every $1 \leq i \leq n$, we have:

$$\int_{\Omega} \frac{\partial v}{\partial x_i}\, w\, dx = -\int_{\Omega} \frac{\partial w}{\partial x_i}\, v\, dx + \int_{\partial\Omega} vw\, \nu_i\, d\sigma,$$

where $\nu_i(x) = \cos(\nu, x_i)$ is the direction cosine between the outward unit normal vector to $\partial\Omega$ at the point x and the axis x_i.

Lemma 1.7 (Green's Formula)

For all $v \in H^2(\Omega)$ and $w \in H^1(\Omega)$, we have:

$$\int_{\Omega} \Delta v\, w\, dx = -\int_{\Omega} \nabla v \cdot \nabla w\, dx + \int_{\partial\Omega} \frac{\partial v}{\partial \nu}\, w\, d\sigma,$$

where $\frac{\partial v}{\partial \nu}$ denotes the normal derivative of v on the boundary $\partial\Omega$.

Lemma 1.8 [21] Let $1 \leq p \leq r \leq q$, and $0 \leq \theta \leq 1$ such that:

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q},$$

then for every $v \in L^p(\Omega) \cap L^q(\Omega)$, we have:

$$\|v\|_{L^r(\Omega)} \leq \|v\|^{\alpha}_{L^p(\Omega)} \|v\|^{1-\alpha}_{L^q(\Omega)}.$$

Lemma 1.9 [21]

If $|\Omega| < \infty$ and $1 \leq p \leq q < \infty$, then we have the continuous embedding $L^q(\Omega) \hookrightarrow L^p(\Omega)$, and:

$$\|v\|_{L^p(\Omega)} \leq |\Omega|^{\frac{1}{p}-\frac{1}{q}} \|v\|_{L^q(\Omega)}.$$

1.6 Some Useful Inequalities

Let $\Omega \subset \mathbb{R}^n$ be an open set.

Theorem (Cauchy-Schwarz Inequality). Let $v, w \in L^2(\Omega)$. Then we have:

$$\left| \int_\Omega v(x)w(x)\,dx \right| \leq \left(\int_\Omega |v(x)|^2 dx \right)^{1/2} \left(\int_\Omega |w(x)|^2 dx \right)^{1/2},$$

which can be written as:

$$\|vw\|_{L^1(\Omega)} \leq \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)}.$$

lemma (Cauchy Inequality with ε).

For all $\varepsilon > 0$ and for all $a, b \in \mathbb{R}$, we have:

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2.$$

Theorem (Hölder's Inequality).

Let $1 \leq p \leq \infty$ and let q be the conjugate exponent defined by:

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $v \in L^p(\Omega)$ and $w \in L^q(\Omega)$, then $vw \in L^1(\Omega)$, and we have:

$$\|vw\|_{L^1(\Omega)} \leq \|v\|_{L^p(\Omega)} \|w\|_{L^q(\Omega)}.$$

Then, Hölder's inequality states:

$$\left|\int_\Omega vw\,dx\right| \leq \left(\int_\Omega |v|^p\,dx\right)^{1/p}\left(\int_\Omega |w|^q\,dx\right)^{1/q}$$

That is:

$$\int_\Omega |vw|\,dx \leq \|v\|_{L^p(\Omega)} \cdot \|w\|_{L^q(\Omega)}$$

Lemma (Algebraic Young's Inequality). For all $a, b \in \mathbb{R}_+$ and $\delta > 0$, we have:

$$|ab| \leq \delta|a|^2 + \frac{1}{4\delta}|b|^2.$$

Theorem (Young's Inequality). For all $(a, b) \in \mathbb{R}^2$, and for real numbers $p, q > 1$ satisfying:

$$\frac{1}{p} + \frac{1}{q} = 1,$$

we have:

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q.$$

Theorem (Gronwall's Inequality). Let $T > 0$, and let $\varphi \in L^1(0, T)$ with $\varphi(t) \geq 0$ almost everywhere. Let $\phi \in L^1(0, T)$ with $\phi(t) \geq 0$ almost everywhere, and assume $\varphi(t)\phi(t) \in L^1(0, T)$. Let $C_1, C_2 \geq 0$.

Suppose that:

$$\phi(t) \leq C_1 + C_2\int_0^t \varphi(s)\phi(s)\,ds, \quad \text{for almost every } t \in (0, T).$$

Then we have:

$$\phi(t) \leq C_1 \exp\left(C_2\int_0^t \varphi(s)\,ds\right), \quad \text{for almost every } t \in (0, T).$$

Theorem (Minkowski's Inequality).

For $1 \leq p \leq \infty$, we have:

$$\|v + w\|_{L^p} \leq \|v\|_{L^p} + \|w\|_{L^p}.$$

1.7 Existence and asymptotic behavior of solution

The study of local existence and uniqueness of solutions of partial differential equations is based on the existence theory for abstract semilinear differential equations.

Definition 1.1 *Let $A : D(A) \subset X \to X$ be a linear operator on a Banach space $(X, \|\cdot\|)$, $F : X \to X$. We consider the following nonhomogeneous Cauchy problem:*

$$\begin{cases} \frac{dv(t)}{dt} + Av(t) = F(v(t)), \quad t > 0, \\ v(0) = v_0. \end{cases} \tag{P}$$

A local solution of (P) is a solution defined in an interval $[0, T)$ for $t < T$.

Maximal existence time

Definition 1.2 *Let $v(t)$ be a weak solution of (P). We define the maximal existence time T of $v(x, t)$ as follows:*

1. If $v(t)$ exists for all $0 \leq t < \infty$, then $T = \infty$.

2. If there exists a $t_0 \in (0, \infty)$ such that $v(t)$ exists for $0 \leq t < t_0$, but doesn't exist at $t = t_0$, then $T = t_0$.

Remark 1.1 *If 1 is satisfied, we say that the solution $v(t)$ is global.*

Stabilization

The purpose of stabilization is to attenuate the vibrations by feedback, thus it consists in guaranteeing the decrease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation. More precisely, the problem of stabilization consists in determining the asymptotic behaviour of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.

They are several types of stabilization:

- Strong stabilization: $E(t) \to 0$ as $t \to \infty$.
- Uniform stabilization: $E(t) \leq Ke^{-ct}$, where $K, c > 0$,forall $t > 0$.
- Polynomial stabilization: $E(t) \leq Kt^{-c}$, where $K, c > 0$, forall $t > 0$.
- Logarithmic stabilization: $E(t) \leq K(\log(t))^{-c}$, where $K, c > 0$, forall $t > 0$.

Chapter 2

Global existence of solution

In this chapter, we prove the global existence of solutions using the energy estimates method. By deriving suitable a priori bounds and constructing appropriate energy functionals, we obtain uniform estimates that ensure the solution exists for all $t \geq 0$.

2.1 Problem Setting

We investigate the following viscoelastic Kirchhoff plate equation with delay and source terms, subject to initial and boundary condition:

$$\begin{cases} v_{tt}(x,t) + \Delta^2 v(x,t) - M(\|\nabla v\|^2)\Delta v(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta^2 v(x,s)\,ds \\ \quad - \mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau) = \mu|v|^{p-2}v, & x \in \Omega,\ t > 0, \\ v(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ v_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{2.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $\frac{\partial v}{\partial \nu}$ denotes the unit outward normal derivative, μ,μ_1 and μ_2 are positive constants, $\tau > 0$ is the time delay, and M, σ, and g are given functions satisfying appropriate assumptions to be specified later. The initial data v_0 and v_1 are assumed to belong to suitable function spaces.

Next, assume the following assumptions:

(A1) $g, \sigma : \mathbb{R}^+ \to \mathbb{R}^+$ are nonincreasing differentiable functions satisfying

$$g(s) \geq 0, \quad l_0 = \int_0^\infty g(s)\, ds < \infty, \quad \sigma(t) > 0, \quad 1 - 2\sigma(t) \int_0^t g(s)\, ds \geq l > 0.$$

with $l = 1 - l_0$.

(A2) There exists a positive differentiable function ξ satisfying

$$g'(t) \leq 0, \quad \sigma'(t) \leq 0, \quad g'(t) \leq -\xi(t)g(t), \quad \text{for } t \geq 0, \quad \lim_{t\to\infty} \frac{\sigma'(t)}{\xi(t)\sigma(t)} = 0.$$

(A3) M is a C^1-function satisfying

$$M(s) \geq -m_0, \quad M(s)s \geq \hat{M}(s), \quad s > 0.$$

where $\hat{M}(s) = \int_0^s M(\tau)\, d\tau$.

(A4) The constants μ_1 and μ_2 satisfy

$$\mu_2 < \mu_1.$$

Assume further that g satisfies

$$\sigma(t) \int_0^\infty g(s)\, ds < \frac{N - 2 - 2M_0}{N - 2 - \frac{1}{2\eta}}.$$

(A5) The constant p satisfies

$$p \geq 2, \quad \text{if } n = 1, 2,$$

and

$$2 < p \leq \frac{n+2}{n-2}, \quad \text{if } n \geq 3.$$

Now,To prove the existence of a unique solution for problem (2.1), we use the following new variable:

$$z(x, \rho, t) = v_t(x, t - \tau\rho), \quad x \in \Omega,\ \rho \in (0, 1),\ t > 0. \tag{2.2}$$

Therefore, we consider:

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad \text{in } \Omega \times (0,1) \times (0,+\infty). \tag{2.3}$$

As a result, problem (2.1) is equivalent to:

$$\begin{cases} v_{tt}(x,t) + \Delta^2 v(x,t) - M(\|\nabla v\|^2)\Delta v(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta^2 v(x,s)\,ds \\ \quad - \mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau) = \mu|v|^{p-2}v, & x \in \Omega,\ t > 0, \\ \tau z_t(x,\rho,\tau) + z_\rho(x,\rho,\tau) = 0, & x \in \Omega \times (0,1) \times \mathbb{R}_+^*, \\ z(x,0,t) = v_t(x,t), & x \in \Omega, t > 0, \\ z(x,\rho,0) = f_0(x-\rho\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{2.4}$$

The first natural question concerns the existence of a solution to problem (2.1). In this section, we provide a sufficient condition that ensures the well-posedness of this problem.

Local Existence

Theorem 2.1(Local existence) Suppose that (A1)–(A5) hold. Then, for any $(v_0, v_1) \in H^2(\Omega) \times L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (0,1))$, there exists a unique weak solution of problem (2.1) satisfying

$$\begin{aligned} v &\in L^\infty(0,T;H^2(\Omega)), \\ v_t &\in L^\infty(0,T;L^2(\Omega)), \\ v_{tt} &\in L^\infty(0,T;L^2(\Omega)) \cap L^\infty(0,T;L^2(\Omega)). \end{aligned}$$

we state the general assumptions on the relaxation function $g : \mathbb{R}^+ \to \mathbb{R}^+$, which is a $\mathcal{C}^1$-class function on $\mathbb{R}^+$. satisfying:

- $g(0) > 0$.
- $1 - \displaystyle\int_0^\infty g(s)\,ds = \ell > 0.$ (H1)

There exists a positive, non-increasing, differentiable function $\xi(t)$ such that:

- $g'(t) \le -\xi(t)g(t), \quad \forall t \ge 0.$ (H2)
- $\displaystyle\int_0^\infty \xi(t)\,dt = \infty.$

lemma 2.1 The following identity holds:

$$\begin{aligned}\sigma(t)\int_0^t g(t-s)\int_\Omega v(s)\,dx\cdot v_t(t)\,dx = &-\frac{d}{dt}\left(\frac{\sigma(t)}{2}(g\circ v)(t)-\frac{\sigma(t)}{2}\|v(t)\|_2^2\int_0^t g(s)\,ds\right)\\ &-\frac{\sigma(t)}{2}g(t)\|v(t)\|_2^2+\frac{\sigma(t)}{2}(g'\circ v)(t)+\frac{\sigma'(t)}{2}(g\circ v)(t)\\ &-\frac{\sigma'(t)}{2}\|v(t)\|_2^2\int_0^t g(s)\,ds.\end{aligned}$$

with

$$(g\circ v)(t)=\int_0^t g(t-s)\int_\Omega |v(s)-v(t)|^2\,dx\,ds.$$

Proof First we rewriting the integral as follows:

$$\begin{aligned}\sigma(t)\int_0^t g(t-s)\int_\Omega v(s)\cdot v_t(t)\,dx\,ds &= \sigma(t)\int_0^t g(t-s)\int_\Omega (v(s)-v(t)+v(t))\cdot v_t(t)\,dx\,ds\\ &= \sigma(t)\int_0^t g(t-s)\int_\Omega (v(s)-v(t))\cdot v_t(t)\,dx\,ds\\ &\quad+\sigma(t)\int_0^t g(t-s)\int_\Omega v(t)\cdot v_t(t)\,dx\,ds.\end{aligned}$$

Now we compute each term separately.

For the first term:

$$\sigma(t)\int_0^t g(t-s)\int_\Omega (v(s)-v(t))\cdot v_t(t)\,dx\,ds=-\frac{\sigma(t)}{2}\frac{d}{dt}(g\circ v)(t)+\frac{\sigma(t)}{2}(g'\circ v)(t),$$

by using the theorem :

$$\frac{1}{2}\frac{d}{dt}(g\circ v)(t)=-\int_0^t g(t-s)\int_\Omega (v(s)-v(t))\cdot v_t(t)\,dx\,ds+\frac{1}{2}(g'\circ v)(t).$$

For the second term:

$$\sigma(t)\int_0^t g(t-s)\,ds\int_\Omega v(t)\cdot v_t(t)\,dx=\sigma(t)\left(\int_0^t g(s)\,ds\right)\frac{1}{2}\frac{d}{dt}\|v(t)\|_2^2.$$

Hence,

$$\sigma(t)\int_0^t g(t-s)\int_\Omega v(s)\cdot v_t(t)\,dx\,ds = -\frac{\sigma(t)}{2}\frac{d}{dt}(g\circ v)(t)+\frac{\sigma(t)}{2}(g'\circ v)(t) + \frac{\sigma(t)}{2}\frac{d}{dt}\left(\|v(t)\|_2^2\int_0^t g(s)\,ds\right)-\frac{\sigma(t)}{2}g(t)\|v(t)\|_2^2.$$

By applying the product and chain rules:

$$\frac{d}{dt}\left(\sigma(t)(g\circ v)(t)\right)=\sigma'(t)(g\circ v)(t)+\sigma(t)\frac{d}{dt}(g\circ v)(t),$$

and

$$\frac{d}{dt}\left(\sigma(t)\|v(t)\|_2^2\int_0^t g(s)\,ds\right)=\sigma'(t)\|v(t)\|_2^2\int_0^t g(s)\,ds+\sigma(t)\frac{d}{dt}\left(\|v(t)\|_2^2\int_0^t g(s)\,ds\right).$$

Hence, we deduce that:

$$\begin{aligned}\sigma(t)\int_0^t g(t-s)\int_\Omega v(s)\,ds\cdot v_t(t)\,dx = &-\frac{d}{dt}\left(\frac{\sigma(t)}{2}(g\circ v)(t)-\frac{\sigma(t)}{2}\|v(t)\|_2^2\int_0^t g(s)\,ds\right)\\ &-\frac{\sigma(t)}{2}g(t)\|v(t)\|_2^2+\frac{\sigma(t)}{2}(g'\circ v)(t)+\frac{\sigma'(t)}{2}(g\circ v)(t)\\ &-\frac{\sigma'(t)}{2}\|v(t)\|_2^2\int_0^t g(s)\,ds.\end{aligned}$$

Lemma 2.2 For $v\in H_0^1(\Omega)$, we have

$$\int_\Omega\left(\int_0^t g(t-s)(v(t)-v(s))\,ds\right)^2 dx\le(1-\ell)C^2(\Delta v)(t),\tag{*}$$

where C is the Poincaré constant and ℓ is given in (H1)

2.2 Global Existence

In this section, we will prove that the solutions established in Theorem 2.1 are global in time. For this purpose, we define the following functionals:

$$\begin{aligned}I(t)=\hat{M}\left(\|\nabla v\|_2^2\right)+\left(1-\sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2+\sigma(t)(g\circ\Delta v)\\ +\int_0^1\|z(\rho,t)\|_2^2\,d\rho-\mu\|v\|_p^p,\end{aligned}\tag{2.6}$$

and

$$J(t) = \frac{1}{2}\left(1 - \sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2 + \frac{1}{2}\sigma(t)(g\circ\Delta v) + \frac{1}{2}\hat{M}\left(\|\nabla v\|_2^2\right) + \frac{\xi}{2}\int_0^1 \|z(\rho,t)\|_2^2\,d\rho - \frac{\mu}{p}\|v\|_p^p. \tag{2.7}$$

Then, it is obvious that

$$E(t) = J(t) + \frac{1}{2}\|v_t\|_2^2 \tag{2.8}$$

lemma 2.3 Assume that (A1)–(A5) hold, and for any $(v_0, v_1) \in H_2^2(\Omega) \times L^2(\Omega)$, such that $I(0) > 0$ and

$$\alpha = \frac{c_p^* c_p^\varrho}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} < 1, \tag{2.9}$$

Then,

$$I(t) > 0, \quad \text{for all } t > 0. \tag{2.10}$$

Proof Since $I(0) > 0$, then by the continuity of v, there exists a time $t_1 > 0$ such that

$$I(t) \geq 0, \quad \forall t \in (0, t_1). \tag{2.11}$$

We assume that

$$\{I(t_0) = 0 \text{ and } I(t) > 0,\ \forall t \in [0, t_0)\}. \tag{2.12}$$

This, together with (2.6),(2.7),(2.8), and assumption (A1), gives that

$$\begin{aligned} J(t) &= \frac{p-2}{2p}\left\langle \hat{M}\|\nabla v\|_2^2\right\rangle + \left(1 - \sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2 + \sigma(t)(g\circ\Delta v)(t) \\ &\quad + \xi\int_0^1 \|z(\rho,t)\|_2^2\,d\rho + \frac{1}{p}I(t) \\ &\geq \frac{p-2}{2p}\left\langle \hat{M}\|\nabla v\|_2^2\right\rangle + \left(1 - \sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2 + \sigma(t)(g\circ\Delta v)(t) \\ &\quad + \xi\int_0^1 \|z(\rho,t)\|_2^2\,d\rho \end{aligned} \tag{2.13}$$

consequently:

$$l\|\Delta v\|_2^2 \leq \left(1-\sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2 \leq \frac{2p}{p-2}J(t)$$
$$\leq \frac{2p}{p-2}E(t) \leq \frac{2p}{p-2}E(0), \quad \forall t \in [0,t_0] \tag{2.14}$$

using:

$$\mu\|v(t_0)\|_p^p \leq c_*^p c_\varrho^p \|\Delta v(t_0)\|_2^p = c_*^p c_\varrho^p \|\Delta v(t_0)\|_{p-2}^2 \|\Delta v(t_0)\|_2^2$$
$$\leq \frac{c_*^p c_\varrho^p}{l}\left[\frac{2p}{l(p-2)}E(0)\right]^{\frac{p-2}{2}} l\|\Delta v(t_0)\|_2^2 = \alpha l\|\Delta v(t_0)\|_2^2$$
$$\leq \left(1-\sigma(t)\int_0^t g(s)\,ds\right)\|\Delta v(t_0)\|_2^2.$$

Therefore, we conclude that $I(t_0) > 0$.

which contradicts (2.12). Thus, $I(t) > 0$ on $[0,T]$.

Theorem 2.1 *Assume that the conditions of Lemma 2.2 hold. Then the solution of (2.1)is global and bounded.*

Proof It suffices to show that

$$\|v_t\|_2^2 + \|\Delta v\|_2^2$$

is bounded independently of t. Using ((2.4), (2.8), and (2.13), we have

$$E(0) \geq E(t) = J(t) + \frac{1}{2}\|v_t\|_2^2 \geq \frac{p-2}{2p}\left(l\|\Delta v\|_2^2\right) + \frac{1}{2}\|v_t\|_2^2. \tag{2.16}$$

Therefore,

$$\|v_t\|_2^2 + \|\Delta v\|_2^2 \leq CE(0), \tag{2.17}$$

where C is a positive constant. This implies that the solution of (2.1) is bounded and global in time.

Chapter 3

General decay of solution

In this chapter, the stability of the Kirchhoff's viscous plate equation with delay and source boundary conditions is studied. The chapter begins with the assumptions and preliminary lemmas. Lyapunov functions are used to analyze the stability of the system, focusing on the evolution of the energy associated with the system and how different conditions affect the mathematical behavior of the solutions. A set of basic assumptions is provided that define the properties of the functions involved in the equation. The chapter also covers the mathematical techniques used to prove results related to the general decay of energy and the stability of the solutions. Additionally, the construction and applications of Lyapunov auxiliary functions in studying stability and energy decay in Kirchhoff's viscous plate equations with delay and source boundary conditions are discussed.

3.1 Assumptions and Preliminary Lemmas

In this section, we will give some notations, sufficient conditions, assumptions, and some preliminary lemmas needed in the proof of our main results. Firstly we have our problem :

$$\begin{cases} v_{tt}(x,t) + \Delta^2 v(x,t) - M(\|\nabla v\|^2)\Delta v(x,t) - \sigma(t)\displaystyle\int_0^t g(t-s)\Delta^2 v(x,s)\,ds \\ \quad - \mu_1 v_t(x,t) - \mu_2 v_t(x,t-\tau) = \mu|v|^{p-2}v, & x \in \Omega,\ t > 0, \\ v(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ v_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega,\ t \in (0,\tau), \end{cases} \tag{2.1}$$

Throughout this chapter, we denote

$$\|\Delta v\|_2 := \|v\|_{H^2(\Omega)},$$

and define the space:

$$H^2(\Omega) = \{v \in H^2(\Omega) \mid v|_\Omega = 0\}.$$

Let c_ϱ be the smallest positive number satisfying

$$\|\nabla v\|_2 \le c_\varrho \|\Delta v\|_2, \quad \forall v \in H^2(\Omega). \tag{3.1}$$

Now, we define the energy associated with problem (2.4) as

$$\begin{aligned} E(t) = \frac{1}{2}\|v_t\|_2^2 + +\frac{1}{2}\left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta v\|_2^2 + \frac{1}{2}\sigma(t)(g \circ \Delta v) + \frac{1}{2}\hat{M}\|\nabla v\|_2^2 \\ + \frac{\xi}{2}\int_0^1 \|z|_2^2 d\rho - \frac{\mu}{p}\|v|_p^p \end{aligned} \tag{3.2}$$

where ξ be a positive constant satisfying

$$\tau\mu_2 \le \xi \le \tau(\mu_1 - \mu_2). \tag{3.3}$$

lemma 3.1 Let v be a solution of problem (2.1) . Then

$$E'(t) \leq -c_0\Big(\|v_t\|_2^2 + \|z\|_2^2\Big) + \frac{\sigma(t)}{2}(g' \circ \Delta v)(t) - \frac{\sigma'(t)}{2}\left(\int_0^t g(s)ds\right)\|\Delta v\|_2^2. \tag{3.4}$$

Proof Multiplying the first equation in (2.4) by v_t, integrating over Ω, and using lemma (2.1), we obtain

$$\begin{aligned}\frac{d}{dt}\Bigg[&\frac{1}{2}\|v_t\|_2^2 + \frac{1}{2}\left(1 - \sigma(t)\int_0^t g(s)ds\right)\|\Delta v\|_2^2 + \frac{1}{2}\sigma(t)(g \circ \Delta v)(t) \\ &+ \frac{1}{2}\hat{M}\|\nabla v\|_2^2\Bigg] \\ &= -\frac{\sigma(t)}{2}g(t)\|\Delta v\|_2^2 + \frac{\sigma(t)}{2}(g' \circ \Delta v)(t) - \mu_1\|v_t\|_2^2 - \mu_2\int_\Omega z v_t\,dx.\end{aligned} \tag{3.5}$$

Multiplying the second equation in (2.4) by ξz and integrating over $\Omega \times (0,1)$, we obtain

$$\begin{aligned}\frac{\xi}{2\tau}\frac{d}{dt}\int_\Omega\int_0^1 |z|^2 d\rho dx &= -\frac{\xi}{2\tau}\int_\Omega\int_0^1 \frac{\partial}{\partial\rho}|z|^2 d\rho dx \\ &= \frac{\xi}{2\tau}\Big(\|v_t\|_2^2 - \|z\|_2^2\Big).\end{aligned} \tag{3.6}$$

Using Young's inequality, we have

$$-\mu_2\int_\Omega z v_t dx \leq \frac{\mu_2}{2}\|z\|_2^2 + \frac{\mu_2}{2}\|v_t\|_2^2. \tag{3.7}$$

Combining (3.5), (3.6), and (3.7), we obtain

$$E'(t) \leq -c_0\big(\|v_t\|_2^2 + \|z\|_2^2\big) + \frac{\sigma(t)}{2}(g' \circ \Delta v)(t) - \frac{\sigma(t)}{2}\left(\int_0^t g(s)ds\right)\|\Delta v\|_2^2, \tag{3.8}$$

where

$$c_0 = \min\left\{\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}, \quad \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right\}, \tag{3.9}$$

which is positive by (3.3).

Lemma 2.2 [3]

Suppose that

$$p \leq \frac{2(n-1)}{n-2}$$

holds. Then, there exists a positive constant $C > 1$ depending only on Ω such that

$$\|w\|_p^s \leq C\big(\|\nabla w\|_2^2\big)$$

for any $w \in H_0^1(\Omega)$, $2 \leq s \leq p$.

3.2 General decay

In this section, we prove the energy decay result by constructing a suitable Lyapunov functionals.

Theorem 2.3. Let $(v_0, v_1) \in H^2(\Omega) \times L^2(\Omega)$ be given. Assume that (A1) $-$ (A5) hold. Then, for each $t > 0$, there exist two positive constants K and p such that, for any solution of problem (2.1), the energy satisfies

$$E(t) \leq Ke^{-k\int_{t_0}^{t} \phi(s)\sigma(s)ds}. \tag{3.10}$$

now, we we assume that $\mu is identically zero, Under this assumption we study the general decay of energy for p$ as follows:

$$L(t) := E(t) + \varepsilon_1\sigma(t)\phi(t) + \varepsilon_2\sigma(t)\psi(t) \tag{3.11}$$

where ε_1 and ε_2 are two positive constants that will be specified later, and

$$\phi(t) = \int_\Omega v v_t \, dx. \tag{3.12}$$

$$\psi(t) = -\int_\Omega v_t \int_0^t g(t-s)(v(t) - v(s)) \, ds \, dx. \tag{3.13}$$

The following lemma expresses the equivalence relation between the two functionals $L(t)$ and $E(t)$.

Lemma 3.1 *There exist two positive constants β_1 and β_2 such that*

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t). \tag{3.14}$$

Proof We have

$$|L(t) - E(t)| = |\varepsilon_1\sigma(t)\phi(t) + \varepsilon_2\sigma(t)\psi(t)| \tag{3.15}$$

which gives

$$|L(t)-E(t)| \le \varepsilon_1|\sigma(t)|\int_\Omega |v||v_t|\,dx + \varepsilon_2|\sigma(t)|\int_\Omega |v_t|\left|\int_0^t g(t-s)(v(t)-v(s))\,ds\right|dx. \tag{3.16}$$

Using Hölder, Young, and Poincaré inequalities, we obtain:

$$\begin{aligned}|L(t)-E(t)| &\le \frac{\varepsilon_1+\varepsilon_2}{2}|\sigma(t)|\|v_t\|^2 + \frac{\varepsilon_2(1-l)}{2}|\sigma(t)|\int_0^t g(t-s)\|v(t)-v(s)\|^2 ds\\ &\le \frac{\varepsilon_1+\varepsilon_2}{2}|\sigma(0)|\|v_t\|^2 + \frac{\varepsilon_2(1-l)}{c_1^2}|\sigma(0)|c_*^2c_\varrho^2(g\circ\Delta v)(t)\\ &\le c(\varepsilon_1,\varepsilon_2)E(t)\end{aligned} \tag{3.17}$$

where

$$c(\varepsilon_1,\varepsilon_2) = \max\left\{\frac{\varepsilon_1+\varepsilon_2}{2}|\sigma(0)| + \frac{\varepsilon_2(1-l)}{c_1^2}|\sigma(0)|c_*^2c_\varrho^2\right\}.$$

Thus, from (3.14), we can obtain (3.17) by choosing sufficiently small ε_1 and ε_2.

Lemma 3.2. The functional $\phi(t)$ defined in (3.12) satisfies:

$$\phi'(t) \le (1+\varepsilon)\|v_t\|_2^2 - c_\varepsilon\|\Delta v\|_2^2 + \frac{1}{4}\sigma(t)(g\circ\Delta v)(t) + \varepsilon\|z(1,t)\|_2^2 + \|v\|_p^p \tag{3.18}$$

Proof Differentiating (3.12) with respect to t and using equation (2.4), we get:

$$\begin{aligned}\phi'(t) &= \|v_t\|_2^2 - M(\|\nabla v\|_2^2)\|\nabla v\|_2^2 - \|\Delta v\|_2^2\\ &\quad + \sigma(t)\int_\Omega\int_0^t g(t-s)(\Delta v(s)-\Delta v(t))\,ds\,dx - \mu_1\int_\Omega vv_t\,dx - \mu_2\int_\Omega zv\,dx + \mu\|v\|^2 + \|v\|_p^p\\ &= \|v_t\|_2^2 - M(\|\nabla v\|_2^2)\|\nabla v\|_2^2 - \|\Delta v\|_2^2 + \mu\|v\|^2 + I_1 + I_2 + I_3.\end{aligned} \tag{3.19}$$

Applying Hölder, Young, Sobolev-Poincaré inequalities, and relation (3.1), we estimate I_1, I_2, and I_3 as follows:

$$I_1 = \int_\Omega\int_0^t g(t-s)(\Delta v(s)-\Delta v(t))\,ds\,dx \le 2\left(\int_0^t g(s)\,ds\right)\|\Delta v\|_2^2 + \frac{1}{4}(g\circ\Delta v)(t). \tag{3.20}$$

$$I_2 = \mu_1\int_\Omega vv_t\,dx \le \varepsilon\|v_t\|^2 + \frac{\mu_1^2c_*^2c_\varrho^2}{4\varepsilon}\|\Delta v\|_2^2, \tag{3.21}$$

$$I_3 = \mu_2\int_\Omega vv_t\,dx \le \varepsilon\|v_t\|^2 + \frac{\mu_2^2c_*^2c_\varrho^2}{4\varepsilon}\|\Delta v\|_2^2, \tag{3.22}$$

Combining inequalities (3.20)-(3.21), and (3.22), we obtain:

$$\phi'(t) \leq (1+\varepsilon)\|v_t\|_2^2 - c_\varepsilon\|\Delta v\|_2^2 + \frac{1}{4}\sigma(t)(g\circ\Delta v)(t) + \varepsilon\|z(1,t)\|_2^2$$

Lemma 3.3 The functional $\psi(t)$ defined in (3.13) satisfies:

$$\begin{aligned}\psi'(t) \leq & \left(w + M_0 c_\rho^2 w + 2w(1-l)^2\right)\|\Delta v\|_2^2 \\ & + \left(\frac{1-l}{4w} + \frac{M_0(1-l)c_\rho^2}{4w} + (2w+\frac{1}{4w})(1-l) + \frac{2(1-l)}{4w}c_\rho^2\right)(g\circ\Delta v)(t) \\ & + (2w - \int_0^t g(s)ds)\|v_t\|_2^2 + \frac{g(0)c_\rho^2}{4w}(g'\circ\Delta v)(t) + w\|v_t\|_2^2 + w\|z\|_2^2 + \|v\|_p^p\end{aligned} \tag{3.23}$$

Proof We differentiate the functional $\psi(t)$ defined in (3.13) with respect to t using equation (2.4), obtaining:

$$\begin{aligned}\psi'(t) = & -\int_\Omega v_{tt}\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & -\int_\Omega v_t\int_0^t g'(t-s)(v(t)-v(s))ds\,dx - \left(\int_0^t g(s)ds\right)\|v_t\|_2^2 \\ = & \int_\Omega \Delta^2 v\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & - M(\|\nabla v\|_a^2)\int_\Omega \Delta v\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & - \delta(t)\int_\Omega\int_0^t g(t-s)\Delta^2 v(s)ds\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & + \mu_1\int_\Omega v_t\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & + \mu_2\int_\Omega v_t(x,t-\tau)\int_0^t g(t-s)(v(t)-v(s))ds\,dx \\ & -\int_\Omega v_t\int_0^t g'(t-s)(v(t)-v(s))ds\,dx - \left(\int_0^t g(s)ds\right)\|v_t\|_2^2 \\ = & I_1 + I_2 + \cdots + I_6 - \left(\int_0^t g(s)ds\right)\|v_t\|_2^2.\end{aligned} \tag{3.24}$$

In what follows, we estimate all the terms in equation (3.24).

Starting with the first term, using Hölder and Young inequalities, condition (A1), we obtain:

$$I_1 = \int_\Omega \Delta^2 v \int_0^t g(t-s)(v(t)-v(s))ds\,dx \leq w\|\Delta v\|_2^2 + \frac{1-l}{4w}(g\circ\Delta v)(t) \tag{3.25}$$

$$\begin{aligned} I_2(t) =& \sigma(t)\int_\Omega \int_0^t g(t-s)\Delta^2 v(s)\,ds \int_0^t g(t-s)(v(t)-v(s))\,ds\,dx && (3.20)\\ \leq& 2w(w-l)^2\sigma(t)\|\Delta v\|_2^2 + \left(2w+\frac{1}{4w}\right)(1-l)\sigma(t)\,(g\circ\Delta v)\,, && (3.26)\end{aligned}$$

Now, since $\|\nabla v\|_2^2 \leq \dfrac{2E(0)}{lc_\varrho^2}$, for all $t \geq 0$, taking

$$M_0 = \sup\left\{M(s),\ 0 < s \leq \frac{2E(0)}{lc_\varrho^2}\right\},$$

we see that $M\left(\|\nabla v\|_2^2\right) \leq M_0 < \infty$. Then, we have

$$\begin{aligned} I_3 &= M\left(\|\nabla v\|_2^2\right)\int_\Omega \Delta v \int_0^t g(t-s)(v(t)-v(s))\,ds\,dx \\ &\leq M_0 w c_\varrho^2\|\Delta v\|_2^2 + \frac{M_0(1-l)c_\varrho^2}{4w}(g\circ\Delta v)(t) \end{aligned} \tag{3.27}$$

Again, using Hölder and Young inequalities, condition (A1), we obtain:

$$I_4 = \mu_1\int_\Omega v_t \int_0^t g(t-s)(v(t)-v(s))ds\,dx \leq w\|v_t\|_2^2 + \frac{(1-l)\mu_1^2 c_\varrho^2 c_*^2}{4w}(g\circ\Delta v) \tag{3.28}$$

$$I_5 = \int_\Omega z(1,t)\int_0^t g(t-s)(v(t)-v(s))ds\,dx \leq w\|z(1,t)\|_2^2 + \frac{(1-l)\mu_2^2 c_\varrho^2 c_*^2}{4w}(g\circ\Delta v) \tag{3.29}$$

Similarly, for I_6, we find:

$$I_6 = \int_\Omega v_t \int_0^t g'(t-s)(v(t)-v(s))ds\,dx \leq w\|v_t\|_2^2 - \frac{g(0)c_\varrho^2 c_*^2}{4w}(g'\circ\Delta v)(t). \tag{3.30}$$

By combining the estimates (3.25)–(3.30), inequality (3.24) becomes:

$$\begin{aligned}\psi'(t) \leq & -\left(\int_0^t g(s)ds - 2w\right)\|v_t\|_2^2 + \left[w + M_0 w c_\varrho^2 + 2w(1-l)^2\sigma(t)\right]\|\Delta v\|_2^2 \\ & + (1-l)\left[\frac{1 + M_0 c_\varrho^2 + (\mu_1^2+\mu_2^2)c_\varrho^2 c_*^2}{4w} + \left(2w + \frac{1}{4w}\right)\sigma(t)\right](g\circ\Delta v)(t) \\ & + \frac{g(0)c_\varrho^2}{4w}(g'\circ\Delta v)(t) + w\|z\|_2^2 + \|v\|_p^p\end{aligned} \tag{3.31}$$

Lemma 3.4. The function $L(t)$ defined in (3.11) satisfies :

$$L'(t) \leq -K_0\sigma(t)E(t) + K_1\sigma(t)(g\circ\Delta v)(t), \quad t > t_0, \tag{3.32}$$

where K_0 and K_1 are some positive constants.

Proof First, since the function g is a positive and continuous function, for any $t_0 > 0$, we have

$$\int_0^t g(s)\,ds \geq \int_0^{t_0} g(s)\,ds := g_0, \quad \text{for all } t \geq t_0. \tag{3.33}$$

Differentiating (3.11) and using Lemmas(3.2) ; (3.2) , we get

$$\begin{aligned}L'(t) &= E'(t) + \varepsilon_1\sigma'(t)\varphi(t) + \varepsilon_1\sigma(t)\varphi'(t) + \varepsilon_2\sigma'(t)\psi(t) + \varepsilon_2\sigma(t)\psi'(t) \\ &\leq E'(t) + \varepsilon_1\sigma(t)\left[(1+\varepsilon)\|v_t\|^2 - c_\varrho\|\Delta v\|^2 + \frac{\sigma(t)}{4}(g\circ\Delta v)(t) + \varepsilon\|\mathfrak{Z}(1,t)\|_2^2\right] \\ &\quad + (1-l)\left[\frac{1 + M_0 c_\varrho^2 + (\mu_1^2+\mu_2^2)c_\varrho^2 c_*^2}{4w} + (2w + \frac{1}{4w})\sigma(t)\right](g\circ\Delta v)(t) \\ &\quad + \varepsilon_2\sigma(t)\left[w + M_0 w c_\varrho^2 + 2w(1-l)^2\sigma(t)\right]\|\Delta w\|^2 + w\sigma(t)\|\mathfrak{Z}(1,t)\|_2^2 \\ &\quad + \varepsilon_2\sigma(t)(2w - \int_0^t g(s)ds)\|v_t\|^2 + \varepsilon_2\sigma(t)\frac{g(0)c_\varrho^2}{4w}(g'\circ\Delta v)(t) \\ &\quad + \varepsilon_1\sigma'(t)\int_\Omega v v_t dx - \varepsilon_2\sigma'(t)\int_\Omega v_t\int_0^t g(t-s)(v(t)-v(s))dsdx + \sigma(t)\mu(\varepsilon_1+\varepsilon_2)\|v\|_p^p.\end{aligned} \tag{3.34}$$

On the other hand, we have

$$\sigma'(t)\int_\Omega v v_t\,dx \leq -\frac{\sigma'(t)}{2}\|v_t\|^2 - \frac{\sigma'(t)}{2}c_\varrho^2 c_*^2\|\Delta v\|_2^2. \tag{3.35}$$

and

$$-\sigma'(t)\int_\Omega v_t\int_0^t g(t-s)(v(t)-v(s))\,ds\,dx \le -\frac{\sigma'(t)}{2}\|v_t\|^2 - \frac{\sigma'(t)}{2c_\varrho^2 c_*^2}\left(\int_0^t g(s)\,ds\right)(g\circ\Delta v)(t). \tag{3.36}$$

Consequently, inequality (3.34) becomes:

$$\begin{aligned} L'(t) \le & -\sigma(t)\left[\frac{\sigma'(t)}{2\sigma(t)}(\varepsilon_1+\varepsilon_2)+\frac{c_0}{\sigma(0)}-\varepsilon_1(1+w)+\varepsilon_2(g_0-2w)\right]\|v_t\|^2 \\ & -\sigma(t)\left[\varepsilon_1 c_\varepsilon+\varepsilon_1\frac{\sigma'(t)}{2\sigma(t)}-\varepsilon_2\left(w+M_0wc_\varrho^2+2w(1-l)^2\sigma(t)\right)\right]\|\Delta v\|^2 \\ & +\sigma(t)\left[\frac{\varepsilon_1\sigma(t)}{4}-\frac{\sigma'(t)}{2\sigma(t)}\varepsilon_2 c_\varrho^2 c_*^2 g_0+\varepsilon_2(1-l)\left(\frac{c_1}{w}+(2w+\frac{1}{4w})\sigma(t)\right)\right](g\circ\Delta v)(t) \\ & -\left[\frac{c_0}{\sigma(0)}-\varepsilon_1\varepsilon-\varepsilon_2 w\right]\|z(1,t)\|^2+\sigma(t)\left[\frac{1}{2}-\frac{g(0)c_\varrho^2}{4w}\right](g'\circ\Delta v)(t)+\sigma(t)\mu(\varepsilon_1+\varepsilon_2)|v|_p^p. \end{aligned} \tag{3.37}$$

now, we define the constants:

$$\begin{aligned} \beta_1 &= -\varepsilon_1(1+w)-\varepsilon_2(2w-\int_0^t g(s)\,ds)>0, \\ \beta_2 &= +\varepsilon_2 c_w-\varepsilon_2 w-M_0c_\varrho^2 w-2w(1-l)^2\sigma(t)>0, \\ \beta_3 &= \varepsilon_2\left(\frac{1-l}{4w}+\frac{M_0(1-l)c_\varrho^2}{4w}+(2w+\frac{1}{4w})(1-l)\sigma(t)+\frac{2(1-l)c_\varrho^2}{4w}\right)>0. \end{aligned}$$

Further, we pick ε_1 and ε_2 so small that (3.14) we remain valid

$$\begin{aligned} \beta_4 &= \frac{g(0)c_\varrho^2\varepsilon_2}{4w}\sigma(t), \\ \beta_5 &= \alpha-\varepsilon_1 w\sigma(t)-w\varepsilon_2\sigma(t)>0. \end{aligned}$$

$\beta_6=\mu(\varepsilon_1+\varepsilon_2)$

Therefore, (3.34) becomes:

$$\begin{aligned} L'(t) \le & -\sigma(t)\left(\beta_1+\frac{\sigma'(t)}{2\varepsilon_1\sigma(t)}-\frac{\sigma'(t)}{2\varepsilon_2\sigma(t)}\right)\|v_t\|^2 \\ & -\sigma(t)\left(\beta_2+\frac{\sigma'(t)}{2\varepsilon_1 c_\varrho^2}\right)\|\Delta v\|^2 \\ & +\sigma(t)\left(\beta_3-\frac{\sigma'(t)}{2\sigma(t)c_\varrho^2}\varepsilon_2\right)(g\circ\Delta v)(t)+\beta_6\sigma(t)\|v\|_p^p \end{aligned} \tag{3.38}$$

Since

$$\lim_{t\to\infty} \frac{-\sigma'(t)}{\sigma(t)} = 0 \tag{3.39}$$

We can choose $t_1 > t_0$ so that (3.38) takes the form :

$$\begin{aligned} L'(t) &\leq -\sigma(t)\left(\beta_1\|v_t\|_2^2 - \beta_2\|\Delta v\|_2^2 +) + \beta_6\|v\|_p^p\right) + \beta_3\sigma(t)(g\circ\Delta v)(t) \\ &\leq -K_0\sigma(t)E(t) + K_1\sigma(t)(g\circ\Delta v)(t), \quad \forall t_1 > t_0, \end{aligned} \tag{3.40}$$

where k_i, $i = 0, 1$ are some positive constants. This completes the proof.

Proof of Theorem 3.2

Multiplying (3.32) by $\xi(t)$, we have:

$$L'(t) \leq -K_0\sigma(t)E(t) + K_1\sigma(t)(g\circ\Delta v)(t) \tag{3.41}$$

Since assumption (A2) holds and using the fact that $-\sigma(t)(g'\circ\Delta v)(t) \leq -2E(t)$ by 3.2, we obtain

$$\begin{aligned} \phi(t)L'(t) &\leq -K_0\sigma(t)\phi(t)E(t) - K_1\sigma(t)(g'\circ\Delta v)(t) \\ &\leq -K_0\sigma(t)\phi(t)E(t) - K_1\left(2E'(t) + \sigma'(t)\left(\int_0^t g(s)\,ds\right)\|\Delta v\|^2\right) \end{aligned} \tag{3.42}$$

From the definition of $E(t)$ and assumption (A2), we know that $l\|\nabla v\|^2 \leq E(t)$, so we get:

$$\begin{aligned} (\phi(t)L(t) + 2P_1E(t)) &\leq -K_0\sigma(t)\phi(t)E(t) - K_1\sigma'(t)\left(\int_0^t g(s)\,ds\right)\|\Delta v\|^2 \\ &\leq -K_0\sigma(t)\phi(t)E(t) - 2lE(t)\sigma'(t) \\ &\leq -\sigma(t)\phi(t)\left(K_0 + \frac{2l\sigma'(t)}{\sigma(t)\phi(t)}\right)E(t) \end{aligned} \tag{3.43}$$

Since

$$\lim_{t\to\infty} \frac{-\sigma'(t)}{\sigma(t)\phi(t)} = 0, \tag{3.43}$$

we can choose $t_1 \geq t_0$ such that

$$K_0 + \frac{2l\sigma'(t)}{\sigma(t)\phi(t)} \quad \text{for } t \geq t_1. \tag{3.44}$$

Finally, let

$$R(t) = \phi(t)L(t) + 2K_1E(t), \tag{3.45}$$

then $L(t)$ is equivalent to $E(t)$, and we arrive at

$$R'(t) \leq -K\phi(t)\sigma(t)R(t) \quad \text{for } t \geq t_1. \tag{3.46}$$

Integrating (3.46) over (t_1, t) with respect to t, we get

$$R(t) \leq R(t_1) \exp\left(-K \int_{t_1}^{t} \phi(s)\sigma(s)\, ds\right), \quad t \geq t_0. \tag{3.47}$$

Consequently, the equivalence relations among $R(t)$, $L(t)$ and $E(t)$ yield the desired result.

$$E(t) \leq Ke^{-k\int_{t_0}^{t} \phi(s)\sigma(s)ds}. \tag{3.48}$$

General Conclusion and Perspectives

In conclusion, this study has examined the global existence and general decay of energy for a viscoelastic Kirchhoff-type plate equation with memory and delay effects, under classical initial and boundary conditions. By employing energy estimates and constructing suitable Lyapunov functionals, we demonstrated that the total energy of the system decays over time, highlighting the stabilizing influence of both memory and delay terms on the system's energy. These findings contribute to a deeper understanding of the asymptotic behavior of such systems and emphasize the importance of combining damping and memory mechanisms to maintain stability. Future work could focus on extending the model to include nonlinear or nonlocal boundary conditions, exploring systems with time-varying delays, or conducting numerical simulations to validate the theoretical results.

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